RECURRENCE RATE IN RAPIDLY MIXING DYNAMICAL SYSTEMS

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ABSTRACT. For measure preserving dynamical systems on metric spaces we study the time needed by a typical orbit to return back close to its starting point. We prove that when the decay of correlation is super-polynomial the recurrence rates and the pointwise dimensions are equal. This gives a broad class of systems for which the recurrence rate equals the Hausdorff dimension of the invariant measure.

1. Introduction

1.1. **Decay of correlations.** Let (X, f, μ) be a measure preserving dynamical system. Recall that the system is said to be *mixing* if for any functions φ, ψ in L^2 the covariance

$$\mathrm{Cov}(\varphi \circ f^n, \psi) := \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \to 0 \quad \text{as } n \to \infty. \tag{1}$$

The decay of the correlation function is, in great generality, arbitrarily slow. The notion of rapid mixing needs a little more structure.

Assume that X is a metric space with metric d, and consider the space $\mathrm{Lip}(X)$ of real Lipschitz functions on X. For many dynamical systems an upper bound for (1) of the form $\|\varphi\|\|\psi\|\theta_n$ has been computed, where $\theta_n \to 0$ with some rate, and $\|\cdot\|$ is a norm on a space of functions with some regularity. Without loss of generality we are considering in this paper the rate of decay of correlations for Lipschitz observables[1].

A broad class of systems enjoy exponential decay of correlations. The main result of the paper (Theorem 3) applies to systems with super-polynomial decay of correlation. This includes for example Axiom A systems with equilibrium states, hyperbolic systems with singularities with their SBR measures such as those considered by Chernov in [7], many systems with a Young tower [17, 18], expanding maps with singularities such as in [14], some non-uniformly expanding maps [1], etc. The main reference for these questions is certainly the book by Baladi [2]. The reader will also find in the review by Luzzatto [12] an exposition of the recent methods for non-uniformly expanding systems and an extensive bibliography on this active field.

1.2. **Recurrence rate and dimensions.** The return time of a point $x \in X$ under the map f in its r-neighborhood is

$$\tau_r(x) = \inf\{n \geq 1 \colon d(f^n x, x) < r\}.$$

We are interested in the behavior as $r \to 0$ of the return time. We define the lower and upper recurrence rate as the limits

$$\underline{R}(x) = \liminf_{r\to 0} \frac{\log \tau_r(x)}{\log(1/r)} \quad \text{and} \quad \overline{R}(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{\log(1/r)}.$$

[1]For example an immediate approximation argument allows easily to go from Hölder or class C^k to Lipschitz.

Whenever $\underline{R}(x) = \overline{R}(x)$ we denote by $R(x)$ the value of the limit.

From now on we assume that X is a Borel subset of a finite dimensional Euclidean space E. Denote by $HD(Y)$ the Hausdorff dimension of a set $Y \subset X$. We define the Hausdorff dimension of a Borel probability measure μ by

$$HD(\mu) = \inf\{HD(Y) \colon Y \text{ Borel set s.t. } \mu(Y) = 1\}$$

We also define a local version of the dimension, namely

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{2}$$

It is well known that the Hausdorff dimension satisfies the relation

$$HD(\mu) = \text{ess-sup } \underline{d}_\mu. \tag{3}$$

Barreira and Saussol established in [4] the following relation

Proposition 1. *Let f be a measurable map and μ be an invariant measure for f. The recurrence rates are bounded from above by the pointwise dimensions :*

$$\underline{R} \le \underline{d}_\mu \quad \textit{and} \quad \overline{R} \le \overline{d}_\mu \quad \mu\textit{-a.e.}$$

We refer to the works by Boshernitzan [6] and Ornstein and Weiss [13] for pioneering related results.

In this paper we are giving conditions under which the opposite inequalities will hold, establishing the equalities

$$\underline{R} = \underline{d}_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\text{-a.e.} \tag{4}$$

1.3. Statement of the results.

Definition 2. *We say that (X, f, μ) has super-polynomial decay of correlations if we have*

$$\left| \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \le \|\varphi\| \|\psi\| \theta_n \tag{5}$$

with $\lim_n \theta_n n^p = 0$ for all $p > 0$, where $\|\cdot\|$ is the Lipschitz norm.

