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Variational Principles for Hyperbolic Flows

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Dedicated to Waldyr Oliva on the occasion of his 70th birthday

Abstract. We establish a conditional variational principle for hyperbolic flows. In particular we provide an explicit expression for the topological entropy of the level sets of Birkhoff averages, and obtain a simple new proof of the corresponding multifractal analysis. One application is that for a geodesic flow φ_t on a compact Riemannian manifold of negative sectional curvature, if there exists a geodesic $\varphi_t x$ with "average" scalar curvature κ, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t K(\varphi_s x)\, ds = \kappa,$$

then there exist uncountably many geodesics with the same "average" scalar curvature κ. The variational principle can also be used to establish the analyticity of several new classes of multifractal spectra for hyperbolic flows.

1 Introduction

In this paper we consider a class of C^1 flows on compact manifolds and study the asymptotic behavior of their orbits. This study reveals an unsuspected complexity behind the already well-known stochastic properties exhibited by certain flows. Instead of considering here the most general situation, we would like to illustrate our statements with an explicit example.

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Consider a compact orientable Riemannian surface M with (sectional) curvature K. The Gauss–Bonnet theorem says that

$$\int_M K\, d\lambda_M = 2\pi\chi(M), \tag{1}$$

where λ_M is the volume measure on M, and $\chi(M)$ denotes the Euler characteristic of M. Let $\Phi = \{\varphi_t\}_t$ denote the corresponding geodesic flow on the unit tangent bundle SM. The geodesic flow preserves the normalized Liouville measure λ_{SM} on SM, induced from the volume on M. By the Birkhoff ergodic theorem the limit

$$\kappa(x) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t}\int_0^t K(\varphi_s x)\, ds \tag{2}$$

exists for λ_{SM}-almost every $x \in SM$. It follows from (1) and (2) that

$$\int_{SM} \kappa\, d\lambda_{SM} = \int_M K\, d\lambda_M = 2\pi\chi(M). \tag{3}$$

Assume now that M has strictly negative curvature K. In this case M must have genus at least 2. The geodesic flow is ergodic, i.e., every set which is invariant under the geodesic flow is either of zero or full measure (see [1] for details). Therefore, besides (3), we have

$$\kappa(x) = \int_M K\, d\lambda_M = 2\pi\chi(M). \tag{4}$$

for λ_{SM}-almost every $x \in SM$. More generally, the identity in (4) holds almost everywhere with respect to any measure preserved by the geodesic flow.

However, it is not at all clear that the function κ may take only the value $2\pi\chi(M)$, i.e., the level sets

$$SM_\alpha = \{x \in SM : \kappa(x) = \alpha\}$$

may very well be nonempty for several values of α. This question appears naturally in the so-called multifractal analysis of dynamical systems (see [8] for details). The following is a consequence of our work in [2].

Theorem 1 *Given a compact orientable surface M with $\chi(M) < 0$, for each metric g in an open set of C^3 metrics on M of strictly negative curvature, there exists an open interval I_g containing $2\pi\chi(M)$ such that $SM_\alpha \subset SM$ is a nonempty proper dense subset with positive topological entropy for every $\alpha \in I_g$.*

See Section 2 for the definition of topological entropy. In particular, Theorem 1 implies that for a metric g as in the theorem, the set SM_α is uncountable for every $\alpha \in I_g$. This reveals an extreme complexity behind the Birkhoff ergodic theorem.

Notice that $SM = N \cup \bigcup_\alpha SM_\alpha$, where

$$N = \left\{ x \in SM : \liminf_{t\to\infty} \frac{1}{t}\int_0^t K(\varphi_s x)\, ds < \limsup_{t\to\infty} \frac{1}{t}\int_0^t K(\varphi_s x)\, ds \right\},$$

and that this union is composed of pairwise disjoint sets. Due to the Birkhoff ergodic theorem the set N has zero measure with respect to *every* invariant measure. This observation strongly contrasts with the following statement, which is also a consequence of our work in [2] (using ideas in [4]).

Theorem 2 *Given a compact orientable surface M with $\chi(M) < 0$, for each metric g in an open set of C^3 metrics on M of strictly negative curvature, the set $N \subset SM$ is a nonempty proper dense subset with topological entropy equal to that of SM.*

There is a good number of rigorous related results, concerning the multifractal analysis of maps. We refer the reader to the book [8] for details and further references. In the case of flows, to our best knowledge, there exist only the works [2] and [9] in the literature.

In this paper we shall provide a simpler new proof of the multifractal analysis for *hyperbolic* flows, or more precisely for C^1 flows with a hyperbolic set (see Section 3.1 for the definition). In particular we shall also consider higher-dimensional manifolds. For example, geodesic flows on compact Riemannian manifolds with negative sectional curvature are hyperbolic. Furthermore, time changes and small C^1 perturbations of hyperbolic flows are also hyperbolic flows.

The new approach to multifractal analysis is based on a conditional variational principle for the topological entropy (see Theorem 5 below). This variational principle is of considerable importance by itself. It virtually contains all known statements of multifractal analysis, and thus unifies and extends all known results concerning the multifractal analysis of topological entropy. In particular it can be used to obtain a much simpler proof of Theorem 1.

We now describe in what consists the conditional variational principle. Let

$$\mathcal{E}(\alpha) = h(\Phi | SM_\alpha)$$

be the topological entropy of the set SM_α. The function $\mathcal{E}$ is called *entropy spectrum*. One of the main objectives of multifractal analysis is to describe the properties of the function $\mathcal{E}$. In most works addressing related problems the function $\mathcal{E}$ is described in terms of a Legendre transform involving the topological pressure (see Section 2 for the definition). We provide a different description, in terms of a variational principle.

Denote by $h_\mu(\Phi)$ the measure-theoretic entropy of the geodesic flow with respect to the measure μ, and by $\mathcal{M}(SM)$ the set of probability measures on SM preserved by the geodesic flow. We also write

$$A = \left\{ \int_{SM} K \, d\mu \colon \mu \in \mathcal{M}(SM) \right\}.$$

Since $\mathcal{M}(SM)$ is weakly compact and convex, the set A is a closed interval. The following is our variational principle for the spectrum $\mathcal{E}$.

Theorem 3 *For a compact orientable surface M and a metric of strictly negative curvature on M, for every $\alpha \in \operatorname{int} A$ we have*

$$\mathcal{E}(\alpha) = \max \left\{ h_\mu(\Phi) : \int_{SM} K \, d\mu = \alpha \text{ and } \mu \in \mathcal{M}(SM) \right\}.$$

The structure of the paper is as follows. In Section 2 we provide some necessary notions from the thermodynamic formalism. In Section 3 we present a much more general version of the conditional variational principle, and describe the main properties of the entropy spectrum. In particular we also consider the so-called "mixed" multifractal spectra (see [3]). Some applications are given in Section 4. Related results in the case of suspension flows are described in Section 5. We also present a higher-dimensional version of the variational principle in Section 6. The proofs are collected in Section 7, and are entirely based on the thermodynamic formalism.

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2 Notions from the thermodynamic formalism

In this section we recall some basic notions from the thermodynamic formalism, in the case of flows. Let $\Phi = \{\varphi_t\}_t$ be a continuous flow on the compact metric space (X, d), that is, a family of transformations $\varphi_t \colon X \to X$ such that $(t, x) \mapsto \varphi_t x$ is continuous, with $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for every t, $s \in \mathbb{R}$, and $\varphi_0 x = x$ for every $x \in X$. Given $x \in X$, $t > 0$, and $\varepsilon > 0$, we define

$$B(x, t, \varepsilon) = \{y \in X : d(\varphi_s y, \varphi_s x) < \varepsilon \text{ for every } 0 \le s \le t\}. \tag{5}$$

Let $u \colon X \to \mathbb{R}$ be a continuous function and write

$$u(x, t, \varepsilon) = \sup \left\{ \int_0^t u(\varphi_s y)\, ds : y \in B(x, t, \varepsilon) \right\}.$$

For each set $Z \subset X$ and each $\alpha \in \mathbb{R}$, we define

$$M(Z, u, \alpha, \varepsilon) = \lim_{T \to \infty} \inf_\Gamma \sum_{(x,t) \in \Gamma} \exp(u(x, t, \varepsilon) - \alpha t),$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i, t_i)\}_i$ such that $(x_i, t_i) \in X \times [T, \infty)$ for each i, and $\bigcup_i B(x_i, t_i, \varepsilon) \supset Z$. One can easily verify that it exists the limit in

$$P_\Phi(u|Z) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \inf\{\alpha : M(Z, u, \alpha, \varepsilon) = 0\}. \tag{6}$$

The number $P_\Phi(u|Z)$ is called the *topological pressure* of u on Z (with respect to the flow Φ). We emphasize that the set Z need not be compact nor Φ-invariant. For simplicity we shall write $P_\Phi(u) = P_\Phi(u|X)$. We also write $h(\Phi|Z) = P_\Phi(0|Z)$ and call $h(\Phi|Z)$ the *topological entropy* of Φ on Z.

