# On fluctuations and the exponential statistics of return times

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**Abstract.** This paper presents some facts related to the exponential statistic of return times. We first investigate the question of computing the speed of convergence to this limiting law. We show that this speed carries some informations about the system under consideration, while via a local analysis we can relate it to some combinatorial property of some orbits. Next, we prove that for an arbitrary dynamical system, the existence of an exponential statistic for the return times implies the equivalence between the fluctuations of the empirical entropies and the repetition times.

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## 1. Introduction

Given a map $T : X \to X$ which preserves a probability measure $\mu$, and a measurable subset $A \subset X$, after Poincaré we know that almost any point from $A$ will return into $A$ again under iterations of $T$. In this setting, it is natural to ask for the quantitative version of this observation. Which proportion of points inside $A$ will return before a time $t$ into $A$ ?

An approximate answer may be obtained by Chebychev's inequality via Kac's theorem, which says that the expectation on $A$ of the return time into $A$ is finite, bounded by $1/\mu(A)$, with equality if for example the measure $\mu$ is ergodic.

To be more precise, given a measurable $A \subset X$ of positive measure we define

$$\tau_A(x) = \inf\{k > 0 : T^k x \in A\}.$$

We call $\tau_A(x)$ the *first return time* or *first entrance time* into $A$ whether $x$ belongs to $A$ or not, respectively. Setting $\mu_A \stackrel{\text{def}}{=} \frac{1}{\mu(A)}\mu|_A$ we can define the *distribution of return times* into the set $A$ by

$$F_A(t) \stackrel{\text{def}}{=} \mu_A\big(x \in A : \tau_A(x) > \frac{t}{\mu(A)}\big).$$

In a wide variety of (sufficiently) mixing dynamical systems it was proven that the distribution of return times is close to an exponential, which means that $F_A(t)$ converges to $\exp(-t)$ as $\mu(A) \to 0$, when $A$ is chosen in a suitable family of sets (e.g.

balls, cylinders). Suppose that for each point $x$ a sequence of neighborhood $(\Omega_\varepsilon(x))_\varepsilon$ with $\mu(\Omega_\varepsilon(x)) \to 0$ as $\varepsilon \to 0$ is given. We say that the return times *on the family* $\{\Omega_\varepsilon(x): x \in X, \varepsilon > 0\}$ are *exponentially distributed* if

$$\mathcal{E}_\mu(\Omega_\varepsilon(x)) \to 0 \text{ as } \varepsilon \to 0, \qquad \text{for } \mu\text{-almost every } x, \tag{1}$$

where for any measurable $A$ of positive measure we set

$$\mathcal{E}_\mu(A) \stackrel{\text{def}}{=} \sup_{t\geq 0} |F_A(t) - \exp(-t)| .$$

This kind of result is proven for Axiom A diffeomorphism [9], some intermittent maps of the interval [10] and some rational maps [8]. The exponential distribution of entrance times — which is implied by the exponential distribution of return times [10] — has also been proven in a variety of different systems. Among them, topological Markov chains [15], intermittent map of the interval [5, 4, 3] some $\varphi$-mixing dynamical systems [7] and some Collet-Eckmann unimodal maps [1].

At this time it seems that this exponential statistic holds very often, thus it makes sense to ask for the reciprocal question: what kind of systems may enjoy an exponential statistic for the return times ?

A partial answer is that $F_A$ is close to the exponential law if and only if $F_A^* \stackrel{\text{def}}{=} \mu(\tau_A > \frac{t}{\mu(A)})$, the distribution of the first entrance time, is close to $F_A$; see (5). This can be viewed as a (kind of) mixing property. Yet it is not easy to relate this property to well known characteristics of the system, like hyperbolicity.

Indeed, all the available explicit estimates of $\mathcal{E}_\mu(A)$ are given in terms of *mixing properties* of the system and *short recurrence* of the set $A$ (see Sections 2 and 4 for details). However, it is unclear if these properties are really essential. That is why we are particularly interested in the effect of slow mixing and short recurrence on the statistic of return times. We give in Section 2 a comprehensive answer to these questions, by showing that the speed of convergence to the exponential law is intimately related to these characteristics. First, this question is addressed in a one-parameter family of intermittent maps (with polynomial rate of mixing), and we show in Subsection 2.1 that this rate of convergence vanishes when the map looses its hyperbolicity. We believe that this rate of convergence can be used to recover some hyperbolic properties of much more general systems. Second, for subshifts of finite type with Gibbs measures (with exponential rate of mixing), we obtain in Subsection 2.2 that the rate of convergence is equal to some quantity defined in terms of a first return time of cylinder sets, and also vanishes for points with short recurrence.

After this attempt to understand which ingredients a system needs to possess in order to have an exponential statistic of return times, we look at an interesting possible application of this statistical property. There is a deep link between recurrence and entropy, as expressed by Ornstein and Weiss theorem [13], which says that a stationary ergodic process will repeats its first $n$ symbols after a time of order $\exp(nh)$ where $h$ is the entropy of the process. This is of particular interest in the theory of compression algorithms and sequences analysis [18, 13, 11, 6].

For numerical analysis of dynamical systems this provides a new way of computing the measure theoretic entropy of a map, simply by looking at a typical orbit. Moreover, no storage of information is needed. Let $\mathcal{Z}$ be a finite partition of $\mathcal{Z}$, and denote by $Z_n^x$ the element of the partition $\mathcal{Z}\vee T^{-1}\mathcal{Z}\vee\cdots\vee T^{-n+1}\mathcal{Z}$ which contains the point $x \in X$. If $\mu$ is ergodic, then the Ornstein-Weiss theorem says that $\frac{1}{n}\log\tau_{Z_n^x}(x)$ converges almost surely to the entropy of $\mu$ relative to $\mathcal{Z}$. Another method to compute the entropy is well known. Take the same partition and compute the measure of the set $Z_n^x$ around the typical point $x$. By Shannon-McMillan-Breiman theorem, $-\frac{1}{n}\log\mu(Z_n^x)$ converges almost surely to the entropy. The problem here is to estimate the measure of $Z_n^x$. With the repetition time there is no need to know the measure, only a typical point is needed, hence it is in principle much simpler to implement.

