SKEW PRODUCTS, QUANTITATIVE RECURRENCE, SHRINKING TARGETS AND DECAY OF CORRELATIONS

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Abstract. We consider toral extensions of hyperbolic dynamical systems. We prove that its quantitative recurrence (also with respect to given observables) and hitting time scale behavior depend on the arithmetical properties of the extension.

By this we show that those systems have a polynomial decay of correlations with respect to C^r observables, and give estimations for its exponent, which depend on r and on the arithmetical properties of the system.

We also show examples of systems of this kind having not the shrinking target property, and having a trivial limit distribution of return time statistics.

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1. Introduction

1.1. **A short overview of hitting and recurrence time.** The study of hitting time or recurrence time in dynamical systems has several facets. Many of them are described by the surveys [1, 32, 2] and references therein, just to mention a few of these aspects:

- information theory in the study of repetitions of words in sequences of symbols [29],
- probability in dynamical Borel-Cantelli lemma [16], shrinking targets problems [22], exponential law, Poisson statistics and extreme value theory [11],
- geometric measure theory in the relation with Hausdorff dimension [3, 13, 12]
- geometry, in logarithm laws for the geodesic flow and similar(see e.g. [35, 25, 17, 28])
- nonlinear analysis in recurrence plots (see e.g. [27]).

Quite recent results showed relations between quantitative indicators representing the scaling behavior of return times and hitting times in small targets, decay of correlations and arithmetical properties. More precisely, let us consider a discrete time dynamical system (X, T, μ), where (X, d) is a metric space and $T : X \to X$ is a measurable map preserving a finite measure μ. Let us consider two points in X and the time which is necessary for the orbit of x to approach y at a distance less than r

$$\tau_r(x, y) = \min\{n \in \mathbb{N}^+ : d(T^n(x), y) < r\}.$$

We consider the behavior of $\tau_r(x, y)$ as $r \to 0$. In many interesting cases this is a power law $\tau_r(x, y) \sim r^R$. When $x = y$ this exponent R gives a quantitative measure of the speed of recurrence of an orbit near to its starting point, and this will be a quantitative recurrence indicator. When $x \neq y$ the exponent is a quantitative measure of how fast the orbit starting from x approaches a point y. It is indeed a measure for the scaling behavior of the hitting time of an orbit starting from x to a sequence of small targets: the balls $B_r(y)$ centered in y of radius r. To extract the exponent the recurrence rate ($R(x, x)$) and hitting time scaling exponent ($R(x, y)$) are defined by

$$\overline{R}(x, y) = \limsup_{r\to 0} \frac{\log \tau_r(x, y)}{-\log r}, \quad \underline{R}(x, y) = \liminf_{r\to 0} \frac{\log \tau_r(x, y)}{-\log r}. \tag{1}$$

A general philosophy is that in "chaotic" systems these scaling exponents are equal to the local dimension. Indeed, several results relate those exponents to the scaling behavior of the measure of the target set $B_r(y)$, the local dimension $d_\mu(y)$ of the measure μ at y (see e.g. [4],[3],[33],[31],[33],[34],[15],[13],[12],[18],[14],[30]). More precisely: in [33] (see also [31]) and [12] it is proved that if a system has superpolynomial decay of correlations with respect to Lipschitz observables (equivalently for C^k observables, see Appendix), then the recurrence and hitting time exponents are equal to the local dimension of the invariant measure (see Theorem 4 for the precise result).

On the other hand the definition of the above scaling exponents also recalls diophantine approximation. Indeed in "non chaotic systems" like Rotations, Interval exchanges or reparametrizations of rotations, their behavior is related to arithmetical properties of the system and not to the dimension (see [5], [23], [24], [19], [3] and Section 2.3 below).

It is worth to remark that there exist mixing systems (with subpolynomial rate, see [19]) where the exponents are not related to dimension but to arithmetical properties.

1.2. **Outline of the results.** The question remained open, if there are polynomially mixing systems (with respect to Lipschitz observables) such that the above defined recurrence-hitting time exponents are different from dimension, and the question whether this can happen for superpolynomially mixing system with respect to smoother observables (C^∞ observables e.g.).

We consider in this paper a skew product of a chaotic map with a rotation with some finite Diophantine type (see below and Section 3 for more precise definitions) and we will show that such a system posses these properties. To be more precise, we are interested in systems of the type

$$(\omega, t) \to (T\omega, t + \varphi(\omega))$$

on a phase space $(\omega, t) \in \Omega \times \mathbb{R}^d/\mathbb{Z}^d$, where T is a piecewise expanding map, and on the second coordinate we have a family of isometries controlled by a certain function φ. The most basic, yet nontrivial example is the case where T is the doubling map on S^1, and on the second coordinate we have circle translations.

Ergodic properties of a skew products as above have been quite well studied in the literature as they are one of the most simple example of systems which is in some sense weakly chaotic. The qualitative ergodic theory of these systems has been studied by Brin. A quantitative result about its speed of mixing was given in [8], where it is proved that provided the rotation's angle has some diophantine properties, such systems have at least polynomial decay of correlations with respect to Lipschitz and C^k observables, moreover they have superpolynomial decay of correlations with respect to C^∞ ones. More recently, in [20] it was shown that under suitable assumptions, including the above cited most basic example, if φ is C^1 and not cohomologous to a piecewise constant function, then the decay of correlations is exponential (for Hölder observables).

In this paper, we investigate the quantitative recurrence and hitting time indicators of such kind of systems and its relations with decay of correlations, in the complementary case where φ *is piecewise constant.* In this case we show the following facts:

- The decay of correlations under Lipschitz or C^k observables is polynomial and we give a concrete estimation for its exponent depending on the arithmetical properties of the system (in the case of multidimensional rotations the linear diophantine type of the angle is involved, see Sections 3.2, 5, 6.1).
- The recurrence and hitting time exponents are related both to the dimension and the arithmetical properties of the rotation and we give estimations for these exponents (see Sections 4,6.1).
- We show that if the Diophantine type is large, the statistics of return time and hitting time has a trivial limiting distribution along some subsequence.

In particular, there is no convergence to the exponential law for these statistics.

- Using a multidimensional rotation with a suitable angle (with intertwined partial quotients) the resulting skew product is an example of a system with *polynomial* decay of correlations, but not satisfying the monotone shrinking target property and having a trivial distribution of return and hitting times. We remark that a mixing system without the monotone shrinking target property was shown in [10]. However the speed of decay of correlations of that example is less than polynomial ([19]), see Section 6.1.

We can hence consider these skew products as a borderline case for the relations between recurrence, hitting time and decay of correlations.

An ingredient of our proofs is an estimate of a discrepancy for random walks generated by multidimensional irrational rotations. This result (which is a generalization of the one given in [34]) is proven in the appendix. There we also prove some other technical results which are more or less known or at least expected but which are, as far as we know, not present in this form in the literature.

2. Background.

2.1. General inequalities for hitting and recurrence time exponents. Recall that the upper and lower local dimension of ν at y are defined as $\overline{d}_\nu(y) = \limsup_{r\to\infty}\frac{\log\nu(B(y,r))}{\log r}$ and $\underline{d}_\nu(y) = \liminf_{r\to\infty}\frac{\log\nu(B(y,r))}{\log r}$. It is relatively easy to obtain that for a general systems ([3],[13]):

Proposition 1. *Let (X,T,ν) be a dynamical system over a separable metric space and ν be an invariant Borel measure. For each y*

$$\underline{R}(x,y) \geq \underline{d}_\nu(y)\ ,\ \overline{R}(x,y) \geq \overline{d}_\nu(y) \tag{2}$$

holds for ν-almost each $x \in X$. Moreover, if X is a measurable subset of $\mathbb{R}^d$ for some d, then

$$\underline{R}(x,x) \leq \underline{d}_\nu(x)\ ,\ \overline{R}(x,x) \leq \overline{d}_\nu(x) \tag{3}$$

for ν-a.e. $x \in X$.

2.2. Rapid mixing and consequences. A sufficient condition for these inequalities to become equalities is that the system is mixing sufficiently rapidly. This is measured by the decay of correlations.

Definition 2 (Decay of correlations)**.** *Let ϕ, $\psi : X \to \mathbb{R}$ be observables on X belonging to the Banach spaces B, B'. Let $\Phi : \mathbb{N} \to \mathbb{R}$ such that $\Phi(n) \underset{n\to\infty}{\to} 0$. A system (X,T,ν) is said to have decay of correlations with speed Φ with respect to observables in B and B' if*

$$\left|\int \phi\circ T^n\psi d\nu - \int \phi d\nu \int \psi d\nu\right| \leq ||\phi||_B\,||\psi||_{B'}\,\Phi(n) \tag{4}$$

where $|| \ ||_B, || \ ||_{B'}$ *are the norms*[1] *in* B *and* B'. *We say that the decay is polynomial with exponent between* a *and* b *if*

$$a \leq \lim \inf_{n\to\infty} \frac{-\log \Phi(n)}{\log n} \leq \lim \sup_{n\to\infty} \frac{-\log \Phi(n)}{\log n} \leq b.$$

We say that the decay of correlation is superpolynomial if $\lim n^{\alpha}\Phi(n) = 0$, $\forall \alpha > 0$.

The decay of correlations has been studied for many systems, mostly those with some form of hyperbolicity. It depends on the class of observables, and it is obvious that if a system has decay of correlations with some given speed with respect to Lipschitz or C^p observables, then it has also with respect to C^k ones when $k \geq p$ (with at least the same speed). There is also a converse which will be used later.

Lemma 3. *If the space where the dynamics act is a manifold* $X \subset \mathbb{R}^d$ *(possibly with boundary) and the system has decay of correlations with respect to* C^p, C^q *observables with speed* $\Phi_{p,q}(n)$ *then it has also with respect to* C^k, C^ℓ *observables with speed* $\Phi_{k,\ell}(n) \leq \Phi_{p,q}(n)^{\frac{1}{\frac{p}{k}+\frac{q}{\ell}-1}}$. *In the case* $k = 1$ *or* $\ell = 1$ *the same is true with Lipschitz (observables and norm) instead of* C^1.

The proof of the above lemma is postponed to the appendix.

Now we can state a result linking decay of correlations and local dimension with recurrence and hitting time ([12],[31]). We see that in rapidly mixing systems the recurrence and hitting time exponents are necessarily equal to the local dimension of the invariant measure.

Theorem 4. *If* (X, T, ν) *has superpolynomial decay of correlations with respect to Lipschitz observables, as above and the local dimension* $d_\nu(y)$ *exists*[2] *then*

$$\overline{R}(x, y) = \underline{R}(x, y) = d_\nu(y) \tag{5}$$

for ν*-almost each* x. *If moreover* $X \subseteq \mathbb{R}^d$ *for some* d, *then*

$$\overline{R}(x) = \underline{R}(x) = d_\nu(x)$$

for ν*-almost each* x.

2.3. **Basic results on circle rotations.** We will see that in the class of system we are interested in, arithmetical properties are important for quantitative recurrence and hitting time scaling behavior. The simplest case of system where this happen is the case of rotations on the circle.

We state some basic results, linking quantitative recurrence, shrinking targets and arithmetical properties in circle rotations.

Definition 5. *Given an irrational number* α *we define the* type *of* α *as the following (possibly infinite) number:*

$$\gamma(\alpha) = \inf\{\beta : \liminf_{q\to\infty} q^\beta \|q\alpha\| > 0\}.$$

[1] It is worth to remark that under reasonable assumptions, a non uniform statement: $\left|\int \phi \circ T^n \psi d\mu - \int \phi d\mu \int \psi d\mu\right| \leq C_{\phi,\psi}\Phi(n)$ can be converted in an uniform one, like (4). See [7].

[2] The limit $d_\nu(y) = \lim_{r\to\infty} \frac{\log \nu(B(y,r))}{\log r}$ exists.

