The Annals of Probability
2014, Vol. 42, No. 6, 2417–2453
DOI: 10.1214/12-AOP808

ON THE LOCAL TIME OF RANDOM PROCESSES IN RANDOM SCENERY[1]

BY FABIENNE CASTELL, NADINE GUILLOTIN-PLANTARD, FRANÇOISE PÈNE AND BRUNO SCHAPIRA

Université Aix-Marseille I, Université de Lyon, Université Européenne de Bretagne and Université de Brest, and Université Paris-Sud 11

Random walks in random scenery are processes defined by $Z_n := \sum_{k=1}^n \xi_{X_1+\cdots+X_k}$, where basically $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z})$ are two independent sequences of i.i.d. random variables. We assume here that X_1 is $\mathbb{Z}$-valued, centered and with finite moments of all orders. We also assume that ξ_0 is $\mathbb{Z}$-valued, centered and square integrable. In this case H. Kesten and F. Spitzer proved that $(n^{-3/4} Z_{[nt]}, t \geq 0)$ converges in distribution as $n \to \infty$ toward some self-similar process $(\Delta_t, t \geq 0)$ called Brownian motion in random scenery. In a previous paper, we established that $\mathbb{P}(Z_n = 0)$ behaves asymptotically like a constant times $n^{-3/4}$, as $n \to \infty$. We extend here this local limit theorem: we give a precise asymptotic result for the probability for Z to return to zero simultaneously at several times. As a byproduct of our computations, we show that Δ admits a bi-continuous version of its local time process which is locally Hölder continuous of order $1/4 - \delta$ and $1/6 - \delta$, respectively, in the time and space variables, for any $\delta > 0$. In particular, this gives a new proof of the fact, previously obtained by Khoshnevisan, that the level sets of Δ have Hausdorff dimension a.s. equal to $1/4$. We also get the convergence of every moment of the normalized local time of Z toward its continuous counterpart.

1. Introduction.

1.1. *Description of the model and of some earlier results.* We consider two independent sequences $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z})$ of independent identically distributed $\mathbb{Z}$-valued random variables. We assume in this paper that X_1 is centered, with finite moments of all orders, and that its support generates $\mathbb{Z}$. We consider the *random walk* $(S_n, n \geq 0)$ defined by

$$S_0 := 0 \quad \text{and} \quad S_n := \sum_{i=1}^n X_i \qquad \text{for all } n \geq 1.$$

Received August 2012.
[1]Supported by the french ANR project MEMEMO2 2010 BLAN 0125.
MSC2010 subject classifications. 60F05, 60F17, 60G15, 60G18, 60K37.
Key words and phrases. Random walk in random scenery, local limit theorem, local time, level sets.

We suppose that ξ_0 is centered, with finite second moment $\sigma^2 := \mathbb{E}[\xi_0^2]$. The sequence ξ is called the *random scenery*.

The *random walk in random scenery* Z is then defined for all $n \geq 1$ by

$$Z_n := \sum_{k=0}^{n-1} \xi_{S_k}.$$

For motivation in studying this process and, in particular, for a description of its connections with many other models, we refer to [5, 10, 14] and the references therein. Denoting by $N_n(y)$ the local time of the random walk S,

$$N_n(y) = \#\{k = 0, \dots, n-1 : S_k = y\},$$

it is straightforward, and important, to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

Kesten and Spitzer [10] and Borodin [2] proved the following functional limit theorem:

$$\left(n^{-3/4} Z_{nt}, t \geq 0\right) \underset{n\to\infty}{\overset{(\mathcal{L})}{\Longrightarrow}} (\sigma \Delta_t, t \geq 0), \tag{1}$$

where:

- $Z_s := Z_n + (s-n)(Z_{n+1} - Z_n)$, for all $n \leq s \leq n+1$,
- Δ is defined by

  $$\Delta_t := \int_{-\infty}^{+\infty} L_t(x)\, d\beta_x$$

  with $(\beta_x)_{x\in\mathbb{R}}$ a standard Brownian motion and $(L_t(x), t \geq 0, x \in \mathbb{R})$ jointly continuous in t and x a version of the local time process of some other standard Brownian motion $(B_t)_{t\geq 0}$ independent of β.

The process Δ is known to be a continuous $(3/4)$-self-similar process with stationary increments, and is called *Brownian motion in random scenery*. It can be seen as a mixture of Gaussian processes, but it is not a Gaussian process.

Let now φ_ξ denote the characteristic function of ξ_0 and let d be such that $\{u : |\varphi_\xi(u)| = 1\} = (2\pi/d)\mathbb{Z}$. In [5] we established the following local limit theorem:

$$\mathbb{P}\left(Z_n = \lfloor n^{3/4} x \rfloor\right) = \begin{cases} d\sigma^{-1} p_{1,1}(x/\sigma) n^{-3/4} + o\left(n^{-3/4}\right), \\ \qquad \text{if } \mathbb{P}\left(n\xi_0 - \lfloor n^{3/4} x \rfloor \in d\mathbb{Z}\right) = 1, \\ 0, \qquad \text{otherwise,} \end{cases} \tag{2}$$

with

$$p_{1,1}(x) := \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[\|L_1\|_2^{-1} e^{-\|L_1\|_2^2 x^2/2}\right]$$

and $\|L_1\|_2 := (\int_{\mathbb{R}} L_1^2(y)\,dy)^{1/2}$ the L^2-norm of L_1. In the particular case when $x = 0$, we get

$$\mathbb{P}(Z_n = 0) = \begin{cases} d\sigma^{-1} p_{1,1}(0) n^{-3/4} + o(n^{-3/4}), & \text{if } n \in d_0\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

with $d_0 := \min\{m \geq 1 : \varphi_\xi(2\pi/d)^m = 1\}$.[2]

Actually the results mentioned above were proved in the more general case when the distributions of the ξ_y's and X_k's are only supposed to be in the basin of attraction of stable laws (see [4, 5] and [10] for details).

1.2. *Statement of the results.*

1.2.1. *Local time of Brownian motion in random scenery.* Let $T_1, \dots, T_k$, be k positive reals. Set

$$\mathcal{D}_{T_1,\dots,T_k} := \det(M_{T_1,\dots,T_k}) \qquad \text{with } M_{T_1,\dots,T_k} = (\langle L_{T_i}, L_{T_j}\rangle)_{1 \leq i,j \leq k},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $L^2(\mathbb{R})$, and

$$\mathcal{C}_{T_1,\dots,T_k} := \mathbb{E}[\mathcal{D}_{T_1,\dots,T_k}^{-1/2}].$$

Our first result is the following:

THEOREM 1. *For any $k \geq 1$, there exist constants $c > 0$ and $C > 0$, such that*

$$c \prod_{i=1}^k (T_i - T_{i-1})^{-3/4} \leq \mathcal{C}_{T_1,\dots,T_k} \leq C \prod_{i=1}^k (T_i - T_{i-1})^{-3/4}$$

for every $0 < T_1 < \cdots < T_k$, with the convention that $T_0 = 0$.

The most difficult (and interesting) part here is the upper bound. The lower bound is obtained directly by using the scaling property of the local time of Brownian motion and the well-known Gram–Hadamard inequality. Concerning the upper bound, we will give more details about its proof in a moment, but let us stress already that even for $k = 2$ the result is not immediate (whereas when $k = 1$ it follows relatively easily from the Cauchy–Schwarz inequality and some basic properties of the Brownian motion; see, e.g., [5]).

A first corollary of this result is the following:

[2]Recall that, for every $n \geq 0$, we have

$$\mathbb{P}(n\xi_0 \in d\mathbb{Z}) > 0 \iff \mathbb{P}(n\xi_0 \in d\mathbb{Z}) = 1 \iff n \in d_0\mathbb{Z}.$$

COROLLARY 2. *For all $k \geq 1$ and all $0 < T_1 < \cdots < T_k$, the random variable $(\Delta_{T_1}, \ldots, \Delta_{T_k})$ admits a continuous density function, denoted by $p_{k,T_1,\ldots,T_k}$, which is given by*

$$p_{k,T_1,\ldots,T_k}(x) := (2\pi)^{-k/2}\mathbb{E}\big[\mathcal{D}_{T_1,\ldots,T_k}^{-1/2} \exp\big(-\tfrac{1}{2}\langle M_{T_1,\ldots,T_k}^{-1} x, x\rangle\big)\big] \qquad \text{for all } x \in \mathbb{R}^k.$$

Theorem 1 also shows that, for every $t \geq 0$, $k \geq 1$ and $x \in \mathbb{R}$, the following quantity (corresponding to the kth moment of the local time of Δ in x at time t; see Theorem 3 below),

$$\mathcal{M}_{k,t}(x) := \int_{[0,t]^k} p_{k,T_1,\ldots,T_k}(x, \ldots, x)\, dT_1 \cdots dT_k \tag{4}$$

is finite. Define now the *level sets* of Δ as the sets of the form

$$\Delta^{-1}(x) := \{t \geq 0 : \Delta_t = x\}$$

for $x \in \mathbb{R}$. We can then state our main application of Theorem 1, which can be deduced by standard techniques:

THEOREM 3. *There exists a nonnegative process $(\mathcal{L}_t(x), x \in \mathbb{R}, t \geq 0)$, such that*:

(i) *a.s. the map $(t, x) \mapsto \mathcal{L}_t(x)$ is continuous and nondecreasing in t. Moreover, for any $\delta > 0$, it is locally Hölder continuous of order $1/4 - \delta$, in the first variable, and of order $1/6 - \delta$, in the second variable*,

(ii) *a.s. for any measurable $\varphi : \mathbb{R} \to \mathbb{R}_+$ and any $t \geq 0$,*

$$\int_0^t \varphi(\Delta_s)\, ds = \int_{\mathbb{R}} \varphi(x)\mathcal{L}_t(x)\, dx,$$

(iii) *for any $T > 0$, we have the scaling property*,

$$\big(\mathcal{L}_{tT}(x), t \geq 0, x \in \mathbb{R}\big) \stackrel{(d)}{=} \big(T^{1/4}\mathcal{L}_t\big(xT^{-3/4}\big), t \geq 0, x \in \mathbb{R}\big),$$

(iv) *for any $x \in \mathbb{R}$, $k \geq 1$ and $t > 0$, the kth moment of $\mathcal{L}_t(x)$ is finite and*

$$\mathbb{E}\big[\mathcal{L}_t(x)^k\big] = \mathcal{M}_{k,t}(x), \tag{5}$$

(v) *a.s. for any $x \in \mathbb{R}$, the support of the measure $d_t\mathcal{L}_t(x)$ is contained in $\Delta^{-1}(x)$.*

The random variable $\mathcal{L}_t(x)$ is called the local time of Δ in x at time t.