When the set Z is compact and Φ-invariant, we can express the topological pressure as

$$\begin{aligned} P_\Phi(u|Z) &= \lim_{\varepsilon \to 0} \liminf_{t \to \infty} \frac{1}{t} \log \inf_\Gamma \sum_{x \in \Gamma} \exp(u(x, t, \varepsilon)) \\ &= \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \inf_\Gamma \sum_{x \in \Gamma} \exp(u(x, t, \varepsilon)), \end{aligned}$$

where the infimum is taken over all finite or countable sets $\Gamma = \{x_i\}_i \subset X$ such that $\bigcup_i B(x_i, t, \varepsilon) \supset Z$. In particular, when Z is compact and Φ-invariant, we can express the topological entropy as

$$h(\Phi|Z) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} \frac{\log N_Z(t, \varepsilon)}{t} = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{\log N_Z(t, \varepsilon)}{t},$$

where $N_Z(t, \varepsilon)$ is the least number of sets $B(x, t, \varepsilon)$ needed to cover Z.

We shall denote by $\mathcal{M}_\Phi(X)$ the set of Φ-invariant Borel probability measures on X, i.e., the Borel probability measures μ on X such that $\mu(\varphi_t A) = \mu(A)$ for every $t \in \mathbb{R}$ and every Borel set $A \subset X$. For every measure $\mu \in \mathcal{M}_\Phi(X)$, the limit

$$h_\mu(\Phi) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \inf\{h(Z, \varepsilon) : \mu(Z) = 1\} \tag{7}$$

is well-defined, where

$$h(Z, \varepsilon) = \inf\{\alpha : M(Z, 0, \alpha, \varepsilon) = 0\}.$$

The number $h_\mu(\Phi)$ is called the *entropy* of Φ with respect to μ. The entropy defined in this way coincides with the "classical" entropy obtained from the entropy of partitions (in the case of metric spaces); this can be shown using similar arguments to those in [8, §11] in the case of discrete time.

Proposition 4 (Variational principle for flows) *For every continuous flow Φ on the compact metric space X and every continuous function $u \colon X \to \mathbb{R}$ we have*

$$P_\Phi(u) = \sup \left\{ h_\mu(\Phi) + \int_X u \, d\mu : \mu \in \mathcal{M}_\Phi(X) \right\}. \tag{8}$$

We say that $\mu \in \mathcal{M}_\Phi(X)$ is an *equilibrium measure* for the function u (with respect to the flow Φ) if the supremum in (8) is attained at this measure, i.e., if

$$P_\Phi(u) = h_\mu(\Phi) + \int_X u \, d\mu.$$

For example, when the map $\mu \mapsto h_\mu(\Phi)$ is upper semi-continuous each continuous function has an equilibrium measure (which need not be unique). In particular, if Φ is expansive then the measure-theoretic entropy is upper semi-continuous. Recall that Φ is *expansive* if there exists $\varepsilon > 0$ such that given any continuous function $s \colon \mathbb{R} \to \mathbb{R}$ with $s(0) = 0$ and point $x, y \in M$ with

$$d(\varphi_t x, \varphi_{s(t)} x) < \varepsilon \quad \text{and} \quad d(\varphi_t x, \varphi_{s(t)} y) < \varepsilon$$

for every $t \in \mathbb{R}$, then $x = y$.

The reader can see [7, 8, 11] for details.

3 Conditional variational principle

3.1 Hyperbolic flows. Let $\Phi = \{\varphi_t\}_t$ be a C^1 flow of the smooth compact manifold M. A closed Φ-invariant set $\Lambda \subset M$ is called *hyperbolic* for Φ if there exists a continuous splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0,$$

and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$ the following properties hold:

1. the vector $\frac{d}{dt}(\varphi_t x)|_{t=0}$ generates $E^0(x)$;
2. $d_x\varphi_t E^s(x) = E^s(\varphi_t x)$ and $d_x\varphi_t E^u(x) = E^u(\varphi_t x)$ for each $t \in \mathbb{R}$;
3. $\|d_x\varphi_t v\| \le c\lambda^t \|v\|$ for every $v \in E^s(x)$ and every $t > 0$;
4. $\|d_x\varphi_{-t} v\| \le c\lambda^t \|v\|$ for every $v \in E^u(x)$ and every $t > 0$.

For example, geodesic flows on compact Riemannian manifolds with strictly negative sectional curvature have the whole unit tangent bundle as a hyperbolic set. Furthermore, time changes and small C^1 perturbations of flows with a hyperbolic set also possess a hyperbolic set.

A closed Φ-invariant hyperbolic set Λ is said to be *locally maximal* if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \varphi_t(\overline{U}).$$

Recall that $\Phi|\Lambda$ is *topologically transitive* if for every nonempty open sets U and V intersecting Λ there exists $t \in \mathbb{R}$ such that $\varphi_t(U) \cap V \cap \Lambda \neq \varnothing$. Furthermore, $\Phi|\Lambda$ is *topologically mixing* if for every nonempty open sets U and V intersecting Λ there exists $t \in \mathbb{R}$ such that $\varphi_s(U) \cap V \cap \Lambda \neq \varnothing$ for every $s > t$.

For a locally maximal hyperbolic set Λ such that $\Phi|\Lambda$ is topologically transitive, each Hölder continuous function $u\colon \Lambda \to \mathbb{R}$ has a unique equilibrium measure. The reader can see [7] for details. Furthermore, the topological pressure, or more precisely the function $u \mapsto P_\Phi(u)$, is analytic in the space of Hölder continuous functions on Λ. This statement is well known in the case of hyperbolic maps (see [11]). One can reduce the case of flows to the case of maps by using a symbolic representation of the flow in terms of a suspension flow over a subshift of finite type (see Section 5), and applying the implicit relation given by Proposition 12 below.

3.2 Conditional variational principle. Let $\Lambda \subset M$ be an invariant set of a continuous flow $\Phi = \{\varphi_t\}_t$ on M. We denote by $C(\Lambda)$ the space of continuous functions $a\colon \Lambda \to \mathbb{R}$. Given a, $b \in C(\Lambda)$ with $b > 0$, we set

$$K_\alpha = K_\alpha(a,b) = \left\{ x \in \Lambda : \liminf_{t\to\infty} \frac{\int_0^t a(\varphi_s x)\,ds}{\int_0^t b(\varphi_s x)\,ds} = \limsup_{t\to\infty} \frac{\int_0^t a(\varphi_s x)\,ds}{\int_0^t b(\varphi_s x)\,ds} = \alpha \right\}.$$

One can easily verify that the set K_α is Φ-invariant. Let also

$$\underline{\alpha} = \underline{\alpha}(a,b) = \inf\left\{ \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \right\}$$

and

$$\overline{\alpha} = \overline{\alpha}(a,b) = \sup\left\{ \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} : \mu \in \mathcal{M}_\Phi(\Lambda) \right\}.$$

The function $\mathcal{F} = \mathcal{F}^{(a,b)}$ defined by

$$\mathcal{F}(\alpha) = h(\Phi|K_\alpha)$$

is called the *entropy spectrum* for the pair of functions (a, b).

We now present the main result of this section.

Theorem 5 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing, and a, $b\colon \Lambda \to \mathbb{R}$ Hölder continuous functions with $b > 0$. Then the following properties hold:*

1. *if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ then $K_\alpha = \varnothing$;*
2. *if $\alpha \in (\underline{\alpha}, \overline{\alpha})$ then $K_\alpha \neq \varnothing$,*

$$\mathcal{F}(\alpha) = \max\left\{ h_\mu(\Phi) : \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda) \right\}, \tag{9}$$

and

$$\mathcal{F}(\alpha) = \min\{P_\Phi(qa - q\alpha b) : q \in \mathbb{R}\}. \tag{10}$$

The identity in (9) is called a *conditional variational principle* for the entropy spectrum. Observe that Theorem 3 is a particular case of Theorem 5. By Theorem 5, if $\underline{\alpha} < \overline{\alpha}$ then $K_\alpha \neq \varnothing$ for every α in $(\underline{\alpha}, \overline{\alpha})$. Theorem 6 below establishes a necessary and sufficient condition, in terms of the functions a and b, so that the strict inequality $\underline{\alpha} < \overline{\alpha}$ takes place.

Let us now explain how to obtain a measure where the maximum in (9) is attained. Let $q(\alpha) \in \mathbb{R}$ be a point where the function $q \mapsto P_\Phi(qa - q\alpha b)$ attains its infimum (it is established in the proof of Theorem 5 that the infimum is indeed attained). Then the unique equilibrium measure μ_α of the function $q(\alpha)(a - \alpha b)$ satisfies (see the proof of Theorem 5 for details):

$$\mathcal{F}(\alpha) = h_{\mu_\alpha}(\Phi) \quad \text{and} \quad \frac{\int_\Lambda a\, d\mu_\alpha}{\int_\Lambda b\, d\mu_\alpha} = \alpha. \tag{11}$$

Using similar arguments to those in [3] in the case of discrete time, we can extend Theorem 5 to the situation when the measure-theoretic entropy is upper semi-continuous, for continuous functions with unique equilibrium measures. For example, for locally maximal hyperbolic sets of topologically mixing C^1 flows the measure-theoretic entropy is upper semi-continuous, and any continuous function with bounded variation has a unique equilibrium measure. Recall that a continuous function $a\colon X \to \mathbb{R}$ has *bounded variation* if there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$\left| \int_0^t a(\varphi_s x)\, ds - \int_0^t a(\varphi_s y)\, ds \right| < \kappa$$

whenever $d(\varphi_s x, \varphi_s y) < \varepsilon$ for every $s < t$.