The main result of the paper is the following.

Theorem 3. *Let (X, f, μ) be a measure preserving dynamical system. If the entropy $h_\mu(f) > 0$, f is Lipschitz (or piecewise Lipschitz with some condition ; see Lemma 13) and the decay of correlation is super-polynomial then*

$$\underline{R} = \underline{d}_\mu \quad \textit{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\textit{-a.e.}$$

We postpone the proof at the end of Section 3. This extends some results by Barreira and Saussol in [4, 5], including the case of Axiom A systems with equilibrium states. The hypotheses in Theorem 3 are satisfied in a number of systems such as those already quoted in the introduction. All these systems have in common some hyperbolic behavior. We now give an example of a relatively different nature, due to the possibility of zero Lyapunov exponents, where one can still apply Theorem 3.

Example 4 (Ergodic toral automorphisms)**.** *Recall that any matrix* $A \in Sl(k, \mathbb{Z})$ *(i.e. the entries of* A *are in* $\mathbb{Z}$ *and* $|\det A| = 1$*) gives rise to an automorphism* f *of the torus* $\mathbb{T}^k$ *by* $f(x) = Ax \mod \mathbb{Z}^k$ *which preserves the Lebesgue measure. The map* f *is ergodic if and only if the matrix* A *has no eigenvalue root of unity. Lind's established* [11] *the exponential decay of correlations (using the algebraic nature and Fourier transform) which is more than enough to apply Theorem 3 and get*

$$R(x) = k \quad \textit{for Lebesgue a.e. } x \in \mathbb{T}^k.$$

for any ergodic automorphism of the torus, even non-hyperbolic*.*

Let f be a diffeomorphism of a compact manifold M and μ be an ergodic invariant measure. By Oseledec's multiplicative ergodic Theorem the Lyapunov exponents

$$\lambda(x, v) = \lim \frac{1}{n} \log |d_x f^n v|$$

are well defined for μ-a.e. $x \in M$ and all nonzero $v \in T_x M$ and take a finite number of values called the Lyapunov exponents of the measure μ. Recall that a measure μ is said to be hyperbolic if none of its Lyapunov exponents are zero. Barreira, Pesin and Schmeling [3] prove the following.

Proposition 5. *Let* f *be a diffeomorphism of a compact manifold and* μ *be an ergodic hyperbolic measure. Then we have*

$$\underline{d}_\mu = \overline{d}_\mu = HD(\mu) \quad \mu\textit{-a.e.}$$

The case of an hyperbolic measure with zero entropy is completely understood.

Proposition 6. *let* f *be a diffeomorphism of a compact manifold and* μ *be an hyperbolic invariant measure. If* $h_\mu(f) = 0$ *then* $R = 0 = HD(\mu)$ μ*-a.e.*

Proof. Barreira and Saussol established in [4] the inequality $\overline{R} \le \overline{d}_\mu$ μ-a.e. and it follows from Ledrappier and Young's work [10] that $HD(\mu) = 0$ if $h_\mu(f) = 0$, which allows to conclude by Proposition 5. □

Corollary 7. *Let* f *be a diffeomorphism of a compact manifold and* μ *be an hyperbolic measure with super-polynomial rate of decay of correlation. Then we have*

$$R = HD(\mu) \quad \mu\textit{-a.e.}$$

Proof. If the entropy is zero then this is the content of Proposition 6. If the entropy is non-zero then this is the content of Theorem 3. □

We point out that in the case of interval maps with nonzero Lyapunov exponent, Saussol, Troubetzkoy and Vaienti prove that $R = HD(\mu)$ μ-a.e. for ergodic measures, under very weak regularity conditions [15]. See Remark 14-(i) for related results.

We now give a sketch of the strategy adopted in this paper.