We believe that the exponents $1/4$ and $1/6$ in part (i) are sharp. One reason is that our proof gives the right critical exponents in the case of the Brownian motion. Another heuristic reason comes from a result proved by Dombry and Guillotin [8], saying that the sum of n i.i.d. copies of the process Δ converges under appropriate normalization toward a fractional Brownian motion with index $3/4$. But the Hölder continuity critical exponents of the local time of the latter process are exactly equal to $1/4$ in the time variable and $1/6$ in the space variable.

An original feature of this theorem is that it gives strong regularity properties of the local time of a process which is neither Markovian nor Gaussian, whereas usually similar results are obtained when at least one of these conditions is satisfied (see, e.g., [9, 16]).

We should notice now that previously only the existence of a process satisfying (ii), (5) for $k \leq 2$ and (v) was known; see [19]. The original motivation of [19] was in fact to study the Hausdorff dimension of the level sets of Δ. Khoshnevisan and Lewis conjectured in [12] that their Hausdorff dimension was a.s. equal to $1/4$, for every $x \in \mathbb{R}$. In [19] Xiao proved the result for almost every x and left open the question to know whether this was true for every x. This was later proved by Khoshnevisan in [11]. With Theorem 3 we can now give an alternative proof, which follows the same lines as standard ones in the case of the Brownian motion:

COROLLARY 4 (Khoshnevisan [11]). *For every $x \in \mathbb{R}$, the Hausdorff dimension of $\Delta^{-1}(x)$ is a.s. equal to $1/4$.*

Actually, Xiao and Khoshnevisan proved their result in the more general setting where the Brownian motion B is replaced by a stable process of index $\alpha \in (1, 2]$. But at the moment it does not seem straightforward for us to adapt our proof to this case.

Now let us give some rough ideas of the proof of Theorem 1. The first thing we use is that $M_{T_1,\dots,T_k}$ is a Gram matrix, and so there is a nice formula for its smallest eigenvalue (8), which shows that to get a lower bound, it suffices to prove that the term $L_{T_1}/T_1^{3/4}$ is far in the L^2-norm from the vector space generated by the terms $(L_{T_j} - L_{T_{j-1}})/(T_j - T_{j-1})^{3/4}$, for $j \geq 2$. Now, by scaling we can always assume that $T_1 = 1$. Next, by using the Hölder regularity of the process L, we can replace the L^2-norm by the L^∞-norm, which is much easier to control. Then we use the Ray–Knight theorem, which says that, if instead of considering the term L_1 we consider L_τ, with τ some appropriate random time, then we get a Markov process. It is then possible to prove that, with high probability, this process is far in the L^∞-norm from any finite-dimensional affine space, from which the desired result follows.

1.2.2. *Random walk in random scenery.* Our first result is a multidimensional extension of our previous local limit theorem. We state it only for return probabilities to 0, to simplify notation, but it works exactly the same if we replace 0 by $[n^{3/4}x]$, for some fixed $x \neq 0$.

THEOREM 5. *Let $k \geq 1$ be some integer and let $0 < T_1 < \cdots < T_k$ be k fixed positive reals. Then for any $n \geq 1$:*

- *If* $[nT_i] \in d_0\mathbb{Z}$, *for all* $i \le k$, *then*
$$\mathbb{P}(Z_{[nT_1]} = \cdots = Z_{[nT_k]} = 0) = (d\sigma^{-1})^k p_{k,T_1,\dots,T_k}(0,\dots,0)n^{-3k/4} + o(n^{-3k/4}).$$
- *Otherwise* $\mathbb{P}(Z_{[nT_1]} = \cdots = Z_{[nT_k]} = 0) = 0$.

Moreover, for every $k \ge 1$ *and every* $\theta \in (0, 1)$, *there exists* $C = C(k, \theta) > 0$, *such that*
$$\mathbb{P}[Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} = 0] \le C(n_1 \cdots n_k)^{-3/4}$$
for all $n \ge 1$ *and all* $n_1, \dots, n_k \in [n^\theta, n]$.

As an application we can prove that the moments of the local time of Z converge toward their continuous counterpart. More precisely, for $z \in \mathbb{Z}$, define the local time of Z in z at time n by
$$\mathcal{N}_n(z) := \#\{m = 1, \dots, n : Z_m = z\}.$$
Then Theorem 5 together with the Lebesgue dominated convergence theorem gives the following:

COROLLARY 6. *For all* $k \ge 1$,
$$\mathbb{E}[\mathcal{N}_n(0)^k] \sim \left(\frac{d}{\sigma d_0}\right)^k \mathcal{M}_{k,1}(0)n^{k/4}$$
as $n \to \infty$, *with* $\mathcal{M}_{k,1}(0)$ *as in* (4).

A natural question now is to know if we could not deduce from this corollary the convergence in distribution of the normalized local time $\mathcal{N}_n(0)/n^{1/4}$ toward $\mathcal{L}_1(0)$. To this end, we should need to know that the law of $\mathcal{L}_1(0)$ is determined by the sequence of its moments. Since this random variable is nonnegative, a standard criterion ensuring this, called Carleman's criterion, is the condition
$$\sum_k \mathcal{M}_{k,1}(0)^{-1/(2k)} = \infty.$$
In particular, a bound for $\mathcal{M}_{k,1}(0)$ in k^{2k} would be sufficient. However, with our proof, we only get a bound in k^{ck}, for some constant $c > 0$. We can even obtain some explicit value for c, but unfortunately it is larger than 2, so this is not enough to get the convergence in distribution; see Remark 14 below. Note that this question is directly related to the question of the dependence in k of the constant C in Theorem 1, which we believe is an interesting question for other problems as well, such as the problem of large deviations for the process $\mathcal{L}$ (see, e.g., [7] in which the case of the fractional Brownian motion is considered).

Another interesting feature of Theorem 5 is that it gives an effective measure of the asymptotic correlations of the increments of Z. Indeed, if we assume to simplify that $k=2$, $\sigma=1$ and $d=1$, then (2) and Theorem 5 (actually its proof) show that

$$\frac{\mathbb{P}(Z_{n+m}-Z_n=0|Z_n=0)}{\mathbb{P}(Z_{n+m}-Z_n=0)} \longrightarrow \frac{\mathbb{E}[\{\|L_1\|_2^2\|\widetilde{L}_t\|_2^2-\langle L_1,\widetilde{L}_t\rangle^2\}^{-1/2}]}{\mathbb{E}[\{\|L_1\|_2\|\widetilde{L}_t\|_2\}^{-1}]} \tag{6}$$

as $n\to\infty$ and $m/n\to t$, for some $t>0$, where L and $\widetilde{L}$ are the local time processes of two independent standard Brownian motions. In particular, the limiting value in (6) is larger than one, which means that the process is asymptotically more likely to come back to 0 at time $n+m$, if we already know that it is equal to 0 at time n.

The general scheme of the proof of Theorem 5 is quite close from the one used for the proof of (3) in [5]. However, in addition to Theorem 1 which is needed here and which is certainly the main new difficulty, some other serious technical problems appear in the multidimensional setting. In particular, at some point we use a result of Borodin [3] giving a strong approximation of the local time of Brownian motion by the random walk local time. This also explains why we need stronger hypotheses on the random walk here. Now concerning the scenery, it is not clear if we can relax the hypothesis of finite second moment, since we strongly use that the characteristic function of $(\Delta_{T_1},\dots,\Delta_{T_k})$ takes the form

$$\psi(\theta_1,\dots,\theta_k)=\mathbb{E}\big[e^{-\sum_{i,j=1}^k a_{i,j}\theta_i\theta_j}\big]$$

with $(a_{i,j})_{i,j}$ some (random) positive symmetric matrix.

Finally, let us mention that in the proof of Theorem 5, we use the following result, which might be interesting on its own. It is a natural multidimensional extension of a result of Kesten and Spitzer [10] on the convergence in distribution of the normalized self-intersection local time of the random walk.

PROPOSITION 7. *Let $k\geq 1$ be given and let $T_1<\cdots<T_k$, be k positive reals. Then*

$$\big(n^{-3/2}\langle N_{n_1+\cdots+n_i},N_{n_1+\cdots+n_j}\rangle\big)_{1\leq i,j\leq k}\overset{(\mathcal{L})}{\Longrightarrow}\big(\langle L_{T_i},L_{T_j}\rangle\big)_{1\leq i,j\leq k}$$

as $n\to\infty$, and $(n_1+\cdots+n_i)/n\to T_i$, for all $i\geq 1$, and where for all p,q,

$$\langle N_p,N_q\rangle:=\sum_{y\in\mathbb{Z}}N_p(y)N_q(y).$$

The paper is organized as follows. In Section 2 we give a short proof of Corollary 2. In Section 3 we prove Theorem 1. Then, in Section 4 we explain how one can deduce Theorem 3 and Corollary 4 from it. Section 5 is devoted to the proof of Theorem 5 and Section 6 to the proof of Corollary 6. Finally, in Section 7 we give a proof of Proposition 7.

We also mention some notational convention that we shall use: if X is some random variable and A some set, then $\mathbb{E}[X,A]$ will mean $\mathbb{E}[X\mathbf{1}_A]$.

2. Proof of Corollary 2. Let $k \geq 1$ be given and let $T_1 < \cdots < T_k$ be some positive reals. The characteristic function $\psi_{T_1,\dots,T_k}$ of $(\Delta_{T_1}, \dots, \Delta_{T_k})$ (with the convention $T_0 = 0$) is given by

$$\begin{aligned}\psi_{T_1,\dots,T_k}(\theta) &= \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_{\mathbb{R}}\left(\sum_{i=1}^{k}\theta_i L_{T_i}(u)\right)^2 du\right)\right]\\ &= \mathbb{E}\left[\exp\left(-\frac{1}{2}\langle M_{T_1,\dots,T_k}\theta, \theta\rangle\right)\right]\end{aligned}$$

with $\theta := (\theta_1, \dots, \theta_k)$. In particular, this function is nonnegative. Moreover, a change of variables (this change is possible since $\mathcal{D}_{T_1,\dots,T_k}$ is almost surely non-null, thanks to Theorem 1) gives

$$\begin{aligned}\int_{\mathbb{R}^k}\psi_{T_1,\dots,T_k}(\theta)\,d\theta &= \mathcal{C}_{T_1,\dots,T_k}\int_{\mathbb{R}^k} e^{-(1/2)\sum_{i=1}^k u_i^2}\,du_1\cdots du_k\\ &= (2\pi)^{k/2}\mathcal{C}_{T_1,\dots,T_k} < \infty.\end{aligned}$$

This implies (see the remark following Theorem 26.2, page 347, in [1]) that $(\Delta_{T_1}, \dots, \Delta_{T_k})$ admits a continuous density function $p_{k,T_1,\dots,T_k}$ given by

$$\begin{aligned}p_{k,T_1,\dots,T_k}(x) &= \frac{1}{(2\pi)^k}\int_{\mathbb{R}^k} e^{-i\langle\theta,x\rangle}\psi_{T_1,\dots,T_k}(\theta)\,d\theta\\ &= (2\pi)^{-k/2}\mathbb{E}\left[\mathcal{D}_{T_1,\dots,T_k}^{-1/2}\exp\left(-\frac{1}{2}\langle M_{T_1,\dots,T_k}^{-1}x, x\rangle\right)\right],\end{aligned}$$

which was the desired result.