It could be that for numerical purpose the (Lempel-Ziv)-Ornstein-Weiss method is not adapted, because the fluctuations of $\frac{1}{n}\log\tau_{Z_n^x}(x)$ could be much stronger than the one of $-\frac{1}{n}\log\mu(Z_n^x)$. We prove in Section 3 that this is never the case when the return time are exponentially distributed on cylinders of $\mathcal{Z}$. Under this assumption, the fluctuations in the two methods are exactly the same (this phenomenon was addressed and prove in [11, 6] in the case of rapidly mixing dynamical systems with log-normal fluctuations).

## 2. Speed of convergence to the exponential law

We consider a measure preserving dynamical system $(X, T, \mu)$ together with a countable measurable partition $\mathcal{Z}$. For $\varphi$-mixing process with $\varphi$ summable, Galves and Schmitt have shown in [7] an upper bound of the type $|F_A^*(t) - \mathrm{e}^{-t}| = \mathcal{O}(\mu(A)^\beta)$. We want here to give a precise meaning to this exponent $\beta$.

**Definition 2.1** *We define the local (lower and upper) rates of convergence of the distribution of return times to the exponential-one law by*

$$\underline{\beta}_{\mathcal{Z}}(x) = \varliminf_{n\to\infty} \frac{\log \mathcal{E}_\mu(Z_n^x)}{\log\mu(Z_n^x)} \quad \text{and} \quad \overline{\beta}_{\mathcal{Z}}(x) = \varlimsup_{n\to\infty} \frac{\log \mathcal{E}_\mu(Z_n^x)}{\log\mu(Z_n^x)}.$$

We recall Proposition 2.3 in [10] which says that for any dynamical system with a measurable partition $\mathcal{Z}$, and for any point $x \in X$, the rate of convergence to the exponential law lies in the interval $[0, 1]$.

$$0 \le \underline{\beta}_{\mathcal{Z}}(x) \le \overline{\beta}_{\mathcal{Z}}(x) \le 1. \tag{2}$$

A rate of convergence $\beta_{\mathcal{Z}} = 1$ is therefore optimal. We will try now to see how this quantity depends on the system under consideration. As already pointed out in the introduction, the estimates for $\mathcal{E}_\mu$ involve a rate of mixing and short recurrence. In Subsection 2.1 we try to understand how $\beta$ evolves while the rate of mixing becomes slower and slower, as a map becomes less and less "hyperbolic". On the other hand, even if the rate of mixing is exponential it is unlikely that the return times statistic around a periodic point is exponential, because too many points would return too fast, hence for these points we should have $\beta_{\mathcal{Z}}(x) = 0$; the effect of short recurrence will be treated in Subsection 2.2.

### *2.1. Speed of convergence versus hyperbolicity*

Let $\alpha \in (0, 1)$ and consider the map $T_\alpha$ defined on $X = [0, 1]$ by

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}], \\ 2x - 1 & \text{otherwise.} \end{cases}$$

The rate of decay of correlation depends strongly on hyperbolic properties of the map, as showed in the family $T_\alpha$. In this sense the non-uniform hyperbolicity of the map reflects in the decay of correlations. We would like to show that the exponent $\beta$ may also contain some informations about the hyperbolicity of a map.

Let $\mathcal{Z}$ be the countable partition generated by the left preimages $a_n$ of 1, $\mathcal{Z} = \{A_m : m \in \mathbb{N}\}$ with $A_m = (a_{n+1}, a_n]$. Let $\mu_\alpha$ denotes the invariant measure absolutely continuous with respect to Lebesgue (see e.g. [17, 12] for the existence and properties). Next proposition shows that for any cylinder of the dynamical partition $\mathcal{Z}$, the tail of the distribution is polynomial.

**Proposition 1** *Let $Z \in \mathcal{Z}_n = \mathcal{Z} \vee T^{-1}\mathcal{Z} \vee \cdots \vee T^{-n+1}\mathcal{Z}$ be any cylinder of the dynamical partition of order $n$. The distribution of entrance time is bounded from below, for some constant $C(Z) > 0$ by*

$$\mu_\alpha(\tau_Z > t) \geq C(Z)\, t^{1-\frac{1}{\alpha}}$$

*for any $t \geq 0$.*

This shows in particular that the first moment of the entrance time does not exist whenever $\alpha > \frac{1}{2}$, since in this case

$$\int \tau_Z d\mu_\alpha \geq t\mu_\alpha(\tau_Z > t) \geq Ct^{2-\frac{1}{\alpha}} \to +\infty \text{ as } t \to \infty.$$

Note that this does not contradict Kac's theorem, which says that the first moment of the return time always exists; the existence of the first moment of the entrance time is in fact equivalent to the existence of the second moment of the return time (Proposition 1.5 in [16]). Despite this polynomial tail, when the cylinders $Z_n^x$ shrink to almost any point $x$ —due to the normalization by $1/\mu_\alpha(Z_n^x)$— this tail goes away, and we recover an exponential distribution of the return times, as proven in [10], with a rate at least $1-\alpha$. These observations are essential to establish the following.