Let q_n being the sequence of convergent denominators of α (see the appendix for some recalls on continued fractions). It is worth to remark that the above definition is equivalent to the following

$$\gamma(\alpha) = \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n}. \tag{6}$$

Every number has type ≥ 1. The set of number of type 1 is of full measure; the set of numbers of type γ has Hausdorff dimension $\frac{2}{\gamma+1}$. There exist numbers of infinite type, called *Liouville* numbers; their set is dense and uncountable and has zero Hausdorff dimension.

The behavior of quantitative recurrence and hitting time in small targets for circle rotations with angle α depend on the type of α, as it is described in the following statement

Theorem 6. *([24], [3]) If T_α is a rotation of the circle, y a point on the circle then for almost every x*

$$\overline{R}(x,y) = \gamma(\alpha), \qquad \underline{R}(x,y) = 1.$$

and for each x

$$\underline{R}(x,x) = \frac{1}{\gamma(\alpha)}, \qquad \overline{R}(x,x) = 1. \tag{7}$$

It is interesting to see that the statement implies the impossibility to construct a system with

$$\underline{R}(x,y) > d_\nu(y) \tag{8}$$

for typical x with a circle rotation, and it is worth to remark that similarly such an example cannot be constructed by an interval exchange [5]. The situation is different if we can consider two dimensional rotations (see Section 6.1). We will exploit this and construct a system having even polynomial decay and still satisfying (8).

2.4. **Statistics of hitting/return time.** Let us consider a family of centered balls $B_r = B_r(y)$ such that $\nu(B_0) = 0$ and $\nu(B_r) \neq 0$ for $r > 0$. We will consider the statistical distribution of return times in these sets. We say that the return time statistics of the system converges to g for the balls B_r, if

$$\lim_{r\to 0} \frac{\nu(\{x \in B_r, \tau_r(x,y) \geq \frac{t}{\nu(B_r)}\})}{\nu(B_r)} = g(t). \tag{9}$$

If $g(t) = e^{-t}$ we say that the system has an exponential return time limit statistics. Such statistics can be found in several systems with some hyperbolic behavior in some class of decreasing sets, however other limit distributions are possible.

Starting from all the space instead of the ball B_r defines the hitting time statistics:

$$\lim_{r\to 0} \nu(\{x \in X, \tau_r(x,y) \geq \frac{t}{\nu(B_r)}\}) = g(t). \tag{10}$$

We will show that in our system there is not exponential statistics, even more, the limiting distribution can be trivial.

3. The class of skew products under consideration

3.1. **Definition.** We will consider a class of systems constructed as follows. The base is a measure preserving system (Ω, T, μ). We assume that T is a piecewise expanding Markov map on a finite-dimensional Riemannian manifold Ω, that is:

- there exists some constant $\beta > 1$ such that $\|D_x T^{-1}\| \leq \beta^{-1}$ for every $x \in \Omega$.
- There exists a collection $\mathcal{J} = \{J_1, \dots, J_p\}$ such that each J_i is a closed proper set and
  (M1) T is a $C^{1+\eta}$ diffeomorphism from $\operatorname{int} J_i$ onto its image;
  (M2) $\Omega = \cup_i J_i$ and $\operatorname{int} J_i \cap \operatorname{int} J_j = \emptyset$ unless $i = j$;
  (M3) $T(J_i) \supset J_j$ whenever $T(\operatorname{int} J_i) \cap \operatorname{int} J_j \neq \emptyset$.

$\mathcal{J}$ is called a Markov partition. It is well known that such a Markov map is semi-conjugated to a subshift of finite type. Without loss of generality we assume that T is topologically mixing, or equivalently that for each i there exists n_i such that $T^{n_i} J_i = \Omega$. We assume that μ is the equilibrium state of some potential $\psi \colon \Omega \to \mathbb{R}$, Hölder continuous in each interior of the J_i's. The sets of the form $J_{i_0,\dots,i_{q-1}} := \bigcap_{n=0}^{q-1} T^{-n} J_{i_n}$ are called cylinders of size q and we denote their collection by $\mathcal{J}_q$.

In this setting it is well known that the pointwise dimension $d_\mu(x)$ exists and $d_\mu(x) = d_\mu$, μ-almost everywhere.

The system is extended by a skew product to a system (M, S) where $M = \Omega \times \mathbb{T}^d$, $\mathbb{T}^d$ is the d-torus and $S : M \to M$ defined by

$$S(\omega, t) = (T\omega, t + \boldsymbol{\alpha}\varphi(\omega)) \tag{11}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{T}^d$ and $\varphi = 1_I$ is the characteristic function of a set $I \subset \Omega$ which is an union of cylinders. In this system the second coordinate is translated by $\boldsymbol{\alpha}$ if the first coordinate belongs to I. We endow (M, S) with the invariant measure $\nu = \mu \times m$ (m is the Haar measure on the torus).

We make the standing assumption that

(NA) for any $u \in [-\pi, \pi]$, the equation $f e^{iu\varphi} = \lambda f \circ T$, where f is Hölder (on the subshift) and $\lambda \in S^1$, has only the trivial solutions $\lambda = 1$ and f constant.

This is equivalent to the fact that the map $(\omega, s) \mapsto (T\omega, s + u\varphi(\omega))$ is weakly-mixing on $\Omega \times \mathbb{T}$. The simple case where I **is a nonempty union of size** 1 **cylinders such that both** I **and** I^c **contain a fixed point fulfills this assumption**.[3]

We will indicate by $\pi_1 : \Omega \times \mathbb{T}^d \to \Omega$ and $\pi_2 : \Omega \times \mathbb{T}^d \to \mathbb{T}^d$ the two canonical projections, moreover, on $\Omega \times \mathbb{T}^d$ we will consider the sup distance.

We will suppose that $\boldsymbol{\alpha}$ is of finite Diophantine type. We introduce two definitions of Diophantine type, which generalize Definition 5 to the higher dimensional cases. The notation $||.||$ will indicate the distance to the nearest integer vector (or number) in $\mathbb{R}^d$ (or $\mathbb{R}$) and $|k| = \sup_{0 \leq i \leq d} |k_i|$ the supremum norm.

Definition 7. *The Diophantine type of* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d)$ *for the* ***linear approximation*** *is*

$$\gamma_l(\boldsymbol{\alpha}) = \inf\{\gamma \ , s.t. \exists c_0 > 0 \ s.t. \|k \cdot \boldsymbol{\alpha}\| \geq c_0 |k|^{-\gamma} \ \forall 0 \neq k \in \mathbb{Z}^d\}$$

[3] Any solution has a modulus $|f|$ constant. The first fixed point gives then $e^{iu} = \lambda$ while the second gives $\lambda = 1$. The existence of such f would then contradicts the ergodicity of T.

Definition 8. *The Diophantine type of* $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d)$ *for the* ***simultaneous approximation*** *is*

$$\gamma_s(\boldsymbol{\alpha}) = \inf\{\gamma\ , s.t. \exists c_0 > 0\ s.t. \|k\boldsymbol{\alpha}\| \geq c_0 |k|^{-\gamma}\ \forall 0 \neq k \in \mathbb{N}\}.$$

3.2. **Upper bound on decay of correlations.** We have an explicit upper bound for the rate of decay of correlations.

Proposition 9. *For Lipschitz observables the rate of decay is* $O(n^{-\frac{1}{2\gamma}})$ *for any* $\gamma > \gamma_l(\alpha)$.

For C^p, C^q *observables, the rate of decay is* $O(n^{-\frac{1}{2\gamma}\max(p,q,p+q-d)})$ *for any* $\gamma > \gamma_l(\alpha)$.

Remark 10. *We remark that the rate of decay of correlations is* superpolynomial *for* C^∞ observables*.*

The proof of the proposition is based on the following statement

Lemma 11. *If the observables* A, B *are respectively of class* C^p *and* C^q *on* M, $p+q > d$ *and* $\int_M A d\nu = 0$*, the correlations satisfy*

$$C_n(A,B) := \int_M AB \circ S^n d\nu \leq C\|A\|_{C^p}\|B\|_{C^q} n^{-\ell+\epsilon}$$

for each $\epsilon > 0$*, where* $\ell = \frac{p+q-d}{2\gamma_l(\alpha)}$*.*

Proof of Proposition 9. Let $x, y \in \mathbb{N}^*$. By Lemma 3 we have $\Phi_{p,q}(n) = O(\Phi_{px,qy}(n)^{\frac{1}{x+y-1}})$. By Lemma 11 we know that $\Phi_{px,py}(n) = O(n^{-\frac{(px+qy-d)}{2\gamma_l(\alpha)}+\epsilon})$ for any $\epsilon > 0$. Hence $\Phi_{p,q}(n) = O(n^{-\ell+\epsilon})$ with

$$\ell = \frac{1}{2\gamma}\frac{px+qy-d}{x+y-1}.$$

Taking the supremum over $x, y \in \mathbb{N}^*$ gives the conclusion. ∎

The idea of the proof of Lemma 11 is to expand in Fourier series the observables with respect to the second variable t and see that $C_n(A,B)$ corresponds to an infinite sum of terms which are similar to correlation integrals of the Fourier coefficients (which are still functions of ω) with respect to a suitable operator.

Let L be the Perron-Frobenius operator for (Ω, T, ψ) defined for, say a bounded function a, by

$$L(a)(\omega) = \sum_{T(\omega')=\omega} e^{\psi(\omega')} a(\omega'), \quad a\colon \Omega \to \mathbb{C}.$$

Without loss of generality we assume that the potential ψ is normalized in the sense that $L1 = 1$. In particular this implies that for any measurable bounded functions a and b, and any integer n,

$$\int_\Omega ab \circ T^n d\mu = \int_\Omega L^n(a) b d\mu.$$

Given $u \in C^0(\mathbb{T}^d, \mathbb{C})$ we denote its Fourier coefficients by

$$\hat{u}(k) = \int_{\mathbb{T}^d} e^{-2i\pi<k,t>} u(t) dm(t), \quad k \in \mathbb{Z}^d$$

and recall that u is equal to its Fourier series

$$u(t) = \sum_{k \in \mathbb{Z}^d} \hat{u}(k) e^{2i\pi<k,t>}, \quad t \in \mathbb{T}^d.$$

Given $u \in \mathbb{R}$ we define the complexified Perron-Frobenius operator L_u

$$L_u(a) = L(e^{iu\varphi}a)$$

where φ is the functions involved in the definition of the skew product, as above. Note that L_u is indeed the Perron Frobenius operator of the potential

$$\psi + iu\varphi.$$

For a given $\omega \in \Omega$ we denote the $k-$th Fourier coefficient of $A(\omega, \cdot)$ by $\hat{A}(\omega, k)$.

Lemma 12. *The correlations can be expressed by the Fourier expansion of the observables and the complexified Perron-Frobenius operator:*

$$C_n(A,B) = \sum_{k\in\mathbb{Z}^d} \int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu.$$

Proof. We have, using the Fourier expansion of B that (denoting $S_n\varphi(\omega) = \sum_{i=0}^{n-1}\varphi(T^i\omega)$)

$$\begin{aligned}
C_n(A,B) &= \sum_{k\in\mathbb{Z}^d} \int_{\Omega\times\mathbb{T}^d} A(\omega,t)\hat{B}(T^n\omega,k)e^{2i\pi\langle k,t+\alpha S_n\varphi(\omega)\rangle}d\mu(\omega)dm(t) \\
&= \sum_{k\in\mathbb{Z}^d} \int_\Omega \hat{B}(T^n\omega,k)e^{2i\pi\langle k,\alpha S_n\varphi(\omega)\rangle} \int_{\mathbb{T}^d} A(\omega,t)e^{2i\pi\langle k,t\rangle}dm(t)d\mu(\omega) \\
&= \sum_{k\in\mathbb{Z}^d} \int_\Omega \hat{B}(T^n\omega,k)e^{2i\pi\langle k,\alpha S_n\varphi(\omega)\rangle}\hat{A}(\omega,-k)d\mu(\omega) \\
&= \sum_{k\in\mathbb{Z}^d} \int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu.
\end{aligned}$$

This proves the lemma. ∎

In the above sum, the Fourier coefficients corresponding to large k will be estimated by the regularity of the observables. To estimate the other coefficients we use the next proposition.