3. Proof of Theorem 1. Let $k \geq 1$ and $0 < T_1 < \cdots < T_k$ be given. Set $t_i := T_i - T_{i-1}$, for $i \leq k$, with the convention that $T_0 = 0$. For every $i = 1, \dots, k$, let $(L_t^{(i)}(x) := L_{T_{i-1}+t}(x) - L_{T_{i-1}}(x), t \in [0, t_i], x \in \mathbb{R})$ be the local time process of $B^{(i)} := (B_{T_{i-1}+t}, t \in [0, t_i])$. Set

$$\widetilde{\mathcal{D}}_{t_1,\dots,t_k} := \det(\widetilde{M}_{t_1,\dots,t_k}) \qquad \text{with } \widetilde{M}_{t_1,\dots,t_k} := \left(\langle L_{t_i}^{(i)}, L_{t_j}^{(j)}\rangle\right)_{1\leq i,j\leq k}$$

and

$$\widetilde{\mathcal{C}}_{t_1,\dots,t_k} := \mathbb{E}[\widetilde{\mathcal{D}}_{t_1,\dots,t_k}^{-1/2}].$$

Since $\widetilde{\mathcal{D}}_{t_1,\dots,t_k} = \mathcal{D}_{T_1,\dots,T_k}$, Theorem 1 is equivalent to proving the existence of constants $c > 0$ and $C > 0$, such that

$$c(t_1\cdots t_k)^{-3/4} \leq \widetilde{\mathcal{C}}_{t_1,\dots,t_k} \leq C(t_1\cdots t_k)^{-3/4} \tag{7}$$

for all positive $t_1, \dots, t_k$.

Let us first notice that $\widetilde{\mathcal{D}}_{t_1,\dots,t_k}$ is a Gram determinant and is thus nonnegative. So $\widetilde{\mathcal{C}}_{t_1,\dots,t_k}$ is well defined as an extended real number.

Now we start with the lower bound in (7). We use the well-known Gram–Hadamard inequality:

$$\widetilde{\mathcal{D}}_{t_1,\dots,t_k} \le \prod_{i=1}^{k} \|L_{t_i}^{(i)}\|_2^2.$$

By using next the scaling property of Brownian motion, we see that $(t_i^{-3/4}\|L_{t_i}^{(i)}\|_2, i \ge 1)$ is a sequence of i.i.d. random variables distributed as $\|L_1\|_2$. Therefore,

$$\mathbb{E}[\widetilde{\mathcal{D}}_{t_1,\dots,t_k}^{-1/2}] \ge c(t_1\cdots t_k)^{-3/4}$$

with $c := (\mathbb{E}[\|L_1\|_2^{-1}])^k > 0$.

We prove now the upper bound in (7), which is the most difficult part. For this purpose, we introduce the new Gram matrix

$$\overline{M}_{t_1,\dots,t_k} := (\langle t_i^{-3/4} L_{t_i}^{(i)}, t_j^{-3/4} L_{t_j}^{(j)}\rangle)_{i,j}.$$

Note that all its eigenvalues are nonnegative and denote by $\overline{\lambda}_{t_1,\dots,t_k}$ the smallest one. We get then

$$\mathcal{D}_{T_1,\dots,T_k} = \widetilde{\mathcal{D}}_{t_1,\dots,t_k} = \left(\prod_{i=1}^{k} t_i^{3/2}\right)\det(\overline{M}_{t_1,\dots,t_k}) \ge \left(\prod_{i=1}^{k} t_i^{3/2}\right)\overline{\lambda}_{t_1,\dots,t_k}^{k}.$$

Thus, we can write

$$\begin{aligned}\widetilde{\mathcal{C}}_{t_1,\dots,t_k} &= \mathbb{E}[\widetilde{\mathcal{D}}_{t_1,\dots,t_k}^{-1/2}]\\ &\le (t_1\cdots t_k)^{-3/4}\mathbb{E}[\overline{\lambda}_{t_1,\dots,t_k}^{-k/2}]\\ &= (t_1\cdots t_k)^{-3/4}\int_0^\infty \mathbb{P}[\overline{\lambda}_{t_1,\dots,t_k}^{-k/2} \ge t]\,dt\\ &\le (t_1\cdots t_k)^{-3/4}\left\{1 + \frac{2}{k}\int_0^1 \mathbb{P}[\overline{\lambda}_{t_1,\dots,t_k} \le \varepsilon]\,\frac{d\varepsilon}{\varepsilon^{1+k/2}}\right\}.\end{aligned}$$

Therefore, Theorem 1 follows from the following proposition:

PROPOSITION 8. *For any $k \ge 1$ and $K > 0$, there exists a constant $C > 0$, such that*

$$\mathbb{P}(\overline{\lambda}_{t_1,\dots,t_k} \le \varepsilon) \le C\varepsilon^K$$

for all $\varepsilon \in (0, 1)$ and all $t_1, \dots, t_k > 0$.

3.1. *Proof of Proposition 8*. Note first that

$$\overline{\lambda}_{t_1,\dots,t_k} = \inf_{u_1^2+\dots+u_k^2=1} \|u_1 t_1^{-3/4} L_{t_1}^{(1)} + \dots + u_k t_k^{-3/4} L_{t_k}^{(k)}\|_2^2. \tag{8}$$

Note next that if $u_1^2 + \dots + u_k^2 = 1$, then $u_{\max} := \max_i |u_i| \geq 1/\sqrt{k}$. Thus, dividing all u_i by $u_{\max}$ leads to

$$\overline{\lambda}_{t_1,\dots,t_k} \geq \frac{1}{k} \min_{i=1,\dots,k} \inf_{(v_j)_{j\neq i}, |v_j|\leq 1} \left\| t_i^{-3/4} L_{t_i}^{(i)} + \sum_{j\neq i} v_j t_j^{-3/4} L_{t_j}^{(j)} \right\|_2^2.$$

Hence, it suffices to bound all terms

$$\mathbb{P}\left[\inf_{(v_j)_{j\neq i}, |v_j|\leq 1} \left\| t_i^{-3/4} L_{t_i}^{(i)} + \sum_{j\neq i} v_j t_j^{-3/4} L_{t_j}^{(j)} \right\|_2^2 \leq k\varepsilon\right] \tag{9}$$

for $i \leq k$. By scaling invariance, and changing t_j by t_j/t_i in (9), one can always assume that $t_i = 1$. It will also be no loss of generality to assume that $i = 1$, the case $i > 1$ being entirely similar. We are thus led to prove that for any $k \geq 1$ and $K > 0$, there exists a constant $C > 0$, such that for all $\varepsilon \in (0, 1)$ and all $t_j > 0$,

$$\mathbb{P}\left[\inf_{(v_j)_{j>1}, |v_j|\leq 1} \left\| L_1^{(1)} + \sum_{j\geq 2} v_j t_j^{-3/4} L_{t_j}^{(j)} \right\|_2^2 \leq \varepsilon\right] \leq C\varepsilon^K. \tag{10}$$

We want now to bound from below the L^2-norm by (some power of) the L^∞-norm using the Hölder regularity of the Brownian local time. To this end, notice that by scaling the constants

$$C_H^{(j)} := \sup_{x\neq y} \frac{|L_{t_j}^{(j)}(x) - L_{t_j}^{(j)}(y)|}{t_j^{3/8}|x-y|^{1/4}}$$

for $j \geq 1$, are i.i.d. random variables. Moreover, the constant of Hölder continuity of order $1/4$ of the jth term of the sum in (10) is smaller than or equal to $C_H^{(j)} t_j^{-3/8}$. Since this can be large, we distinguish between indices j such that t_j is small from the other ones. More precisely, we define $J = \{j : t_j \leq \varepsilon^4\}$, and

$$\mathcal{E}_J := \bigcup_{j\in J} \operatorname{supp}(L_{t_j}^{(j)}),$$

where $\operatorname{supp}(f)$ denotes the support of a function f. Set also

$$\mathcal{E}'_J := \{x \in \mathbb{R} : d(x, \mathcal{E}_J) < \varepsilon\}.$$

To simplify notation, set now for all $x \in \mathbb{R}$ and $v = (v_j)_{j\geq 2}$,

$$F_v(x) := L_1^{(1)}(x) + \sum_{j\notin J, j\geq 2} v_j t_j^{-3/4} L_{t_j}^{(j)}(x) \quad \text{and} \quad G_v(x) := \sum_{j\in J} v_j t_j^{-3/4} L_{t_j}^{(j)}(x).$$

Notice that $G_v = 0$ on $\mathcal{E}_J^c$ and that

$$\sup_{x \neq y} \frac{|F_v(x) - F_v(y)|}{|x-y|^{1/4}} \leq \varepsilon^{-3/2} \sum_j C_H^{(j)}.$$

Thus, if for some $x \notin \mathcal{E}'_J$, $|F_v(x)| \geq \varepsilon$, and if in the same time $\sum_j C_H^{(j)} \leq 1/(2\varepsilon^{1/4})$, then

$$\begin{aligned}\|F_v + G_v\|_2^2 &\geq \int_{x-\varepsilon^{11}}^{x+\varepsilon^{11}} F_v(y)^2\,dy \geq \int_{x-\varepsilon^{11}}^{x+\varepsilon^{11}} \Bigl(\varepsilon - \varepsilon^{-3/2} \sum_j C_H^{(j)} \varepsilon^{11/4}\Bigr)_+^2 dy \\ &\geq \frac{1}{2}\varepsilon^{13}.\end{aligned}$$

