TOPOLOGICAL SYNCHRONISATION OR A SIMPLE ATTRACTOR?

Abstract. A few recent papers introduced the concept of topological synchronisation. We refer in particular to [9], where the theory was illustrated by a skew product system, coupling two logistic maps. In this case, we show that the topological synchronisation could be easily explained as the birth of an attractor for increasing values of the coupling strength and the mutual convergence of two marginal empirical measures. Numerical computations based on a careful analysis of the Lyapunov exponents suggest that the attractor supports an absolutely continuous physical measure.

1. Introduction

The recent paper [9], which also garnered some press attention [10, 11], introduced the concept of topological synchronisation which occurs when in a dynamical system it is possible to identify two or more attractors which become very similar when the system evolves. This situation is for instance met in coupled lattice map, where each site of the lattice brings its own attractor. It is written in [9] that *during the gradual process of topological adjustment in phase space, the multifractal structures of each strange attractor of the two coupled oscillators continuously converge, taking a similar form, until complete topological synchronization ensues.* As an example of this process of synchronisations, the authors in [9] studied a skew system whose base is a logistic (master) map T of the interval $[-1,1]$ and the other map (the slave), is another logistic map on the same interval which is coupled with the master in a convex way in order to be confined to the interval $[-1,1]$. As an indicator of the *closeness* of the attractors of the master and slave maps when the coupling strength, say k, increases, the authors in [9] used the spectrum D_q of generalized dimensions. They showed in particular the interesting phenomenon, which they called the *zipper effect*, that the dimensions begin to synchronise at negative q with low values of k before getting similar for positive values of q when k arrives at the threshold of complete synchronisation of the attractors. They interpreted it by saying that *the road to complete synchronization starts at low coupling with topological synchronization of the sparse areas in the attractor and continues with topological synchronizations of much more dense areas in the attractor until complete topological synchronization is reached for high enough coupling.*

The object of our note is to show that, in the case of the skew system where the master and the slave map are the logistic one, if we denote with $\{x_n\}_{n\geq 0}$ the trajectory of the master system and with $\{y_n\}_{n\geq 0}$ that of the slave system, the topological synchronisation is easily interpreted as the presence of an invariant set in the neighborhood of the diagonal of the square $[-1,1]^2$ to which $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ converge when the coupling strength tends to 1, in the sense of (6). Moreover, we show that the empirical measure computed along the trajectories of the slave system approaches, in the limit as the number of the iterations

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tends to infinity, the physical measure of the master map. We compute numerically the Lyapunov exponent of the master map T and we show that it is positive for the parameter values considered in [9], which implies that the attractor in the master space is a finite union of intervals. We finally discuss the real occurrence of a multifractal spectrum for the empirical measure with these values.

2. The attractor

The skew system studied in [9] is defined on the square $[-1,1]^2$ and has the form for $0<k<1$

$$\begin{cases} x_{n+1} = T_1(x_n) \\ y_{n+1} = (1-k)T_2(y_n) + kT_1(x_n), \end{cases} \tag{1}$$

where T_1 and T_2 are two maps of the interval $[-1,1]$ into itself. Set

$$\Delta_n := |x_n - y_n|;$$

it is immediate to see that for any $n \geq 1$:

$$\Delta_n \leq 2(1-k) \sup_{i=1,2} \sup_{x\in[-1,1]} |T_i(x)|, \tag{2}$$

and therefore the sequences x_n, y_n approach each other when $k \to 1$. We now specialize to the example investigated in [9] and show how to improve the previous bound. The skew system now reads:

$$\begin{cases} x_{n+1} = T(x_n) = c_1(1-2x_n^2) \\ y_{n+1} = (1-k)c_2(1-2y_n^2) + c_1 k(1-2x_n^2). \end{cases} \tag{3}$$

We have

$$\Delta_{n+1} = |(1-k)c_1(1-2x_n^2) - (1-k)c_2(1-2y_n^2)| \leq \\ (1-k)|c_1 - 2c_1x_n^2 - c_2 + 2c_2y_n^2|.$$

We add and subtract the term $2c_1y_n^2$ and we easily obtain

$$\Delta_{n+1} \leq (1-k)|(c_1-c_2)(1-2y_n^2) + 2c_1(y_n^2 - x_n^2)|.$$

We now put $\Delta_c := |c_1 - c_2|$. Since x_n, y_n are in the interval $[-1,1]$, we have

$$\Delta_{n+1} \leq (1-k)\Delta_c|1-2y_n^2| + 4c_1(1-k)\Delta_n \leq (1-k)\Delta_c + 4c_1(1-k)\Delta_n.$$

We now iterate it and we finally get

$$\Delta_{n+1} \leq (1-k)\Delta_c \sum_{l=0}^{n} (1-k)^l(4c_1)^l + (1-k)^{n+1}(4c_1)^{n+1}.$$

We then require

$$k > 1 - \frac{1}{4c_1}, \tag{4}$$

and we define the quantity

$$W_\infty(k) := \Delta_c(1-k)\frac{1}{1-(1-k)4c_1}, \text{ such that } \lim_{k\to 1} W_\infty(k) = 0. \tag{5}$$