**Theorem 2** *For any $\alpha \in (0,1)$ we have for the system $([0,1], T_\alpha, \mu_\alpha)$*

$$1-\alpha \leq \underline{\beta}_{\mathcal{Z}} \leq \overline{\beta}_{\mathcal{Z}} \leq \frac{1}{\alpha} - 1 \qquad \mu_\alpha\text{-a.e..}$$

*In particular, if $\alpha > 1/2$ then $\overline{\beta}_{\mathcal{Z}} < 1$ almost everywhere, and $\|\overline{\beta}_{\mathcal{Z}}\|_{L^\infty(\mu_\alpha)}$ vanishes as $\alpha$ tends to one.*

Note that the upper bound is interesting only when $\alpha \in (\frac{1}{2}, 1)$. This non optimal rate of convergence to the exponential law corresponds also to the range of parameter $\alpha$ where the central limit theorem may fail to hold (due to a non-summable decay of correlations, see Theorem 4.1 in [12]).

### *2.2. Speed of convergence versus short recurrence*

From now on, let $(X, T)$ be a topologically mixing subshift of finite type. The statistic will be done with respect to the Gibbs measure $\mu$ associated to an Hölder continuous potential $\varphi : X \to \mathbb{R}$ (see [2] for details). Let $\mathcal{Z}$ be the finite partition of $X$ into 1-cylinders. We focus on the dependence of the speed of convergence with respect to some local quantity defined in terms of *first return time of cylinders*, the first return time of a set $A \subset X$ being defined by $\tau(A) = \inf_{x\in A} \tau_A(x)$.

**Definition 2.2** *We define the local (lower and upper) rate of return for cylinders by*

$$\underline{R}_{\mathcal{Z}}(x) = \varliminf_{n\to\infty} \frac{\tau(Z_n^x)}{n} \quad \text{and} \quad \overline{R}_{\mathcal{Z}}(x) = \varlimsup_{n\to\infty} \frac{\tau(Z_n^x)}{n}.$$

Note that $\overline{R}_{\mathcal{Z}} \leq 1$ by specification, and for periodic points one has $\underline{R}_{\mathcal{Z}} = \overline{R}_{\mathcal{Z}} = 0$. By an easy construction it can be shown that the set of points $x$ for which $0 < \underline{R}_{\mathcal{Z}}(x) < \overline{R}_{\mathcal{Z}}(x) = 1$ is dense in $X$.

**Definition 2.3** *A point $x \in X$ is called, say, $\varphi$-regular if it is a generic point for the function $\varphi$, which means $\frac{1}{n}\sum_{k=0}^{n-1}\varphi(T^k x)$ converges as $n \to \infty$.*

We emphasize the fact that a lot of points are $\varphi$-regular; for example, by Birkhoff theorem almost all points with respect to any invariant probability measure are $\varphi$-regular. Note also that the set of $\varphi$-regular point is invariant. When $\varphi$ is (cohomologous to) a constant, that is to say that $\mu$ is the measure of maximal entropy, then obviously all the points are regular.

**Theorem 3** *The rate of convergence of the distribution of return times to the exponential-one law and the rate of return for cylinders are the same for $\varphi$-regular points, which means that if $x \in X$ is $\varphi$-regular then*

$$\underline{\beta}_{\mathcal{Z}}(x) = \underline{R}_{\mathcal{Z}}(x) \quad \text{and} \quad \overline{\beta}_{\mathcal{Z}}(x) = \overline{R}_{\mathcal{Z}}(x).$$

Theorem 3 not only says that for the points with neighborhoods with long recurrence (i.e. $\underline{R}_{\mathcal{Z}}(x) > 0$) the convergence to the exponential law is fast (exponential), but also that this is a necessary condition. We believe that this phenomenon appears in much more general situations. This explains why to establish the exponential statistic of the return time one really has to consider the problem of short recurrence.

**Remark 4** *From the equality given by Theorem 3, it is clear that $\underline{\beta}_{\mathcal{Z}}$ (resp. $\overline{\beta}_{\mathcal{Z}}$) has the same properties than $\underline{R}_{\mathcal{Z}}$ (resp. $\overline{R}_{\mathcal{Z}}$) on $\varphi$-regular points. Hence, by Proposition 3.6 in [10] we obtain that for any ergodic invariant measure $\nu$ we have $\nu$-a.e. $x \in X$*

$$\underline{\beta}_{\mathcal{Z}}(x) = \underline{R}_{\mathcal{Z}}(x) = \text{const} \quad \text{and} \quad \overline{\beta}_{\mathcal{Z}}(x) = \overline{R}_{\mathcal{Z}}(x) = \text{const}.$$

An easy adaptation of the proof of Proposition 3.7 in [10] yields to the following theorem, which was stated in the same paper in this special setting.

**Theorem 5 ([10])** *For every Gibbs measure $\nu$ with Hölder potential,*

$$\underline{R}_{\mathcal{Z}} = \overline{R}_{\mathcal{Z}} = 1 \quad \nu\text{-a.e.}.$$

Collecting the results of Theorem 5 and Theorem 3 we find out that almost all points with respect to Gibbs measures with Hölder potential have an optimal rate of convergence ($\beta_{\mathcal{Z}} = 1$) to the exponential law.