3.3 Analyticity of the spectrum. We say that a function $a\colon \Lambda \to \mathbb{R}$ is Φ-*cohomologous* to a function $b\colon \Lambda \to \mathbb{R}$ if there exists a bounded measurable function $q\colon \Lambda \to \mathbb{R}$ such that

$$a(x) - b(x) = \lim_{t \to 0} \frac{q(\varphi_t x) - q(x)}{t}$$

for every $x \in \Lambda$. The following theorem shows that whenever a is not Φ-cohomologous to a multiple of b the spectrum $\mathcal{F}$ is real analytic on its domain.

Theorem 6 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing, and a, $b\colon \Lambda \to \mathbb{R}$ Hölder continuous functions with $b > 0$. Then the following properties hold:*

1. *if a is Φ-cohomologous to a multiple cb of b, then $\underline{\alpha} = \overline{\alpha} = c$ and $K_c = \Lambda$;*
2. *if a is Φ-cohomologous to no multiple of b, then $\underline{\alpha} < \overline{\alpha}$ and the function $\mathcal{F}$ is real analytic on the interval $(\underline{\alpha}, \overline{\alpha})$.*

One can easily verify that if a is Φ-cohomologous to the multiple cb of b, then $c = \int_\Lambda a\, d\mu / \int_\Lambda b\, d\mu$ for every $\mu \in \mathcal{M}_\Phi(\Lambda)$.

In the special case when $b = 1$ the statement in Theorem 6 was established in [2] with a different method.

We now show that "most" Hölder continuous functions satisfy the second alternative in Theorem 6. Let $C^\alpha(\Lambda)$ be the space of Hölder continuous functions on Λ with Hölder exponent $\alpha \in (0, 1]$. We also denote by $C^\alpha_+(\Lambda)$ the subset of $C^\alpha_+(\Lambda)$ composed of the strictly positive functions. Given $a \in C^\alpha(\Lambda)$ we define its norm by

$$\|a\|_\alpha = \sup\{|a(x)| : x \in \Lambda\} + \sup\left\{ \frac{|a(x) - a(y)|}{d(x,y)^\alpha} : x,\, y \in \Lambda \text{ and } x \neq y \right\},$$

where d denotes the distance on M.

Theorem 7 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically transitive. Then, for each $\alpha \in (0,1)$, the set of functions $(a,b) \in C^\alpha(\Lambda) \times C^\alpha_+(\Lambda)$ such that a is Φ-cohomologous to no multiple of b is open and dense in $C^\alpha(\Lambda) \times C^\alpha_+(\Lambda)$.*

Combining Theorems 5, 6, and 7 we readily obtain the following statement, formulated without using the notion of cohomology.

Theorem 8 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing. Given $\alpha \in (0,1]$, for each $(a,b) \in C^\alpha(\Lambda) \times C^\alpha_+(\Lambda)$ in an open and dense set, the entropy spectrum $\mathcal{F}$ is real analytic on the nonempty interval $(\underline{\alpha}, \overline{\alpha})$, and satisfies the identities* (9) *and* (10) *for every $\alpha \in (\underline{\alpha}, \overline{\alpha})$.*

3.4 Relation with multifractal analysis. The above study can be used to provide a new proof of the multifractal analysis of the entropy spectrum effected by Barreira and Saussol in [2] when $b = 1$. This new proof avoids symbolic dynamics (unlike the approach in [2] or the related approach in [9], for which symbolic dynamics in crucial), and in particular does not require the so-called Markov systems introduced by Bowen and Ratner. Instead it provides a simple minded approach entirely based on the thermodynamic formalism.

We set $b = 1$ and write

$$\mathcal{E}(\alpha) = h(\Phi|K_\alpha(a,1)).$$

The function $\mathcal{E}$ is called the *entropy spectrum for the Birkhoff averages of a*. Note that

$$K_\alpha(a,1) = \left\{ x \in \Lambda : \liminf_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s x)\, ds = \limsup_{t\to\infty} \frac{1}{t}\int_0^t a(\varphi_s x)\, ds = \alpha \right\}.$$

Set $T(q) = P_\Phi(qa)$. It follows from (10) in Theorem 5 that

$$\mathcal{E}(\alpha) = \min\{T(q) - q\alpha : q \in \mathbb{R}\},$$

i.e., $\mathcal{E}$ is the Legendre transform of the T. Taking the derivative of the function $q \mapsto T(q) - q\alpha$ we obtain

$$\mathcal{E}(T'(q)) = T(q) - qT'(q).$$

Furthermore, the function $q \mapsto T'(q)$ is the inverse of the function $\alpha \mapsto q(\alpha)$ in Section 3.2 (note that when $b = 1$ the number $q(\alpha)$ is uniquely defined; this is in general not the case for an arbitrary function b). It follows from (11) that $\mathcal{E}(T'(q)) = h_{\nu_q}(\Phi)$, where ν_q is the unique equilibrium measure of qa.

Putting these observations together and using Theorem 6 we obtain the following statement.

Theorem 9 **(**[2]**)** *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing, and $a\colon \Lambda \to \mathbb{R}$ a Hölder continuous function. Then the following properties hold:*

1. *$\mathcal{E}(T'(q)) = T(q) - qT'(q) = h_{\nu_q}(\Phi)$ for every $q \in \mathbb{R}$;*
2. *if a is not Φ-cohomologous to a constant on Λ, then $\mathcal{E}$ and T are real analytic strictly convex functions (forming a Legendre pair).*

We emphasize that Theorem 9 is not new (although in [2] it was formulated for $C^{1+\varepsilon}$ flows). On the other hand, the way in which the theorem is obtained here (as a simple application of Theorem 5) is indeed new and illustrates how the conditional variational principle obtained in Theorem 5 can be used in a straightforward manner to recover the multifractal analysis of the entropy spectra for hyperbolic flows.

4 Applications

In this section we present applications of the above results, in the case of multifractal spectra obtained from local entropies and from Lyapunov exponents.

4.1 Multifractal spectra for local entropies. Let $\Lambda \subset M$ be an invariant set of a continuous flow $\Phi = \{\varphi_t\}_t$ on M, and ν a Φ-invariant Borel probability measure on M. For each point $x \in M$ we define the *lower* and *upper ν-local entropies of Φ at x* by

$$\underline{h}_\nu(\Phi, x) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \liminf_{t\to\infty} -\frac{1}{t} \log \nu(B(x,t,\varepsilon))$$

and

$$\overline{h}_\nu(\Phi, x) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \limsup_{t\to\infty} -\frac{1}{t} \log \nu(B(x,t,\varepsilon)),$$

with $B(x,t,\varepsilon)$ as in (5). Whenever $\underline{h}_\nu(\Phi,x) = \overline{h}_\nu(\Phi,x)$, the common value is denoted by $h_\nu(\Phi,x)$ and is called the *ν-local entropy of Φ at x*. By the Shannon–McMillan–Breiman theorem, the ν-local entropy of Φ is well-defined ν-almost everywhere. In addition, if ν is ergodic then $h_\nu(\Phi,x) = h_\nu(\Phi)$ for ν-almost every $x \in M$.

The *entropy spectrum for local entropies* is defined by

$$\mathcal{H}(\alpha) = h(\Phi | K_\alpha^h),$$

where

$$K_\alpha^h = \{x \in M : \underline{h}_\nu(\Phi,x) = \overline{h}_\nu(\Phi,x) = \alpha\}.$$

In the case of hyperbolic flows we have

$$K_\alpha^h = \left\{ x \in M : \liminf_{t\to\infty} -\frac{1}{t}\log\nu(B(x,t,\varepsilon)) = \limsup_{t\to\infty} -\frac{1}{t}\log\nu(B(x,t,\varepsilon)) = \alpha \right\}$$

for all sufficiently small $\varepsilon > 0$, i.e., the limits in ε are not necessary (compare with Proposition 11 in [2]).

Again for hyperbolic flows, there exists a unique measure m_E of maximal entropy, i.e., a unique invariant probability measure such that $h(\Phi) = h_{m_E}(\Phi)$. We write

$$\underline{\alpha}^h = \inf\left\{ -\int_\Lambda a\, d\mu : \mu \in \mathcal{M}_\Phi(\Lambda) \right\} \quad \text{and} \quad \overline{\alpha}^h = \sup\left\{ -\int_\Lambda a\, d\mu : \mu \in \mathcal{M}_\Phi(\Lambda) \right\}.$$

The following can be readily obtained from Theorems 5 and 6 setting $b = -1$.