Theorem 8 states that under sufficiently rapid mixing the recurrence rates equal the pointwise dimensions a.e. on the set where $\underline{R} > 0$. Indeed, mixing implies that $\mu(B \cap f^{-n}B) = O(\mu(B)^2)$ for large n. If now we consider the set $B \cap \left(f^{-n}B \cup f^{-n-1}B \cup \cdots \cup f^{-n-\ell}B\right)$ then its measure is bounded by $O(\ell\mu(B)^2)$. If $\ell \le \mu(B)^{-1+\varepsilon}$ then we get that the proportion of points inside

B that enter in B in the time interval $[n, n+\ell]$ is bounded by $O(\mu(B)^\varepsilon)$. Using the decay of correlations we are able to prove that this last statement is true for n of the order $\mathrm{diam}(B)^{-\delta}$ for some small $\delta > 0$, whenever B is a ball. A Borel Cantelli argument then shows that typical points do not enter again in the ball B in the time interval $[\mathrm{diam}(B)^{-\delta}, \mu(B)^{-1+\varepsilon}]$ (see Lemma 9 for precise statement). This is what we call the *long flight property.*

In addition, for systems which are not too wild (e.g. finite Lyapunov exponents, see Lemma 13) and with nonzero metric entropy, a symbolic coding (see Lemma 12) allows to use Ornstein-Weiss' theorem on repetition time of symbolic sequences to prove that the return time of a typical point in a ball B is larger than $\mathrm{diam}(B)^{-\delta}$; see Lemma 11. This together with the long flight property establishes Equation (4).

The structure of the paper is as follows. We state and prove in Section 2 the core result, Theorem 8. In Section 3 we provide some conditions under which the recurrence rate is nonzero.

2. Rapid mixing implies long flights

Theorem 8. *Assume that the rate of decay of correlations is super-polynomial. Then on the set* $\{\underline{R} > 0\}$ *we have*

$$\underline{R} = \underline{d}_\mu \quad \textit{and} \quad \overline{R} = \overline{d}_\mu \quad \mu\textit{-a.e.}$$

Proof. By Proposition 1 we know that $\underline{R} \le \underline{d}_\mu$ and $\overline{R} \le \overline{d}_\mu$. Furthermore, the first inequality implies that for any $a > 0$ we have $\{\underline{R} > a\} \subset \{\underline{d}_\mu > a\}$ μ-a.e. But on the set $\{\underline{R} > a\}$ we have $\tau_r(x) \ge r^{-a}$ provided r is sufficiently small. By Lemma 9 below with $\delta = a$ and $\varepsilon > 0$ we get that $\tau_r(x) \ge \mu(B(x,r))^{-1+\varepsilon}$ provided r is sufficiently small, for μ-a.e. $x \in \{\underline{R} > a\}$. Thus $\underline{R} \ge (1-\varepsilon)\underline{d}_\mu$ and $\overline{R} \ge (1-\varepsilon)\overline{d}_\mu$ μ-a.e. on $\{\underline{R} > a\}$. The conclusion follows by taking $\varepsilon > 0$ arbitrary small. □

The following lemma expresses that the orbit of a typical point has the long flight property.

Lemma 9. *Let* $X_a = \{\underline{d}_\mu > a\}$ *for some* $a > 0$. *For any* $\delta, \varepsilon > 0$, *for* μ*-a.e.* $x \in X_a$ *there exists* $r(x) > 0$ *such that for any* $r \in (0, r(x))$ *and any integer* n *in* $[r^{-\delta}, \mu(B(x,r))^{-1+\varepsilon}]$ *we have* $d(f^n x, x) \ge r$.

Proof. Let D be the dimension of the Euclidean space $E \supset X$. Fix $b > 0$, $c = a\varepsilon/2$ and consider for $r_0 > 0$ the set $G = G_1 \cap G_2 \cap G_3$ where

$$\begin{aligned} G_1 &= \{x \in X_a \colon \forall r \le r_0, \mu(B(x,2r)) \le r^a\} \\ G_2 &= \{x \in X \colon \forall r \le r_0, \mu(B(x,r/2)) \ge r^{D+b}\} \\ G_3 &= \{x \in X \colon \forall r \le r_0, \mu(B(x,r/2)) \ge \mu(B(x,4r)) r^c\}. \end{aligned}$$