By sending $n \to \infty$ we finally get

$$\limsup_{n\to\infty} \Delta_n \leq W_\infty(k). \tag{6}$$

We now use the following values taken in [9]:

$$c_1 = 0.89, \;\; c_2 = 0.8373351.$$

First of all we note that with these values (4) gives $k > 0.72$, which is consistent with what was used in [9]. As in the latter we now take $k = 0.9$, which is the value where, according to [9], the system reaches complete topological synchronization. By substituting into $W_\infty(k)$ we get

$$W_\infty(0.9) = 0.0082,$$

which implies that the projections x_n and y_n are really very close. The bound (2) instead gives, still for $k = 0.9$,

$$\sup_{n\geq 1} |\Delta_n| \leq (1-k)[c_1 + c_2] = 0.17273351.$$

3. The measures

In order to justify the closeness of the asymptotic behaviors of the master and slave dynamics, the paper [9] uses the spectrum of the generalized dimensions. These dimensions are defined in terms of a probability measure, see, e.g., [6] and [12, 3] for a rigorous treatment. Roughly speaking, if μ denotes a probability measure, and $B(x,r)$ a ball of center x and radius r on the phase space M, the generalized dimensions D_q are defined by the following scaling of the correlation integral

$$\int_M \mu(B(x,r))^{q-1} d\mu \sim r^{D_q(q-1)}, \;\; r \to 0. \tag{7}$$

The importance of the generalized dimension is that in several cases, see [12] and the references therein for rigorous results, if we denote by

$$d_\mu(x) := \lim_{r\to 0} \frac{1}{\log r} \log \mu(B(x,r)),$$

the local dimension of the measure μ at the point x, provided the limit exists, then

$$D_q(q-1) = \inf_\alpha \{q\alpha - f(\alpha)\}, \tag{8}$$

where $f(\alpha)$ denotes the Hausdorff dimension of the set of points for which $d_\mu(x) = \alpha$[1]. The master map has several probability invariant measures; we pick one, namely the *physical measure* μ which is given by the weak limit of the probability measures

$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}, \tag{9}$$

where x is chosen Lebesgue almost everywhere on the unit interval, see for instance [8], Chapter V.1. In the following we will forget about the initial condition x, provided it

[1] It is worth noticing that in the next section we will compute the D_q spectrum in a few cases by using the position (8) and *not* the definition (7) in terms of the correlation integral.

is taken Lebesgue almost everywhere, and simply write $x_i = T^i(x)$. The *slave* sequence $\{y_n\}_{n\geq 0}$ could be seen as a non-autonomous, or *sequential*, dynamical system and it is not clear what probability measure we should associate to it. We argue that in [9] the authors used the sequence of probability measures

$$\nu_n := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{y_i},$$

where y_i is the point associated to x_i in (3). We call μ_n and ν_n the *empirical measures*. There are now two questions: (i) does the sequence ν_n converge weakly? and, in the affirmative case, (ii) is that weak limit point equal to μ? This is in fact what the numerical simulations on the generalized dimensions seem to indicate in [9].[2]

To study weak convergence we have to integrate the probability measures against continuous function on the interval $[-1, 1]$. Let f be one of this function; since it is also uniformly continuous, given $\varepsilon > 0$, call δ_ε the quantity such that $|f(x) - f(y)| < \frac{\varepsilon}{2}$, when $|x - y| < \delta_\varepsilon$. Let $k_\varepsilon \in (0, 1)$ such that

$$2W_\infty(k_\varepsilon) < \delta_\varepsilon. \tag{10}$$

For values of k such that $k_\varepsilon < k < 1$ we define $n_{k,\varepsilon}$ as

$$(1-k)^{n_{k,\varepsilon}+1}(4c_1)^{n_{k,\varepsilon}+1} \leq W_\infty(k),$$

such that for all $n > n_{k,\varepsilon}$, $\Delta_n \leq \delta_\varepsilon$. By weak-compactness there will be a subsequence n_l for which $(\nu_{n_l})_{l\geq 1}$ will converge weakly to a probability measure μ^*. Then for any continuous function f on the unit interval and for n_l sufficiently large, say $n_l > n^*$, we have that $|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu^*(f)| \leq \varepsilon/2$. Then

$$|\mu^*(f) - \mu(f)| \leq \left|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu^*(f)\right| + \left|\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f)\right|$$

We now estimate the second piece on the right hand side:

$$\frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f) = \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(y_i) - \mu(f) + \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(x_i) - \frac{1}{n_l}\sum_{i=0}^{n_l-1} f(x_i).$$

[2] We point out, however, that it is in general not enough to have a weak convergence of the measures to ensure the convergence of the D_q spectrum. Suppose for instance that the master system has an absolutely continuous invariant measure $d\mu(x) = h(x)dx$ and that, for k close enough to 1, so does the measure of the slave system $d\mu_k(x) = h_k(x)dx$. If $h(x) \sim_{x_0} \text{const}|x - x_0|^\alpha$, with $-1 < \alpha < 0$ as, for instance, it is the case for some quadratic map at 0, then the local dimension of μ at x_0 is $\alpha + 1 < 1$ and it is easily seen (see the detailed computations in the section 4.2) that the D_q spectrum is not constant. Moreover, if we further assume that, for all $k < 1$, h_k is a piecewise constant function converging in L^1 to h, it is easy to see that the D_q spectrum for μ_k is constant equal to 1 for all $k < 1$, so that there is no convergence to the spectrum of the master map.