Theorem 10 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing, and ν an equilibrium measure of a Hölder continuous function $a\colon \Lambda \to \mathbb{R}$ such that $P_\Phi(a) = 0$. Then the following properties hold:*

1. *if $\alpha \notin [\underline{\alpha}^h, \overline{\alpha}^h]$ then $K_\alpha^h = \varnothing$;*

2. *if* $\alpha \in (\underline{\alpha}^h, \overline{\alpha}^h)$ *then* $K_\alpha^h \neq \varnothing$, *and*

$$\begin{aligned}\mathcal{H}(\alpha) &= \max\left\{h_\mu(\Phi) : -\int_\Lambda a\,d\mu = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda)\right\}\\ &= \min\{P_\Phi(qa) + q\alpha : q \in \mathbb{R}\};\end{aligned}$$

3. *if* $\nu = m_E$, *i.e.,* a *is* Φ*-cohomologous to zero, then* $\underline{\alpha}^h = \overline{\alpha}^h = c$ *and* $K_c^h = \Lambda$;
4. *if* $\nu \neq m_E$, *i.e.,* a *is not* Φ*-cohomologous to zero, then* $\underline{\alpha}^h < \overline{\alpha}^h$ *and the function* $\mathcal{H}$ *is real analytic on the interval* $(\underline{\alpha}^h, \overline{\alpha}^h)$.

The second statement in the theorem shows that the spectrum $\mathcal{H}$ is the Legendre transform of the function $q \mapsto P_\Phi(qa)$.

4.2 Multifractal spectra for Lyapunov exponents. We now consider the case of Lyapunov exponents. We recall that the C^1 flow $\Phi = \{\varphi_t\}_t$ on M is said to be *conformal* on a hyperbolic set Λ if the maps

$$d_x\varphi_t|E^s(x)\colon E^s(x) \to E^s(\varphi_t x) \quad \text{and} \quad d_x\varphi_t|E^u(x)\colon E^u(x) \to E^u(\varphi_t x)$$

are multiples of isometries for each $x \in \Lambda$ and $t \in \mathbb{R}$.

Let Z_s and Z_u be, respectively, the sets of points $x \in M$ for which there exist the limits

$$\lambda_s(x) \stackrel{\text{def}}{=} \lim_{t\to+\infty} \frac{1}{t}\log\|d_x\varphi_t|E^s(x)\| \quad \text{and} \quad \lambda_u(x) \stackrel{\text{def}}{=} \lim_{t\to+\infty} \frac{1}{t}\log\|d_x\varphi_t|E^u(x)\|.$$

By Kingman's subadditive ergodic theorem, we have $\nu(M \setminus Z_s) = \nu(M \setminus Z_u) = 0$. Pesin and Sadovskaya observed in [9] that when Φ is conformal on Λ we have

$$\lambda_s(x) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \zeta_s(\varphi_\tau x)\,d\tau \quad \text{and} \quad \lambda_s(x) = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \zeta_u(\varphi_\tau x)\,d\tau,$$

where

$$\zeta_s(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial t}\log\|d_x\varphi_t|E^s(x)\|\bigg|_{t=0} = \lim_{t\to 0}\frac{1}{t}\log\|d_x\varphi_t|E^s(x)\| \tag{12}$$

and

$$\zeta_u(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial t}\log\|d_x\varphi_t|E^u(x)\|\bigg|_{t=0} = \lim_{t\to 0}\frac{1}{t}\log\|d_x\varphi_t|E^u(x)\|. \tag{13}$$

Since the distributions $x \mapsto E^s(x)$ and $x \mapsto E^u(x)$ are Hölder continuous, the functions ζ_s and ζ_u are Hölder continuous on Λ provided that the flow Φ is of class $C^{1+\varepsilon}$ for some $\varepsilon > 0$.

The *stable* and *unstable entropy spectra for Lyapunov exponents* are defined by

$$\mathcal{L}_s(\alpha) = h(\Phi|K_\alpha^s) \quad \text{and} \quad \mathcal{L}_u(\alpha) = h(\Phi|K_\alpha^u),$$

where

$$K_\alpha^s = \{x \in Z_s : \lambda_s(x) = \alpha\} \quad \text{and} \quad K_\alpha^u = \{x \in Z_u : \lambda_u(x) = \alpha\}. \tag{14}$$

We write

$$\underline{\alpha}_s = \inf\left\{\int_\Lambda \zeta_s\,d\mu : \mu \in \mathcal{M}_\Phi(\Lambda)\right\} \quad \text{and} \quad \overline{\alpha}_s = \sup\left\{\int_\Lambda \zeta_s\,d\mu : \mu \in \mathcal{M}_\Phi(\Lambda)\right\},$$

and

$$\underline{\alpha}_u = \inf\left\{\int_\Lambda \zeta_u\,d\mu : \mu \in \mathcal{M}_\Phi(\Lambda)\right\} \quad \text{and} \quad \overline{\alpha}_u = \sup\left\{\int_\Lambda \zeta_u\,d\mu : \mu \in \mathcal{M}_\Phi(\Lambda)\right\}.$$

The following can be readily obtained from Theorems 5 and 6 setting $a = \zeta_s$ and $b = 1$.

Theorem 11 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a $C^{1+\varepsilon}$ flow Φ with $\varepsilon > 0$ such that $\Phi|\Lambda$ is conformal and topologically mixing. Then the following properties hold:*

1. *if $\alpha \notin [\underline{\alpha}_s, \overline{\alpha}_s]$ then $K^s_\alpha = \varnothing$;*
2. *if $\alpha \in (\underline{\alpha}_s, \overline{\alpha}_s)$ then $K^s_\alpha \neq \varnothing$, and*

$$\begin{aligned}\mathcal{L}_s(\alpha) &= \max\left\{h_\mu(\Phi) : \int_\Lambda \zeta_s \, d\mu = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda)\right\}\\ &= \min\{P_\Phi(q\zeta_s) - q\alpha : q \in \mathbb{R}\};\end{aligned}$$

3. *if ζ_s is Φ-cohomologous to zero, then $\underline{\alpha}_s = \overline{\alpha}_s = c$ and $K^s_c = \Lambda$;*
4. *if ζ_s is not Φ-cohomologous to zero, then $\underline{\alpha}_s < \overline{\alpha}_s$ and the function $\mathcal{L}_s$ is real analytic on the interval $(\underline{\alpha}_s, \overline{\alpha}_s)$.*

We can also formulate corresponding statements in the case of the spectrum $\mathcal{L}_s$, obtained from Theorems 5 and 6 setting $a = \zeta_u$ and $b = 1$.

In [9], Pesin and Sadovskaya obtained a complete multifractal analysis of the spectrum $\mathcal{L}_s$. Using similar arguments to those in Section 3.4, we can use Theorem 11 to obtain a new proof of the multifractal analysis of the spectrum $\mathcal{L}_s$. A similar remark applies to the spectrum $\mathcal{L}_u$.

5 Suspension flows

5.1 Preliminaries. It is well-known that hyperbolic flows can be modeled by suspension flows over subshifts of finite type, using the so-called Markov systems introduced by Bowen [5] and Ratner [10]. In this section we shall show how to use this representation in order to describe the variational principle in Theorem 5 in terms of the discrete time dynamics in the base of the suspension. Instead of formulating our results only for suspension flows obtain from hyperbolic flows we shall instead consider arbitrary suspension flows over subshifts of finite type.

We briefly recall some notions concerning suspensions. Let $T\colon X \to X$ be a homeomorphism of the compact metric space X, and $\tau\colon X \to (0,\infty)$ a Lipschitz function. Consider the space

$$Y = \{(x,s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\},$$

with the points $(x, \tau(x))$ and $(Tx, 0)$ identified for each $x \in X$. One can introduce in a natural way a topology on Y which makes Y a compact topological space. This topology is induced by a distance introduced by Bowen and Walters in [6] (see the appendix in [2] for details).

The *suspension flow over T with height function τ* is the flow $\Psi = \{\psi_t\}_t$ on Y where $\psi_t\colon Y \to Y$ is defined by

$$\psi_t(x,s) = (x, s+t).$$

We extend τ to a function $\tau\colon Y \to \mathbb{R}$ by

$$\tau(y) = \min\{t > 0 : \psi_t y \in X \times \{0\}\},$$

and extend T to a map $T\colon Y \to X \times \{0\}$ by

$$T(y) = \psi_{\tau(y)} y.$$

Since there is no danger of confusion we continue to use the symbols τ and T for the extensions. Given a continuous function $a\colon Y \to \mathbb{R}$ we define a new function

$\Delta_a : Y \to \mathbb{R}$ by

$$\Delta_a(y) = \int_0^{\tau(y)} a(\psi_s y)\, ds.$$

Combining the variational principles for the topological pressures of the flow and the base map, we obtain a relation between the two topological pressures.

Proposition 12 *If Ψ is a suspension flow on Y over $T\colon X \to X$, and $a\colon Y \to \mathbb{R}$ is a continuous function, then*

$$P_T(\Delta_a - P_\Psi(a)\tau) = 0.$$

Recall that two functions $A\colon X \to \mathbb{R}$ and $B\colon X \to \mathbb{R}$ are said to be T-*cohomologous* if there exists a bounded measurable function $q\colon X \to \mathbb{R}$ such that

$$A(x) - B(x) = q(Tx) - q(x)$$

for every $x \in X$. The following is established in [2].