**Remark 6** *In a forthcoming paper with S. Vaienti and S. Troubetzkoy we show that indeed for any invariant measure $\nu$ with non zero entropy we have $\underline{R}_{\mathcal{Z}} \geq 1$, $\nu$-a.e..*

## 3. Convergence in law of the repetition time

In this section, $(X, \mathcal{B}, \mu)$ is a probability space, $T : X \to X$ is a measurable map, and $\mathcal{Z}$ is a countable measurable partition of $X$. We define the $n$-repetition time of $x$ by $R_n(x) = \tau_{Z_n^x}(x)$. $R_n(x)$ is the smallest time $k > 0$ such that $x$ and $T^k x$ lies in the same component of $\mathcal{Z}_n$. Let $\mathbf{a} = (a_n)_{n\in\mathbb{N}}$ be some sequence converging to $+\infty$ and let $h \geq 0$. We define the law of the fluctuations of the measure of $n$-atoms and of the $n$-repetition time by

$$\mathcal{M}_{\mathbf{a}}^{(n)}(t) \stackrel{\text{def}}{=} \mu\left\{x \in X : \frac{-\log\mu(Z_n^x) - nh}{a_n} > t\right\},$$

$$\mathcal{R}_{\mathbf{a}}^{(n)}(t) \stackrel{\text{def}}{=} \mu\left\{x \in X : \frac{\log R_n(x) - nh}{a_n} > t\right\}.$$

We also define the corresponding limiting law by

$$\underline{\mathcal{M}}_{\mathbf{a}}(t) = \varliminf_{n\to\infty} \mathcal{M}_{\mathbf{a}}^{(n)}(t),\ \overline{\mathcal{M}}_{\mathbf{a}}(t) = \varlimsup_{n\to\infty} \mathcal{M}_{\mathbf{a}}^{(n)}(t),\ \text{ and } \mathcal{M}_{\mathbf{a}}(t) = \lim_{n\to\infty} \mathcal{M}_{\mathbf{a}}^{(n)}(t)$$

whenever the last limit exists. We do the same for the repetition time $\mathcal{R}_{\mathbf{a}}$.

We are now able to give our results saying that the fluctuations of the measure are equal to the fluctuations of the repetition time when the map has an exponential statistic for the return times.

**Theorem 7** *If the return times are exponentially distributed on cylinders of $\mathcal{Z}$ (see (1)) then the following properties hold:*

*(i) for all $t \in \mathbb{R}$ we have*

$$\underline{\mathcal{M}}_{\mathbf{a}}(t+) \le \underline{\mathcal{R}}_{\mathbf{a}}(t) \le \underline{\mathcal{M}}_{\mathbf{a}}(t-) \quad \text{and} \quad \overline{\mathcal{M}}_{\mathbf{a}}(t+) \le \overline{\mathcal{R}}_{\mathbf{a}}(t) \le \overline{\mathcal{M}}_{\mathbf{a}}(t-);$$

*(ii) in particular for all but countably many points we have*

$$\underline{\mathcal{R}}_{a} = \underline{\mathcal{M}}_{\mathbf{a}} \quad \text{and} \quad \overline{\mathcal{R}}_{\mathbf{a}} = \overline{\mathcal{M}}_{\mathbf{a}};$$

*(iii) moreover, the limit $\mathcal{M}_{\mathbf{a}}$ exists on a dense set if and only if $\mathcal{M}_{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{a}}$ exists for all but countably many points, and in this case the two limits are equal;*

*(iv) finally, one can exchange $\mathcal{R}$ and $\mathcal{M}$ is all of these statements.*

It is possible to sharpen this result when the distributions are continuous.

**Theorem 8** *Suppose that the return time on cylinders of $\mathcal{Z}$ are exponentially distributed (see (1)).*

*The limiting distribution $\mathcal{M}_{\mathbf{a}}$ exists and is continuous if and only if the limiting distribution $\mathcal{R}_{\mathbf{a}}$ exists and is continuous. In this case, $\mathcal{R}_{\mathbf{a}} = \mathcal{M}_{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{a}}^{(n)}$ converges uniformly to $\mathcal{R}_{\mathbf{a}}$ if and only if $\mathcal{M}_{\mathbf{a}}^{(n)}$ converges uniformly to $\mathcal{M}_{\mathbf{a}}$.*

**Remark 9** *We emphasize that in Theorem 7 and Theorem 8:*

*(i) the measure $\mu$ does not have to be invariant;*

*(ii) it is in fact not necessary that the sequence of partition $\mathcal{Z}_n$ is given by $\mathcal{Z} \vee T^{-1}\mathcal{Z} \vee \cdots \vee T^{-n+1}\mathcal{Z}$ for some fixed partition $\mathcal{Z}$. Actually all the results remain true when $(\mathcal{Z}_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of countable measurable partitions, such that the return times are exponentially distributed on atoms of $(\mathcal{Z}_n)_n$.*

We now discuss the case of an ergodic measure preserving dynamical system $(X, T, \mu)$. Suppose that $H_\mu(\mathcal{Z}) = -\sum_{Z\in\mathcal{Z}} \mu(Z)\log\mu(Z)$ is finite. Under these assumptions, the Shannon-McMillan-Breiman theorem tells us that $-\frac{1}{n}\log\mu(Z_n^x)$ converges $\mu$-almost surely to the metric entropy relative to $\mathcal{Z}$ denoted by $h_\mu(T, \mathcal{Z})$. A similar result holds for the repetition time, as proven by Ornstein and Weiss [13]. Namely, if $\mathcal{Z}$ is finite, $\frac{1}{n}\log R_n$ converges $\mu$-almost surely to $h_\mu(T, \mathcal{Z})$. Hence we must set $h = h_\mu(T, \mathcal{Z})$ in order to obtain a non-trivial law. Theorem 8 applies, hence the fluctuations of the measure and of the repetition time are equal if they are given by a continuous distribution.

For log-normal fluctuations, taking $a_n = \sigma\sqrt{n}$ for some $\sigma \in (0, \infty)$ and $h = h_\mu(T, \mathcal{Z})$ we get the immediate corollary of Theorem 8.