Consider now the difference, for $n_l \geq n_{k,\varepsilon}+2$:

$$\frac{1}{n_l}\sum_{i=0}^{n_l-1}[f(y_i)-f(x_i)] = \frac{1}{n_l}\sum_{i=0}^{n_{k,\varepsilon}}[f(y_i)-f(x_i)] + \frac{1}{n_l}\sum_{i=n_{k,\varepsilon}+1}^{n_l-1}[f(y_i)-f(x_i)].$$

By exploiting the uniform continuity of f on the unit interval we have

$$\left|\frac{1}{n_l}\sum_{i=n_{k,\varepsilon}+1}^{n_l-1}[f(y_i)-f(x_i)]\right| \leq \frac{n_l-n_{k,\varepsilon}-2}{n_l}\varepsilon/2.$$

The other piece gives

$$\left|\frac{1}{n_l}\sum_{i=0}^{n_{k,\varepsilon}}[f(y_i)-f(x_i)]\right| \leq 2\max|f|\frac{n_{k+\varepsilon}+1}{n_l}.$$

By sending $l\to\infty$, we finally get that $|\mu^*(f)-\mu(f)|\leq\varepsilon$, and this result is independent of the subsequence we choose. We thus have

Proposition 3.1. *(i) For any* $\varepsilon>0$*, let* $k_\varepsilon\in(0,1)$ *be given as in (10); then for all* k *such that* $k_\varepsilon<k<1$*, we have*

$$\left|\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f(y_i)-\mu(f)\right|\leq\varepsilon.$$

(ii) As a consequence we get:

$$\inf_{0<k<1}\left|\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f(y_i)-\mu(f)\right|=0.$$

This is the best result we could get without further information on the system and it justifies the numerical evidence that the empirical measures constructed along the x and y axis become very close to each other when $k\to1$.

4. The Lyapunov exponent

4.1. **A short review on quadratic maps.** We said above that μ, the invariant measure for the master map T, is a physical measure; the paper [9] claims that such a measure has a multifractal structure for the prescribed values of c_1, where *the master has a dense strange attractor*, [ibid]. Before exploring and commenting such a possibility, we should remind a few important properties of the quadratic maps: first that they usually depend upon a parameter, in our case c since the map in [9] is of the form

$$[-1,1]\ni x\longmapsto T(x)=c(1-2x^2)\in[-1,1], \tag{11}$$

with $0<c\leq1$. We refer in particular to the nice review paper by Thunberg [13], which contains a clear and exhaustive list of all the relevant results on unimodal maps and a reach bibliography. First of all we define the attractor Ω_c of the map T as the unique set of accumulation points of the orbit of the point x, whenever this point is chosen Lebesgue almost everywhere. Then it is well known, see [2] or Theorem 6 in [13], that for our kind

of logistic maps, the attractor could be of three types:
(1) an attracting periodic orbit; (2) a Cantor set of measure zero; (3) a finite union of intervals with a dense orbit.
Still in the quadratic case, we could classify the preceding three different types of attractors in terms of the set of parameters c. Following section 2.2 in [13] we have:
(1) $\mathcal{P} := \{c \in \mathbb{R} : \ \Omega_c$ is a periodic cycle$\}$ is open and dense in parameter space and consists of countably infinitely many nontrivial intervals.
(2) $\mathcal{C} := \{c \in \mathbb{R} : \Omega_c$ is a Cantor set$\}$ is a completely disconnected set of Lebesgue measure zero.
(3) $\mathcal{I} := \{c \in \mathbb{R} : \Omega_c$ is a union of intervals$\}$ is a completely disconnected set of positive Lebesgue measure.
The physical measures, constructed according to the prescription (9) exist and are parametrized by c in the following way:
(1) If $c \in \mathcal{P}$, the physical measure consists of normalized point masses on the periodic cycle Ω_c.
(2) If $c \in \mathcal{C}$, the support of the physical measure equals the Cantor attractor Ω_c, and it is singular with respect to Lebesgue measure.
(3) (a) There is a full-measure subset $\mathcal{S} \subset \mathcal{I}$ such that for all $c \in \mathcal{I}$, the physical measure is absolutely continuous with respect to Lebesgue measure and its support equals the interval attractor Ω_c.
(b) There are uncountably many parameters in $\mathcal{I} \setminus \mathcal{C}$ for which the physical measure may fail to exist.

We now dispose of a very efficient numerical test to determine the nature of a physical measure. It is based on the following two rigorous results:
(i) the first says that if T has a non-flat critical point, as in our case, and it admits an absolutely continuous invariant probability measure μ, then it is the weak-limit of the sequence μ_n given in (9) and therefore it is a physical measure, see Chapter V.1 in [8].
(ii) The second result is taken from the paper [7]. Let us define the number

$$\lambda_T := \limsup_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)|.$$

This quantity exists for x chosen *Leb*-almost everywhere and it is strictly positive if and only if T has an absolutely continuous invariant measure.
From the joint use of (i) and (ii) it follows immediately that if we can show that the sequence

$$\lambda_T^n := \frac{1}{n} \log |(T^n)'(x)| = \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(x_i)| \tag{12}$$

has a positive limit for *Leb*-a.e. x, then the sequence of empirical measures μ_n in (9) converges weakly to an absolutely continuous invariant probability measure and therefore the attractor Ω_c will be a finite union of intervals and not a Cantor set.

In fig. 1, we represent the bifurcation diagram of T, and its Lyapunov exponent for different values of parameter c. This quantity is non-positive whenever the attractor is a periodic cycle or a Cantor set.

We computed in particular the limit of λ_T^n for $c = c_1 = 0.89$, still called λ_T, and we got a positive value of ≈ 0.35, confirming the fact that μ *is not supported on a Cantor set*. We performed the same computation with $c = c_2$ and, denoting from now on by $\tilde{\mu}$ the associated physical measure, there are strong numerical evidences that it is again absolutely continuous, with a strictly positive Lyapunov exponent.