Proposition 13 *If $\Psi = \{\psi_t\}_t$ is a suspension flow on Y over $T\colon X \to X$, and $a\colon Y \to \mathbb{R}$ and $b\colon Y \to \mathbb{R}$ are continuous functions, then the following properties are equivalent:*

1. *a is Ψ-cohomologous to b on Y with*

$$a(y) - b(y) = \lim_{t\to 0} \frac{q(\psi_t y) - q(y)}{t} \text{ for every } y \in Y;$$

2. *Δ_a is T-cohomologous to Δ_b on $X \times \{0\}$ with*

$$\Delta_a(y) - \Delta_b(y) = q(Ty) - q(y) \text{ for every } y \in X \times \{0\}.$$

This proposition allows us to characterize the cohomology of the flow Ψ entirely in terms of the cohomology of the map T on the base.

We also have the following statement.

Proposition 14 *Let $\Psi = \{\psi_t\}_t$ be a suspension flow on Y over $T\colon X \to X$, and $a\colon Y \to \mathbb{R}$ and $b\colon Y \to \mathbb{R}$ continuous functions. If $x \in X$ and $s \in [0, \tau(x)]$, then*

$$\liminf_{t\to\infty} \frac{\int_0^t a(\psi_\tau(x,s))\, d\tau}{\int_0^t b(\psi_\tau(x,s))\, d\tau} = \liminf_{m\to\infty} \frac{\sum_{i=0}^m \Delta_a(T^i x)}{\sum_{i=0}^m \Delta_b(T^i x)}$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t a(\psi_\tau(x,s))\, d\tau}{\int_0^t b(\psi_\tau(x,s))\, d\tau} = \limsup_{m\to\infty} \frac{\sum_{i=0}^m \Delta_a(T^i x)}{\sum_{i=0}^m \Delta_b(T^i x)}.$$

These propositions will allow us to obtain a conditional variational principle with respect to the base.

5.2 Conditional variational principle. Let Ψ be a suspension flow on Y over a homeomorphism $T\colon X \to X$ of the compact metric space X, and μ a T-invariant probability measure in X. The measure μ induces a Ψ-invariant probability measure ν in Y such that

$$\int_Y a\, d\nu = \frac{\int_X \Delta_a\, d\mu}{\int_X \tau\, d\mu} \tag{15}$$

for every continuous function $a\colon Y \to \mathbb{R}$. Furthermore, any Ψ-invariant measure ν in Y is of this form for some T-invariant probability measure μ in X.

Given continuous functions a, $b\colon Y \to \mathbb{R}$ with $b > 0$, we define

$$K_\alpha \stackrel{\text{def}}{=} \left\{ x \in Y : \liminf_{t\to\infty} \frac{\int_0^t a(\psi_s x)\,ds}{\int_0^t b(\psi_s x)\,ds} = \limsup_{t\to\infty} \frac{\int_0^t a(\psi_s x)\,ds}{\int_0^t b(\psi_s x)\,ds} = \alpha \right\}.$$

It follows from Proposition 14 that the set K_α is composed of the points $(x, s) \in Y$ such that

$$\liminf_{m\to\infty} \frac{\sum_{i=0}^m \Delta_a(T^i x)}{\sum_{i=0}^m \Delta_b(T^i x)} = \limsup_{m\to\infty} \frac{\sum_{i=0}^m \Delta_a(T^i x)}{\sum_{i=0}^m \Delta_b(T^i x)} = \alpha$$

and $s \in [0, \tau(x)]$. Let also

$$\underline{\alpha} \stackrel{\text{def}}{=} \inf \left\{ \frac{\int_Y a\,d\nu}{\int_Y b\,d\nu} : \nu \in \mathcal{M}_\Psi(Y) \right\} = \inf \left\{ \frac{\int_X \Delta_a\,d\mu}{\int_X \Delta_b\,d\mu} : \mu \in \mathcal{M}_T(X) \right\}$$

and

$$\overline{\alpha} \stackrel{\text{def}}{=} \sup \left\{ \frac{\int_Y a\,d\nu}{\int_Y b\,d\nu} : \nu \in \mathcal{M}_\Psi(Y) \right\} = \inf \left\{ \frac{\int_X \Delta_a\,d\mu}{\int_X \Delta_b\,d\mu} : \mu \in \mathcal{M}_T(X) \right\},$$

where $\mathcal{M}_\Psi(Y)$ (respectively $\mathcal{M}_T(X)$) denotes the set of Ψ-invariant Borel probability measures on Y (respectively the set of T-invariant Borel probability measures on X).

The function $\mathcal{F}$ defined by

$$\mathcal{F}(\alpha) = h(\Psi|K_\alpha)$$

is again called the *entropy spectrum* for the pair of functions (a, b).

We now consider the special case when T is a subshift of finite type.

Theorem 15 *Let Ψ be a suspension flow on Y over a topologically mixing two-sided subshift of finite type, and $a\colon Y \to \mathbb{R}$ and $b\colon Y \to \mathbb{R}$ Hölder continuous functions. Then the following properties hold:*

1. *if $\alpha \notin [\underline{\alpha}, \overline{\alpha}]$ then $K_\alpha = \varnothing$;*
2. *if $\alpha \in (\underline{\alpha}, \overline{\alpha})$ then $K_\alpha \neq \varnothing$ and*

$$\begin{aligned} \mathcal{F}(\alpha) &= \max \left\{ \frac{h_\mu(T)}{\int_X \tau\,d\mu} : \frac{\int_X \Delta_a\,d\mu}{\int_X \Delta_b\,d\mu} = \alpha \text{ and } \mu \in \mathcal{M}_T(X) \right\} \\ &= \min \left\{ \sup_{\mu \in \mathcal{M}_T(X)} \frac{h_\mu(T) + \int_X \Delta_{qa - q\alpha b}\,d\mu}{\int_X \tau\,d\mu} : q \in \mathbb{R} \right\}; \end{aligned}$$

3. *if a is Ψ-cohomologous to a multiple cb of b, i.e., if Δ_a is T-cohomologous to a multiple $c\Delta_b$ of b, then $\underline{\alpha} = \overline{\alpha} = c$ and $K_c = \Lambda$;*
4. *if a is Ψ-cohomologous to no multiple of b, i.e., if Δ_a is T-cohomologous to no multiple of Δ_b, then $\underline{\alpha} < \overline{\alpha}$ and the function $\mathcal{F}$ is real analytic on the interval $(\underline{\alpha}, \overline{\alpha})$.*

6 Higher dimensional multifractal spectra

The purpose of this section is to obtain a higher dimensional version of the conditional variational principle in Theorem 5. Essentially, instead of considering Birkhoff averages (or ratios of Birkhoff averages) we want to consider vectors of Birkhoff averages.

Let again $\Lambda \subset M$ be an invariant set of a continuous flow $\Phi = \{\varphi_t\}_t$. Let a_1, …, $a_d \colon \Lambda \to \mathbb{R}$ and b_1, …, $b_d \colon \Lambda \to \mathbb{R}$ be continuous functions, such that $b_i > 0$ for each $i = 1$, …, d. Set

$$A = \left\{ \left(\frac{\int_\Lambda a_1\, d\mu}{\int_\Lambda b_1\, d\mu}, \ldots, \frac{\int_\Lambda a_d\, d\mu}{\int_\Lambda b_d\, d\mu} \right) : \mu \in \mathcal{M}_\Phi(\Lambda) \right\},$$

and for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ let

$$K_\alpha = \left\{ x \in \Lambda : \lim_{t\to\infty} \left(\frac{\int_0^t a_1(\varphi_s x)\, ds}{\int_0^t b_1(\varphi_s x)\, ds}, \ldots, \frac{\int_0^t a_d(\varphi_s x)\, ds}{\int_0^t b_d(\varphi_s x)\, ds} \right) = \alpha \right\}, \tag{16}$$

in the sense that the limit in (16) exists and equals α. The function $\mathcal{F} = \mathcal{F}^{(a,b)}$ defined by

$$\mathcal{F}(\alpha) = h(\Phi | K_\alpha)$$

is called the *entropy spectrum* for $(a, b) = (a_1, \ldots, a_d, b_1, \ldots, b_d)$. The following establishes a conditional variational principle for the spectrum $\mathcal{F}$, thus providing a higher dimensional version of Theorem 5.

Theorem 16 *Let $\Lambda \subset M$ be a locally maximal hyperbolic set of a C^1 flow Φ such that $\Phi|\Lambda$ is topologically mixing, and a_1, …, $a_d \colon \Lambda \to \mathbb{R}$ and b_1, …, $b_d \colon \Lambda \to \mathbb{R}$ Hölder continuous functions with $b_i > 0$ for each $i = 1$, …, d. Then the following properties hold:*

1. *if $\alpha \notin A$ then $K_\alpha = \varnothing$;*
2. *if $\alpha \in \operatorname{int} A$ then $K_\alpha \neq \varnothing$ and*

$$\begin{aligned} \mathcal{F}(\alpha) &= \max \left\{ h_\mu(\Phi) : \left(\frac{\int_\Lambda a_1\, d\mu}{\int_\Lambda b_1\, d\mu}, \ldots, \frac{\int_\Lambda a_d\, d\mu}{\int_\Lambda b_d\, d\mu} \right) = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda) \right\} \\ &= \min \left\{ P_\Phi \left(\sum_{i=1}^d (q_i a_i - q_i \alpha_i b_i) \right) : (q_1, \ldots, q_d) \in \mathbb{R}^d \right\}. \end{aligned}$$

One can also obtain higher dimensional versions of the remaining theorems in Section 3.