Strictly speaking, to finish the proof we have to work on the subshift of finite type generated by the Markov partition. There is nothing essential, but at some point we will need that the Perron-Frobenius operator preserves the space of Hölder functions of exponent η. For convenience we will keep the same notation (Ω, T, μ) for the symbolic dynamical system, and leave the details to the reader.

Proposition 13 ([21])**.** *There exists two constants $c_0 > 0$ and $c_1 > 0$ such that, as an operator acting on Hölder functions of exponent η, for any n and $u \in [-\pi, \pi]$ the operator norm $|||\cdot|||_{\eta,\eta}$ satisfies*

$$|||L^n_u|||_{\eta,\eta} \leq c_0 e^{-c_1 n u^2}.$$

Proof. This property follows classically from quasi-compactness of the Perron-Frobenius operators [21]:

First, by a perturbative argument and since φ is not cohomologous to a constant, there exists $\beta > 0$ and some constants c_0 and c_1 such that if $u \in (-\beta, \beta)$ then $|||L^n_u|||_{\eta,\eta} \leq c_0 e^{-c_1 n u^2}$.

Next whenever $u \neq 0 \mod 2\pi$, the operator L_u has no eigenvalues of modulus (larger than or) equal to one by the assumption (NA), hence the spectral radius is smaller than one, which gives $\lim_{n\to\infty} |||L^n_u|||^{1/n}_{\eta,\eta} < 1$. An uniform exponential contraction follows then by compacity of $[-\pi,\pi]\setminus(\beta,\beta)$. Changing if necessary the values of the constants, the advertised upper bound also holds in the case $u\notin(-\beta,\beta)$. ∎

Proposition 14. *For any $\gamma>\gamma_l(\alpha)$ there exists $c_2>0$ such that for any non-zero $k\in\mathbb{Z}^d$*

$$|||L^n_{2\pi\langle k,\alpha\rangle}|||_{\eta,\eta}\le c_0\exp(-c_2|k|^{-2\gamma}n).$$

Proof. Let $\gamma>\gamma_l(\alpha)$. By definition of the Diophantine type of α, we have $\|\langle k,\alpha\rangle\|\ge c|k|^{-\gamma}$ for some constant $c>0$ and any $k\in\mathbb{Z}^d$. The result follows by Proposition 13. ∎

We are now able to prove the main Lemma.

Proof of Lemma 11. We split the estimate into two parts, depending whether k is large or not. Let N be an integer which will be chosen later.

When k is large the estimate is simple: Since $A\in C^p$ and $B\in C^q$ we have $\|\hat{A}(\omega,k)\|_\infty\le\|A\|_p|k|^{-p}$ and $\|\hat{B}(\omega,k)\|_\infty\le\|B\|_q|k|^{-q}$. Hence

$$\left|\int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu\right|\le\|A\|_p\|B\|_q|k|^{-p-q}.$$

Therefore, since the operator $L^n_{2\pi\langle k,\alpha\rangle}$ does not expand the sup-norm, for any integer N we get

$$\left|\sum_{|k|>N}\int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu\right|\le C_1\|A\|_p\|B\|_qN^{d-p-q}. \tag{12}$$

The case $k=0$ is done independently: Indeed, we have

$$\int_\Omega\hat{A}(\cdot,0)d\mu=\int_{\Omega\times\mathbb{T}^d}A\ d\mu\otimes m=0$$

by hypothesis. Then the usual decorrelation estimate gives that for some $C,\lambda>0$ we have

$$\left|\int_\Omega L^n(\hat{A}(\cdot,0))\hat{B}(\cdot,0)d\mu\right|\le C_2\|A\|_1\|B\|_\infty\lambda^n \tag{13}$$

(recall that $||.||_1$ here is the C^1 norm). On the other hand, for small $k\neq0$ we make use of Proposition 14:

Observe that$\|\hat{A}(\cdot,k)\|\le\|A\|_p|k|^{1-p}$ (here $||.||$ is the Holder norm).

$$\left|\int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu\right|\le\|A\|_p|k|^{1-p}c\exp(-c|k|^{-2\gamma}n)\|B\|_q|k|^{-q}.$$

Therefore

$$\sum_{1\le|k|\le N}\left|\int_\Omega L^n_{2\pi\langle k,\alpha\rangle}(\hat{A}(\cdot,-k))\hat{B}(\cdot,k)d\mu\right|\le c\|A\|_p\|B\|_q\sum_{1\le|k|\le N}|k|^{1-p-q}\exp(-cn|k|^{-2\gamma}). \tag{14}$$

which is easily seen to be superpolynomially decaying in n, if we choose $N=n^{1/2\gamma-\epsilon}$, since

$$\sum_{1\le|k|\le N}|k|^{1-p-q}\exp(-cn|k|^{-2\gamma})\le N^{d+1-p-q}\exp(-cnN^{-2\gamma}).$$

Putting together the three estimates (12), (13) and (14) gives

$$C_n(A,B) \le const\|A\|_p\|B\|_q n^{-\ell},$$

for any $\ell < \frac{p+q-d}{2\gamma}$. ∎

4. Quantitative recurrence, shrinking targets and arithmetical properties

In this section we give estimations on the quantitative recurrence and hitting time exponent of skew products, as defined before. These results show that both the quantitative recurrence and the hitting time exponent depend on the arithmetical properties of α.

4.1. **Deterministic and random multidimensional rotations.**

The quantitative recurrence and hitting time exponents of our system is influenced by the underlying rotation. Multidimensional rotations have not been completely investigated from this point of view. We shall start with some general considerations on recurrence, hitting times and discrepancy on multidimensional rotations and random walks generated by such rotations. These results will be used in the following.

Let us consider a rotation $x \to x+\alpha$ on $\mathbb{T}^d$, $\alpha = (\alpha_1, ...\alpha_d)$. It is easy to see that for this map, the recurrence rates $\underline{R}(x,x)$ and $\overline{R}(x,x)$ do not depend on x, hence there are constants that we denote by

$$\underline{R}(x,x) = \underline{rec}(\alpha), \quad \overline{R}(x,x) = \overline{rec}(\alpha).$$

Reproducing the proof of [3] for one dimensional rotations, it is easy to see that

Proposition 15. *Let us consider a rotation $x \to x+\alpha$ on $\mathbb{T}^d$, $\alpha = (\alpha_1, ...\alpha_d)$. We have*

$$\underline{rec}(\alpha) = \frac{1}{\gamma_s(\alpha)}.$$

Concerning the hitting time exponents $\underline{R}(x,y)$ and $\overline{R}(x,y)$ a similar thing happens (using that $R(x,y) = R(x-y,0)$ for any x, y, together with the invariance $R(x+\alpha,0) = R(x,0)$): there exists also two constants $\underline{hit}(\alpha)$ and $\overline{hit}(\alpha)$ such that for any y and a.e. x

$$\underline{R}(x,y) = \underline{hit}(\alpha), \quad \overline{R}(x,y) = \overline{hit}(\alpha).$$

The following proposition is quite similar to the transference principle between homogeneous and inhomogeneous diophantine exponents (see [6]).

Proposition 16. *Let us consider a rotation $x \to x+\alpha$ on $\mathbb{T}^d$, $\alpha = (\alpha_1, ...\alpha_d)$. We have*

$$\overline{hit}(\alpha) \ge \gamma_l(\alpha).$$

Proof. Let $\gamma < \gamma_l(\alpha)$. Then there is a sequence of vectors $k_p = (k_{1,p}, ..., k_{d,p})$ with $|k_p| \to \infty$ such that $||k_p \cdot \alpha|| < |k_p|^{-\gamma}$. By choosing a subsequence we can also suppose without loss of generality that $p \le \frac{1}{2}\sqrt{\log(|k_p|)}$.

Let $r_p = \frac{|k_p|^{-1}}{2p^2}$. For our purpose it is sufficient to prove that for a full measure set of $t \in \mathbb{T}^d$, eventually (as $p \to \infty$)

$$||n\alpha_1 + t_1|| < r_p, ..., ||n\alpha_d + t_d|| < r_p \implies n > \frac{d|k_p|^\gamma}{\log|k_p|}.$$

Let us fix p and choose $t=(t_1,...,t_d)$ such that

$$||t_1k_{1,p}+t_2k_{2,p}+...+t_dk_{d,p}||>\frac{d}{p^2}.$$

Now suppose n is such that $||n\alpha_1+t_1||<r_p,...,||n\alpha_d+t_d||<r_p$ and remark that

$$\begin{aligned}\frac{d}{p^2} &< ||k_p\cdot t|| = ||k_p\cdot(-n\alpha)+k_p\cdot(t+n\alpha)||\\ &\leq n||k_p\cdot\alpha||+d|k_p|||t+n\alpha||\\ &\leq n|k_p|^{-\gamma}+d|k_p|r_p\\ &\leq n|k_p|^{-\gamma}+\frac{d}{2p^2}\end{aligned}$$

and thus $n\geq\frac{d|k_p|^\gamma}{2p^2}\geq\frac{d|k_p|^\gamma}{\log|k_p|}$.

Now, it remains to show that the set of t such that $||t\cdot k_p||>\frac{d}{p^2}$ eventually has full measure.

Indeed, let us consider the $\mathbb{Z}^d$-periodic set

$$C_p=\{t\in\mathbb{R}^d\ s.t.\ ||t\cdot k_p||\leq\frac{d}{p^2}\}.$$

Now, choose an integer nonsingular matrix such that $Ae_1=k_p$ (e_1 is the first vector of the canonical basis of $\mathbb{R}^d$) then $C_p=\{t\in\mathbb{R}^d\ s.t.\ ||A^tt\cdot e_1||\leq\frac{d}{p^2}\}$. Now the measure of these sets must be evaluated on the torus $T^d=\mathbb{R}^d/\mathbb{Z}^d$. Let us denote by π the canonical projection $\mathbb{R}^d\to T^d$. Let us now consider the set $C'_p=\{x\in\mathbb{R}^d\ s.t.\ ||x\cdot e_1||\leq\frac{d}{p^2}\}$. First let us remark that $\mathrm{Leb}(\pi(C'_p))=\frac{2}{p^2}$. Now remark that being a nonsingular integer matrix, A^t induces a measure preserving map $\overline{A}^t$ on the torus and $\pi(C_p)=(\overline{A}^t)^{-1}[\pi(C'_p)]$, and then $\mathrm{Leb}(\pi(C_p))=\frac{2}{p^2}$. This allows us to apply Borel Cantelli lemma and deduce that the set of points belonging to C_p for infinitely many values of p is a zero measure set. ∎

We recall the classical notion of discrepancy for a rotation of angle $\alpha\in\mathbb{R}^d$. Let $\mathcal{P}^d$ be the set of rectangles $R=\prod_i[a_i,b_i]\subset[0,1]^d$. Let $|R|=\prod_i(b_i-a_i)$.

$$D_n(\alpha)=\sup_{R\in\mathcal{P}^d}\left|\frac{1}{n}\operatorname{card}\left\{0\leq k\leq n-1\colon k\alpha\in R+\mathbb{Z}^d\right\}-|R|\right|.$$

It is governed by the linear Diophantine type of α

Proposition 17 ([26]). *The discrepancy satisfies, for any $\gamma>\gamma_l(\alpha)$,*

$$D_n(\alpha)=O\left(n^{-1/\gamma}\right).$$

In addition, the second coordinate of the skew product evolves as a random rotation, thus we will need a notion of discrepancy of the random walk on the torus $\alpha S_n\varphi$ driven by the Gibbs measure μ, defined by

$$D_n^\mu(\alpha)=\sup_{R\in\mathcal{P}^d}\left|\mu\left(\{\omega\mid\alpha S_n\varphi(\omega)\in R+\mathbb{Z}^d\}\right)-|R|\right|.$$

In this random setting the upper bound become

Proposition 18 (After [34]). *The discrepancy satisfies, for any $\gamma > \gamma_l(\alpha)$,*

$$D_n^\mu(\alpha) = O\left((\sqrt{n})^{-1/\gamma}\right).$$

The proof is postponed to the appendix.