4.2. **Multifractal spectrum for absolutely continuous measures.** Let us summarize: by choosing the parameter c with positive (Lebesgue) probability, we could get a periodic cycle or union of intervals. On the other hand Dirac measures with finitely many masses on the periodic cycles cannot have a multifractal spectrum. Finally the Lyapunov exponent for $c = c_1$ is positive showing that the attractor could not be a Cantor set and the physical measure will be absolutely continuous. The question is, therefore, if such a measure μ could exhibit a multifractal spectrum. Let us consider unimodal maps of Benedicks-Carleson type, which are known to preserve an absolutely continuous invariant measure μ [4]. Under certain conditions (see [5] for details), their density has the form [5]:

$$h(x) = \psi_0(x) + \sum_{k\geq 1} \frac{\varphi_k(x)}{\sqrt{|x - c_k|}},$$

with ψ_0 bounded and for all $k \geq 1$, φ_k is piece-wise C^1, and such that $||\varphi_k||_\infty \leq e^{-ak}$ for some $a > 0$.

Theo: Est on sur que l'application 11 a ces propriétés?

Proposition 4.1. *Suppose f satisfies the hypothesis of Theorem 2.7 in* [5]*. Then, the generalized dimensions spectrum of μ is given by:*

$$D_q = \begin{cases} 1 \ \textit{if } q < 2 \\ \frac{q}{2(q-1)} \ \textit{otherwise.} \end{cases} \tag{13}$$

Proof. We start be noticing that since h is bounded away from 0 [4], the local dimensions are all smaller than or equal to 1. Now, the measure of a ball centered at x of radius r is given by

$$\mu(B(x,r)) := \int_{x-r}^{x+r} \psi_0(y)dy + \sum_{k\geq 1} \int_{x-r}^{x+r} \frac{\varphi_k(y)}{\sqrt{|y - c_k|}} dy.$$

Let us take $m_n = \frac{n}{2a}$ and $\delta_n = m_n^{-\log n}$. Let

$$\Gamma_n = \{x : \exists k < m_n \text{ such that } |x - c_k| < \delta_n\}.$$

Given $r > 0$, we take n the smaller integer such that $r < e^{-n}$. Since the functions φ_k are bounded, the integrals in the sum are bounded above by $c_1\sqrt{r}||\varphi_k||_\infty$ when $|x - c_k| < \delta_n$ and by $c_2 r/\sqrt{\delta_n}||\varphi_k||_\infty$ otherwise , where $c_1, c_2 > 0$. We get:

$$\mu(B(x,r)) \leq 2r||\psi_0||_\infty + c_1\sqrt{r} \sum_{k:|x-c_k|<\delta_n} ||\varphi_k||_\infty + c_2 \frac{r}{\sqrt{\delta_n}} \sum_{k:|x-c_k|>\delta_n} ||\varphi_k||_\infty \tag{14}$$

For $x \notin \Gamma_n$, the first sum starts at least at m_n is therefore at most $c_3 e^{-am_n} \leq c_3\sqrt{r}$; The second geometric sum is bounded par c_3. Thus there exists $c_4 > 0$ such that,

$$\mu(B(x,r)) \leq c_4 r + c_4 \frac{r}{\sqrt{\delta_n}}.$$

If $x \notin \Gamma_n$ for n large enough, then $d_\mu(x) = 1$, since $-\log \delta_n$ is of order $(\log\log(1/r))^2 << \log(1/r)$, so the second term does not affect the dimension. Therefore $d_\mu(x) = 1$ in the set

$$G = \bigcup_p \bigcap_{n>p} \Gamma_n^c.$$

Let $\Gamma = G^c = \bigcap_p \bigcup_{n>p} \Gamma_n$, the set of x such that there exists an infinity of n such that $x \in \Gamma_n$. Γ is covered by the union of balls

$$\bigcup_n \bigcup_{k<m_n} B(c_k, \delta_n).$$

Now, for all $\varepsilon > 0$, we have

$$\sum_n \sum_{k<m_n} \delta_n^\varepsilon = \sum_n (\frac{n}{2a})^{1-\varepsilon \log n} < \infty.$$

So the Hausdorff measure $H^\varepsilon(\Gamma)$ is finite, which show that $\dim_H(\Gamma) \leq \varepsilon$.

It is easily seen from (14) that for all $x \in [-1,1]$,

$$\mu(B(x,r)) \leq c_5\sqrt{r}. \tag{15}$$

This shows that the infimum of the local dimensions is larger or equal to 1/2. On the other hand, since for all k, φ_k is C^1 by part, the singularities are of type $|x - c_k|^{-1/2}$, so that

$$\mu(B(c_k,r)) \geq c_6\sqrt{r}. \tag{16}$$

Combining the last two estimates, we get

$$d_\mu(c_k) = 1/2.$$

We can now compute the generalized dimensions. $D_q(q-1)$ is defined as the Legendre transform of the function $f(\alpha) := d_H\{x; d(x) = \alpha\}$, where d_H denotes the Hausdorff dimension. In our case, we have $f(1) = 1$ and, for all $\alpha < 1$, either $f(\alpha) = 0$, or $f(\alpha)$ is not defined, so that

$$D_q = (q-1)^{-1} \inf_\alpha \{q\alpha, q-1\}.$$

Since the density is bounded from below [4], the local dimensions are bounded above by 1, and since $\min(d_\mu) = 1/2$, we obtain our result.