Moreover, it is known that the $C_H^{(j)}$ have finite moments of any order. Therefore, it suffices to prove that for any $k \geq 1$ and $K > 0$, there is a constant $C > 0$, such that for all $\varepsilon \in (0,1)$ and all $t_2, \dots, t_k > 0$,

$$\mathbb{P}\Bigl(\inf_{v \in \mathbb{R}^{k-1}} \sup_{x \notin \mathcal{E}'_J} |F_v(x)| \leq \varepsilon\Bigr) \leq C\varepsilon^K. \tag{11}$$

This will follow from the next two lemmas, that we shall prove in the next subsections:

LEMMA 9. *Let $(L_t(x), t \geq 0, x \in \mathbb{R})$ be a continuous in (t,x) version of the local time process of a standard Brownian motion B. Then for any $K > 0$ and $k \geq 0$, there exist $N \geq 1$ and $C > 0$, such that, for any $\varepsilon \in (0,1)$, one can find N points $x_1, \dots, x_N \in \mathbb{R}$, satisfying $|x_i - x_j| \geq \varepsilon^{1/8}$ for all $i \neq j$, and*

$$\mathbb{P}\bigl(\#\{j \leq N : L_1(x_j) > \varepsilon^{1/4}\} \leq k\bigr) \leq C\varepsilon^K. \tag{12}$$

LEMMA 10. *For any $K > 0$ and $k \geq 1$, there exist a constant $C > 0$ and an integer $M \geq 1$, such that for all $x \in \mathbb{R}$, $\varepsilon \in (0,1)$ and $t_2, \dots, t_k > 0$,*

$$\mathbb{P}\Bigl(L_1^{(1)}(x) > \varepsilon^{1/4}, \inf_{v \in \mathbb{R}^{k-1}} \sup_{|y-x| \leq M\varepsilon} |F_v(y)| \leq \varepsilon\Bigr) \leq C\varepsilon^K.$$

Indeed, we can first always assume that $\mathcal{E}_J$ is included in the union of at most k intervals of length ε, since for any $j \in J$ and $K \geq 1$, by scaling there exists $C > 0$, such that

$$\begin{aligned}\mathbb{P}\Bigl[\sup_{s \leq t_j} |B_s^{(j)} - B_0^{(j)}| \geq \varepsilon/2\Bigr] &\leq \mathbb{P}\Bigl[\sup_{s \leq \varepsilon^4} |B_s| \geq \varepsilon/2\Bigr] \\ &= \mathbb{P}\Bigl[\sup_{s \leq 1} |B_s| \geq \varepsilon^{-1}/2\Bigr] \leq C\varepsilon^K.\end{aligned} \tag{13}$$

Thus, among any $k+1$ points at distance larger than $\varepsilon^{1/8}$ from each other, at least one of them must be at distance larger than $M\varepsilon$ from $\mathcal{E}'_J$, at least if ε is small enough. Therefore, Lemma 9 shows that for any $K \ge 1$, there exists $C > 0$, such that

$$\begin{aligned}
&\mathbb{P}\Big[\inf_v \sup_{y \notin \mathcal{E}'_J} |F_v(y)| \le \varepsilon\Big] \\
&\qquad \le C\varepsilon^K + \sum_{m=1}^{N} \mathbb{P}\Big[L_1^{(1)}(x_m) > \varepsilon^{1/4}, \inf_v \sup_{|y-x_m|\le M\varepsilon} |F_v(y)| \le \varepsilon\Big],
\end{aligned}$$

where $(x_1, \dots, x_N)$ are given by Lemma 9. Then (11) follows from the above inequality and Lemma 10. This concludes the proof of Proposition 10.

3.2. *Proof of Lemma* 9. We first prove the result for $k = 0$. Assume without loss of generality that K is an integer larger than 1 and set

$$\mathcal{X}_0 = \{j\varepsilon^{1/8} : -8K \le j \le 8K\}.$$

Set also $s_0 := 0$ and for every $m \ge 1$,

$$s_m := \inf\{s > 0 : |B_s| \ge m\varepsilon^{1/8}\}.$$

Note already that there exists $C > 0$, such that for all $\varepsilon \in (0, 1)$,

$$\mathbb{P}(s_{8K} > 1) \le \mathbb{P}\Big(\sup_{s\le 1} |B_s| \le 8K\varepsilon^{1/8}\Big) \le C\varepsilon^K$$

by using, for instance, [17], Proposition 8.4, page 52. Thus, it suffices to prove that

$$\mathbb{P}(L_{s_{8K}}(x) \le \varepsilon^{1/4}\ \forall x \in \mathcal{X}_0) \le \varepsilon^K \tag{14}$$

for all $\varepsilon \in (0, 1)$. By using the Markov property, and noting that $s_m \ge s_{m-1} + s_1 \circ \theta_{s_{m-1}}$ (where θ is the usual shift on the trajectories), we get a.s. for every $m \ge 1$,

$$\mathbb{P}(L_{s_m}(B_{s_{m-1}}) - L_{s_{m-1}}(B_{s_{m-1}}) \le \varepsilon^{1/4} | \mathcal{F}_{s_{m-1}}) \le \mathbb{P}(L_{s_1}(0) \le \varepsilon^{1/4}).$$

By the scaling property of the Brownian motion, we know that $L_{s_1}(0)$ has the same law as $\varepsilon^{1/8} L'_1(0)$, with $L'_1(0)$ the local time of a standard Brownian motion taken at the first hitting time of $\{\pm 1\}$. Moreover, it is known that $L'_1(0)$ is an exponential random variable with parameter 1 (see, e.g., [18], Exercise (4.12), Chapter VI, page 265). Therefore, a.s. for every $m \ge 1$,

$$\mathbb{P}(L_{s_m}(B_{s_{m-1}}) - L_{s_{m-1}}(B_{s_{m-1}}) \le \varepsilon^{1/4} | \mathcal{F}_{s_{m-1}}) \le \mathbb{P}(L'_1(0) \le \varepsilon^{1/8}) \le \varepsilon^{1/8}.$$

Then we get by induction,

$$\begin{aligned}
&\mathbb{P}(L_{s_{8K}}(x) \le \varepsilon^{1/4}\ \forall x \in \mathcal{X}_0) \\
&\qquad \le \mathbb{P}(L_{s_m}(B_{s_{m-1}}) - L_{s_{m-1}}(B_{s_{m-1}}) \le \varepsilon^{1/4}\ \forall m \le 8K) \le \varepsilon^K,
\end{aligned}$$

proving (14). This concludes the proof of the lemma for $k = 0$.

Now we prove the result for general $k \geq 0$. For $m \in \mathbb{Z}$, consider the set

$$\mathcal{X}_m := m(16K + 1)\varepsilon^{1/8} + \mathcal{X}_0.$$

Then the proof above shows similarly that for any $0 \leq m \leq k$,

$$\mathbb{P}\big(L_1(x) \leq \varepsilon^{1/4}\ \forall x \in \mathcal{X}_m \cup \mathcal{X}_{-m}\big) \leq C\varepsilon^K.$$

The lemma follows immediately.

3.3. *Proof of Lemma* 10. Let $K > 0$ be fixed, and assume without loss of generality that $x \geq 0$. Fix also $M \geq 1$ some integer to be chosen later.

For every affine subspace V of $\mathbb{R}^M$, we denote by V_ε the set

$$V_\varepsilon := \{v \in \mathbb{R}^M : d(v, V) \leq \varepsilon\},$$

where $d(v, V) = \min\{|v - y|_\infty : y \in V\}$. Then we can write

$$\mathbb{P}\Big(L_1^{(1)}(x) > \varepsilon^{1/4}, \inf_{v \in \mathbb{R}^{k-1}} \sup_{|y-x| \leq M\varepsilon} |F_v(y)| \leq \varepsilon\Big)$$
$$\leq \mathbb{P}\big[L_1^{(1)}(x) > \varepsilon^{1/4}, \big(L_1^{(1)}(x+\varepsilon), \dots, L_1^{(1)}(x+M\varepsilon)\big) \in \mathcal{V}_\varepsilon\big] := \mathcal{P}_\varepsilon,$$

where

$$\mathcal{V} := \mathrm{Vect}\big(\big(L_{t_j}^{(j)}(x + \ell\varepsilon)\big)_{\ell=1,\dots,M}, j \in I\big)$$

with $I := \{j > 1 : j \notin J\}$. Set now

$$\tau := \inf\{s > 0 : L_s^{(1)}(x) > \varepsilon^{1/4}\}$$

and for $y \geq 0$, $Y(y) := L_\tau^{(1)}(x + y)$. It follows from the second Ray–Knight theorem (see [18], Theorem (2.3), page 456) that Y is equal in law to a squared Bessel process of dimension 0 starting from $\varepsilon^{1/4}$. Moreover, with this notation, we can write

$$\mathcal{P}_\varepsilon = \mathbb{P}\big[\tau < 1 \text{ and } \big(Y(\varepsilon), \dots, Y(M\varepsilon)\big) \in \mathcal{V}_\varepsilon^*\big] \tag{15}$$

with

$$\mathcal{V}^* := \big(Y(\ell\varepsilon) - L_1^{(1)}(x + \ell\varepsilon)\big)_{\ell=1,\dots,M} + \mathcal{V},$$

which is an affine space of $\mathbb{R}^M$, of dimension at most $k - 1$.

Observe now that even on the event $\{\tau < 1\}$, the space $\mathcal{V}^*$ is not independent of Y and τ, since its law depends a priori on τ. However, if this was true (and we will see below how one can reduce the proof to this situation), then $\mathcal{P}_\varepsilon$ would be dominated by

$$\sup_V \mathbb{P}\big[\big(Y(\varepsilon), \dots, Y(M\varepsilon)\big) \in V_\varepsilon\big]$$

with the sup taken over all affine subspaces $V \subseteq \mathbb{R}^M$ of dimension at most $k - 1$. This last term in turn is controlled by the following lemma, whose proof is postponed to the next subsections.