4.2. **Hitting time.** We start with an easy consequence of the proof of Proposition 16.

Theorem 19. *Suppose that $\gamma_l(\alpha) > \overline{d}_\mu(\pi_1(y)) + d$. Then, there exists a sequence $r_n \to 0$ such that the hitting time statistics to balls $B(y, r_n)$ has a trivial limiting distribution.*

In particular these systems are polynomially mixing but the distribution of hitting time to balls does not converge to the exponential law.

Proof. We assume without loss of generality that $\pi_2(y) = 0$. Let $\gamma < \gamma_l(\alpha)$ and $\epsilon > 0$ such that $\gamma - 2\epsilon > \overline{d}_\mu(\pi_1(y)) + d$. In the proof of Proposition 16 it is shown that there exists a sequence $r_p \to 0$ such that the hitting time to the r_p-neighborhood of $\mathbb{Z}^d$ by the sequence $t + n\alpha$ is at least $r_p^{-\gamma+\epsilon}$, for all $t \in \mathbb{T}^d$ outside a set of measure $2/p^2$. Since $S_n\varphi \le n$, this implies that $\tau_{r_p}(x, y) \ge r_p^{-\gamma+\epsilon}$. Let $s > 0$. For any r sufficiently small $\frac{s}{\nu(B(y,r))} < r_p^{-\gamma+\epsilon}$. Therefore, for any $s > 0$,

$$\nu\left(x \colon \tau_r(x, y) \le \frac{s}{\nu(B(y, r))}\right) \le \frac{2}{p^2}$$

provided p is sufficiently large. ∎

The preceding result allows us to estimate the hitting time exponents in our skew product.

Theorem 20. *In the skew product (M, S, ν) described above, at each target point $y \in M$ it hold*

(15)

$$\overline{R}(x, y) \ge \max(\overline{d}_\mu(\pi_1(y)) + d, \overline{hit}(\alpha)) \quad \text{and} \quad \underline{R}(x, y) \ge \max(\underline{d}_\mu(\pi_1(y)) + d, \underline{hit}(\alpha))$$

for ν-a.e. $x \in M$ (recall that π_i are the two canonical projections, as defined above).

Proof. The inequality $\overline{R}(x, y) \ge \overline{d}_\mu(\pi_1(y)) + d$ follows for ν-a.e. x from the general inequality (2), since the invariant measure is the product measure of μ and the Haar measure on the torus, then $\overline{d}_{\mu\times m}(y) = \overline{d}_\mu(\pi_1(y)) + d$. In particular $\tau_r(x, y) \to \infty$ as $r \to 0$.

Furthermore, for ν-a.e. $x = (\omega, t)$, the upper hitting time exponent $R(t, 0)$ for the rotation by angle α is equal to $\overline{hit}(\alpha)$. Let $h > \overline{hit}(\alpha)$. There exists a sequence $r_p \to 0$ such that $\|t + n\alpha\| > r_p$ for $n = 1, \ldots, \lfloor r_p^{-h} \rfloor$. Since $0 < S_n\varphi(\omega) \le n$ for any n sufficiently large and $\tau_{r_p}(x, y) \to \infty$, we get that $\tau_{r_p}(x, y) \ge r_p^{-h}$ for any p sufficiently large. hence $\overline{R}(x, y) \ge h$. The first inequality follows since h is arbitrary.

The corresponding statement for $\underline{R}$ can be done with a simplification of the same proof (since now the estimates will be valid for any r and not only for a sequence r_p). ∎

The question if the above general inequality in Theorem 20 is sharp arises naturally. We show that if μ is a Bernoulli measure and $d = 1$ this is the case.

Proposition 21. *We assume that all the branches of the Markov map T are full, i.e. $T(J_i) = \Omega$ for all i, that μ is a Bernoulli measure i.e. $\mu([a_1 \dots a_n]) = \mu([a_i]) \cdots \mu([a_n])$, and I depends only on the first symbol, i.e. I is an union of 1-cylinders (recall that $\varphi = 1_I$).*

Then for any y, for ν-a.e. x we have

$$\overline{R}(x, y) \leq \max(\overline{d}_\mu(\pi_1(y)) + d, d\gamma_l(\alpha)).$$

Remark 22. *Under the assumptions of the proposition:*

- *If $\gamma_l(\alpha) > d_\mu + d$ then the hitting time exponent is larger than the dimension of the invariant measure a.e.*
- *In dimension $d = 1$ we get the equality for all y*

$$\overline{R}(x, y) = \max(d_\mu(\pi_1(y)) + 1, \gamma(\alpha)),$$

for ν-a.e. x.

Proof. Without loss of generality we assume that $y = (\omega', 0)$.

Let $\xi = \max(\overline{d}_\mu(\omega') + d, d\gamma_l(\alpha))$ and let $\epsilon > 0$. Let $\gamma = \gamma_l(\alpha) + \epsilon/d$.

For $r > 0$ set $N_r = r^{-\xi-\epsilon}$. For all t we have by Proposition 17 that $D_{N_r}(\alpha) \leq N_r^{-\frac{1}{\gamma}}$ whenever r is sufficiently small, hence

$$\begin{aligned} \operatorname{card}\{k \leq N_r \colon \|t + k\alpha\| < r\} &\geq N_r \left((2r)^d - D_{N_r}(\alpha)\right) \\ &\geq r^{-\xi-\epsilon}(2^d r^d - r^{\frac{\xi+\epsilon}{\gamma}}) \\ &\geq r^{d-\xi-\epsilon}. \end{aligned} \tag{16}$$

Denote by $K_r = \{k_r^1 < k_r^2 < \cdots < k_r^p\}$ the set of integers $k < N_r$ such that $\|t + k\alpha\| < r$. Notice that k_r^1 is the first hitting time of $[-r, r]$ by the orbit of t under the rotation of angle α, therefore by Theorem 6, for a.e. t we have $\liminf_{r\to 0} \log k_r^1 / \log(1/r) \geq 1$. In particular for any r sufficiently small,

$$k_r^1 \geq -\log r. \tag{17}$$

We now fix some typical t and consider r so small that (16) and (17) hold. Let $b \in (0, \mu(A))$.

Let $G = \{\omega \in \Omega \colon \forall n = k_r^1, ..., N_r, S_n\varphi(\omega) > bn\}$. Let q be the minimal integer such that any q-cylinder has a diameter less than $r/2$. Note that $q = O(-\log r)$. Let B be the union of all q-cylinders that intersect the ball $B(\omega', r/2)$. When $\omega \in G$, if $S_n\varphi(\omega) \in K$ and $T^n(\omega) \in B$ then $\tau_r(\omega) := \tau_r((\omega, t), (\omega', 0)) \leq n \leq N_r/b$.

Let $t_j(\omega)$ be the smallest integer n such that $S_n\varphi(\omega) = k_j$. This means $S_n\varphi(\omega) = k_j > S_{n-1}\varphi(\omega)$. Since the increments of $S_n\varphi$ are 0 or 1 the sets $\Gamma_{(n_j)} := \{t_1 = n_1, \dots, t_p = n_p\}$, $n_1 < \cdots < n_p$, form a partition of Ω. Therefore

$$\mu(\tau_r > \frac{1}{b} N_r) \leq \mu(\Omega \setminus G) + \underbrace{\sum_{n_1 < \cdots < n_p} \mu(\Gamma_{(n_j)} \cap \bigcap_{j=1}^p T^{-n_j} B^c)}_{(\star)}.$$

A large deviations estimates show that there exists $c \in (0, 1)$ such that $\mu(S_n\varphi > bn) \leq c^n$ for any n sufficiently large. With (17) this gives the estimate for the first term

$$\mu(\Omega \setminus G) \leq \sum_{n=k_r^1}^{N_r} \mu(S_n \leq bn) \leq c^{k_r^1}/(1 - c) \leq r^{-\log c}/(1 - c).$$

We estimate the second term $(\star)$. Set $n_1' = n_1, n_2' = n_2 - n_1, \ldots, n_p' = n_p - n_{p-1}$ and define (k_j') in the same way. The set $\Gamma_{(n_j)} \cap \bigcap_j T^{-n_j} B^c$ is equal to the intersection of the sets

$$\begin{aligned}
&\{S_{n_1'}\varphi = k_1' > S_{n_1'-1}\varphi\},\\
&T^{-n_1}(B^c \cap \{S_{n_2'}\varphi = k_2' > S_{n_2'-1}\varphi\}),\\
&\vdots\\
&T^{-n_{p-1}}(B^c \cap \{S_{n_p'}\varphi = k_p' > S_{n_p'-1}\varphi\}),\\
&T^{-n_p}B^c.
\end{aligned}$$

Since φ depends only on the first symbol, these sets depend on different coordinates. Thus by the Bernoulli property the measure of the intersection is the product of the measures:

$$(\star) = \mu(\{S_{n_1'}\varphi = k_1' > S_{n_1'-1}\varphi\}) \prod_{j=2}^{p-1} \mu(B^c \cap \{S_{n_p'}\varphi = k_p' > S_{n_p'-1}\varphi,\})\mu(B^c).$$

Since for any $j = 1, \ldots, p$ we have

$$\sum_{n_j'=1}^{\infty} \mu(B^c \cap \{S_{n_j'}\varphi = k_j' > S_{n_j'-1}\varphi\}) = \mu(B^c) = 1 - \mu(B).$$

A summation over $n_1 < \cdots < n_p$ then gives

$$(\star) = (1 - \mu(B))^p.$$

By (16) we have $p \geq r^{d-\xi-\epsilon}$. Whenever r is sufficiently small we have

$$\mu(B) \geq \mu(B(\omega'^{\overline{d}_\mu(\omega')+\epsilon/2}.$$

Therefore

$$(1 - \mu(B))^p \leq \left(1 - r^{\overline{d}_\mu(\omega')+\frac{\epsilon}{2}}\right)^{\left[r^{-(\overline{d}_\mu(\omega')+\frac{\epsilon}{2})}r^{-\frac{\epsilon}{2}}\right]} \leq \exp(-r^{-\frac{\epsilon}{2}}).$$

Setting $r_n = e^{-n}$ we get that $\sum_n \mu(\tau_{r_n} > \frac{1}{b}N_{r_n}) < \infty$. By Borel-Cantelli we conclude that for μ-a.e. ω, if n sufficiently large then

$$\tau_{r_n}((\omega, t), (\omega', 0)) \leq \frac{1}{b} r_n^{-\xi-\epsilon}.$$

This implies that $\overline{R}((\omega, t), (\omega', 0)) \leq \xi + \epsilon$. The conclusion follows since ε is arbitrary. ∎

A bound for $\underline{R}(x, y)$ similar to the one given in Proposition 20 also hold. Since this is related with the dynamical Borel-Cantelli lemma and similar topics we postopone it to Section 6.1.

4.3. **Quantitative recurrence.**

Theorem 23. *We have $\underline{R}(x, x) \leq d_\mu + \frac{1+d_\mu}{\gamma_s(\alpha)}$ for ν-a.e. x.*

Proof. Since the recurrence does not depend on its initial value, we fix arbitrarily some $t \in \mathbb{T}^d$. Let $\epsilon > 0$, set $\gamma = \gamma_s(\alpha) - \epsilon$ and take $\delta = d_\mu + 2\epsilon$.