We claim that $\mu(G) \to \mu(X_a)$ as $r_0 \to 0$. Indeed, by definition of the lower pointwise dimension we have $\mu(G_1) \to \mu(X_a)$. In addition since $\overline{d}_\mu \le D$ a.e. we have $\mu(G_2) \to 1$ and since E is Euclidean the measure μ is weakly diametrically regular (see Lemma 1 in [4]), thus $\mu(G_3) \to 1$ as well. Let $r \le r_0$ and define the set

$$A_\varepsilon(r) = \{y \in X \colon \exists n \in [r^{-\delta}, \mu(B(y,3r))^{-1+\varepsilon}], d(f^n y, y) < r\}.$$

Let $x \in G$. By the triangle inequality we get the inclusion

$$B(x,r)\cap A_\varepsilon(r) \subset \{y \in B(x,r)\colon \exists n \in [r^{-\delta}, \mu(B(x,2r))^{-1+\varepsilon}], d(f^n y, x) < 2r\}$$
$$= \bigcup_{r^{-\delta}\le n\le \mu(B(x,2r))^{-1+\varepsilon}} B(x,r)\cap f^{-n}B(x,2r).$$

Let $\eta_r\colon [0,\infty)\to\mathbb{R}$ be the r^{-1}-Lipschitz map such that $1_{[0,r]}\le \eta_r \le 1_{[0,2r]}$ and set $\varphi_{x,r}(y) = \eta_r(d(x,y))$. Clearly $\varphi_{x,r}$ is also r^{-1}-Lipschitz. By the assumption on the decay of correlation we obtain

$$\begin{aligned}\mu(B(x,r)\cap f^{-n}B(x,2r)) &\le \int \varphi_{x,r}\varphi_{x,2r}\circ f^n d\mu\\ &\le \|\varphi_{x,r}\|\|\varphi_{x,2r}\|\theta_n + \int \varphi_{x,r}d\mu\int\varphi_{x,2r}d\mu\\ &\le r^{-2}\theta_n + \mu(B(x,2r))\mu(B(x,4r)).\end{aligned}$$

Choose $p>1$ such that $\delta(p-1)-2\ge D+2b$ and take r_0 so small that $n\ge r_0^{-\delta}$ implies $\theta_n\le (p-1)(n+1)^{-p}$. Since $\sum_{n\ge q} n^{-p}\le \frac{1}{p-1}(q-1)^{1-p}$ we obtain for $r\in(0,r_0)$

$$\begin{aligned}\mu(B(x,r)\cap A_\varepsilon(r)) &\le r^{\delta(p-1)-2} + \mu(B(x,2r))^\varepsilon\mu(B(x,4r))\\ &\le \mu(B(x,r/2))\left(r^b + r^{\varepsilon a - c}\right).\end{aligned}$$

Let $B\subset G$ be a maximal r-separated set[2]. Since $(B(x,r))_{x\in B}$ covers G we have

$$\begin{aligned}\mu(G\cap A_\varepsilon(r)) &\le \sum_{x\in B}\mu(B(x,r)\cap A_\varepsilon(r))\\ &\le \sum_{x\in B}\mu(B(x,r/2))(r^b + r^{\varepsilon a-c})\\ &\le r^b + r^{\varepsilon a - c}\end{aligned}$$

since the balls $(B(x,r/2))_{x\in B}$ are disjoints. This implies that

$$\sum_m \mu(G\cap A_\varepsilon(\mathrm{e}^{-m}))<\infty,$$

thus by Borel-Cantelli Lemma we obtain that for μ-a.e. $y\in G$ there exists $m(y)$ such that for every $m\ge m(y)$ we have $y\notin A_\varepsilon(\mathrm{e}^{-m})$. For any r sufficiently small (i.e. $r\le \mathrm{e}^{-m(y)}$), taking m such that $\mathrm{e}^{-m-1}<r\le \mathrm{e}^{-m}$ implies that $\mathrm{e}^{\delta m}\le r^{-\delta}$ and $3\mathrm{e}^{-m}<3\mathrm{e}r$ hence there exists no $n\in[r^{-\delta},\mu(B(y,3\mathrm{e}r))^{-1+\varepsilon}]$ such that $d(f^n y, y)<r$. By weak diametric regularity the factor $3\mathrm{e}$ in the radius is irrelevant and this proves the lemma. □

Remark 10. *Observe that we only use that the decay of correlation is at least n^{-p} for some $p>\frac{D+2}{\delta}+1$. If in addition (5) holds with the first norm $\|\varphi\|$ taken to be the $L^1(\mu)$ norm (e.g. expanding maps) then $p>\frac{D+1}{\delta}+1$ suffices.*

[2]that is if $x\ne x'\in B$ then $d(x,x')\ge r$ and maximal in the sense that for any $y\in G$ there exists $x\in B$ such that $d(x,y)<r$.