**Corollary 10** *Let $(X, T, \mu)$ be an ergodic measure preserving dynamical system. Assume that the return times are exponentially distributed on cylinders of $\mathcal{Z}$. For any $\sigma \in (0, \infty)$ we have the equivalence:*

*(i) the random variable* $\frac{\log R_n - nh}{\sigma\sqrt{n}}$ *converges in law to* $\mathcal{N}(0,1)$*;*

*(ii) the random variable* $\frac{-\log\mu(Z_n)-nh}{\sigma\sqrt{n}}$ *converges in law to* $\mathcal{N}(0,1)$*.*

In [6] an explicit formula is given in the case where the variance $\sigma$ vanishes. However, in this full generality we do not know what happens when the variance is zero or infinity.

The case of log-normal distribution was treated in some papers using the following sketch (see [6, 14]), and inspired us to formulate Theorem 8 :

(i) the measure of the cylinders have log-normal fluctuations;

(ii) the law of *entrance time* in cylinders is exponential, with some sharp control on the rate of convergence to the limiting law,

$$\left|\mu(\tau_{Z_n^x} > t/\mu(Z_n^x)) - \exp(-t)\right| = \mathcal{O}(\mu(Z_n^x)^\beta);$$

(iii) small returns have small probabilities.

Another approach of a more probabilistic nature was developed in [11] and was essentially a strong approximation between the measure of a cylinder and the repetition time for strongly mixing stationary processes.

## 4. Proofs of the results in Section 2

Before the proofs, we recall the definition of the quantities involved in the paper [10] and the related results.

**Proposition 11 (Theorem 2.1 and Lemma 2.4 in [10])** *Given a dynamical system* $(X,T,\mu)$ *and a measurable set* $U$ *with positive measure we define*

$$\begin{aligned} a_N(U) &= \mu_U(\tau_U \le N),\\ b_N(U) &= \sup\left\{|\mu_U(T^{-N}V) - \mu(V)|; V \text{ measurable}\right\},\\ c(U) &= \sup_{k\ge 0}|\mu_U(\tau_U > k) - \mu(\tau_U > k)|,\\ \mathcal{E}_\mu(U) &= \sup_{t\ge 0}|\mu_U(\tau_U > t/\mu(U)) - \mathrm{e}^{-t}|. \end{aligned}$$

*The distance to the exponential law of the distribution of return time on* $U$ *is bounded from above by*

$$\mathcal{E}_\mu(U) \le 4\mu(U) + c(U)(1-\log c(U))\,; \tag{3}$$

*and conversely*

$$c(U) \le 2\mu(U) + \mathcal{E}_\mu(U)(2 - \log \mathcal{E}_\mu(U)); \tag{4}$$

*while the quantity* $c(U)$ *may be estimated by*

$$c(U) \le \inf\{a_N(U) + b_N(U) + N\mu(U) : N \text{ is an integer }\}. \tag{5}$$

### *4.1. Proofs of the results in Subsection 2.1*

**Proof of Proposition 1** Let $\alpha \in (0,1)$ and denote by $\mu = \mu_\alpha$ the $T_\alpha$-invariant measure absolutely continuous with respect to Lebesgue. Let $Z$ be a cylinder of the dynamical partition $\mathcal{Z}_n$ and let $m = m(Z) \geq n$ be the minimal integer such that $T^m Z = X$. This implies that $Z \cap [0, a_m) = \emptyset$, hence for any integer $k$, any point inside $[0, a_{m+k})$ has an entrance time in $Z$ bigger than $k$, which gives

$$\mu(\tau_Z > k) \geq \mu([0, a_{m+k})).$$

An easy modification of Lemma 3.2 in [10] gives $a_p \geq c_1 p^{-\frac{1}{\alpha}}$, while the density $h_\alpha$ of the invariant measure $\mu_\alpha$ satisfies $h_\alpha(x) \geq c_2 x^{-\alpha}$ (see [17]). Hence

$$\mu([0, a_{m+k})) = \int_0^{a_{m+k}} h_\alpha(x)dx \geq c_1 c_2 (m+k)^{1-\frac{1}{\alpha}} \geq c_1 c_2 (m+1)^{1-\frac{1}{\alpha}} k^{1-\frac{1}{\alpha}}$$

which proves the proposition, with $C(Z) = c_1 c_2 (m(Z)+1)^{1-\frac{1}{\alpha}}$. □

**Proof of Theorem 2** Let $\alpha \in (0,1)$ and set $\mu = \mu_\alpha$. Let $x \in X$ be such that $\underline{\beta}_{\mathcal{Z}}(x) \geq 1 - \alpha > 0$ and $\lim_{n\to\infty} -\frac{1}{n}\log \mu(Z_n^x) = h_\mu > 0$; by Theorem 3.8 in [10] and the Shannon-McMillan-Breiman theorem this concerns $\mu$-a.e. points. Let $\varepsilon \in (0, 1-\alpha)$ be arbitrary. Setting $Z = Z_n^x$, by (4) in Proposition 11 we get for any $k > 0$

$$\begin{aligned}\mu(\tau_Z > k) - \mathrm{e}^{-k\mu(Z)} &\leq |\mu(\tau_Z > k) - \mu_Z(\tau_Z > k)| + |\mu_Z(\tau_Z > k) - \mathrm{e}^{-k\mu(Z)}| \\ &\leq 2\mu(Z) + \mathcal{E}_\mu(Z)(3 - \log \mathcal{E}_\mu(Z)).\end{aligned}$$

With $k = 1/\mu(Z_n^x)^{1+\varepsilon}$, Proposition 1 yields to

$$\overline{\beta}_{\mathcal{Z}}(x) \leq \overline{\lim_{n\to\infty}} \frac{\log[C(Z_n^x)\mu(Z_n^x)^{(\frac{1}{\alpha}-1)(1+\varepsilon)} - \exp(-\mu(Z_n^x)^{-\varepsilon})]}{\log \mu(Z_n^x)}. \tag{6}$$