We consider a set K with $\mu(K) > 1 - \epsilon$ arbitrarily close to 1, such that for any r there exists a cover of K by balls of diameter r with cardinality less than $cr^{-d_\mu - \epsilon}$ for some constant $c > 0$ (see e.g. [3] for a detailed construction).

Fix some $q \in \mathbb{N}^*$ such that $\|q\alpha\| < q^{-\gamma}$. At this time q the rotation makes by definition a close return to the origin. The idea is that many of its multiples kq will inherit this property. More precisely, let $r = r(q) = q^{-\frac{\gamma}{1+\delta}}$. For any integer $k < r^{-\delta}$ we still have

$$\|kq\alpha\| \leq k\|q\alpha\| \leq r^{-\delta} q^{-\gamma} = r^{-\delta+\delta+1} = r.$$

Since $S^n(\omega, t) = (T^n(\omega), t + \alpha S_n\varphi(\omega))$, if an integer n satisfies (i) $S_n\varphi(\omega) \leq qr^{-\delta}$, (ii) $S_n\varphi(\omega)$ is a multiple of q and (iii) n is a r-return for T then n is an r-return for S.

Let F be the finite extension of T defined on $\Omega \times \mathbb{Z}_q$ by

$$F(\omega, z) = (T(\omega), z + \varphi(\omega)).$$

(we put the discrete metric on $\mathbb{Z}_q$). This map preserves the probability measure $\mu \times \frac{1}{q} H^0$ where H^0 is the counting measure on $\mathbb{Z}_q$.

Note that if $n \leq qr^{-\delta}$ is an r-return for F then $S_n\varphi(\omega) = 0 \mod q$, therefore n is also an r-return for S. Therefore if $\tau_r^F(\omega, z)$ denotes the r-return for F and $\tau_r^S(\omega, t)$ is the r-return for S then $\tau_r^S(\omega, t) \leq \tau_r^F(\omega, z)$ whenever the latter is less than $qr^{-\delta}$.

We now take a cover of K by balls $\{B_i\}$ of diameter r which has the properties mentioned before.

We have, repeating the computation in [3] (in particular using Kăc's lemma for F)

$$\begin{aligned}
\mu(\omega \in K \colon \tau_r^S(\omega, t) > qr^{-\delta}) &\leq \mu \times \frac{1}{q} H^0((\omega, z) \colon \tau_r^F(\omega, z) > qr^{-\delta}) \\
&\leq \sum_i \sum_{j \in \mathbb{Z}_q} \mu \times \frac{1}{q} H^0((\omega, z) \in B_i \times \{j\} \colon \tau^F_{B_i \times \{j\}}(\omega, z) > qr^{-\delta}) \\
&\leq \sum_i \sum_{j \in \mathbb{Z}_q} \frac{r^\delta}{q} \int_{B_i \times \{j\}} \tau^F_{B_i \times \{j\}} d\mu \times \frac{1}{q} H^0 \\
&\leq cr^\epsilon.
\end{aligned}$$

We now take a sequence $q_n \in \mathbb{N}^*$ going to infinity such that $\|q_n\alpha\| \leq q_n^{-\gamma}$ and $\sum_n r(q_n)^\epsilon < \infty$. By Borel-Cantelli we get that for μ-a.e. $\omega \in K$ we have

$$\tau_{r(q_n)}(\omega, t) \leq r(q_n)^{-\delta - \frac{1+\delta}{\gamma}} \tag{18}$$

for any n sufficiently large. Thus, writing $x = (\omega, t)$ we get

$$\underline{R}(x, x) \leq \delta + \frac{1+\delta}{\gamma},$$

for ν-a.e. $x \in K \times \mathbb{T}^d$. The conclusion follows letting $\epsilon \to 0$. ∎

We remark in particular that when γ_s is large, the lower quantitative recurrence exponent becomes smaller than the dimension of the measure ν, which is $d_\mu + d$.

The fact that this rapid recurrence occurs at the *same scale* for μ-a.e. points enables us to deduce the following result.

Theorem 24. *Suppose that* $\gamma_s(\alpha) > \frac{1+d_\mu}{d}$.
For ν*-a.e.* x*, there exists a subsequence* $r_n \to 0$ *such that the return time statistics in balls* $B(x, r_n)$ *has a trivial limiting distribution.*

In particular, these systems are polynomially mixing but the return time distribution to balls does not converge to the exponential law.

Proof. We use the same notation of the proof of Theorem 23. Consider $\epsilon > 0$ so small that $\delta + \frac{1+\delta}{\gamma} + 3\epsilon < d_\mu + d$. Take n_0 so large that the set H of points $\omega \in \Omega$ such that (18) holds for any $n > n_0$ has a measure $\mu(H) > 1 - \epsilon$.

Let $\omega \in \Omega$ be such that for any r sufficiently small $r^{d_\mu+\epsilon} \leq \mu(B(\omega, r)) < r^{d_\mu-\epsilon}$ and $\mu(B(\omega, r) \cap H)/\mu(B(\omega, r)) \to 1$ as $r \to 0$. The fact that the pointwise dimension is μ-a.e. equal to d_μ and Lebesgue density theorem shows that this concerns μ-a.e. points of H.

Given $r > 0$ we set $L_r = \lceil \log^2 r \rceil$. For any r sufficiently small we have
$$\frac{\mu(B(\omega, 2L_r r))}{\mu(B(\omega, L_r r))} \leq r^{-3\epsilon}.$$
Hence there exists an integer $k_r \in [L_r, 2L_r]$ such that
$$\mu(B(x, k_r r)) - \mu(B(x, (k_r - 1)r)) \leq \frac{1}{-\log r} \mu(B(x, k_r r)), \tag{19}$$
otherwise
$$\mu(B(\omega, 2L_r r))(1 - \frac{1}{-\log r})^{L_r} \geq \mu(B(\omega, L_r r))$$
which would contradict the previous inequality provided r is sufficiently small.

Let $r_n = k_{r(q_n)} r(q_n)$. If $\omega' \in H \cap B(x, r_n) \setminus B(x, r_n - r(q_n))$ we have by (18) that
$$\tau_{r_n}(\omega', t) \leq r(q_n)^{-d_\mu + d - 3\epsilon} \leq \mu(B(x, r_n)) r_n^\epsilon$$
for any n sufficiently large. For any $s > 0$, once $r_n \leq s$ we get, setting $x = (\omega, t)$,
$$\nu_{B(x,r_n)}(y \colon \tau_{r_n}(y) \geq \frac{s}{\mu(B(x, r_n))}) \leq \mu_{B(\omega,r_n)}(H^c \cap B(\omega, r_n - r(q_n))).$$
Using (19) and the fact that ω is a Lebesgue density point of H we conclude that the upper bound goes to zero as $n \to \infty$. ∎

Theorem 25. *In a skew product as above the lower recurrence rate is bounded from below by*
$$\underline{R}(x, x) \geq \min\left(\frac{d_\mu}{1 - \frac{1}{2\gamma_l(\alpha)}}, d_\mu + d\right) \quad \nu - a.e.x.$$

Proof. Let $\gamma > \gamma_l(\alpha)$. Let $\epsilon > 0$. Set $\Delta = \min\left((d_\mu - 3\epsilon)/(1 - 1/2\gamma), d_\mu + d\right)$.

We take a set K of measure $\mu(K) > 1 - \epsilon$ and $r_0 > 0$ such that for any $\omega \in K$ and $r \in (0, r_0)$ we have $\mu(B(\omega, 2r)) \leq r^{d_\mu - \epsilon}$.

Given $r > 0$ we fix k as the smallest integer such that any cylinder $Z \in \mathcal{J}_k$ has a diameter less than r. Let $Z \in \mathcal{J}_k$. Let $B(Z, r)$ be the union of balls $\bigcup_{\omega' \in Z} B(\omega', r)$. Hence if $Z \cap K \neq \emptyset$ then $\mu(B(Z, r)) \leq r^{d_\mu - \epsilon}$.

Given an integer n, let
$$W(r, k, n) = \{\omega \in \Omega \colon T^n(\omega) \in B(\omega, r) \text{ and } \|\alpha S_n \varphi(\omega)\| < r\}.$$

Recall that $\varphi = 1_I$ is constant on m-cylinders for some integer $m \geq 1$. Assume that $n \geq m + k$ and let us decompose its Birkhoff sum as

$$S_n\varphi = S_{k-m}\varphi + S_m\varphi \circ T^{k-m} + S_{n-k-m}\varphi \circ T^k + S_m\varphi \circ T^{n-m}.$$

Denote by E the range of $S_m\varphi$. We have $\operatorname{card} E \leq p^{2m} < \infty$. The sum $S_{k-m}\varphi$ is constant on Z and we denote its common value by q_Z. Then for any ω in Z we have $S_n\varphi(\omega) = q_Z + u + S_{n-k-m}\varphi(T^k\omega) + v$ for some $u, v \in E$. Notice that $S_{n-k-m}\varphi(\omega)$ is $\sigma(T^{-k}\mathcal{J}_{n-k})$ measurable, thus the ψ-mixing property[4] of the measure μ implies that

$$\begin{aligned}
\mu(K \cap W(r,k,n)) &= \sum_{Z \in \mathcal{J}_k} \mu(Z \cap K \cap W(r,k,n)) \\
&\leq \Psi(0)^2 \sum_{Z \in \mathcal{J}_k, Z \cap K \neq \emptyset} \sum_{u,v \in E} \\
&\qquad \mu(Z \cap T^{-k}\{\|\alpha(q_Z + u + v + S_{n-k-m}\varphi)\| < r\} \cap T^{-n}B(Z,r)) \\
&\leq \Psi(0)^2 \sum_{Z \in \mathcal{J}_k, Z \cap K \neq \emptyset} \sum_{u,v \in E} \\
&\qquad \mu(Z)\mu(\|\alpha(q_Z + u + v + S_{n-k-m}\varphi)\| < r)\mu(B(Z,r)) \\
&\leq \Psi(0)^2 p^{4m} r^{d_\mu - \epsilon} \left(D^\mu_{n-k-m}(\alpha) + (2r)^d\right).
\end{aligned}$$

Take $\delta \in (0, \min(d_\mu, \Delta))$. When $r^{-\delta} \leq n \leq r^{-\Delta}$ we have, for some constant c,

$$\mu(K \cap W(r,k,n)) \leq c r^{d_\mu - \epsilon} \left[n^{\frac{-1}{2\gamma}} + r^d\right].$$

Therefore

$$\sum_{r^{-\delta} \leq n \leq r^{-\Delta}} \mu(K \cap W(r,k,n)) = O\left(c r^{d_\mu - \epsilon}\left[(r^{-\Delta})^{1-1/2\gamma} + r^d\right]\right) = O(r^\epsilon),$$

by our choice of Δ. This shows that the probability that for some $\omega \in K$ there is a r-return of $x = (\omega, t)$ between the times $r^{-\delta}$ and $r^{-\Delta}$ is $O(r^\epsilon)$. By a Borel Cantelli argument, for μ-a.e. $\omega \in K$, there are no returns in this time interval. On the other hand, by Theorem 4 the recurrence rate for the base map T only is equal to $d_\mu > \delta$, therefore there are no returns in the time interval $1, \ldots, r^{-\delta}$ also. The result follows since ϵ is arbitrary. ∎

4.4. **Observed systems.** In the previous sections, we studied the return time and hitting time of the skew-product and we showed that their exponents depend on the arithmetical properties of α. A natural question is to study only the return of the second coordinate instead of the whole system. We are going to study the Poincaré recurrence for a specific observation [31], the canonical projection $\pi_2 : \Omega \times \mathbb{T}^d \to \mathbb{T}^d$ and prove that the recurrences rates for the observation do not depend on T.