LEMMA 11. *Let Y be a squared Bessel process of dimension* 0 *starting from* $\varepsilon^{1/4}$. *For any $M \geq 1$ and $k \geq 1$, there exists $C > 0$, such that for all $\varepsilon \in (0, 1)$,*

$$\sup_V \mathbb{P}\big[\big(Y(\varepsilon), \dots, Y(M\varepsilon)\big) \in V_\varepsilon\big] \leq C\varepsilon^{(5M-4(k-1))/8}, \tag{16}$$

where the sup *is over all affine subspaces $V \subseteq \mathbb{R}^M$ of dimension at most $k-1$.*

So at this point we are just led to see how one can solve the problem of the dependence between $\mathcal{V}^*$ and τ. To this end, we introduce the time τ' spent by $B^{(1)}$ above x before time τ, which by the occupation times formula (see [18], Theorem (2.3), page 456) is equal to

$$\tau' := \int_0^\tau \mathbf{1}_{\{B_s^{(1)} \geq x\}}\, ds = \int_0^\infty Y(y)\, dy.$$

Moreover, τ' is also equal in law to the first hitting time of $\varepsilon^{1/4}/2$ by a Brownian motion (see the proof of Theorem (2.7), page 243, in [18]). In particular,

$$\mathbb{P}\big(\tau' \leq \varepsilon^{3/4}\big) = \mathcal{O}\big(\varepsilon^K\big). \tag{17}$$

Next, instead of using Lemma 11, we will need the following refinement:

LEMMA 12. *Let $M \geq 1$ be some integer. Let Y be a squared Bessel process of dimension* 0 *starting from* $\varepsilon^{1/4}$. *Set*

$$A_\varepsilon := \left\{\big|Y(M\varepsilon) - \varepsilon^{1/4}\big| \leq \varepsilon^{1/2} \text{ and } \int_0^{M\varepsilon} Y(y)\, dy < \varepsilon\right\}.$$

Then $\mathbb{P}(A_\varepsilon^c) = \mathcal{O}(\varepsilon^K)$. Moreover, for any $M \geq 1$ and $k \geq 1$, there exists $C > 0$, such that for any affine space V of dimension at most $k-1$, almost surely for all $\varepsilon \in (0, 1)$,

$$\begin{aligned}\mathbf{1}_{\{\int_0^\infty Y(y)\, dy \geq \varepsilon^{3/4}\}} &\mathbb{P}\left[\big(Y(\varepsilon), \dots, Y(M\varepsilon)\big) \in V_\varepsilon, A_\varepsilon \,\middle|\, \int_0^\infty Y(y)\, dy\right] \\ &\leq C\varepsilon^{(5M-4(k-1))/8}.\end{aligned}$$

We postpone the proof of this lemma to the next subsections, and we conclude now the proof of Lemma 10. Let M be an integer larger than $(4(k-1) + 8K)/5$. According to (15), (17) and the first part of Lemma 12, we get

$$\begin{aligned}\mathcal{P}_\varepsilon &\leq \mathbb{P}\big(\tau < 1, \tau' \geq \varepsilon^{3/4}, \big(Y(\varepsilon), \dots, Y(M\varepsilon)\big) \in \mathcal{V}_\varepsilon^*, A_\varepsilon\big) + \mathcal{O}\big(\varepsilon^K\big) \\ &\leq \mathbb{E}\big[\mathbf{1}_{\{\tau<1, \tau' \geq \varepsilon^{3/4}\}} \mathbb{E}\big[f\big(Y, \mathcal{V}^*\big) | \tau, \tau'\big]\big] + \mathcal{O}\big(\varepsilon^K\big)\end{aligned}$$

with

$$f(y, V) := \mathbf{1}_{\{(y(\varepsilon), \dots, y(M\varepsilon)) \in V_\varepsilon\} \cap \{|y(M\varepsilon) - \varepsilon^{1/4}| \leq \varepsilon^{1/2} \text{ and } \int_0^{M\varepsilon} y(s)\, ds < \varepsilon\}}.$$

Now, since on the one hand Y and $\mathcal{V}^*$ are independent conditionally to (τ, τ'), and on the other hand Y and τ are independent conditionally to τ', we have

$$\begin{aligned}\mathbb{E}[f(Y, \mathcal{V}^*)|\tau, \tau'] &= \int \mathbb{E}[f(Y, V)|\tau, \tau']\, d\mathbb{P}_{\mathcal{V}^*|(\tau,\tau')}(V) \\ &= \int \mathbb{E}[f(Y, V)|\tau']\, d\mathbb{P}_{\mathcal{V}^*|(\tau,\tau')}(V).\end{aligned}$$

Since, moreover, $\mathcal{V}^*$ and τ' are independent conditionally to τ, we get

$$\mathbb{E}[f(Y, \mathcal{V}^*)|\tau, \tau'] = \int \mathbb{E}[f(Y, V)|\tau']\, d\mathbb{P}_{\mathcal{V}^*|\tau}(V). \tag{18}$$

Hence, according to our choice of M,

$$\begin{aligned}\mathcal{P}_\varepsilon &\le \mathbb{E}\biggl[\mathbf{1}_{\{\tau<1,\tau'\ge\varepsilon^{3/4}\}} \int \mathbb{E}[f(Y, V)|\tau']\, d\mathbb{P}_{\mathcal{V}^*|\tau}(V)\biggr] + \mathcal{O}(\varepsilon^K) \\ &\le \mathbb{E}\biggl[\mathbf{1}_{\{\tau<1,\tau'\ge\varepsilon^{3/4}\}} \int \mathbf{1}_{b(V,\varepsilon,\tau')}\, d\mathbb{P}_{\mathcal{V}^*|\tau}(V)\biggr] + \mathcal{O}(\varepsilon^K)\end{aligned}$$

with

$$b(V, \varepsilon, t') := \{\mathbf{1}_{\{\tau'\ge\varepsilon^{3/4}\}}\mathbb{E}[f(Y, V)|\tau' = t'] > C\varepsilon^{(5M-4(k-1))/8}\}.$$

The second part of Lemma 12 insures that, for every affine subspace V of dimension at most $k-1$ of $\mathbb{R}^M$, we have

$$\mathbf{1}_{b(V,\varepsilon,t')} = 0 \qquad \text{for } \mathbb{P}_{\tau'}\text{-almost every } t' > 0.$$

However, since $b(V, \varepsilon, \tau')$ depends a priori on V, we cannot conclude directly. But it is well known that τ' admits a positive density function on $(0, +\infty)$ [see (24) below for an explicit expression]. Therefore, for every V,

$$\mathbf{1}_{b(V,\varepsilon,t')} = 0 \qquad \text{for Lebesgue almost every } t' > 0. \tag{19}$$

Now it follows from the excursion theory that τ' and $\tau - \tau'$ are independent and identically distributed. Therefore, (τ, τ') admits a continuous density function h on $(0, +\infty)^2$ and we have

$$\begin{aligned}\mathcal{P}_\varepsilon &\le \mathcal{O}(\varepsilon^K) + \int_0^1 \biggl(\int_0^t \biggl(\int \mathbf{1}_{b(V,\varepsilon,t')}\, d\mathbb{P}_{\mathcal{V}^*|\tau=t}(V)\biggr) h(t, t')\, dt'\biggr) dt \\ &\le \mathcal{O}(\varepsilon^K) + \int_0^1 \biggl(\int \biggl(\int_0^t \mathbf{1}_{b(V,\varepsilon,t')} h(t, t')\, dt'\biggr) d\mathbb{P}_{\mathcal{V}^*|\tau=t}(V)\biggr) dt \qquad (20) \\ &\le \mathcal{O}(\varepsilon^K),\end{aligned}$$

the last term of (20) being equal to zero according to (19). This concludes the proof of Lemma 10.

It remains now to prove Lemma 12. Its proof uses Lemma 11, so let us start with the proof of the latter.

3.4. *Proof of Lemma* 11. We first prove the following result:

LEMMA 13. *For every $K>0$ and $M\ge 1$, there exists $C>0$, such that for all $\varepsilon\in(0,1)$,*

$$\mathbb{P}\big[\exists \ell\in\{1,\dots,M\}:\big|Y(\ell\varepsilon)-\varepsilon^{1/4}\big|>\varepsilon^{1/2}\big]\le C\varepsilon^K.$$

PROOF. Recall that Y is the solution of the stochastic differential equation

$$Y(y)=\varepsilon^{1/4}+2\int_0^y\sqrt{Y(u)}\,d\beta_u \qquad \text{for all } y\ge 0,$$

where β is a Brownian motion (see [18], Chapter XI). In particular, Y is stochastically dominated by the square of a one-dimensional Brownian motion starting from $\varepsilon^{1/8}$. Then it follows that, for some constant $C>0$, whose value may change from line to line, but depending only on K and M,

$$\begin{aligned}
&\mathbb{P}\big[\exists \ell\in\{1,\dots,M\}:\big|Y(l\varepsilon)-\varepsilon^{1/4}\big|>\varepsilon^{1/2}\big]\\
&\quad\le \mathbb{P}\Big[\sup_{s\le M\varepsilon}\big|Y(s)-\varepsilon^{1/4}\big|>\varepsilon^{1/2}\Big]\\
&\quad\le C\varepsilon^{-4K}\mathbb{E}\Big[\Big(\int_0^{M\varepsilon}Y(u)\,du\Big)^{4K}\Big]\\
&\qquad\text{by the Burkholder–Davis–Gundy inequality,}\\
&\quad\le C\varepsilon^{-1}\int_0^{M\varepsilon}\mathbb{E}\big[Y(u)^{4K}\big]\,du\le C\varepsilon^{-1}\int_0^{M\varepsilon}\mathbb{E}\big[\big(\varepsilon^{1/8}+B_u\big)^{8K}\big]\,du\le C\varepsilon^K
\end{aligned}$$

with B some standard Brownian motion. This concludes the proof of the lemma. $\square$

We continue now the proof of Lemma 11. Set

$$\mathcal{B}_\infty(\varepsilon^{1/4},\varepsilon^{1/2}):=\big\{(y_1,\dots,y_M)\in\mathbb{R}^M:\big|y_\ell-\varepsilon^{1/4}\big|\le\varepsilon^{1/2}\ \forall\ell\in\{1,\dots,M\}\big\}.$$

Lemma 13 shows that for any V of dimension at most $k-1$,

$$\begin{aligned}
&\mathbb{P}\big[\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in V_\varepsilon\big]\\
&\qquad\le\mathbb{P}\big[\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in\mathcal{B}_\infty\big(\varepsilon^{1/4},\varepsilon^{1/2}\big)\cap V_\varepsilon\big]+C\varepsilon^K.
\end{aligned}\tag{21}$$

Next observe that $B_\infty(\varepsilon^{1/4},\varepsilon^{1/2})\cap V_\varepsilon$, can be covered by $\mathcal{O}(\varepsilon^{-(k-1)/2})$ balls of radius ε. It follows that

$$\begin{aligned}
&\mathbb{P}\big[\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in\mathcal{B}_\infty\big(\varepsilon^{1/4},\varepsilon^{1/2}\big)\cap V_\varepsilon\big]\\
&\qquad\le C\varepsilon^{-(k-1)/2}\sup_{x\in\mathcal{B}_\infty(\varepsilon^{1/4},\varepsilon^{1/2})}\mathbb{P}\big[\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in\mathcal{B}_\infty(x,\varepsilon)\big].
\end{aligned}\tag{22}$$