Note that $Z_n^x = Z^x_{m(Z_n^x)}$ where $m(Z_n^x)$ is as in the proof of Proposition 1 and that $\lim_{n\to\infty} -\frac{1}{n}\log\mu(Z_n^x) > 0$. Consequently, $\lim_{n\to\infty} \log \frac{C(Z_n^x)}{\mu(Z_n^x)} = 0$, hence (6) gives us

$$\overline{\beta}_{\mathcal{Z}}(x) \leq (\frac{1}{\alpha} - 1)(1+\varepsilon).$$

The conclusion follows since $\varepsilon > 0$ was arbitrary. □

### *4.2. Proofs of the results in Subsection 2.2*

In the sequel we suppose without loss of generality that the potential $\varphi$ is normalized, so that its pressure is zero, and the invariant density is the constant function $\mathbb{1}$. We refer the reader to [2] for details. Let $P$ be the Perron-Frobenius operator of $(X, T, \varphi)$, which satisfies $P\mathbb{1} = \mathbb{1}$, and $\mu(f \circ Tg) = \mu(fPg)$ for all continuous $f, g : X \to \mathbb{R}$.

**Proposition 12** *For any $\varphi$-regular point $x \in X$ and $0 \leq r \leq 1$, we have*

$$\lim_{n\to\infty} \frac{\log \mu(Z^x_{[rn]})}{\log \mu(Z_n^x)} = r.$$

**Proof of Proposition 12** If $r = 0$ then the result is obvious, because $\mathcal{Z}$ generates; so let us suppose that $r \neq 0$.

As usual, we write $S_n\varphi = \sum_{k=0}^{n-1} \varphi \circ T^k$. Let $x \in X$ be a $\varphi$-regular point, and $\varphi_*(x)$ be the limit of the Birkhoff average. By the Gibbs property, it is easy to see that $\lim_{n\to\infty} \frac{1}{n}\log\mu(Z_n^x) = \varphi_*(x)$, which implies

$$\frac{\log \mu(Z^x_{[rn]})}{\log \mu(Z_n^x)} = \frac{r\frac{1}{rn}\log\mu(Z^x_{[rn]})}{\frac{1}{n}\log\mu(Z_n^x)} \xrightarrow[n\to\infty]{} r.$$

□

**Lemma 13** *There exists a constant $c_3$ such that for any $x \in X$, and any integer $n > 0$, the following bound holds:*

$$c(Z_n^x) \leq c_3 n \mu(Z_{nr_n}^x).$$

*whenever $r_n = \tau(Z_n^x)/n \leq 1$, where $c(Z_n^x)$ is defined in Proposition 11.*

**Proof of Lemma 13** We first compute $a_N(Z_n^x)$:

$$\begin{aligned} a_N(Z_n^x) &\leq \sum_{k=nr_n}^{N} \frac{1}{\mu(Z_n^x)} \mu(Z_n^x \cap T^{-k} Z_n^x) \\ &= \sum_{k=nr_n}^{N} \frac{1}{\mu(Z_n^x)} \int_{Z_n^x} P^k \mathbf{1}_{Z_n^x} d\mu \\ &\leq N \| P^{nr_n} \mathbf{1}_{Z_n^x} \|_\infty \leq N \exp(S_{nr_n}\varphi(x)) \leq c_1 N \mu(Z_{nr_n}^x). \end{aligned}$$

The last two inequalities follow from the bounded distortion and the Gibbs property.

For $b_N(Z_n^x)$, we observe that since our system enjoys an exponential decay of correlations, there exists some $\Lambda < 1$ such that $b_N(Z_n^x) \leq c_2 \Lambda^{N-n}$. Thus, taking $N = (K+1)n$, with $K$ sufficiently large (but which does not depend on $n$) leads to $b_N(Z_n^x) \leq \Lambda^{Kn} \leq \mu(Z_{nr_n}^x)$. And finally, with this particular choice of $N$, applying (5) in Proposition 11 yields to the result. □

**Proof of Theorem 3** Let $x \in X$ be a $\varphi$-regular point.

We first show that $\overline{\beta}_{\mathcal{Z}}(x) \geq \overline{R}_{\mathcal{Z}}(x)$. We use the estimate (3) in Proposition 11, then we apply Lemma 13, and finally Proposition 12 (we set $r_n = \tau(Z_n^x)/n$)

$$\begin{aligned} \overline{\beta}(x) = \overline{\lim_{n\to\infty}} \frac{\log \mathcal{E}_\mu(Z_n^x)}{\log \mu(Z_n^x)} &\geq \overline{\lim_{n\to\infty}} \frac{\log c(Z_n^x)}{\log \mu(Z_n^x)} \\ &\geq \overline{\lim_{n\to\infty}} \frac{\log \mu(Z_{nr_n}^x)}{\log \mu(Z_n^x)} = \overline{\lim_{n\to\infty}}\, r_n = \overline{R}_{\mathcal{Z}}(x). \end{aligned}$$

The case $\underline{\beta}_{\mathcal{Z}}(x) \geq \underline{R}_{\mathcal{Z}}(x)$ follows in the same way.