Definition 26. *Let $x \in \Omega \times \mathbb{T}^d$, $r > 0$ and $p \in \mathbb{N}$. We define the p-non-instantaneous return time for the observation*

$$\tau^{obs}_{r,p}(x,x) = \inf\left\{k > p,\ \pi_2\left(S^k(x)\right) \in B\left(\pi_2(x), r\right)\right\}.$$

[4]The ψ-mixing property is the following: there exists some sequence $\Psi(n) \searrow 0$ such that for any k, any $A \in \sigma(\mathcal{J}_k)$ and any measurable set B we have

$$|\mu(A \cap T^{-k-n}B) - \mu(A)\mu(B)| \leq \Psi(n)\mu(A)\mu(B).$$

As previously, we define the lower and upper non-instantaneous recurrence rates for the observation

$$\underline{R}^{obs}(x,x)=\lim_{p\to\infty}\liminf_{r\to 0}\frac{\log\tau_{r,p}^{obs}(x,x)}{-\log r}$$

and

$$\overline{R}^{obs}(x,x)=\lim_{p\to\infty}\limsup_{r\to 0}\frac{\log\tau_{r,p}^{obs}(x,x)}{-\log r}.$$

In the following proposition, we will prove that the recurrence rates for this observation only depends on the underlying rotation.

Proposition 27. *For ν-almost every $x\in M$*

$$\underline{R}^{obs}(x,x)=\underline{rec}(\alpha)(=\frac{1}{\gamma_s(\alpha)})\qquad \textit{and}\qquad \overline{R}^{obs}(x,x)=\overline{rec}(\alpha).$$

Proof. First of all, we can remark that for every $x=(\omega,t)\in\Omega\times\mathbb{T}^d$, every $r>0$ and every $p\in\mathbb{N}^*$

$$\tau_{r,p}^{obs}(x,x)\geq\tau_r(t,t)$$

where $\tau_r(t,t)$ is the return time of t with respect to the rotation with angle α. This gives us that

$$\underline{R}^{obs}(x,x)\geq\underline{rec}(\alpha)\qquad\text{and}\qquad\overline{R}^{obs}(x,x)\geq\overline{rec}(\alpha).$$

On the other hand, we have

$$\begin{aligned}\tau_{r,p}^{obs}(x,x) &= \inf\left\{k>p,\,S^k(\omega,t)\in\Omega\times B(t,r)\right\}\\ &= \inf\left\{k>p,\,t+\alpha S_k\varphi(\omega)\in B(t,r)\right\}\\ &= \inf\left\{k>p,\,\|\alpha S_k\varphi(\omega)\|\leq r\right\}.\end{aligned}\tag{20}$$

Birkhoff ergodic theorem gives for μ-almost every $\omega\in\Omega$

$$\lim_{k\to+\infty}\frac{1}{k}S_k\varphi(\omega)=\mu(I)=:a.$$

Then, for μ-almost every $\omega\in\Omega$ there exists $N\in\mathbb{N}$ such that $\forall k>N$, $S_k\varphi(\omega)>\frac{a}{2}k$.

By definition of $\underline{rec}(\alpha)$, there exists a sequence $r_n\to 0$ and a sequence of integers $q_n\geq p$ such that $\tau_{r_n}(t,t)\leq q_n$ and $\lim\frac{q_n}{-\log r_n}=\underline{rec}(\alpha)$. For all $n>Na/2$ we have $S_{\lceil\frac{2}{a}q_n\rceil}\varphi(\omega)>q_n$ and thus

$$\tau_{r_n,p}^{obs}(x,x)\leq\frac{2}{a}q_n.$$

Letting n goes to infinity and then p to infinity proves that

$$\underline{R}^{obs}(x,x)\leq\underline{rec}(\alpha).$$

We proceed similarly with $\overline{R}^{obs}$. ∎

The hitting time of the observed system may be defined similarly,

Definition 28. *Let $x,y\in\Omega\times\mathbb{T}^d$, $r>0$. We define the hitting time for the observation*

$$\tau_r^{obs}(x,y)=\inf\left\{k>0,\,\pi_2\left(S^k(x)\right)\in B\left(\pi_2(y),r\right)\right\}.$$

As previously, we define the lower and upper hitting time exponents for the observation

$$\underline{R}^{obs}(x,y) = \liminf_{r\to 0} \frac{\log \tau_r^{obs}(x,y)}{-\log r}$$

and

$$\overline{R}^{obs}(x,y) = \limsup_{r\to 0} \frac{\log \tau_r^{obs}(x,y)}{-\log r}.$$

Also for the hitting time, with an obvious modification of the proof for return times, one can show that the exponents do not depend on T:

Proposition 29. *For every $y \in M$ and for ν-almost every $x \in M$*

$$\underline{R}^{obs}(x,y) = \underline{hit}(\alpha) \qquad \textit{and} \qquad \overline{R}^{obs}(x,y) = \overline{hit}(\alpha).$$

5. Decay of correlations, lower bounds

In [19, Corollary 3] the following relation between decay of correlations and hitting time exponent is proved :

Proposition 30. *If a map on a finite dimensional Riemannian manifold M, has an absolutely continuous invariant measure with strictly positive density at some point y_0[5] and s.t. $\overline{R}(x,y_0) = R_0$, x-a.e., then the speed of decay of correlations (over Lipschitz observables) of the system is at most a power law with lower exponent (see Definition 2)*

$$p = \lim\inf_{n\to\infty} \frac{-\log \Phi(n)}{\log n} \le \frac{2\dim M + 2}{R_0 - \dim M}.$$

We remark that the assumptions on the absolute continuity of the invariant measure can be largely relaxed (see [19]). By this proposition and Proposition 20 we have easily:

Proposition 31. *The decay of correlations with respect to Lipschitz observables of a skew product on $M = \Omega \times \mathbb{T}^d$, such that μ is absolutely continuous with positive density is a power law with exponent*

$$p \le \frac{2\dim M + 2}{\max(\dim M, \gamma_l(\alpha)) - \dim M}. \tag{21}$$

These bounds are probably not sharp but the advantage is that they clearly show the dependence on the Diophantine type of α. A comparison with the upper bound given in Proposition 9 gives that for the skew product of the doubling map and a circle rotation endowed with the Lebesgue measure, the exponent p satisfies

$$\frac{1}{2\gamma(\alpha)} \le p \le \frac{6}{\max(2,\gamma(\alpha)) - 2}.$$

[5]The density is greater than some positive number in a neighborhood of y_0.

6. Systems with $\underline{R}(x,y) > d$ and consequences

6.1. **A skew product with lower hitting time exponent larger than the dimension.** With a suitable torus translation it is possible to obtain a system [19] where the *lower* hitting time exponent is bigger than the local dimension ($\underline{R}(x,y) > d_\nu$ for typical x, y) this leads to some other consequence as the lack of a dynamical Borel-Cantelli property, or the triviality of the limit return time distribution.

Using a rotation of this kind in a skew product we obtain

Theorem 32. *There exists a system which has polynomial decay of correlation over C^r observables, superpolynomial decay w.r.t. C^∞ ones and*

- *its lower exponent of decay of correlations with respect to Lipschitz observables satisfies $\frac{1}{16} \leq \liminf_{n\to\infty} \frac{-\log \Phi(n)}{\log n} \leq \frac{8}{13}$;*
- *the lower hitting time indicator $\underline{R}(x,y)$ is bigger than the local dimension;*
- *it has no Monotone Shrinking Target property;*
- *it has trivial limit return time statistics.*

Let us describe this example. Let us consider a rotation $T_{\boldsymbol{\alpha}}$, $\alpha{=}(\alpha_1,\alpha_2)$ on the torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ by the angle α. Suppose γ_1, γ_2 are respectively the types of α_1 and α_2 and denote by q_n and q'_n the partial convergent denominators of α_1 and α_2.

To obtain a rotation with large lower hitting time indicator let us take $\xi > 2$ and let $Y_\xi \subset \mathbf{R}^2$ be the class of couples of irrationals (α_1, α_2) given by the following conditions on their convergents to be satisfied eventually:

$$q'_n \geq q_n^\xi;$$
$$q_{n+1} \geq {q'_n}^\xi.$$

We note that each Y_ξ is uncountable and dense in $[0,1]\times[0,1]$ and each irrational of the couple is of type at least ξ^2. Indeed if α_1 is such a number: $q_{n+1} \geq q_n^{\xi^2}$ (then $q_n \geq q_1^{\xi^{2n}}$) and then $\gamma(\alpha_1) \geq \limsup_{n\to\infty} \frac{\log q_n^{\xi^2}}{\log q_n} = \xi^2$. We also remark that $\gamma_l(\alpha) \geq \max(\gamma(\alpha_i)) \geq \xi^2$.

With some more work (see Propositions 43 and 44 in Subsection 6.4) it is possible to obtain the following:

Proposition 33. *There is an angle $\alpha \in Y_4$ which is of finite Diophantine type for the linear approximation. More precisely $\gamma_l(\alpha) = 16$.*

This implies that the following example exists.

Example 34. *Let us consider the skew product (M,S) where $M = [0,1]\times\, \mathbb{T}^2$ with $\alpha \in Y_4$, such that $\gamma_l(\alpha) = 16$ and T preserves an absolutely continuous invariant measure μ with strictly positive density.*

We will see below that the example satisfies all items of Theorem 32.

The reason we take a rotation with angle in Y_ξ is that the lower hitting time indicator is bounded from below by ξ.

Theorem 35 ([19]). *If $T_{(\alpha_1,\alpha_2)}$ is a rotation of the two torus by a vector $(\alpha_1,\alpha_2) \in Y_\xi$ and $y \in \mathbb{T}^2$, then for almost every $x \in \mathbb{T}^2$*

$$\underline{R}(x,y) \geq \xi.$$

In particular we remark that if $\xi > 2$ then in this example the lower hitting time indicator is bigger than the local dimension. Using Theorem 20 we get:

Proposition 36. *In a skew product with a rotation by an angle included in Y_ξ, at each target point y it hold*

$$\underline{R}(x,y) \geq \max(\underline{d}_\mu(\pi_1(y)) + 2, \xi) \tag{22}$$

for ν-a.e. x.

Now we can obtain the first item in Theorem 32.

Proposition 37. *In a skew product on $M = \Omega \times T^2$ with a rotation by an angle $\alpha \in Y_4$ with $\gamma_l(\alpha) = 16$ as above, the decay of correlations with respect to Lipschitz observables satisfies*

$$\frac{1}{16} \leq \liminf_{n\to\infty} \frac{-\log \Phi(n)}{\log n} \leq \frac{8}{13}.$$

Proof. The second inequality is obtained putting $d = 2, \overline{R} \geq 4^2$,(see Theorem 20, recalling that if $\alpha = (\alpha_1, \alpha_2)$ then $\gamma_l(\alpha) \geq \max_i(\gamma(\alpha_i)))$ in Proposition 30. The first inequality is obtained by Proposition 9. ∎

Obviously the requirement on the dimension of μ is independent from the others, and such system exists. Hence we have the second item of Theorem 32.

Proposition 38. *In the above example $\underline{R}(x,y) > d_\nu(y)$ almost everywhere.*

Proof. Recall that we consider $\beta \in Y_4$ $\underline{R}(x,y) \geq 4$ for each y and almost each x. Here $d_\nu(y) = 3$ almost everywhere. ∎

In the next subsections we clarify and discuss the other items of the proposition and finally prove the existence of the angles mentioned in Proposition 33.

6.2. **No dynamical Borel-Cantelli property and monotone shrinking targets.** Let us consider a family of balls $B_i = B_{r_i}(y)$ with $i \in \mathbb{N}$ centered in y and such that $r_i \to 0$. In several systems, mostly having some sort of fast decay of correlations or generical arithmetic properties, the following generalization of the second Borel-Cantelli lemma can be proved:

$$\sum \nu(B_i) = \infty \Rightarrow \nu(\limsup_i T^{-i}(B_i)) = 1 \tag{23}$$

or equivalently $T^i(x) \in B_i$ for infinitely many i, when x is typical with respect to ν. We recall that there are mixing systems where the above statement does not hold (an example was given in [10], nevertheless in this example the speed of decay of correlations is less than polynomial, see [19]). From what it is said above it easily follows that in our examples (which are polynomially mixing with respect to C^r observables and superpolynomial mixing with respect to C^∞ ones) the above property is also violated hence proving the second item of Theorem 32.