3. Non-zero recurrence rate

We proceed now to find conditions under which the recurrence rate does not vanish. Denote by $\xi(x)$ the unique element of a partition ξ containing the point x and by $\xi^n = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n+1}\xi$ the dynamical partition, for any integer n .

3.1. **Coding by symbolic systems.** Next lemma, which proof is fairly simple, is the key-observation which gives to Theorem 8 all its interest.

Lemma 11. *Assume that ξ is a partition such that*

> ***[Large interior property]*** *for μ-a.e. x there exists $\chi = \chi(x) < \infty$ such that $B(x, \mathrm{e}^{-\chi n}) \subset \xi^n(x)$ for all n sufficiently large.*

If furthermore the entropy $h_\mu(f, \xi) > 0$ then $\underline{R} > 0$ μ-a.e.

Proof. Let ξ be such a partition. Define

$$R_n(x, \xi) = \min\{k > 0 \colon f^k x \in \xi^n(x)\}.$$

Ornstein and Weiss [13] prove that if ξ is a finite partition with entropy $h_\mu(f, \xi)$ then

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(x, \xi) = h_\mu(f, \xi) \quad \mu\text{-a.e.}$$

Since ξ has large interior, for μ-a.e. $x \in X$ there exists a number $\chi = \chi(x)$ such that $B(x, e^{-\chi n}) \subset \xi^n(x)$. Thus[3]

$$\underline{R}(x) = \liminf_{n\to\infty} \frac{\log \tau_{\mathrm{e}^{-n\chi(x)}}(x)}{n\chi(x)} \geq \liminf_{n\to\infty} \frac{\log R_n(x, \xi)}{n\chi(x)} = \frac{h_\mu(f, \xi)}{\chi(x)} > 0 \quad \mu\text{-a.e.}$$

□

Combining Lemma 11 and Theorem 8 we get that if we have super-polynomial decay of correlations and a partition of positive entropy with large interior then $\underline{R} = \underline{d}_\mu$ and $\overline{R} = \overline{d}_\mu$. This includes for example the case of loosely Markov dynamical systems and we recover Urbanski's result in [16][4]. The rest of the section consists in finding sufficient conditions for the existence of such a partition.

3.2. **Reasonable dependence on initial condition.**

Lemma 12. *Assume that the system (X, f, μ) satisfies the following condition*

> ***[Reasonable sensitivity]*** *for μ-a.e. x there exists $\gamma, \lambda > 0$ such that f^n is $\mathrm{e}^{\lambda n}$-Lipschitz on the ball $B(x, \mathrm{e}^{-\gamma n})$ for all n sufficiently large.*

If furthermore the entropy $h_\mu(f) > 0$ then there exists a partition which satisfies the conditions in Lemma 11 (i.e. large interior and nonzero entropy).

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[3] Note that by monotonicity of τ_r the liminf may be attained by any subsequence of the form $\mathrm{e}^{-\chi n}$.

[4] Strictly speaking, these systems may only enjoy a *local* super-polynomial decay of correlations, in the sense that there exists a partition (modulo μ) into open sets V_i and constants θ_n^i such that (5) holds whenever $\operatorname{supp} \varphi \subset V_i$ and $\operatorname{supp} \psi \subset V_i$, where $\lim_n \theta_n^i n^p = 0$ for all $p > 0$. It is clear from the proof of Lemma 9 that this local property suffices.