□

Notice that although this spectrum is not differentiable and constant for $q < 2$, its numerical approximation shows a smooth behavior, see Fig. 5.1 in [1]. It is enough for the measure of the slave to have a density bounded away form 0 as k approaches 1, to yield $D_q = 1$ for negative q. This could explain the observed *zipper effect* described in [9].

Theo: Apparement ils observent le zipper effect aussi pour d'autre systemes, e.g. Rossler, ou les D_q sont lisses et ont un comportement non trivial

It is possible to construct a density which has a non trivial singularity spectrum, defined in an interval of positive Lebesgue measure, showing that even absolutely continuous invariant measures can have all sort of multifractal spectra. We present this example in the appendix.

4.3. **A random analog.** Our proposition 3.1 suggests that the sequence of empirical measures ν_n for the slave non-autonomous evolution converges weakly to μ. Such an evolution could be understood in another way. Consider the logistic master map T; at each step x_n we now add a number $(1-k)\omega_n$, where $\omega_n \in [-1, 1]$ is *taken with the probability distribution given by the invariant measure $\tilde{\mu}$ of the master map $\tilde{T}(x) = c_2(1-2x^2)$*, see above. Suppose moreover that the $\{\omega_n\}_{n\geq 1}$ are mutually independent [3]. We thus get a random dynamical system perturbed with additive noise

$$x_{n+1} = kx_n + (1-k)\omega_n.$$

It is well known that such random dynamical systems admits a *stationary* probability measure ν_s. For a large class of maps admitting invariant sets and supporting eventually singular measures, the noise has a regularizing effect making very often the stationary measure absolutely continuous. Moreover the stationary measure is the weak-limit of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{kx_i+(1-k)\omega_i},$$

for ν_s almost all initial condition x_0 and almost all realization $\{\omega_n\}_{n\geq 1}$. Therefore ν_s could be considered as the weak limit of the sequence of empirical measures ν_n constructed in the previous section,[4] and therefore the latter converge to an absolutely continuous measure. This is confirmed by Fig. 6, which shows the support of the limiting measure of the ν_n; for $k > 0.2$ the histogram is compatible with the presence of a smooth density[5].

[3]This is of course not true when if ω_n is distributed as $\tilde{T}^n(x)$, with x chosen *Leb*-a.e., but it becomes asymptotically true since T mixes exponentially fast with respect to $\tilde{\mu}$.

[4]Notice that in the limit of zero noise ($k \to 1$ in our case), the smooth measure ν_s could converge weakly to an eventual singular measure and this is coherent with Proposition 3.1. This is a typical weak stochastic stability result.

[5]It is however interesting to observe that the empirical measure is not always absolutely continuous, although it continues to converge weakly to the physical measure μ of the master map. This is shown in Fig. 7.

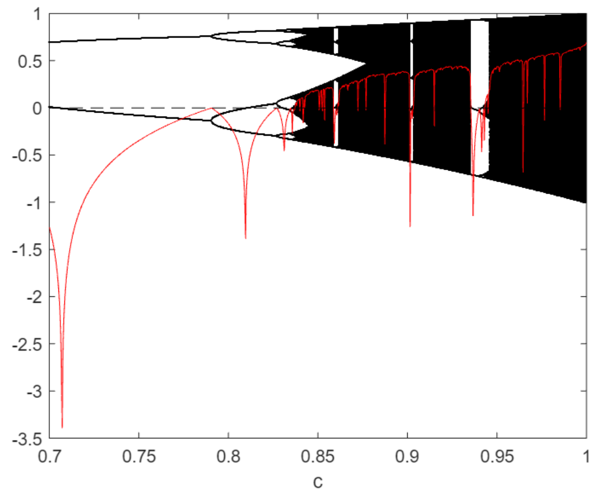

FIGURE 1. Bifurcation diagram for the map $T(x) = c(1 - 2x^2)$ and its associated λ_T computed over different values of the parameter c.

To gauge the convergence of the measure ν_n to μ, we plot in figure 4 the evolution of the empirical Lyapunov exponent (we set $\tilde{\lambda}_T$ the limiting value):

$$\tilde{\lambda}_T^n := \frac{1}{n} \sum_{i=0}^{n-1} \log |T'(y_i)|, \;\; y_i = (1-k)c_2(1-2y_{i-1}^2) + c_1 k(1-2x_{i-1}^2), i \geq 1,$$

with respect to the master parameter c_1.

For the values of c_1 and c_2 prescribed in [9], the dependence of $\tilde{\lambda}_T$ vs k is made explicit in Fig. 5. We see that when $k \to 1$ the empirical Lyapunov exponent $\tilde{\lambda}_T$ converges to λ_T.

This supports the conclusions of Proposition 3.1, although in principle it could not be applied to $\log |T'|$, which is not even bounded on $[-1, 1]$.

In figure 5, we represent the densities associated with the measure ν_n at $k = 0$ and $k = 0.5$ and $k = 1$ (at which $\nu_n = \mu$). For all these values of k. the density seems to be unbounded on an irregular set, which may be compatible with the simple formal models presented in section 4.2 and in the Appendix and therefore with the findings of a non trivial D_q spectrum, as found in [9].