Definition 39. *We say that the system has the monotone shrinking target property if* (23) *holds for every decreasing sequence of balls in X with the same center.*

In [16] the following fact is proved:

Theorem 40. *Assume that there is no atom in X, if the system has the monotone shrinking target property, then for each y we have*

$$\liminf_{r\to 0} \frac{\log \tau_r(x,y)}{-\log \nu(B_r(y))} = 1 \quad \textit{for } \nu\textit{-a.e. } x.$$

We hence easily have:

Proposition 41. *In a skew product as above over $M = \Omega \times T^2$, if $\boldsymbol{\alpha} \in Y_\xi$ and $\xi > d_\mu + d$ then the system has not the monotone shrinking target property.*

Proof. Let us consider y such that the local dimension $d_\mu(\pi_2(y)) = d_\mu$ (this is a full measure set). The statement follows from the definition of local dimension: it holds $\frac{\log \mu(B_r(y))}{\log r} \to d_\mu(\pi_1(y)) + d$. Moreover

$$\liminf_{r\to 0}\Big(\frac{\log \tau_{B_r(y)}(x)}{-\log \mu(B_r(y))}\frac{\log \mu(B_r(y))}{\log r}\Big) = \underline{R}(x,y) \tag{24}$$

by definition. Since in our examples $\underline{R}(x,y) > d_\mu + d$. Then $\lim \frac{\log \tau_{B_r(y)}(x)}{-\log \mu(B_r(y))} > 1$ and then by Theorem 40 the system has not the monotone shrinking target property. ∎

6.3. **Trivial return and hitting time distribution.** The following statement shows that if the logarithm law (hitting time exponent=dimension) does not hold then the return time statistic has a trivial limit (compare with Theorem 24 where the result holds for a subsequence).

Theorem 42. *([15]) If (X, T, ν) is a finite measure preserving system over a metric space X and*

$$\underline{R}(x,y) > \overline{d}_\nu(y) \tag{25}$$

a.e., then the system has trivial limit return time statistic in sequence $B_r(y)$. That is, the limit in (9) exists for each $t > 0$ and $g(t) = 0$.

The mixing system without logarithm law given in [19] hence has trivial return limit statistic in each centered sequence of balls, this gives an example of a smooth mixing systems with trivial limit return time statistics. As said before, this example, as the Fayad example has slower than polynomial decay of correlations. Since in Example 34 we found a system with polynomial decay of correlations with respect to Lipschitz observables but lower hitting time indicator is bigger than the local dimension hence the third item of Theorem 32 is established.

6.4. **Construction of a diophantine angle with intertwined partial quotients.** We briefly recall the basic definitions and properties of continued fractions (for a general reference see e.g. [6]) that will be needed in the sequel. Let α be an irrational number, denote by $[a_0; a_1, a_2, \ldots]$ its continued fraction expansion:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} =: [a_0; a_1, a_2, \ldots].$$

The integers $a_0, a_1, a_2 \ldots$ are called partial quotients of α and are all positive except for a_0. As usual, we define inductively the sequences p_n and q_n by:

$$\begin{aligned} p_{-1} &= 1, \quad p_0 = 0, \quad p_{k+1} = a_{k+1}p_k + p_{k-1} \text{ for } k \geq 0; \\ q_{-1} &= 0, \quad q_0 = 1, \quad q_{k+1} = a_{k+1}q_k + q_{k-1} \text{ for } k \geq 0. \end{aligned}$$

The fractions p_n/q_n are called the *convergents* of α, as they do in fact converge to it. Moreover they can be seen as *best approximations* of α in the following sense.

As usual denote by $\|x\| := \min_{n\in\mathbb{Z}} |x-n|$ the distance of a real number form the integers. Then $q = q_n$ for some n if and only if

$$\|q\alpha\| < \|q'\alpha\| \text{ for every positive } q' < q$$

and p_n is the integer such that $\|q_n\alpha\| = |q_n\alpha - p_n|$.

We will see that there are 2 dimensional angles which are of finite type for the linear approximation, but have intertwined partial quotients, in a way that they belong to some nontrivial class Y_ξ, with $\xi = 4$, which is big enough to have an example which can be used in Section 6.1. The proof is quite similar to the ones given in [9, Section 7]. Since we need a slightly different result and we need to change some step we write it explicitly here.

Proposition 43. *There are α and α'such that their partial quotients q_n, q'_n satisfy for any n:*

(1) $q'^4_{n-1} \le q_n \le 4q'^4_{n-1}$
(2) $q^4_n \le q'_n \le 4q^4_n$
(3) $q_n \wedge q'_{n-1} = 1$
(4) $q'_n \wedge q_n = 1$

where $a \wedge b = 1$ means that a and b are relatively prime.

Proof. We will construct α, α' and their partial quotients q_n, q'_n by constructing the appropriate coefficients a_n, a'_n.

Now let us proceed by induction, assuming that $a_0, ..., a_{n-1}$, $a'_0, ..., a'_{n-1}$ have been constructed satisfying items 1–4, now let us construct a_n.

Since $q'_{n-1} \wedge q_{n-1} = 1$ then there is an integer $\tau_n < q'_{n-1}$ such that

$$\tau_n q_{n-1} \equiv -q_{n-2} (\text{mod } q'_{n-1}),$$

so that q'_{n-1} divides $\tau_n q_{n-1} + q_{n-2}$. Now, choose ρ_n is such that $\rho_n \wedge q'_{n-1} = 1$ and

$$q'^4_{n-1} \le \rho_n q_{n-1} \le 2q'^4_{n-1}$$

and define $a_n = \tau_n + \rho_n$. With this choice of a_n

$$q_n = a_n q_{n-1} + q_{n-2} = \rho_n q_{n-1} + \tau_n q_{n-1} + q_{n-2}.$$

Since $\tau_n \le q'_{n-1}$ we have $q'^4_{n-1} \le q_n \le 4q'^4_{n-1}$ and item 1) is satisfied at step n. On the other hand, from the inductive hypotesis $q_{n-1} \wedge q'_{n-1} = 1$ and from our choice of ρ_n it follows $\rho_n q_{n-1} \wedge q'_{n-1} = 1$, on the other hand $\tau_n q_{n-1} + q_{n-2}$ is a multiple of q'_{n-1}. Consequently $q_n \wedge q'_{n-1} = 1$, and item 3) of the inductive step is proved at step n.

Now we construct a'_n in the same way: $a'_n = \tau'_n + \rho'_n$ with

$$\tau'_n q'_{n-1} \equiv -q'_{n-2} (\text{mod } q_n),$$

and ρ'_n is such that $\rho'_n \wedge q_n = 1$ and $q^4_n \le \rho'_n q'_{n-1} \le 2q^4_n$ (recall that q_n was already constructed just above) then

$$q'_n = a'_n q'_{n-1} + q'_{n-2} = \rho'_n q'_{n-1} + \tau'_n q'_{n-1} + q'_{n-2},$$

and hence

$$q^4_n \le q'_n \le 4q^4_n,$$

proving Item 2). Finally, we use the already proved relation: $q_n \wedge q'_{n-1} = 1$ to obtain $q'_n \wedge q_n = 1$ (Item 4) at step n as before. ∎

Proposition 44. *The vector (α, α') described above is such that*

$$\gamma_l((\alpha, \alpha')) = 16.$$

Proof. As remarked before, it is easy to see that $\gamma(\alpha) = \gamma(\alpha') = \xi^2 = 16$, hence $\gamma_l((\alpha, \alpha')) \geq 16$. For the opposite inequality, we have to show that for any $(k, l) \in \mathbb{Z}^2$, with $\max(|k|, |l|)$ sufficiently large

$$||k\alpha + l\alpha'|| \geq \frac{1}{(\max(|k|, |l|))^{16}}.$$

If k or $l = 0$ then the problem is reduced to a one dimensional one, and the exponent is the diophantine exponent of α and α' which was already remarked to be 16.

Now, let us suppose $l, k \neq 0$ and $q'_{n-1} \leq \max(|k|, |l|) \leq q'_n$ for some n.

Let us suppose $q'_{n-1} \leq \max(|k|, |l|) \leq q_n$ (the case $q_n \leq \max(|k|, |l|) \leq q'_n$ will be treated below).

We recall some general properties of continued fractions:

$$\begin{aligned} |\alpha - \frac{p_n}{q_n}| &\leq \frac{1}{q_n q_{n+1}}, \\ |\alpha' - \frac{p'_{n-1}}{q'_{n-1}}| &\leq \frac{1}{q'_{n-1} q'_n}. \end{aligned}$$

As $|k| \leq q_n$, $|l| \leq q_n$

$$\begin{aligned} |k\alpha + l\alpha' - k\frac{p_n}{q_n} - l\frac{p'_{n-1}}{q'_{n-1}}| &\leq \frac{1}{q_{n+1}} + \frac{q_n}{q'_n q'_{n-1}} \\ &\leq \frac{1}{q_n^{16}} + \frac{q_n}{q_n^4 q'_{n-1}} \\ &= o(\frac{1}{q_n q'_{n-1}}). \end{aligned}$$

for n (and consequently $\max(|k|, |l|)$) sufficiently large. On the other hand, since $q_n \wedge q'_{n-1} = 1$ and $q_n \wedge p_n = 1$, $k \leq q_n$ implies

$$||k\frac{p_n}{q_n} - l\frac{p'_{n-1}}{q'_{n-1}}|| \geq \frac{1}{q'_{n-1} q_n}.$$

With the above estimation we have that for large n

$$||k\alpha - l\alpha'|| \geq \frac{1}{2q'_{n-1} q_n}$$

thus using again the inequalities between the various q_n and q'_n we have

$$||k\alpha - l\alpha'|| \geq \frac{1}{8q'^5_{n-1}}.$$

Since $q'_{n-1} \leq \max(|k|, |l|)$ we obtain

$$||k\alpha - l\alpha'|| \geq \frac{1}{8(\max(|k|, |l|))^5}. \tag{26}$$

Now let us consider the case $q_n \leq \max(|k|, |l|) \leq q'_n$:

again

$$\begin{aligned}|\alpha - \frac{p_n}{q_n}| &\leq \frac{1}{q_n q_{n+1}},\\ |\alpha' - \frac{p'_n}{q'_n}| &\leq \frac{1}{q'_n q'_{n+1}}.\end{aligned}$$

As $|k| \leq q'_n$, $|l| \leq q'_n$

$$\begin{aligned}|k\alpha + l\alpha' - k\frac{p_n}{q_n} - l\frac{p'_n}{q'_n}| &\leq \frac{q'_n}{q_n q_{n+1}} + \frac{1}{q'_{n+1}}\\ &\leq \frac{1}{q_n'^8} + \frac{q'_n}{q_n'^4 q_n}\\ &= o(\frac{1}{q_n q'_n}).\end{aligned}$$

On the other hand, since $q_n \wedge q'_{n-1} = 1$ and $q_n \wedge p_n = 1$, $k \leq q'_n$ implies

$$||k\frac{p_n}{q_n} - l\frac{p'_n}{q'_n}|| \geq \frac{1}{q'_n q_n}.$$

With the above estimation we have that for large n

$$||k\alpha - l\alpha'|| \geq \frac{1}{2q'_n q_n}$$

thus using again the inequalities between the various q_n and q'_n we have

$$||k\alpha - l\alpha'|| \geq \frac{1}{8q_n^5}.$$

Since $q_n \leq \max(|k|, |l|)$ we obtain

$$||k\alpha - l\alpha'|| \geq \frac{1}{8(\max(|k|, |l|))^5}. \tag{27}$$

Finally, (26) together with (27) imply that for any $(k, l) \in \mathbb{Z}^2$, with $\max(|k|, |l|)$ sufficiently large

$$||k\alpha - l\alpha'|| \geq \frac{1}{8(\max(|k|, |l|))^5} \geq \frac{1}{(\max(|k|, |l|))^{16}}.$$

∎

7. Appendix

7.1. **Changing regularity of the observables.** We prove the lemma which allows to give a better bound on the decay of correlations, by passing through higher regularity observables.