We now compute the other bound: $\overline{\beta}_{\mathcal{Z}}(x) \leq \overline{R}_{\mathcal{Z}}(x)$. Suppose that $\overline{R}_{\mathcal{Z}}(x) < 1$, otherwise there is nothing to prove; see (2). There exists a subsequence of integers such that $r_n = \tau(Z_n^x)/n \to \overline{R}_{\mathcal{Z}}(x)$, with $r_n < 1$. Fix such an integer $n$, and let $k = nr_n$ and $C = Z_n^x$. Let us put $D = Z_k^x \cap T^{-k} C$. One can check that $D \subset C$, and any point in $D$ returns in $C$ after exactly $k$ iterations. We shall prove then that the measure $\mu_C(D)$ is quite big. The Gibbs property tells us that for any $y \in D$, the following holds:

$$\begin{aligned} \mu_C(D) \geq c_4 \frac{\exp(S_{n+k}\varphi(y))}{\exp(S_n\varphi(y))} &= c_4 \frac{\exp(S_k\varphi(y))\exp(S_n\varphi(T^k y))}{\exp(S_n\varphi(y))} \\ &\geq c_5 \exp(S_k\varphi(x)) \geq c_6 \mu(Z_k^x). \end{aligned}$$

Here we used the fact that both $y$ and $T^k y$ belong to $Z_n^x$, and the bounded distortion inequality. This ensures that between $[nr_n, nr_n + 1]$, the distribution of return times has a jump

$$\mu_{Z_n^x}(\tau_{Z_n^x} > nr_n) - \mu_{Z_n^x}(\tau_{Z_n^x} > nr_n + 1) \geq \mu_C(D) \geq c_6 \mu(Z_{nr_n}^x),$$

while the jump of the exponential $\mathrm{e}^{-k\mu(Z_n^x)}$ in this interval is only

$$\exp(-nr_n \mu(Z_n^x)) - \exp(-[nr_n + 1]\mu(Z_n^x)) = \mathcal{O}(\mu(Z_n^x)).$$

This proves that

$$\overline{\beta}_{\mathcal{Z}}(x) = \overline{\lim_{n\to\infty}} \frac{\log \mathcal{E}_\mu(Z_n^x)}{\log \mu(Z_n^x)} \le \overline{\lim_{n\to\infty}} \frac{\log \mu(Z^x_{r_n n})}{\log \mu(Z_n^x)} = \overline{\lim_{n\to\infty}}\, r_n = \overline{R}_{\mathcal{Z}}(x).$$

The case $\underline{\beta}_{\mathcal{Z}}(x) \le \underline{R}_{\mathcal{Z}}(x)$ follows in the same way. □

## 5. Proofs of the results in Section 3

The following simple lemma will be of interest for the proof of Theorem 7, it shows up a very interesting property of limiting distributions (notice that distributions are decreasing functions).

**Lemma 14** *If $u_n : \mathbb{R} \to [0,1]$ is a sequence of decreasing functions, then we have the equivalence*

*(i) $u_n$ converges on some dense set;*

*(ii) $u_n$ converges except possibly for countably many points.*

**Proof of Lemma 14** We first define $\underline{u}(t) = \underline{\lim}\, u_n(t)$ and $\overline{u}(t) = \overline{\lim}\, u_n(t)$. Both $\underline{u}$ and $\overline{u}$ are decreasing, hence $v = \overline{u} - \underline{u} \ge 0$ is of bounded variation. If $u_n$ converges on some dense set then $v = 0$ on the same dense set. This implies that $\{x : v(x) > 1/n\}$ is finite for all integer $n$, hence $\{v > 0\}$ is at most countable, which proves the implication. The reciprocal is trivial. □

**Proof of Theorem 7**

(i) Let $\varepsilon > 0$ and define

$$K_{\varepsilon,N} = \{x : \mathcal{E}_\mu(Z_n^x) < \varepsilon \text{ for all } n \ge N\}.$$

By assumption the set $E = \cap_{\varepsilon>0} \cup_{N\in\mathbb{N}} K_{\varepsilon,N}$ is of full $\mu$-measure. Choose then $N(\varepsilon)$ sufficiently large such that for all $n > N(\varepsilon)$ we have $\mu(K_{\varepsilon,n}) > 1 - \varepsilon$ and $\exp(-a_n) < \varepsilon$.

Let $n > N(\varepsilon)$ and $t \in \mathbb{R}$. Define $A_n(t) = \{x \in X : \frac{\log R_n(x) - nh}{a_n} > t\}$ and set $\delta_n = \log a_n / a_n$. We are interested in the limit of $\mathcal{R}^{(n)}_{\mathbf{a}}(t) = \mu(A_n(t))$. We first decompose the measure along cylinders $Z \in \mathcal{Z}_n$ according to whether or not they intersect $K_{\varepsilon,n}$:

$$\mathcal{R}^{(n)}_{\mathbf{a}}(t) = \sum_{Z\in\mathcal{Z}_n, Z\cap K_{\varepsilon,n}\neq\emptyset} \mu(Z\cap A_n(t)) + \sum_{Z\in\mathcal{Z}_n, Z\cap K_{\varepsilon,n}=\emptyset} \mu(Z\cap A_n(t)) \quad (7)$$

The second term is bounded by $\mu(X \setminus K_{\varepsilon,n}) \le \varepsilon$, while whenever $Z \cap K_{\varepsilon,n} \neq \emptyset$, we have $\mathcal{E}_\mu(Z) \le \varepsilon$. Thus if we notice that $R_n(x) = \tau_Z(x)$ when $x \in Z$ we get that for these $Z$

$$\begin{aligned}\mu(Z\cap A_n(t)) &= \mu(Z)\mu_Z(\tau_Z > \exp(ta_n + nh)) \\ &\le \mu(Z)\left[\exp\left[-\mu(Z)\exp(ta_n+nh)\right] + \varepsilon\right].\end{aligned}$$

Accordingly, we have

$$\mathcal{R}^{(n)}_{\mathbf{a}}(t) \le 2\varepsilon + \int_X \exp\left[-\mu(Z_n^x)\exp(ta_n + nh)\right] d\mu(x). \quad (8)$$