5. Conclusions

The paper [9] used the spectrum of the generalized dimensions to follow the process of synchronisation in master/slave systems. We showed that for the parameter values of the quadratic map considered in the aforementioned paper, the master map has an absolutely continuous invariant measure and the attractor is not a Cantor set. We did not find in the literature any result on the multifractal spectrum of such a measure. We instead gave examples of densities allowing a multifractal structure. In those cases the function $q \to D_q$ is continuous but not smooth, which is not what was observed in [9], unless smoothness was a consequence of numerical approximations. Moreover our examples suggest that the dimensions are constant for negative q, since the invariant densities are bounded away from

zero, which supports the presence of the zipper effect.
We presented a detailed study of the Lyapunov exponent and we believe that it is a much more reliable technique, besides to be more theoretically founded, to describe the synchronisation process.

6. Appendix

Let C be the ternary Cantor set in the unit interval. Take $h(x) = \frac{1}{H} dist(C,x)^\alpha$, with $\frac{\log 2}{\log 3} - 1 < \alpha < 0$ and H a normalizing constant. The complement of C in I is the union over $n \geq 1$ of the union of 2^{n-1} gaps G_n of length 3^{-n}. Let x_1 be the middle point of such a gap and x_2 be a boundary point. We have

$$\int_{G_n} dist(x,C)^\alpha dx = 2\int_{x_1}^{x_2} |x - x_2|^\alpha dx = \frac{2^{-\alpha}}{\alpha+1} 3^{-n(\alpha+1)} \tag{17}$$

The contribution of the 2^{n-1} gaps G_n of size 3^{-n} is

$$2^{n-1}\frac{2^{-\alpha}}{\alpha+1}3^{-n(\alpha+1)} = c(\frac{2}{3^{\alpha+1}})^n.$$

Taking the union over all the gap sizes, we get

$$\int_I h(x)dx = \frac{c}{H}\sum_{n\geq 1}(\frac{2}{3^{\alpha+1}})^n < \infty,$$

since $\alpha > \frac{\log 2}{\log 3} - 1$. The density h is then integrable on I.

Now, we have that for $a \in C$,

$$\mu(B(a,r)) = \frac{1}{H}\int_{a-r}^{a+r} d(x,C)^\alpha dx \geq \frac{1}{H}\int_{a-r}^{a+r} r^\alpha dx = \frac{1}{H}2(\alpha+1)^{-1}r^{\alpha+1}, \tag{18}$$

since for $x \in B(a,r)$, $dist(x,C) < r$ and $\frac{\log 2}{\log 3} - 1 < \alpha < 0$.

On the other hand, we have

$$\mu(B(a,r)) = \int_{B(a,r)} h(x)dx \leq \sum_{G\in S_1}\int_G h(x)dx + \sum_{G\in S_2}\int_{G\cap B(a,r)} h(x)dx, \tag{19}$$

where S_1 is the set of gaps of C of size strictly smaller than r intersecting $B(a,r)$ and S_2 is the set of gaps of C of size greater or equal to r intersecting $B(a,r)$.

Note that for all $a \in C$, $card(S_2) \leq 2$, and for each of the gaps G in S_2,

$$\int_{G\cap B(a,r)} h(x)dx \leq 2\int_{x_1}^{x_1+r} |x_1 - x|^\alpha dx \leq c_1 r^{\alpha+1},$$

where $x_1 \in B(a,r)$ is a boundary point of G and c_1 is a constant, so that the second term in eq. 19 is bounded above by

$$2c_1 r^{\alpha+1}. \tag{20}$$

Now the gaps in S_1 are of length smaller than r; for any $k > 0$ there are $O(r3^k)$ gaps of size 3^{-k} intersecting $B(a,r)$.

Each of these gaps have a mass of $\frac{2^{-\alpha}}{\alpha+1}3^{-k(\alpha+1)}$, so that there will be constants c_2 and c_3 such that

$$\sum_{G\in S_1}\int_G h(x)dx \le c_2\frac{1}{H}\frac{2^{-\alpha}}{\alpha+1}\frac{r}{3}\sum_{k=\lfloor-\frac{\log r}{\log 3}\rfloor}^{+\infty}\frac{1}{3^{\alpha k}} \le c_3 r^{\alpha+1} \tag{21}$$

Combining eqs. (19), (20) and (21), we get that

$$\mu(B(a,r)) \le (2c_1+c_3)r^{\alpha+1}.$$

Together with 18, we can show that the local dimension of μ at the point $a\in C$ is

$$d_\mu(a) = \lim_{r\to 0}\frac{\log\mu(B(a,r))}{\log r} = \alpha+1.$$

If now $a\notin C$, it is easy to see that $d_\mu(a)=1$. Indeed, a lies in a gap of size 3^{-k} for some $k\ge 1$. Taking r small enough, the ball $B(a,r)$ contains no singularities of the density, which is continuous, and $\mu(B(a,r)) = O(r)$.

Following the usual procedure in multifractal analysis (see above), we can compute the singularity spectrum $f(z)$ of μ, which gives the Haussdorf dimension of the set of points in I with local dimension z:

$$\begin{cases} f(\alpha+1) = \frac{\log 2}{\log 3} \\ f(1) = 1 \\ f \text{ is not defined otherwise.}\end{cases} \tag{22}$$

Therefore

$$D_q = (q-1)^{-1}\inf_{s=\alpha+1,1}\{qs - f(s)\} = (q-1)^{-1}\inf\{q(\alpha+1)-\frac{\log 2}{\log 3}, q-1\}. \tag{23}$$

Finally,

$$D_q = \begin{cases} \frac{q(\alpha+1)-\frac{\log 2}{\log 3}}{q-1} \text{ if } q > q_0 = \frac{\frac{\log 2}{\log 3}-1}{\alpha} \\ 1 \text{ otherwise.}\end{cases} \tag{24}$$

D_q is not differentiable in q_0, but exhibits a non trivial behavior for $q>q_0$, that could be detected by numerical methods. A visual representation of this D_q spectrum in available in figure 2.