We now illustrate Theorem 16 in the case of the Lyapunov exponents introduced in Section 4.2. Namely, instead of considering separately the level sets K^s_α and K^u_α in (14), we consider the level sets $K^s_{\alpha_s} \times K^u_{\alpha_u}$ of the pair (λ_s, λ_u).

Let ζ_s and ζ_u be the functions defined in (12) and (13), and take $d = 2$, $a_1 = \zeta_s$, $a_2 = \zeta_u$, and $b_1 = b_2 = 1$. We assume that for each $(q_s, q_u) \in \mathbb{R}^2$ the function $q_s\zeta_s + q_u\zeta_u$ is Φ-cohomologous to no constant. One can verify that this assumption ensures that $A = \overline{\operatorname{int} A}$. For each point (α_s, α_u) in the dense set $\operatorname{int} A$ it follows from Theorem 16 that $K^s_{\alpha_s} \times K^u_{\alpha_u} \neq \varnothing$ and

$$\begin{aligned} \mathcal{F}(\alpha_s, \alpha_u) &= \max \left\{ h_\mu(\Phi) : \left(\int_\Lambda \zeta_s\, d\mu, \int_\Lambda \zeta_u\, d\mu \right) = (\alpha_s, \alpha_u) \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda) \right\} \\ &= \min \left\{ P_\Phi(q_s\zeta_s + q_u\zeta_u) - q_s\alpha_s - q_u\alpha_u : (q_s, q_u) \in \mathbb{R}^2 \right\}. \end{aligned}$$

In particular, the functions $(\alpha_s, \alpha_u) \mapsto \mathcal{F}(\alpha_s, \alpha_u)$ and $(q_s, q_u) \mapsto P_\Phi(q_s\zeta_s + q_u\zeta_u)$ form a Legendre pair.

7 Proofs

Proof of Theorem 3 This is an immediate consequence of Theorem 5. □

Proof of Theorem 5 Suppose that $K_\alpha \neq \varnothing$, and let $x \in K_\alpha$. The sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ on Λ such that

$$\int_\Lambda u\,d\mu_n = \frac{1}{n}\int_0^n u(\varphi_s x)\,ds$$

for every $u \in C(\Lambda)$ has an accumulation point $\mu \in \mathcal{M}_\Phi(\Lambda)$. Therefore

$$\alpha = \lim_{t\to\infty}\frac{\int_0^t a(\varphi_s x)\,ds}{\int_0^t b(\varphi_s x)\,ds} = \lim_{n\to\infty}\frac{\int_\Lambda a\,d\mu_n}{\int_\Lambda b\,d\mu_n} = \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} \in [\underline{\alpha}, \overline{\alpha}].$$

This establishes the first statement.

We now consider the second statement. By Proposition 4, for every $\alpha \in \mathbb{R}$ we have

$$\begin{aligned}\inf_{q\in\mathbb{R}} P_\Phi(qa - q\alpha b) &= \inf_{q\in\mathbb{R}}\sup\left\{h_\mu(\Phi) + \int_\Lambda (qa - q\alpha b)\,d\mu : \mu \in \mathcal{M}_\Phi(\Lambda)\right\}\\ &\geq \sup\left\{h_\mu(\Phi) : \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda)\right\}.\end{aligned} \tag{17}$$

For each $\alpha \in (\underline{\alpha}, \overline{\alpha})$ there exist measures ν_- and ν_+ in $\mathcal{M}_\Phi(\Lambda)$ such that

$$\frac{\int_\Lambda a\,d\nu_-}{\int_\Lambda b\,d\nu_-} < \alpha < \frac{\int_\Lambda a\,d\nu_+}{\int_\Lambda b\,d\nu_+}.$$

Furthermore, for any $q \in \mathbb{R}$ and $\mu \in \mathcal{M}_\Phi(\Lambda)$ we have

$$P_\Phi(qa - q\alpha b) \geq h_\mu(\Phi) + q\left(\int_\Lambda a\,d\mu - \alpha\int_\Lambda b\,d\mu\right),$$

and hence,

$$\liminf_{q\to\pm\infty} P_\Phi(qa - q\alpha b) \geq h_{\nu_\pm}(\Phi) + \liminf_{q\to\pm\infty} q\left(\int_\Lambda a\,d\nu_\pm - \alpha\int_\Lambda b\,d\nu_\pm\right) = +\infty.$$

In particular, the map $q \mapsto P_\Phi(qa - q\alpha b)$ attains its infimum at some $q = q(\alpha) \in \mathbb{R}$. Furthermore this map is analytic (see [11] for details). Denoting by μ_α the equilibrium measure of the function $q(\alpha)(a - \alpha b)$ we obtain

$$0 = \frac{d}{dq}P_\Phi(qa - q\alpha b)\bigg|_{q=q(\alpha)} = \int_\Lambda a\,d\mu_\alpha - \alpha\int_\Lambda b\,d\mu_\alpha.$$

Consequently (17) is in fact an equality and

$$\begin{aligned}\inf_{q\in\mathbb{R}} P_\Phi(qa - q\alpha b) &= h_{\mu_\alpha}(\Phi)\\ &= \max\left\{h_\mu(\Phi) : \frac{\int_\Lambda a\,d\mu}{\int_\Lambda b\,d\mu} = \alpha \text{ and } \mu \in \mathcal{M}_\Phi(\Lambda)\right\}.\end{aligned} \tag{18}$$

Since Λ is hyperbolic, the flow $\Phi|\Lambda$ is expansive, and thus it is easy to verify that the identity in (7) simplifies to give

$$h_\mu(\Phi) = \inf\{h(\Phi|Z) : \mu(Z) = 1\}$$

(i.e., the limits in ε in (6) and (7) are not necessary provided that ε is made sufficiently small). Since μ_α is ergodic we have $\mu_\alpha(K_\alpha) = 1$, and thus

$$h_{\mu_\alpha}(\Phi) \le \mathcal{F}(\alpha). \tag{19}$$

By (18) and (19) it remains to show that

$$\mathcal{F}(\alpha) \le \inf_{q\in\mathbb{R}} P_\Phi(qa - q\alpha b).$$

Assume that this is not the case. Then there exist $q \in \mathbb{R}$, $\delta > 0$, and $c > 0$ such that

$$\mathcal{F}(\alpha) - \delta > c > P_\Phi(qa - q\alpha b). \tag{20}$$

We set $u = qa - \alpha qb$ and

$$K_{\alpha,\delta,\tau} = \left\{ x \in \Lambda : \left| \int_0^t u(\varphi_s x)\, ds \right| < \delta t \text{ for every } t \ge \tau \right\}.$$

Then $K_\alpha \subset \bigcup_{\tau\in\mathbb{N}} K_{\alpha,\delta,\tau} \stackrel{\text{def}}{=} K_{\alpha,\delta}$. From the basic properties of topological entropy (see [8] for details) we obtain

$$\lim_{\tau\to+\infty} h(\Phi|K_{\alpha,\delta,\tau}) = h(\Phi|K_{\alpha,\delta}) \ge h(\Phi|K_\alpha) = \mathcal{F}(\alpha).$$

In particular, there exists $\tau \in \mathbb{N}$ such that

$$c + \delta < h(\Phi|K_{\alpha,\delta,\tau}). \tag{21}$$

For every $y \in B(x,t,\varepsilon)$ and every $s \in [0,t]$, we have $d(\varphi_s x, \varphi_s y) < \varepsilon$ and thus

$$|u(x,t,\varepsilon)| \le \left| \int_0^t u(\varphi_s y)\, ds \right| + \eta(\varepsilon)t,$$

where

$$\eta(\varepsilon) \stackrel{\text{def}}{=} \sup\left\{ |u(x) - u(y)| : d(x,y) < \varepsilon \right\}.$$

Furthermore, if $B(x,t,\varepsilon) \cap K_{\alpha,\delta,\tau} \ne \varnothing$ then there exists $y \in B(x,t,\varepsilon)$ such that

$$\left| \int_0^t u(\varphi_s y)\, ds \right| < \delta t$$

whenever $t \ge \tau$. These estimates yield

$$|u(x,t,\varepsilon)| \le [\delta + \eta(\varepsilon)]t$$

whenever $B(x,t,\varepsilon) \cap K_{\alpha,\delta,\tau} \ne \varnothing$ and $t \ge \tau$. Hence

$$\begin{aligned} M(K_{\alpha,\delta,\tau}, u, c, \varepsilon) &= \lim_{T\to\infty} \inf_\Gamma \sum_{(x,t)\in\Gamma} \exp(u(x,t,\varepsilon) - ct) \\ &\ge \lim_{T\to\infty} \inf_\Gamma \sum_{(x,t)\in\Gamma} \exp(-[\delta + \eta(\varepsilon)]t - ct) \\ &= M(K_{\alpha,\delta,\tau}, 0, c + \delta + \eta(\varepsilon), \varepsilon), \end{aligned}$$

where the infimum is taken over all finite or countable sets $\Gamma = \{(x_i,t_i)\}_i$ such that $(x_i,t_i) \in X \times [T,\infty)$ for each i, and $\bigcup_i B(x_i,t_i,\varepsilon) \supset K_{\alpha,\delta,\tau}$. Since u is continuous we have $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and by the definition of topological entropy (21) implies that

$$M(K_{\alpha,\delta,\tau}, u, c, \varepsilon) > 0$$

for all sufficiently small $\varepsilon > 0$. Therefore

$$c \le P_\Phi(qa - q\alpha b|K_{\alpha,\delta,\tau}) \le P_\Phi(qa - q\alpha b),$$

which contradicts the assumption in (20). This establishes the second statement. $\square$