Now for $y > 0$, denote by Y_y a squared Bessel process with dimension 0 starting from y. An explicit expression of its semigroup is given just after Corollary (1.4), page 441, in [18]. In particular, when $y > \varepsilon^{1/4}/2$, the law of $Y_y(\varepsilon)$ is the sum of a Dirac mass at 0 with some negligible weight and of a measure with density

$$z \mapsto q_\varepsilon(y,z) := (2\varepsilon)^{-1}\sqrt{\frac{y}{z}}\exp\left(-\frac{y+z}{2\varepsilon}\right)I_1\left(\frac{\sqrt{yz}}{\varepsilon}\right),$$

where I_1 is the modified Bessel function of index 1. Moreover, it is known (see (5.10.22) or (5.11.10) in [13]) that $I_1(z) = \mathcal{O}(e^z/\sqrt{z})$, as $z \to \infty$. Thus,

$$\sup_{|y-\varepsilon^{1/4}|\le\varepsilon^{1/2}} \sup_{|z-\varepsilon^{1/4}|\le\varepsilon^{1/2}} q_\varepsilon(y,z) = \mathcal{O}(\varepsilon^{-3/8}).$$

It follows that

$$\sup_{|x-\varepsilon^{1/4}|\le\varepsilon^{1/2}} \sup_{|y-\varepsilon^{1/4}|\le\varepsilon^{1/2}} \mathbb{P}\big[\big|Y_y(\varepsilon)-x\big|\le\varepsilon\big] = \mathcal{O}(\varepsilon^{5/8}).$$

Then by using the Markov property and Lemma 13, we get by induction

$$\sup_{x\in\mathcal{B}_\infty(\varepsilon^{1/4},\varepsilon^{1/2})} \mathbb{P}\big[\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in\mathcal{B}_\infty(x,\varepsilon)\big] \le C\varepsilon^{5M/8}. \tag{23}$$

Since all the constants in our estimates are uniform in V, Lemma 11 follows from (21), (22) and (23).

3.5. *Proof of Lemma* 12. Let $K > 0$ be given. Lemma 13 shows, in particular, that

$$\mathbb{P}\big[\big|Y(M\varepsilon)-\varepsilon^{1/4}\big| > \varepsilon^{1/2}\big] = \mathcal{O}(\varepsilon^K).$$

Next, recall that Y is stochastically dominated by the square of a one-dimensional Brownian motion starting from $\varepsilon^{1/8}$. It follows that

$$\mathbb{P}\left(\int_0^{M\varepsilon} Y(y)\,dy \ge \varepsilon\right) = \mathcal{O}(\varepsilon^K),$$

and this already proves the first part of the lemma.

It remains to prove the second part. We deduce it from Lemma 11. To simplify notation, from now on we will denote the integral of Y on $[0,\infty)$ by $\int_0^\infty Y$. Likewise, $\int_0^{M\varepsilon} Y$ and $\int_{M\varepsilon}^\infty Y$ will have analogous meanings. For any affine subspace $V \subseteq \mathbb{R}^M$ of dimension at most $k-1$, set

$$A'_\varepsilon(V) := \big\{\big(Y(\varepsilon),\dots,Y(M\varepsilon)\big)\in V_\varepsilon\big\}\cap A_\varepsilon.$$

Then for any nonnegative bounded measurable function ϕ supported on $[\varepsilon^{3/4},\infty)$, we can write

$$\begin{aligned}\mathbb{E}\left[\phi\left(\int_0^\infty Y\right)\mathbb{P}\left[A'_\varepsilon(V)\Big|\int_0^\infty Y\right]\right] &= \mathbb{E}\left[\phi\left(\int_0^\infty Y\right), A'_\varepsilon(V)\right]\\ &= \mathbb{E}\left[\phi\left(\int_0^{M\varepsilon} Y + \int_{M\varepsilon}^\infty Y\right), A'_\varepsilon(V)\right].\end{aligned}$$

Now we recall that if Y_y denotes a squared Bessel process of dimension 0 starting from some $y > 0$, then $\int_0^\infty Y_y$ is equal in law to the first hitting time of $y/2$ by some Brownian motion, and thus has density given by

$$f_y(t) := \frac{y}{2}(2\pi t^3)^{-1/2}\exp(-(y/2)^2/2t) \qquad \text{for all } t > 0 \text{ and } y > 0; \tag{24}$$

see, for instance, [18], page 107. In particular,

$$\sup_{t\ge\varepsilon^{3/4}}\sup_{t'\le\varepsilon}\sup_{|y-\varepsilon^{1/4}|\le\varepsilon^{1/2}}\frac{f_y(t-t')}{f_{\varepsilon^{1/4}}(t)} < \infty.$$

Then by using the Markov property and Lemma 11, we get

$$\begin{aligned}\mathbb{E}\Big[\phi\Big(\int_0^\infty Y\Big)\mathbb{P}\Big[A'_\varepsilon(V)\Big|\int_0^\infty Y\Big]\Big] &= \mathbb{E}\Big[\int_{\varepsilon^{3/4}}^\infty \phi(t) f_{Y(M\varepsilon)}\Big(t-\int_0^{M\varepsilon} Y\Big)\,dt, A'_\varepsilon(V)\Big]\\ &\le C\mathbb{P}[A'_\varepsilon(V)]\mathbb{E}\Big[\phi\Big(\int_0^\infty Y\Big)\Big]\\ &\le C\varepsilon^{(5M-4(k-1))/8}\mathbb{E}\Big[\phi\Big(\int_0^\infty Y\Big)\Big].\end{aligned}$$

Since this holds for any ϕ, this proves the second part of Lemma 12, as wanted.

REMARK 14. As mentioned in the Introduction, a careful look at the proof shows that the constant $C = C(k)$ appearing in the upper bound of the theorem can be taken of the form k^{ck}, for some universal constant $c > 0$. Indeed, the main estimate we use along the proof is that $e^{-1/\epsilon}$ is bounded by $k^k\epsilon^k$, and so, for instance, the constant C in (13) is a $\mathcal{O}(K^K)$.

4. Proof of Theorem 3 and Corollary 4. We start with the proof of Theorem 3. We follow the general strategy which is used in the case of the Brownian motion, as, for instance, in Le Gall's course ([15], Chapter 2).

Consider the regularizing function

$$p_\varepsilon(y) := \frac{1}{\sqrt{2\pi\varepsilon}}\exp\Big(-\frac{y^2}{2\varepsilon}\Big)\varepsilon > 0, \qquad y \in \mathbb{R},$$

and recall that by Fourier inversion

$$p_\varepsilon(y) = \frac{1}{2\pi}\int_{\mathbb{R}}\exp\Big(iy\xi - \frac{1}{2}\varepsilon|\xi|^2\Big)\,d\xi.$$

Define then for all $\varepsilon \in (0, 1]$, $t > 0$ and $x \in \mathbb{R}$,

$$\mathcal{L}(\varepsilon, t, x) := \int_0^t p_\varepsilon(\Delta_s - x)\,ds.$$

As explained in [15], it suffices to control the three terms

$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon,t,x')\big)^{2p}\big],\qquad \mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon',t,x)\big)^{2p}\big],$$
$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon,t',x)\big)^{2p}\big].$$

For the first term, some elementary computation shows that

$$\begin{aligned}&\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon,t,x')\big)^{2p}\big]\\ &\quad\le c_p\int_{\mathbb{R}^{2p}} d\xi_1\cdots d\xi_{2p}\prod_{j=1}^{2p}\big|e^{-ix\xi_j}-e^{-ix'\xi_j}\big|\\ &\qquad\times\int_{\Xi_p} ds_1\cdots ds_{2p}\big|\mathbb{E}\big[e^{i\sum\xi_j\Delta_{s_j}}\big]\big|\end{aligned}\tag{25}$$

with c_p some positive constant (whose value may change in the following lines) and $\Xi_p:=\{s_1\le\cdots\le s_{2p}\le t\}$. We use next that for any $\gamma\in(0,1]$ and any y, $y'\in\mathbb{R}$,

$$\big|e^{iy}-e^{iy'}\big|\le c\big|y-y'\big|^{\gamma}$$

for some constant $c>0$. Moreover, if $\eta_j=\xi_j+\cdots+\xi_{2p}$, and $t_j=s_j-s_{j-1}$, for all $j\ge 1$ (with the convention $s_0=0$), then

$$\mathbb{E}\big[e^{i\sum\xi_j\Delta_{s_j}}\big]=\mathbb{E}\big[e^{-(1/2)\sum_{i,j}\eta_i\eta_j\langle L^{(i)}_{t_i},L^{(j)}_{t_j}\rangle}\big]=\mathbb{E}\big[e^{-\langle\widetilde{M}_{t_1,\dots,t_{2p}}\eta,\eta\rangle/2}\big]$$

with $\eta=(\eta_1,\dots,\eta_{2p})$. Therefore, a change of variables in (25) gives

$$\begin{aligned}&\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon,t,x')\big)^{2p}\big]\\ &\quad\le c_p\big|x-x'\big|^{2\gamma p}\\ &\qquad\times\int_{\mathbb{R}^{2p}} d\eta\prod_{j=1}^{2p}|\eta_{j+1}-\eta_j|^{\gamma}\Big(\int_{[0,t]^{2p}} dt_1\cdots dt_{2p}\,\mathbb{E}\big[e^{-\langle\widetilde{M}_{t_1,\dots,t_{2p}}\eta,\eta\rangle/2}\big]\Big)\end{aligned}$$

with the convention $\eta_{2p+1}=0$. Now we make another change of variables: $(\eta_1,\dots,\eta_{2p})\to(\eta_1/t_1^{3/4},\dots,\eta_{2p}/t_{2p}^{3/4})$. Then we fix some $T>0$, and by using also that for all j, and $t\le T$,

$$\big|t_{j+1}^{-3/4}\eta_{j+1}-t_j^{-3/4}\eta_j\big|\le c\max\big(t_{j+1}^{-3/4},t_j^{-3/4}\big)|\eta|\le ct_{j+1}^{-3/4}t_j^{-3/4}T^{3/4}|\eta|$$

for some constant $c>0$, we get for all $t\le T$,

$$\begin{aligned}&\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x)-\mathcal{L}(\varepsilon,t,x')\big)^{2p}\big]\\ &\quad\le c_{p,T}\big|x-x'\big|^{2\gamma p}\\ &\qquad\times\int_{[0,t]^{2p}} dt_1\cdots dt_{2p}\Big(\prod_{j=1}^{2p}t_j^{-3/4(1+2\gamma)}\Big)\int_{\mathbb{R}^{2p}} d\eta\,\mathbb{E}\big[e^{-\langle\overline{M}_{t_1,\dots,t_{2p}}\eta,\eta\rangle/2}\big]|\eta|^{2\gamma p}\end{aligned}$$

for some constant $c_{p,T} > 0$. Now Proposition 8 shows that all moments of $1/\overline{\lambda}_{t_1,\dots,t_{2p}}$ are bounded by positive constants, uniformly in $(t_1,\dots,t_{2p})$. Therefore, by using that for all η,

$$\langle \overline{M}_{t_1,\dots,t_{2p}}\eta,\eta\rangle \geq \overline{\lambda}_{t_1,\dots,t_{2p}}|\eta|^2$$

and the change of variables $\eta \to \eta/(\overline{\lambda}_{t_1,\dots,t_{2p}})^{1/2}$, we get for all $\gamma < 1/6$,

$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x) - \mathcal{L}(\varepsilon,t,x')\big)^{2p}\big] \leq c_{p,T}\big|x-x'\big|^{2\gamma p}$$

for all $t \leq T$.