It is in fact possible to construct an example of a density having a singularity spectrum defined in an interval of positive Lebesgue measure.
Let T be an application of the interval I with three branches, coded by 0, 1 and 2. Each $w\in\{0,1,2\}^{\mathbb{N}}$ encodes a unique point $x\in I$. We note $u=-\log|T'|$ and K the Cantor set constituted of the set of points whose codes do not contain 1. Let p be the pressure of u on K and μ_u the Gibbs measure of u on K. We fix $\alpha>0$ and define the density, for $x\notin K$, as

$$h(w) = \exp[-n(p+\alpha)],$$

where x is coded by w and n is the smallest integer such that $w_n = 1$. The measure μ with density h with respect to Lebesgue is finite, and has the following properties: for w coding a point in K :

(1) The measure of a cylinder Z_n^w is about $\mu_u(Z_n^w)\exp(-\alpha n)$.

(2) The diameter of this cylinder is about $\exp(np)\mu_u(Z_n^w)$.

If this measure has a non trivial multifractal spectrum for the entropy, the set E_β of points of K where

$$(-1/n)\log \mu_u(Z_n^w) \to \beta$$

has a Hausdorff dimension $g(\beta)$ which is non trivial in an interval of values of β, and for these points the local dimension, [(*) computed as the limit of log(1)/ log(2)] is exactly

$$(-\beta - \alpha)/(p - \beta).$$

We obtain

$$f((-\beta - \alpha)/(p - \beta)) = g(\beta),$$

for an interval of values of β.

(*) a faire plus finement si l'on veut la vraie dimension avec $\log \mu([x - r, x + r])/ \log r$ puisqu'il faudrait comparer intervalles et [union de] cylindres.

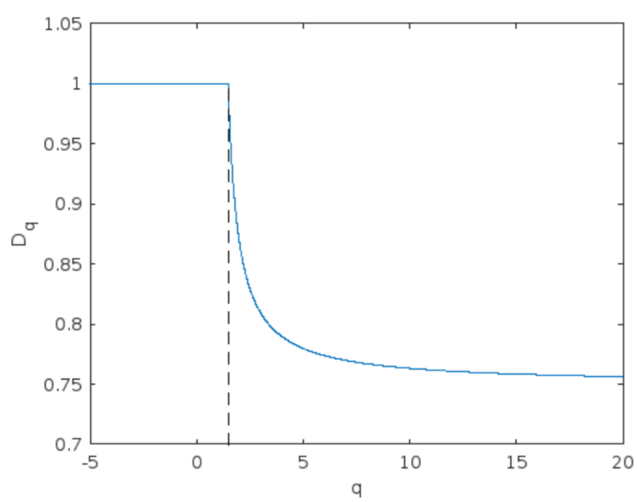

FIGURE 2. D_q spectrum (24) for the example in the Appendix.

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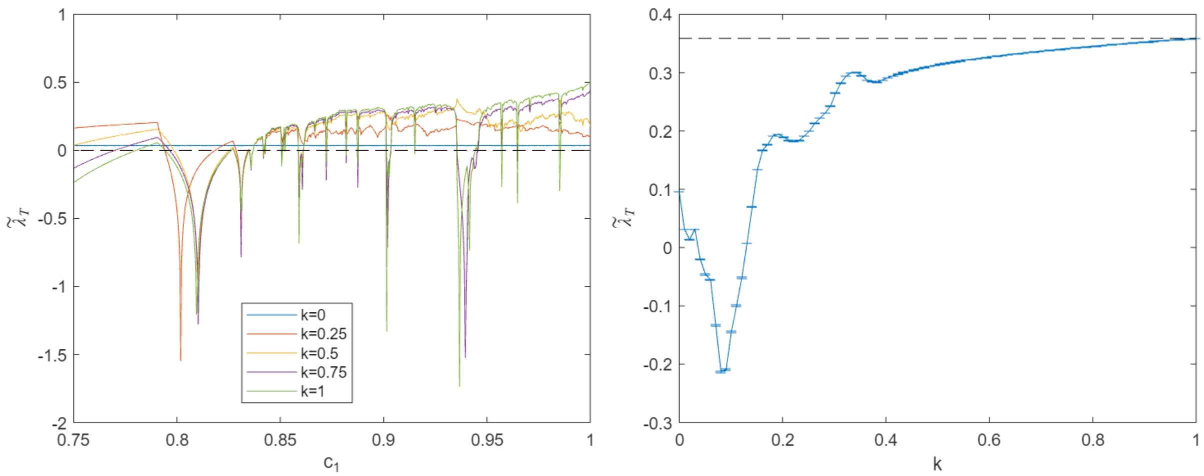
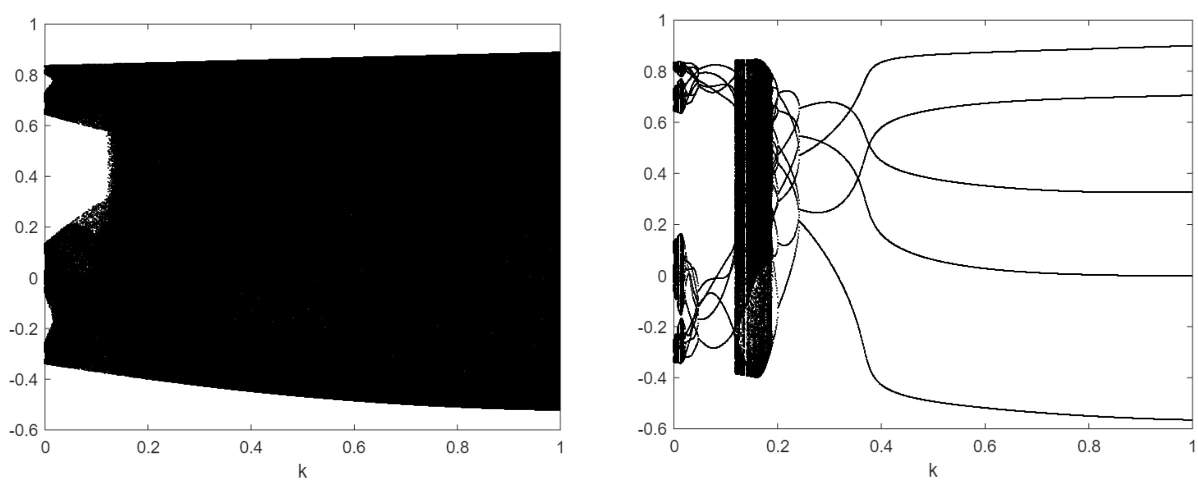