Proof of Lemma 3. Let $k \leq p$ and $\ell \leq q$. Let $f \in C^k$ and $g \in C^\ell$ be Lipschitz observables such that $\int f d\mu = \int g d\mu = 0$. Let us consider a regularization by convolution. As usual, let us consider a function $\rho \in C^\infty(\mathbb{R}^d)$, $\rho \geq 0$ having support in $B_1(0)$ and such that $\int \rho dx = 1$. Let $\epsilon > 0$. Then consider $\rho_\epsilon(x) = \epsilon^{-d}\rho(\frac{x}{\epsilon})$, this function has support in $B_\epsilon(0)$ and still $\int \rho_\epsilon dx = 1$. Let us consider a multiindex $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$, then remark that

$$D^\alpha \rho_\epsilon = \epsilon^{-d} D^\alpha \left(\rho\left(\frac{x}{\epsilon}\right)\right) = \epsilon^{-d}\epsilon^{-|\alpha|}\left(D^\alpha \rho\right)\left(\frac{x}{\epsilon}\right)$$

hence

$$\|D^\alpha \rho_\epsilon\|_{L^1} = \int |D^\alpha \rho_\epsilon| dx = \epsilon^{-|\alpha|} \int |D^\alpha \rho| dx.$$

Now consider f_ϵ defined by

$$f_\epsilon(x) = f * \rho_\epsilon(x) = \int f(y)\rho(x-y)dy$$

and $g_\epsilon = g * \rho_\epsilon$ (the convolution of g and ρ_ϵ with respect to Lebesgue measure on $\mathbb{R}^d$). For any $\beta \le \alpha$ such that $|\beta| \le k$ we get

$$D^\alpha f_\epsilon(x) = \int D^\beta f(y)\ D^{\alpha-\beta}\rho_\epsilon(x-y)dy \le \|f\|_{C^k}\|D^{\alpha-\beta}\rho_\epsilon\|_{L^1}.$$

This implies that

$$\begin{aligned}
\|f_\epsilon\|_{C^p} &\le \|f\|_{C^k} \sup_{|\alpha|\le p-k} \|(D^\alpha \rho_\epsilon)\|_{L^1} \\
&\le \|f\|_{C^k} \sup_{|\alpha|\le k} \epsilon^{-(p-k)} \int |D^\alpha(\rho)| dx \\
&\le C\epsilon^{-(p-k)}\ \|f\|_{C^k}
\end{aligned}$$

where C is a constant depending on the function ρ.

Moreover, for any ϵ and δ, it holds that $||f - f_\epsilon||_\infty \le \epsilon^k\ ||f||_{C^p}$, $||g - g_\delta||_\infty \le \delta^\ell\ ||g||_{C^\ell}$. Now let us estimate the decay of correlations of f and g by their regularized functions:

$$\begin{aligned}
&\left|\int f\circ T^n g d\nu\right| \\
\le\ & \left|\int (f\circ T^n + f_\epsilon\circ T^n - f_\epsilon \circ T^n)\ (g + g_\delta - g_\delta) d\nu\right| \\
\le\ & \int |(f\circ T^n - f_\epsilon\circ T^n)\ (g - g_\delta)|\, d\nu + \int |(f\circ T^n - f_\epsilon\circ T^n)\ (g_\delta)|\, d\nu \\
& + \int |(f_\epsilon\circ T^n)\ (g-g_\delta)| d\nu + \int |(f_\epsilon\circ T^n)\ (g_\delta)| d\nu \\
\le\ & \epsilon^k\delta^\ell\ ||f||_{C^k}\ ||g||_{C^\ell} + \epsilon^k\ ||f||_{C^k}\ ||g||_1 + \delta^\ell\ ||f||_1\ ||g||_{C^\ell} + \|f_\epsilon\|_{C^p}\|g_\delta\|_{C^q}\Phi_{k,\ell}(n) \\
\le\ & \epsilon^k\delta^\ell\ ||f||_{C^k}\ ||g||_{C^\ell} + \epsilon^k\ ||f||_{C^k}\ ||g||_1 + \delta^\ell\ ||f||_1\ ||g||_{C^\ell} + C\epsilon^{k-p}\delta^{q-\ell}\ \|f\|_{C^k}\|g\|_{C^\ell}\ \Phi_{p,q}(n) \\
\le\ & ||f||_{C^p}\ ||g||_{C^q}(\epsilon^k\delta^\ell + (\epsilon^k + \delta^\ell)\ (1 + \mathrm{diam}(X)) + C\epsilon^{k-p}\delta^{\ell-q}\Phi_{p,q}(n)).
\end{aligned}$$

since $||f||_1 \le \|f\|_{C^p}(1+\mathrm{diam}(X))$. This will be essentially minimized when $\epsilon^k = \delta^\ell = \epsilon^{k-p}\delta^{\ell-q}\Phi_{p,q}(n)$. This gives

$$\left|\int f\circ T^n g d\nu\right| \le ||f||_{C^k}\ ||g||_{C^\ell}\Phi(n)^{\frac{k\ell}{p\ell+qk-k\ell}}.$$

In the case $k=1$ or $\ell = 1$ the same estimate is valid for Lipschitz observables[6] with the C^1 norm replaced by the Lipschitz one. ∎

[6]We recall that, by Rademacher Theorem, a Lipschitz function has derivatives defined almost-everywhere and the derivatives are a.e. bounded by the Lipschitz constant of the function. This clearly suffices to make the proof works.

7.2. **Discrepancy estimates.** We prove here Proposition 18.

Proof. We follow the line of [34] (Theorem 5.5) who established this result in the case of the simple symmetric random $\pm\alpha$ on the circle (that is $d=1$ there and the increments where $\pm\alpha$ with independent equal probability).

The discrepancy of a probability distribution Q on the torus is defined by the maximal difference, among all rectangles R of $\mathcal{P}^d$, between $Q(R)$ and its Lebesgue measure. Recall Erdös-Túran-Koksma inequality [26] who estimates the discrepancy in terms of the Fourier transform Φ_Q:

$$D(Q, Leb) \leq 3^s \left(\frac{2}{H+1} + \sum_{0<|h|_\infty \leq H} \frac{1}{r(h)} |\Phi_Q(h)| \right), \tag{28}$$

where $r(h) = \prod_{i=1}^d \max(1, |h_i|)$. Let $D = \{1, \ldots, d\}$. We apply it with Q the distribution of $\alpha S_N \varphi$ under μ which satisfies

$$|\Phi_Q(h)| = \left| \int_\Omega e^{-2i\pi\langle h, \alpha S_N\varphi\rangle} d\mu \right| = \left| \int_\Omega L^N_{2\pi\langle h,\alpha\rangle}(1) d\mu \right| \leq c_0 e^{-c_1 N \langle h,\alpha\rangle^2},$$

by Proposition 13.

Let $\gamma > \gamma_l(\alpha)$. There exists some constant c_2 such that for any $h \neq 0$, $\|\langle h, \alpha\rangle\| > c_2 |h|^{-\gamma}$.

For $K \subset D$ and $H \in \mathbb{N}$ let

$$H^K = \{h \in \mathbb{N}^d \colon 0 \leq h_i \leq H \text{ if } i \in K, h_i = 0 \text{ otherwise}\}$$

and

$$\Gamma_N(H^K) = \sum_{0 \neq h \in H^K} \frac{1}{r(h)} e^{-c_1 N \langle h, \alpha\rangle^2}.$$

We will show that

$$\sum_{0<|h|_\infty \leq H} \frac{1}{r(h)} |\Phi_Q(h)| = O(\frac{H^{\gamma-1}}{\sqrt{N}}). \tag{29}$$

According to (28), the proposition will follow from (29) with the choice $H = N^{\frac{1}{2\gamma}}$. Clearly it suffices to show that for $K = D$

$$\Gamma_N(H^K) = O(\frac{H^{\gamma-1}}{\sqrt{N}}). \tag{30}$$

We prove it by induction on the cardinality of $K \subset D$. For $K = \emptyset$ there is nothing to prove. Let $\ell \leq d-1$ and assume that it is true for any $K \subset D$ of cardinality ℓ. Let $K \subset D$ with cardinality $\ell+1$. The sum for $h \in H^K$ restricted a face $h_i = 0$, for some $i \in K$, corresponds to the ℓ-dimensional situation, where the estimate holds. Therefore, it suffices to prove the estimate for the sum restricted to $h_i \geq 1$ for any $i \in K$. In this case $r(h)$ is simply the product $\prod_{i\in K} h_i$.

Given $a \colon \mathbb{N}^d \to \mathbb{R}$, $n \in \mathbb{N}^d$ and $i \in D$ let $a_i(n) = a(n) - a(n+e_i)$, $a^i(n) = \sum_{k_i=1}^{n_i} a(n + (k_i - n_i)e_i)$. We extend this notation to multiindices $I \subset \{1, \ldots, d\}$, defining $a^I(n)$ and $a_I(n)$ in the obvious way. Note $n - I = n - \sum_{i\in I} e_i$. We use the following multidimensional Abel's summation formula, for any $a, b \colon \mathbb{N}^d \to \mathbb{R}$:

$$(ab)^K = \sum_{I \subset K} (b_I a^K)^I (n - I). \tag{31}$$

We apply this formula to $b(h) = \prod_{i\in K} h_i^{-1}$ and $a(h) = e^{-c_1 N\|\langle h,\alpha\rangle\|}$.

We first estimate $a^K(h)$:

$$a^K(h) = \sum_{1\le k\le h} e^{-c_1 N\|\langle k,\alpha\rangle\|},$$

where $1 \le k \le h$ means that $\forall i \in K$, $1 \le k_i \le h_i$ and $k_i = 0$ otherwise. Take $\eta = c_0|2h|_\infty^{-\gamma}$. Whenever $0 \le k \ne k' \le h$ we have

$$\big|\|\langle k,\alpha\rangle\| - \|\langle k',\alpha\rangle\|\big| \ge \min(\|\langle k \pm k',\alpha\rangle\|) > \eta.$$

Therefore, each of the interval $[0,\eta)$, $[\eta, 2\eta)$, ..., contains at most one $\|\langle k,\alpha\rangle\|$ with $1 \le k \le h$, and the first interval does not contain any of them since $\langle 0,\alpha\rangle \in [0,\eta)$. Thus, setting $c = c_1c_2$,

$$a^K(h) \le \sum_{j=1}^{\infty} e^{-cN(j\eta)^2} \le \int_0^\infty e^{-cN\eta^2u^2}du = \frac{1}{\eta\sqrt{N}}\int_0^\infty e^{-cv^2}dv,$$

with the change of variable $v = \eta u\sqrt{N}$. That is

$$a^K(h) \le C\frac{1}{\sqrt{N}}|h|_\infty^\gamma$$

for some constant C. On the other hand, we have $b_I(h) = b(h)\prod_{i\in I}(h_i+1)^{-1}$. This implies that, for some constant C' and any $I \subset K$,

$$(b_I a^K)^I(n^{I,K,H}) \le C'\frac{1}{\sqrt{N}}H^{\gamma-1},$$

where $n_i^{I,K,H} = H-1, H, 0$ according to $i \in I, K, D\setminus K$. The inducing step (30) follows by (31). ∎

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