Let $f_n(t,x) = \mu(Z_n^x)\exp(ta_n + nh)$ and $B_n(t) = \{x \in X : f_n(t,x) < a_n\}$. We decompose the integral in (8) according to whether or not $x \in B_n(t)$. First,

$$\int_{X\setminus B_n(t)} \mathrm{e}^{-f_n(t,x)} d\mu(x) \le \sup_{X\setminus B_n(t)} \mathrm{e}^{-f_n(t,x)} \le \mathrm{e}^{-a_n} \le \varepsilon. \quad (9)$$

On the other hand,

$$\int_{B_n(t)} \mathrm{e}^{-f_n(t,x)} d\mu(x) \le \mu(B_n(t))$$
$$\le \mu\left(\{x : \frac{-\log\mu(Z_n^x) - nh}{a_n} > t - \frac{\log a_n}{a_n}\}\right)$$
$$\le \mathcal{M}_{\mathbf{a}}^{(n)}(t-\delta_n). \tag{10}$$

Using estimates (8), (9) and (10) we finally end up with

$$\mathcal{R}_{\mathbf{a}}^{(n)}(t) \le \mathcal{M}_{\mathbf{a}}^{(n)}(t-\delta_n) + 3\varepsilon.$$

If we consider the complement of $X \setminus A_n(t)$ instead of $A_n(t)$ itself, a similar computation yields to $1 - \mathcal{R}_{\mathbf{a}}^{(n)}(t) \le 1 - \mathcal{M}_{\mathbf{a}}^{(n)}(t+\delta_n) + 3\varepsilon$. Thus

$$\mathcal{M}_{\mathbf{a}}^{(n)}(t+\delta_n) - 3\varepsilon \le \mathcal{R}_{\mathbf{a}}^{(n)}(t) \le \mathcal{M}_{\mathbf{a}}^{(n)}(t-\delta_n) + 3\varepsilon. \tag{11}$$

Since $\delta_n \to 0$ as $n \to \infty$ and one-sided limits of $\underline{\mathcal{M}}_{\mathbf{a}}$ exists we get

$$\underline{\mathcal{M}}_{\mathbf{a}}(t+) - 3\varepsilon \le \underline{\mathcal{R}}_{\mathbf{a}}(t) \le \underline{\mathcal{M}}_{\mathbf{a}}(t-) + 3\varepsilon.$$

As $\varepsilon$ was arbitrary, this finishes the first part of the proof. The limit $\overline{\mathcal{R}}_{\mathbf{a}}$ van be treated in the same way.

(ii) It suffices to remark that decreasing functions can have at most a countable number of discontinuities.

(iii) Lemma 14 ensures that if $\mathcal{M}_{\mathbf{a}}$ exists on a dense set then it exists for all reals except for an at most countable set $C$. Moreover, $\mathcal{M}_{\mathbf{a}}|_{X\setminus C}$ is decreasing, hence it is continuous except may be on countably many points $D$. Hence $\mathcal{M}_{\mathbf{a}}$ exists and is continuous on $X \setminus (C \cup D)$ where $C \cup D$ is at most countable. The first statement of the theorem implies then that $\mathcal{R}_{\mathbf{a}}$ exists and is equal to $\mathcal{M}_{\mathbf{a}}$ on $X \setminus (C \cup D)$.

(iv) From Inequality (11) applied once with $t = s - \delta_n$ then with $t = s + \delta_n$ we get

$$\mathcal{R}_{\mathbf{a}}^{(n)}(s+\delta_n) - 3\varepsilon \le \mathcal{M}_{\mathbf{a}}^{(n)}(s) \le \mathcal{R}_{\mathbf{a}}^{(n)}(s-\delta_n) + 3\varepsilon.$$

The rest of the proof follows exactly in the same way.

□

**Proof of Theorem 8** When $\mathcal{M}_{\mathbf{a}}$ or $\mathcal{R}_a$ is continuous Theorem 7 gives us that $\mathcal{M}_a = \mathcal{R}_{\mathbf{a}}$. Remark then that the distribution is uniformly continuous, because the limits in $\pm\infty$ exist. Then for all $\varepsilon > 0$ it is possible to find a $\delta = \delta(\varepsilon)$ independent of $t, s \in \mathbb{R}$ such that $|\mathcal{M}_{\mathbf{a}}(t) - \mathcal{M}_{\mathbf{a}}(s)| \le \varepsilon$ whenever $|t-s| \le \delta$. In addition, if the convergence of $\mathcal{M}_{\mathbf{a}}^{(n)}$ to $\mathcal{M}_{\mathbf{a}}$ is uniform, taking $M(\varepsilon)$ sufficiently large we get that for any $n > M(\varepsilon)$ and real $s$, $|\mathcal{M}_{\mathbf{a}}^{(n)}(s) - \mathcal{M}_{\mathbf{a}}(s)| < \varepsilon$ and $\delta_n = \log a_n/a_n < \delta$. Thus for all real $t$,

$$\mathcal{M}_{\mathbf{a}}(t) - 2\varepsilon \le \mathcal{M}_{\mathbf{a}}^{(n)}(t+\delta_n) \le \mathcal{M}_{\mathbf{a}}^{(n)}(t-\delta_n) \le \mathcal{M}_{\mathbf{a}}(t) + 2\varepsilon.$$

Let $N(\varepsilon)$ be as in the proof of Theorem 7. Inequality (11) in the proof of Theorem 7 gives that for all real $t$ and for any $n > \max\{N(\varepsilon), M(\varepsilon)\}$

$$\mathcal{M}_{\mathbf{a}}(t) - 5\varepsilon \le \mathcal{R}_{\mathbf{a}}^{(n)}(t) \le \mathcal{M}_{\mathbf{a}}(t) + 5\varepsilon.$$

This completes the proof. □

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