(A) (B)

Figure 3. Evolution of $\tilde{\lambda}_T$ with c_1 for different values of k (left) and with k for the fixed value of $c_1 = 0.89$ (right). For both figures, we took for $c_2 = 0.8373351$.



(A) $c_1 = 0.89$; $c_2 = 0.8373351$ (B) $c_1 = 0.901502$; $c_2 = 0.8373351$

Figure 4. Bifurcation diagram of the dynamics of y_n with the parameter k, for two different values of c_1. On the left, the master measure is supported on an interval, and on the right on a set of 5 points.

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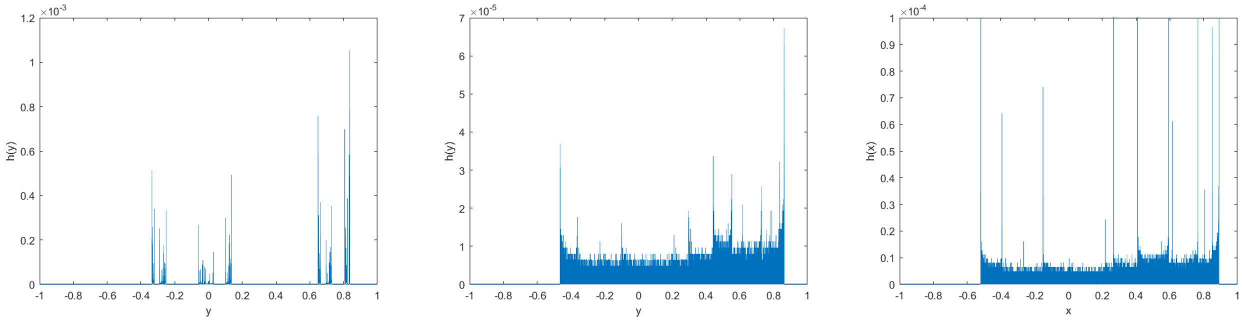

(A) $k = 0$ (B) $k = 0.5$ (C) $k = 1$

FIGURE 5. Numerical estimation of the density of the measure ν for different values of k.

REFERENCES

[1] Leif Anderson, Improved Methods for Calculating the Multifractal Spectrum for Small Data Sets, PhD Thesis, Colorado State University Fort Collins, Colorado Spring 2014, https://ui.adsabs.harvard.edu/abs/2014PhDT.......106A/abstract

[2] A. M. Blokh and M. Yu. Lyubich, *Measurable dynamics of S-unimodal maps of the interval*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, 24 (1991), pp. 545–573.

[3] Th. Caby, D. Faranda, G. Mantica, S. Vaienti, P. Yiou, *Generalized dimensions, large deviations and the distribution of rare events*, PHYSICA D, Vol. 40015, 132143, (2019).

[4] L. -S. Young, Decay of correlations for certain quadratic maps, Communications in Mathematical Physics volume 146, pages123–138 (1992)

[5] V. Baladi, D. Smania, Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps, Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 45 (2012) no. 6, pp. 861-926.

[6] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, *Fractal measures and their singularities: The characterization of strange sets.* Phys. Rev. A 33(2), 1141 (1986).

[7] G. Keller, *Exponents, attractors and Hopf decompositions for interval maps*, Ergodic Theory Dynam. Systems, 10 (1990), pp. 717–744.

[8] W. de Melo and S. van Strien, One-Dimensional Dynamics, Springer-Verlag, Berlin, Heidelberg, 1993.

[9] Nir Lahav1, Irene Sendina-Nadal, Chittaranjan Hens, Baruch Ksherim, Baruch Barzel, Reuven Cohen, Stefano Boccaletti, *Topological synchronization of chaotic systems*, Scientific Reports, (2022) 12:2508

[10] *Topological synchronization of chaotic systems*, https://phys.org/news/2022-04-topological-synchronization-chaotic.html

[11] *Scientists reveal for first time the exact process by which chaotic systems synchronize*, https://phys.org/news/2019-01-scientists-reveal-exact-chaotic-synchronize.html

[12] Y. Pesin, Dimension Theory in Dynamical Systems, The University of Chicago Press, Chicago, 1997.

[13] Hans Thunberg. *Periodicity versus Chaos in One-Dimensional Dynamics*, SIAM Review 43.1 (2001), pp. 3–30.