Proof of Theorem 6 Assume that there exists a constant c such that a is Φ-cohomologous to cb on Λ. We obtain

$$\begin{aligned}\left|\int_0^t a(\varphi_\tau x)\,d\tau - c\int_0^t b(\varphi_\tau x)\,d\tau\right| &= \lim_{s\to 0}\frac{1}{s}\left|\int_s^{s+t} q(\varphi_\tau x)\,d\tau - \int_0^t q(\varphi_\tau x)\,d\tau\right| \\ &= \lim_{s\to 0}\frac{1}{s}\left|\int_t^{s+t} q(\varphi_\tau x)\,d\tau - \int_0^s q(\varphi_\tau x)\,d\tau\right| \\ &\le 2\sup|q|,\end{aligned} \tag{22}$$

and thus

$$\left|\frac{\int_0^t a(\varphi_\tau x)\,d\tau}{\int_0^t b(\varphi_\tau x)\,d\tau} - c\right| \le \frac{2\sup|q|}{t\inf|b|}$$

for every $x \in \Lambda$ and $t > 0$. Therefore $K_c = \Lambda$. Furthermore, by (22), if $\mu \in \mathcal{M}_\Phi(\Lambda)$ then

$$\begin{aligned}0 &= \lim_{t\to\infty}\int_\Lambda \left(\frac{1}{t}\int_0^t a(\varphi_\tau x)\,d\tau - \frac{c}{t}\int_0^t b(\varphi_\tau x)\,d\tau\right)\,d\mu(x) \\ &= \int_\Lambda a\,d\mu - c\int_\Lambda b\,d\mu,\end{aligned}$$

and $\underline{\alpha} = \overline{\alpha} = c$. This establishes the first statement.

For the second statement, consider functions a and b such that a is Φ-cohomologous to no multiple of b. We assume that $\underline{\alpha} = \overline{\alpha} = c$. In this case the function

$$\mu \mapsto \int_\Lambda a\,d\mu - c\int_\Lambda b\,d\mu$$

is constant and equal to zero. In particular, when μ is the invariant measure supported on the periodic orbit of a point $x = \varphi_T x$ we obtain

$$\frac{1}{T}\int_0^T a(\varphi_s x)\,ds = c\frac{1}{T}\int_0^T b(\varphi_s x)\,ds.$$

By Livshitz's theorem for flows (see, for example, Theorem 19.2.4 in [7]), the functions a and cb must be Φ-cohomologous. This contradiction ensures that $\underline{\alpha} < \overline{\alpha}$.

We now proceed in a similar way to that in [3] to establish the analyticity of the spectrum.

Lemma 1 *If $a - \alpha b$ is Φ-cohomologous to no constant for every $\alpha \in \mathbb{R}$ then the spectrum $\mathcal{F}$ is real analytic on the interval $(\underline{\alpha}, \overline{\alpha})$.*

Proof of the lemma Let $\alpha \in (\underline{\alpha}, \overline{\alpha})$ and put

$$F(q, \alpha) = P_\Phi(qa - q\alpha b).$$

By Theorem 5 the number $\mathcal{F}(\alpha)$ coincides with $\min_{q\in\mathbb{R}} F(q, \alpha)$. It is well known that F is real analytic in both variables. We want to apply the implicit function theorem to show that the minimum is attained at some $q = q(\alpha)$ which is real analytic in α.

We have

$$\partial_q F(q, \alpha) = \int_\Lambda (a - \alpha b)\,d\nu_{q,\alpha},$$

where $\nu_{q,\alpha}$ is the equilibrium measure of $qa - q\alpha b$. By Theorem 5 there exists $q = q(\alpha) \in \mathbb{R}$ at which the function $q \mapsto P_\Phi(qa - q\alpha b)$ attains a minimum. We have $\partial_q F(q(\alpha), \alpha) = 0$. Furthermore, the function $q \mapsto F(q, \alpha)$ is strictly convex because $a - \alpha b$ is Φ-cohomologous to no constant. Hence $q = q(\alpha)$ is the unique number satisfying $\partial_q F(q, \alpha) = 0$. Again since $a - \alpha b$ is Φ-cohomologous to no constant, the derivative $\partial_q^2 F$ does not vanish (see [11]). It follows from the implicit function theorem that the function $\alpha \mapsto q(\alpha)$ is real analytic. This completes the proof of the lemma. □

Assume now that a is not Φ-cohomologous to cb for every $c \in \mathbb{R}$. By Lemma 1, it remains to consider the case when there exist $c, d \in \mathbb{R}$ with $d \neq 0$, and a bounded measurable function $q\colon \Lambda \to \mathbb{R}$ such that

$$a(x) - cb(x) = d + \lim_{t \to 0} \frac{q(\varphi_t x) - q(x)}{t} \tag{23}$$

for every $x \in \Lambda$. One can easily show that $x \in K_\alpha(a, b)$ if and only if $x \in K_{d/(\alpha - c)}(b, 1)$. Furthermore, it follows from (22) and (23) that

$$\left| \int_0^t a(\varphi_\tau x)\, d\tau - c \int_0^t b(\varphi_\tau x)\, d\tau - dt \right| \leq 2 \sup|q|.$$

Since $b > 0$ and $d \neq 0$ we conclude that $c \neq \alpha$ for every $\alpha \in \mathbb{R}$ such that $K_\alpha(a, b) \neq \varnothing$. Hence, the function $\alpha \mapsto d/(\alpha - c)$ is real analytic on $(\underline{\alpha}, \overline{\alpha})$.

Observe that b cannot be Φ-cohomologous to a constant $\gamma \in \mathbb{R}$. Otherwise the function a would be Φ-cohomologous to $cb + d = (c + d/\gamma)b$ (since $b > 0$ the constant γ would be positive), which is a contradiction. Hence we can apply Lemma 1 to the pair of functions $(b, 1)$ to conclude that $\mathcal{F}^{(b,1)}$ is real analytic on the nonempty interval $(\underline{\kappa}, \overline{\kappa})$, where

$$\underline{\kappa} = \inf \left\{ \int_\Lambda b\, d\mu : \mu \in \mathcal{M}_\Phi(\Lambda) \right\} \quad \text{and} \quad \overline{\kappa} = \sup \left\{ \int_\Lambda b\, d\mu : \mu \in \mathcal{M}_\Phi(\Lambda) \right\}.$$

Since $b > 0$ we have $\underline{\kappa} > 0$.

The function $\mathcal{F}^{(a,b)}$ is the composition of the real analytic functions $\alpha \mapsto d/(\alpha - c)$ and $\mathcal{F}^{(b,1)}$, and thus it is real analytic. Furthermore,

$$(\underline{\alpha}, \overline{\alpha}) = \begin{cases} (c + d/\overline{\kappa}, c + d/\underline{\kappa}) & \text{when } d > 0 \\ (c + d/\underline{\kappa}, c + d/\overline{\kappa}) & \text{when } d < 0 \end{cases}.$$

This completes the proof of the theorem. □

Proof of Theorem 7 Set $H = C^\alpha(\Lambda) \times C_+^\alpha(\Lambda)$, and denote by $G \subset H$ the set of pairs $(a, b) \in H$ such that a is not Φ-cohomologous to any multiple of b. Let $(a, b) \in H \setminus G$, that is, a is Φ-cohomologous to some multiple cb of b. Let also Γ_i be distinct periodic orbits of points $x_i = \varphi_{T_i} x_i$ for $i = 0$, 1. We write

$$\langle g \rangle_i = \frac{1}{T_i} \int_0^{T_i} g(\varphi_t x_i)\, dt$$

for each continuous function $g\colon \Lambda \to \mathbb{R}$ and $i = 0$, 1. Choose a function $h \in C^\alpha(\Lambda)$ with $h|_{\Gamma_0} = \langle b \rangle_0$ and $h|_{\Gamma_1} = \langle b \rangle_1 + 1$. This is always possible since Γ_0 and Γ_1 are closed and disjoint. We consider the new pair of functions

$$(\tilde{a}, \tilde{b}) = (a, b) + (\varepsilon h, 0) \in H,$$

where ε is a positive constant. For each $\tilde{c} \in \mathbb{R}$ we have

$$\tilde{a} - \tilde{c}\tilde{b} = a - cb + (c - \tilde{c})b + \varepsilon h.$$

Thus if $\tilde{a} - \tilde{c}\tilde{b}$ is Φ-cohomologous to zero we have

$$0 = \langle \tilde{a} - \tilde{c}\tilde{b} \rangle_0 = (c - \tilde{c} + \varepsilon)\langle b \rangle_0 \quad \text{and} \quad 0 = \langle \tilde{a} - \tilde{c}\tilde{b} \rangle_1 = (c - \tilde{c} + \varepsilon)\langle b \rangle_1 + \varepsilon.$$

Since $\langle b \rangle_0 \geq \min b > 0$, we must have $c - \tilde{c} + \varepsilon = 0$. But this is impossible in view of the second identity and since $\varepsilon \neq 0$. This contradiction implies that $(\tilde{a}, \tilde{b}) \in G$. Since ε is arbitrary, the pair of functions (a, b) can be arbitrarily approximated in H by pairs in G, and thus G is dense in H.