A similar computation leads to an analogous estimate for the second term, except that this time we need to choose $\gamma < 1/12$: for all $p \geq 1$, $T > 0$ and $\gamma < 1/12$, there exists some constant $c'_{p,T} > 0$, such that for all $x \in \mathbb{R}$, all $\varepsilon, \varepsilon' > 0$ and all $t \leq T$,

$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x) - \mathcal{L}(\varepsilon',t,x)\big)^{2p}\big] \leq c'_{p,T}\big|\varepsilon-\varepsilon'\big|^{2p\gamma}.$$

Now the estimate of the last term is easier. After some calculation and by using Theorem 1, we get for $t < t'$,

$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x) - \mathcal{L}(\varepsilon,t',x)\big)^{2p}\big] \leq c_p \int_{t\leq s_1\leq\cdots\leq s_{2p}\leq t'} \frac{ds_1\cdots ds_{2p}}{\prod(s_j-s_{j-1})^{3/4}},$$

which shows that

$$\mathbb{E}\big[\big(\mathcal{L}(\varepsilon,t,x) - \mathcal{L}(\varepsilon,t',x)\big)^{2p}\big] \leq c_p\big|t'-t\big|^{p/2}.$$

Then parts (i) and (ii) in Theorem 3 follow from Kolmogorov's criterion (see [15] for details). For (iii), first observe that (ii) implies that a.s. for any $t > 0$ and $x \in \mathbb{R}$,

$$\mathcal{L}_t(x) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{\{\Delta_s\in[x-\varepsilon,x+\varepsilon]\}}\,ds.$$

Then (iii) immediately follows from this equation and the property of self similarity of Δ. For (iv), we can observe that by using the above computations and the dominated convergence theorem, we get

$$\mathbb{E}\big[\mathcal{L}_t(x)^k\big] = \lim_{\varepsilon\to 0}\mathbb{E}\big[\mathcal{L}(\varepsilon,t,x)^k\big].$$

Part (iv) follows. Part (v) is immediate and was already observed in [19].

Concerning Corollary 4, the upper bound was already proved in [19] and [11] and was considered there as the easiest part. So we only care about the lower bound here. For this we can use Frostman's lemma together with Theorem 3, which directly proves the result (see [15], e.g.).

5. Proof of Theorem 5. In most of this section, $t_1, \dots, t_k$ are fixed positive reals. Moreover, by convention a function $f(n_1, \dots, n_k)$ is said to be a $o_k(g(n))$, for some function g, if it converges to 0 after multiplication by $1/g(n)$, when $n \to \infty$ and $n_i/n \to t_i$ for all $i \geq 1$. Analogous convention is used for the notation $\mathcal{O}_k(g(n))$.

Recall that $(S_m, m \geq 0)$ denotes the random walk. For every $i = 1, \dots, k$, let $(N_m^{(i)}(x), 1 \leq m \leq n_i, x \in \mathbb{Z})$ be the local time process of $(S_m^{(i)} := S_{n_1+\cdots+n_{i-1}+m}, 0 \leq m \leq n_i - 1)$. In other words,

$$\begin{aligned} N_m^{(i)}(x) &:= \#\{k = 0, \dots, m-1 : S_{n_1+\cdots+n_{i-1}+k} = x\} \\ &= N_{n_1+\cdots+n_{i-1}+m}(x) - N_{n_1+\cdots+n_{i-1}}(x) \end{aligned}$$

for all $i \leq k$. Set also

$$D_{n_1,\dots,n_k} := \det(\langle N_{n_i}^{(i)}, N_{n_j}^{(j)} \rangle)_{1 \leq i,j \leq k},$$

where here $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\ell_2(\mathbb{Z})$.

5.1. *Inverse Fourier transform and a periodicity issue.* The first step in local limit theorems is often the use of the Fourier inverse transform. This is essentially the content of the next lemma. Before stating it, let us introduce some new notation. Recall that φ_ξ denotes the characteristic function of ξ_0. Let now $\varphi_{n_1,\dots,n_k}$ be the characteristic function of $(Z_{n_1+\cdots+n_i} - Z_{n_1+\cdots+n_{i-1}})_{i=1,\dots,k}$. Since $(\xi_y)_{y \in \mathbb{Z}}$ is a sequence of i.i.d. random variables, which is independent of S, we have for all $(\theta_1, \dots, \theta_k) \in \mathbb{R}^k$

$$\begin{aligned} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k) &:= \mathbb{E}\left[\prod_{y \in \mathbb{Z}} \varphi_\xi\left(\sum_{j=1}^k \theta_j \left(N_{n_1+\cdots+n_j}(y) - N_{n_1+\cdots+n_{j-1}}(y)\right)\right)\right] \\ &= \mathbb{E}\left[\prod_{y \in \mathbb{Z}} \varphi_\xi\left(\sum_{j=1}^k \theta_j N_{n_j}^{(j)}(y)\right)\right]. \end{aligned} \tag{26}$$

We can now state the announced lemma.

LEMMA 15. *If $n_i \in d_0\mathbb{Z}$ for all $i \leq k$, then*

$$\begin{aligned} &\mathbb{P}(Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} = 0) \\ &\quad = \left(\frac{d}{2\pi}\right)^k \int_{[-\pi/d,\pi/d]^k} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\, d\theta_1 \cdots d\theta_k. \end{aligned}$$

Otherwise, $\mathbb{P}(Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} = 0) = 0$.

PROOF. Since Z is $\mathbb{Z}$-valued, we immediately get

$$\begin{aligned} \mathbb{P}(Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} = 0) &= \mathbb{P}(Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} - Z_{n_1+\cdots+n_{k-1}} = 0) \\ &= \frac{1}{(2\pi)^k} \int_{[-\pi,\pi]^k} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\, d\theta_1 \cdots d\theta_k. \end{aligned}$$

Notice now that $e^{2i\pi\xi_0/d} = \varphi_\xi(2\pi/d)$ almost surely and that $\varphi_\xi(2\pi/d)^d = 1$. Hence, for any integer $m \geq 0$ and any $u \in \mathbb{R}$,

$$\varphi_\xi(2m\pi/d + u) = \varphi_\xi(2\pi/d)^m \varphi_\xi(u).$$

We deduce that, for every $(l_1, \dots, l_k) \in \mathbb{Z}^k$, we have

$$\begin{aligned}
&\varphi_{n_1,\dots,n_k}\biggl(\theta_1 + \frac{2l_1\pi}{d}, \dots, \theta_k + \frac{2l_k\pi}{d}\biggr) \\
&\quad = \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} \varphi_\xi\biggl(\sum_{j=1}^k \biggl(\theta_j + \frac{2l_j\pi}{d}\biggr) N_{n_j}^{(j)}(y)\biggr)\biggr] \\
&\quad = \mathbb{E}\biggl[\prod_{y\in\mathbb{Z}} \varphi_\xi(2\pi/d)^{\sum_{j=1}^k l_j N_{n_j}^{(j)}(y)} \varphi_\xi\biggl(\sum_{j=1}^k \theta_j N_{n_j}^{(j)}(y)\biggr)\biggr] \\
&\quad = \varphi_\xi(2\pi/d)^{\sum_{j=1}^k l_j n_j} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k),
\end{aligned}$$

since $\sum_y N_{n_j}^{(j)}(y) = n_j$. But, if $n_j \in d_0\mathbb{Z}$ for all $j \leq k$, then $\varphi_\xi(2\pi/d)^{\sum_{j=1}^k n_j l_j} = 1$, for all $(l_1, \dots, l_k) \in \mathbb{Z}^k$, and the result follows with a change of variables. If not, let j be such that $n_j \notin d_0\mathbb{Z}$. Then $\varphi_\xi(2\pi/d)^{n_j}$ is a nontrivial dth root of unity and we can write

$$\begin{aligned}
&\mathbb{P}(Z_{n_1} = \cdots = Z_{n_1+\cdots+n_k} = 0) \\
&\quad = \frac{1}{(2\pi)^k} \biggl(\sum_{l_j=0}^{d-1} \varphi_\xi(2\pi/d)^{n_j l_j}\biggr) \\
&\qquad \times \int_{[-\pi,\pi]^{k-1}} \biggl[\int_{-[\pi/d,\pi/d]} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\, d\theta_j\biggr] d\theta_1 \cdots d\theta_{j-1}\, d\theta_{j+1} \cdots d\theta_k \\
&\quad = 0.
\end{aligned}$$

This concludes the proof of the lemma. □

5.2. *A typical behavior for random walks.* We want to argue that typically the simple random walk visits roughly $\sqrt{n}$ sites before time n, spends time of order at most $\sqrt{n}$ on each of them, and that its local time process is Hölder continuous of order $1/2$, with a Hölder constant in $\mathcal{O}(n^{1/4})$. This is true with high probability if we allow some correction of order n^γ, with $\gamma > 0$. This is the content of the next lemma, which can be proved as Lemma 6 in [5] and is standard. Set for all $i \leq k$,

$$N_i^* := \sup_y N_{n_i}^{(i)}(y) \quad \text{and} \quad R_i := \#\{y : N_{n_i}^{(i)}(y) > 0\}.$$

LEMMA 16. *For every $n \ge 1$ and $\gamma > 0$, set* $\Omega_{n_1,\dots,n_k} := \Omega^{(1)}_{n_1,\dots,n_k} \cap \Omega^{(2)}_{n_1,\dots,n_k}$, *where*

$$\Omega^{(1)}_{n_1,\dots,n_k} := \{R_i \le n_i^{1/2+\gamma}\ \forall i \le k\}$$

and

$$\Omega^{(2)}_{n_1,\dots,n_k} := \left\{\sup_{y\neq z} \frac{|N^{(i)}_{n_i}(y) - N^{(i)}_{n_i}(z)|}{|y-z|^{1/2}} \le n_i^{1/4+\gamma}\ \forall i \le k\right\}.$$

Then, for every p, $\mathbb{P}(\Omega^c_{n_1,\dots,n_k}) = o(\min_i n_i^{-p})$.