Proof. **Claim :** For any $x \in X$, $s > 0$ there exists $\rho \in (s, 2s)$ such that for all n

$$\mu(\{y \in X : \rho - 4^{-n}s < d(x,y) < \rho + 4^{-n}s\}) \leq \frac{1}{2^{n-1}}\mu(B(x,2s)). \tag{6}$$

Indeed, let m be the measure on the interval $(0,2)$ defined by $m([0,t)) = \mu(B(x,st))$. We construct a sequence of open intervals I_n starting from $I_0 = (1,2)$. If I_n is an interval of length 4^{-n} we divide it into 4 pieces of equal length and choose I_{n+1} the left of the right central piece of smallest measure. We have $m(I_{n+1}) \leq \frac{1}{2}m(I_n)$. I_n is a decreasing sequence of intervals with $\overline{I}_{n+1} \subset I_n$ thus $\cap_n I_n$ contains one point, say $\bar\rho$. Since $\bar\rho \in I_n$ we have $\bar\rho \pm 4^{-n} \in I_{n-1}$ thus $m((\bar\rho - 4^{-n}, \bar\rho + 4^{-n})) \leq m(I_{n-1}) \leq \frac{1}{2^{n-1}}m(I_0)$. Proving the claim with $\rho = s\bar\rho$.

Fix $s > 0$ so small that any partition made by sets of diameter less than $2s$ has nonzero entropy (see [8]). Choose a maximal s-separated set E. For any $x \in E$ take $\rho_x \in (s,2s)$ such that (6) in the claim holds. Let $E = \{x_1, x_2, \ldots\}$ be an enumeration of the (at most) countable set E. Put $B_i = B(x_i, \rho_{x_i})$ and define $Q_1 = B_1$, $Q_2 = B_2 \setminus Q_1$, $Q_3 = B_3 \setminus (Q_1 \cup Q_2)$, ... By maximality the collection of sets $\xi = \{Q_1, Q_2, \ldots\}$ is a partition of X (modulo μ) and since $\partial\xi \subset \cup_i \partial B_i$ we get

$$\begin{aligned}\mu(\{x \in X : d(x,\partial\xi) < 4^{-n}s\}) &\leq \mu(\cup_i\{x \in X : \rho_{x_i} - 4^{-n} < d(x_i,x) < \rho_{x_i} + 4^{-n}\}) \\ &\leq \frac{1}{2^{n-1}}\sum_i \mu(B(x_i,2s)).\end{aligned}$$

Since the x_i are s-separated and E is Euclidean there are at most $c(E) = c(\dim E)$ balls of radius $2s$ that can intersect, thus the last sum is bounded by $\frac{c(E)}{2^{n-1}}$. This proves that for some constants $a, c > 0$ and all $\varepsilon > 0$

$$\mu(x \in X : d(x,\partial\xi) < \varepsilon) < c\varepsilon^a.$$

Thus for any $b > 0$ we have by the invariance of μ

$$\sum_n \mu(\{x \in X : d(f^n x, \partial\xi) < \mathrm{e}^{-bn}\}) \leq \sum_n c\mathrm{e}^{-abn} < \infty.$$

This implies by Borel-Cantelli Lemma that for μ-a.e. x there exists $n(x) < \infty$ such that $d(f^n x, \partial\xi) \geq \mathrm{e}^{-bn}$, hence $B(f^n x, \mathrm{e}^{-bn}) \subset \xi(f^n x)$, for any $n \geq n(x)$. Taking $c(x) \in (0,1)$ sufficiently small we have $B(f^n x, c(x)\mathrm{e}^{-bn}) \subset \xi(f^n x)$ for all integer n.

Fix $x \in X$ where the reasonable sensitivity condition holds. Without loss of generality, and changing if necessary $c(x)$ into a smaller constant we assume that f^n is $\mathrm{e}^{\lambda n}$-Lipschitz on the ball $B(x, c(x)\mathrm{e}^{-\gamma}n)$ for all integer n and that $\lambda > \gamma + b$.

We show then by induction that $B(x, c(x)\mathrm{e}^{-\lambda n}) \subset \xi^k(x)$ for any $k \leq n$. Indeed, this is trivially true for $k = 1$, and if this holds for some $k \leq n-1$ then we have

$$f^k(B(x, c(x)^2\mathrm{e}^{-\gamma n}) \subset B(f^k x, c(x)\mathrm{e}^{\lambda k - \gamma n}) \subset B(f^k x, \mathrm{e}^{-bn}) \subset \xi(f^k x).$$

Hence $B(x, c(x)^2\mathrm{e}^{-\gamma n}) \subset \xi^{k+1}(x)$. □

We finally provide a sufficient condition for reasonable sensitivity.