We now show that G is open. Let $(a, b) \in G$. Since $b > 0$ there exists a unique $c = c(a, b) \in \mathbb{R}$ such that $P_\Phi(a - cb) = P_\Phi(0)$. By Livshitz's theorem for flows (see, for example, Theorem 19.2.4 in [7]) there exists a periodic orbit Γ_0 such that $\langle a - cb \rangle_0 \neq 0$. Choose $\varepsilon \in (0, \min b/2)$ and take $(\tilde{a}, \tilde{b}) \in H$ such that $\|a - \tilde{a}\|_\alpha + \|b - \tilde{b}\|_\alpha < \varepsilon$. We have

$$|P_\Phi(\tilde{a} - c\tilde{b}) - P_\Phi(0)| \leq \|\tilde{a} - a - c(\tilde{b} - b)\|_\alpha < (1 + |c|)\varepsilon. \tag{24}$$

Let now $\tilde{c} \in \mathbb{R}$ be the unique number such that $P_\Phi(\tilde{a} - \tilde{c}\tilde{b}) = P_\Phi(0)$. Observe that if $\tilde{a}$ was Φ-cohomologous to some multiple of $\tilde{b}$, then $\tilde{a}$ would be Φ-cohomologous to $\tilde{c}\tilde{b}$ and to no other multiple of $\tilde{b}$. Since $\tilde{b} > \min b/2 > 0$ it follows from (24) that

$$|c - \tilde{c}| \leq \frac{1}{\min \tilde{b}} |P_\Phi(\tilde{a} - \tilde{c}\tilde{b}) - P_\Phi(\tilde{a} - c\tilde{b})| < \frac{2(1 + |c|)\varepsilon}{\min b}.$$

Therefore

$$\begin{aligned}
|\langle \tilde{a} - \tilde{c}\tilde{b} \rangle_0| &\geq |\langle a - cb \rangle_0| - |\langle \tilde{a} - a - (\tilde{c}\tilde{b} - cb) \rangle_0| \\
&\geq |\langle a - cb \rangle_0| - \|\tilde{a} - a\|_\alpha - |\tilde{c} - c| \cdot \|\tilde{b}\|_\alpha - |c| \cdot \|\tilde{b} - b\|_\alpha \\
&\geq |\langle a - cb \rangle_0| - \left(1 + \frac{2(1 + |c|)(\|b\|_\alpha + \varepsilon)}{\min b} + |c|\right)\varepsilon > 0,
\end{aligned}$$

provided that ε is chosen sufficiently small (but only depending on a and b, since c is uniquely determined once a and b are fixed). This implies that $\tilde{a}$ is not Φ-cohomologous to $\tilde{c}\tilde{b}$. Hence the ball of radius ε centered at (a, b) is contained in G. This shows that G is open. $\square$

Proof of Proposition 12 The Abramov entropy formula says that with ν and μ as in (15) we have

$$h_\nu(\Psi) = \frac{h_\mu(T)}{\int_X \tau \, d\mu}. \tag{25}$$

Using (15) and (25) we obtain

$$\begin{aligned}
P_T(\Delta_a - P_\Psi(a)\tau) &= \sup \left\{ h_\mu(T) + \int_X (\Delta_a - P_\Psi(a)\tau) \, d\mu : \mu \in \mathcal{M}_T(X) \right\} \\
&= \sup \left\{ \left(h_\nu(\Psi) + \int_Y a \, d\nu - P_\Psi(a) \right) \int_X \tau \, d\mu : \nu \in \mathcal{M}_\Psi(Y) \right\}.
\end{aligned}$$

The desired identity follows now immediately from

$$\sup \left\{ h_\nu(\Psi) + \int_Y a \, d\nu - P_\Psi(a) : \nu \in \mathcal{M}_\Psi(Y) \right\} = 0,$$

and the fact that τ is continuous and positive. $\square$

Proof of Proposition 14 The proof is a modification of Proposition 6 in [2]. Given $m \in \mathbb{N}$, define a function $\tau_m \colon Y \to \mathbb{R}$ by

$$\tau_m(x) = \sum_{i=0}^{m-1} \tau(T^i x).$$

When $x \in Y$ and $m \in \mathbb{N}$, one can easily verify that

$$\int_0^{\tau_m(x)} a(\psi_s x)\, ds = \sum_{i=0}^{m-1} \Delta_a(T^i x). \tag{26}$$

Given $t > 0$ there exists a unique $m \in \mathbb{N}$ such that $\tau_m(x) \le t < \tau_{m+1}(x)$. One can write $t = \tau_m(x) + \kappa$ for some $\kappa \in (\inf \tau, \sup \tau)$ and thus

$$\begin{aligned}
&\left| \frac{\int_0^t a(\psi_s x)\, ds}{\int_0^t b(\psi_s x)\, ds} - \frac{\int_0^{\tau_m(x)} a(\psi_s x)\, ds}{\int_0^{\tau_m(x)} b(\psi_s x)\, ds} \right| \\
&= \left| \frac{\int_{\tau_m(x)}^t a(\psi_s x)\, ds \int_0^{\tau_m(x)} b(\psi_s x)\, ds - \int_0^{\tau_m(x)} a(\psi_s x)\, ds \int_{\tau_m(x)}^t b(\psi_s x)\, ds}{\int_0^t b(\psi_s x)\, ds \int_0^{\tau_m(x)} b(\psi_s x)\, ds} \right| \\
&\le \frac{\kappa \sup|a| \cdot \tau_m(x) \sup b + \tau_m(x) \sup|a| \cdot \kappa \sup b}{\tau_m(x) \sup b \cdot \tau_m(x) \sup b} \\
&= \frac{2\kappa \sup|a|}{\tau_m(x) \sup b}.
\end{aligned}$$

When $t \to \infty$, we have $m \to \infty$ and $\tau_m(x) \to \infty$. Hence, it follows from (26) that

$$\left| \frac{\int_0^t a(\psi_s x)\, ds}{\int_0^t b(\psi_s x)\, ds} - \frac{\sum_{i=0}^{m-1} \Delta_a(T^i x)}{\sum_{i=0}^{m-1} \Delta_b(T^i x)} \right| \to 0$$

as $t \to \infty$. This implies the desired statement. ☐

Proof of Theorem 15 It follows from (15) that

$$\frac{\int_Y a\, d\nu}{\int_Y b\, d\nu} = \frac{\int_X \Delta_a\, d\mu}{\int_X \Delta_b\, d\mu}. \tag{27}$$

By (25) and (27), and using similar arguments to those in the proof of Theorem 5 we obtain the first and second statements in the theorem.

The two last statements follows from Proposition 13, using similar arguments to those in the proof of Theorem 6. ☐

Proof of Theorem 16 The proof of the first statement can be obtained in a similar manner to that of the first statement in Theorem 5.

For the second statement, we give a brief description of what changes are required in the proof of Theorem 5 when $d > 1$. For any $\alpha = (\alpha_1, \ldots, \alpha_d) \in \operatorname{int} A$ and any $q = (q_1, \ldots, q_d) \in \mathbb{R}^d \setminus \{0\}$ we can choose measures ν_-^q and μ_+^q such that

$$\sum_{i=1}^d q_i \left(\int_\Lambda a_i\, d\nu_-^q - \alpha_i \int_\Lambda b_i\, d\nu_-^q \right) < 0 < \sum_{i=1}^d q_i \left(\int_\Lambda a_i\, d\nu_+^q - \alpha_i \int_\Lambda b_i\, d\nu_+^q \right).$$

These measures play the role of the measures ν_- and ν_+ in the proof of Theorem 5. In fact a similar argument to that in the proof of Theorem 5 shows that

$$\liminf_{\|q\|\to\infty} P_\Phi\left(\sum_{i=1}^d (q_i a_i - q_i \alpha_i b_i)\right) = +\infty,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$. This ensures that the function

$$F\colon (q_1,\dots,q_d) \mapsto P_\Phi\left(\sum_{i=1}^d (q_i a_i - q_i \alpha_i b_i)\right)$$

attains its infimum at some point $q(\alpha) \in \mathbb{R}^d$, and $\nabla_q F(q(\alpha)) = 0$. This property allows one to use essentially the same argument as that in the proof of Theorem 5 by replacing a and b by the vectors $(a_1,\dots,a_d)$ and $(b_1,\dots,b_d)$. $\square$

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