Lemma 13. *Assume that f is Lipschitz or that the following condition holds*

*[**Piecewise Lipschitz with finite exponent**] there exists a partition $\mathcal{A}$ (modulo μ) into open sets such that on each $A \in \mathcal{A}$ the map f is Lipschitz with constant $L_f(A)$, the singularity set $\partial\mathcal{A} = \cup_{A\in\mathcal{A}}\partial A$ is such that $\mu(\{x \in X \colon d(x,\partial\mathcal{A}) < \epsilon\}) \leq c\epsilon^a$ for some constants $c > 0$ and $a > 0$ and the average Lipschitz exponent $\log L_f := \sum_{A\in\mathcal{A}} \log^+ L_f(A)\mu(A)$ is finite.*

Then the first condition of Lemma 12 is satisfied (i.e. the system is reasonably sensitive).

Proof. We prove the piecewise case, the other one is obvious. Let $\lambda > \log L_f$. By the Birkhoff Ergodic Theorem, for μ-a.e. x there exists $m(x)$ such that

$$L_f(\mathcal{A}(x))L_f(\mathcal{A}(fx))\cdots L_f(\mathcal{A}(f^{n-1}x)) \leq \mathrm{e}^{\lambda n}$$

for all $n \geq m(x)$. Replacing if necessary the upper bound by $\mathrm{e}^{\lambda n}/c(x)$ for some constant $c(x) \geq 1$ the inequality will hold for any integer n. Proceeding as in the last part of the proof of Lemma 12 we get that for any $b > 0$, changing $c(x)$ if necessary, we have $B(f^n x, c(x)\mathrm{e}^{-bn}) \subset \mathcal{A}(f^n x)$ for any integer n. We then conclude similarly that $B(x, c(x)^2\mathrm{e}^{-bn}\mathrm{e}^{-\lambda n}) \subset \mathcal{A}^n(x)$. This concludes the proof taking $\gamma = b + \lambda$. □

The proof of Theorem 3 follows now easily from the preceding results.

Proof of Theorem 3. By Lemma 13 the map is reasonably sensitive. This implies by Lemma 12 the existence of a partition with large interior. By Lemma 11 we find that $\underline{R} > 0$ a.e. and the conclusion follows from Theorem 8. □

Remark 14. *(i) We remark that if f is C^2 on a compact manifold, or more generally if f is piecewise $C^{1+\alpha}$ with reasonable singularity set such as in [9], and μ is an ergodic measure with nonzero entropy, then the exponents λ and γ in Lemma 12 can be taken arbitrarily close to the largest Lyapunov exponent λ_μ^+ of the measure[5]. Thus the exponent χ in Lemma 11 may also be taken arbitrarily close to λ_μ^+. This readily implies that*

$$\underline{R} \geq h_\mu/\lambda_\mu^+ \quad \mu\text{-a.e.} \tag{7}$$

This is optimal in dimension one or more generally for conformal maps, where under mild assumptions we have $HD(\mu) = h_\mu/\lambda_\mu$.

In the case of diffeomorphisms a similar argument may be applied also with backward iterates and the lower bound in (7) easily generalizes[6] to

$$\underline{R} \geq h_\mu \left(\frac{1}{\lambda_\mu^+} - \frac{1}{\lambda_\mu^-}\right) \quad \mu\text{-a.e.}$$

where λ_μ^- is the smallest Lyapunov exponent of the measure μ. Note that for C^2 surface diffeomorphisms (e.g. Henon maps) with hyperbolic measures this is optimal and shows that

$$R = HD(\mu) \quad \mu\text{-a.e.}$$

[5] to see this, consider a Lyapunov chart whose local chart at x has a diameter $\rho(x)$, where ρ is η-slowly varying. A choice like $\lambda = \lambda_\mu^+ + 2\eta$ and $\gamma = \lambda + \eta$ would do the job.

[6] it essentially amounts to consider a two-sided version of Lemma 11 ; we leave the details to the reader.

(iii) Combining the above observation with Remark 10 shows that the assumption on the super-polynomial decay of correlations in Theorem 8 may be reduced to a decay at a rate n^{-p} *for some* $p > \frac{D+2}{h_\mu}\lambda_\mu^+ + 1$.

References

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