Note that on $\Omega_{n_1,\dots,n_k}$, for every i, we have

$$N_i^* \le n_i^{1/2+\gamma} \quad \text{and} \quad V^{(i)}_{n_i} := \sum_y \bigl(N^{(i)}_{n_i}(y)\bigr)^2 \le n_i^{3/2+3\gamma}.$$

5.3. *Scheme of the proof.* We follow roughly the same lines as for the proof of Theorem 1 in [5]. However, the situation is more complicated here, since we consider multiple times in a non-Markovian context. Moreover, we want upper bounds which are uniform in $n_1, \dots, n_k$, and this also requires some additional care.

First we have to see that the main contribution in the estimate comes from the integral near the origin. Recall, in particular, the notation from (26).

PROPOSITION 17. *Let $\eta \in (0, 1/8)$ be given. Then, for every $t_1, \dots, t_k \in (0, 1)$, we have*

$$\int_{U(\eta)} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\, d\theta_1 \cdots d\theta_k = \left(\frac{\sqrt{2\pi}}{\sigma}\right)^k \mathcal{C}_{t_1,\dots,t_k} n^{-3k/4} + o_k\bigl(n^{-3k/4}\bigr),$$

where $U(\eta) := \{|\theta_i| \le n_i^{-1/2-\eta}\ \forall i \le k\}$. Moreover, for every $\theta \in (0, 1)$,

$$\sup_{n\ge 1} \sup_{n^\theta \le n_1,\dots,n_k \le n} \left(\prod_{i=1}^k n_i^{3/4}\right) \left|\int_{U(\eta)} \varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\, d\theta_1 \cdots d\theta_k\right| < \infty.$$

The next two propositions show that the rest of the integral is negligible.

PROPOSITION 18. *Let $\eta \in (0, 1/8)$ be given. Then, for every $t_1, \dots, t_k \in (0, 1)$, we have*

$$\int_{V(\eta)} \bigl|\varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\bigr|\, d\theta_1 \cdots d\theta_k = o_k\bigl(n^{-3k/4}\bigr),$$

where $V(\eta) := \{|\theta_i| \le n_i^{-1/2+\eta}\ \forall i \le k\} \cap \{\exists j : |\theta_j| \ge n_j^{-1/2-\eta}\}$. Moreover, for every $\theta \in (0, 1)$,

$$\sup_{n\ge 1} \sup_{n^\theta \le n_1,\dots,n_k \le n} \left(\prod_{i=1}^k n_i^{3/4}\right) \int_{V(\eta)} \bigl|\varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\bigr|\, d\theta_1 \cdots d\theta_k < \infty.$$

PROPOSITION 19. *Let $\eta \in (0, 1/2)$ and $\theta \in (0, 1)$ be given. Then there exists $c > 0$ such that*

$$\sup_{n \geq 1} \sup_{n^\theta \leq n_1, \dots, n_k \leq n} \int_{\{\exists i \,:\, |\theta_i| > n_i^{-1/2+\eta}\}} |\varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)| \, d\theta_1 \cdots d\theta_k = o(e^{-n^c}).$$

This last proposition can be proved by using exactly the same argument as in the proof of Proposition 10 in [5]. The only difference is that, if say $|\theta_i| > n_i^{-1/2+\eta}$, then after having defined peaks for $S^{(i)}$, we need to work also conditionally to all $N_{n_j}^{(j)}$, for $j \neq i$. But this does not change anything to the proof. Since it would be fastidious to reproduce the argument, we will not prove this proposition here and we refer the reader to [5] for details.

Note that Theorem 5 readily follows from these propositions and Lemma 15.

5.4. *Proof of Proposition* 17. We will use Borodin's result [3] on approximations of Brownian local time by random walks local time. He proved, in particular (see Remark 1.3 in [3]) that under some moment condition on the random walk, and on a suitable probability space, for all $T > 0$ and all $\gamma > 0$, there exist constants $C > 0$ and $\delta > 0$, such that for all $n \geq 1$,

$$\mathbb{P}(E_n^c) \leq C n^{-1-\delta}$$

with

$$E_n := \left\{ \sup_{(t,x) \in [0,T] \times \mathbb{R}} |N_{[nt]}([\sqrt{n}x]) - \sqrt{n} L_t(x)| \leq C n^{1/4} \ln n, \right.$$
$$\left. \left| B_1 - \frac{S_n}{\sqrt{n}} \right| \leq n^{-1/4+\gamma} \right\},$$

where N and L are the local time processes, respectively, of the random walk S and of the Brownian motion B. But a careful look at his proof shows actually that if the random walk increments have finite moments of any order, then the above holds for any $\delta > 0$ [see Formulas (3.8), (3.9) and Lemma 3.2].

By using now this result, Lemma 16 and the Markov property of the random walk and of Brownian motion, we deduce the following:

LEMMA 20. *Let $\gamma \in (0, 1/4)$ and $k \geq 1$ be given. Then for every $n \geq 1$ and every $0 \leq n_1, \dots, n_k \leq n$, it is possible to construct the Brownian motion and the random walk on a suitable probability space, such that for all $p > 0$,*

$$\mathbb{P}(F_{n,n_1,\dots,n_k}^c) = \mathcal{O}\left(\left(\min_i n_i \right)^{-p} \right),$$

where $F_{n,n_1,\dots,n_k} = F_1(n,n_1,\dots,n_k) \cap \cdots \cap F_4(n,n_1,\dots,n_k)$, *and* (*with* $t_i = n_i/n$ *for* $i \le k$)

$$F_1(n,n_1,\dots,n_k) := \Big\{ \sup_{x\in\mathbb{R}} \big| N_{n_i}^{(i)}(\sqrt{n}x) - \sqrt{n} L_{t_i}^{(i)}(x) \big| \le n_i^{1/4+\gamma} \ \forall i \le k \Big\},$$

$$F_2(n,n_1,\dots,n_k) := \big\{ \sup\{ |x - S_0^{(i)}| : N_{n_i}^{(i)}(x) \neq 0 \} \le t_i^{1/2} n^{1/2+\gamma} \ \forall i \le k \big\},$$

$$F_3(n,n_1,\dots,n_k) := \big\{ \sup\{ |x - B_0^{(i)}| : L_{t_i}^{(i)}(x) \neq 0 \} \le t_i^{1/2} n^{\gamma} \ \forall i \le k \big\},$$

$$F_4(n,n_1,\dots,n_k) := \Big\{ \sup_k N_{n_i}^{(i)}(k) \le t_i^{1/2} n^{1/2+\gamma} \text{ and } \sup_x L_{t_i}^{(i)}(x) \le t_i^{1/2} n^{\gamma} \ \forall i \le k \Big\}.$$

The proof of this result is elementary and left to the reader. Define now for all $\varepsilon > 0$, the set

$$\widetilde{\Omega}_{n_1,\dots,n_k}(\varepsilon) := \Bigg\{ \Bigg(\prod_{i=1}^k n_i^{-3/2} \Bigg) D_{n_1,\dots,n_k} \ge \varepsilon \Bigg\}. \tag{27}$$

We then obtain the following:

LEMMA 21. *Let* $\theta \in (0,1)$ *and* $\theta_0 \in (0, \theta/4)$ *be given. Then for every* $L > 0$, *we have*

$$\sup_{n\ge 1} \sup_{n^\theta \le n_1,\dots,n_k \le n} \sup_{\varepsilon \ge n^{-\theta_0}} \varepsilon^{-L} \mathbb{P}\big(\widetilde{\Omega}^c_{n_1,\dots,n_k}(\varepsilon)\big) < \infty$$

and for every $p > 0$,

$$\sup_{n\ge 1} \sup_{n^\theta \le n_1,\dots,n_k \le n} \mathbb{E}\Bigg[\Bigg(\prod_{i=1}^k n_i^{3p/2} \Bigg) D^{-p}_{n_1,\dots,n_k}, \widetilde{\Omega}_{n_1,\dots,n_k}(n^{-\theta_0}) \Bigg] < \infty.$$

PROOF. Let $\gamma > 0$ be such that $\theta_0 < (\theta/4) - 3\gamma k$ and let $L > 0$ be fixed. Thanks to the previous lemma, we can assume that the Brownian motion B and the random walk S are constructed on a space, where

$$\mathbb{P}\big(F^c_{n,n_1,\dots,n_k}\big) = \mathcal{O}\big(n^{-p}\big)$$

for all $p > 0$. Now for all i, j, set

$$A_{i,j}^{(n)} := (n_i n_j)^{-3/4} \sum_y N_{n_i}^{(i)}(y) N_{n_j}^{(j)}(y)$$

and

$$\mathcal{A}_{i,j} := (t_i t_j)^{-3/4} \int_{\mathbb{R}} L_{t_i}^{(i)}(x) L_{t_j}^{(j)}(x)\, dx$$

with $t_i = n_i/n$ and $t_j = n_j/n$. First, we rewrite $A_{i,j}^{(n)}$ as follows:

$$A_{i,j}^{(n)} = (t_i t_j)^{-3/4} \int_{\mathbb{R}} \frac{N_{n_i}^{(i)}(\lfloor \sqrt{n} x \rfloor)}{\sqrt{n}} \frac{N_{n_j}^{(j)}(\lfloor \sqrt{n} x \rfloor)}{\sqrt{n}} \, dx.$$

Observe next that, on $F_{n,n_1,\dots,n_k}$, for all i, j and $n_i, n_j \le n$, we have

$$A_{i,i}^{(n)} \le n^{3\gamma}, \qquad \mathcal{A}_{i,i} \le n^{3\gamma} \tag{28}$$

and

$$\begin{aligned} t_i^{-3/2} \int_{\mathbb{R}} \left| \frac{N_{n_i}^{(i)}(\lfloor \sqrt{n} x \rfloor)}{\sqrt{n}} - L_{t_i}^{(i)}(x) \right|^2 dx &\le t_i^{-3/2} 2 t_i^{1/2} n^{\gamma} t_i^{1/2} n^{-1/2+2\gamma} \\ &\le 2 t_i^{-1/2} n^{-1/2+3\gamma}. \end{aligned}$$

Hence, with the use of the Cauchy–Schwarz inequality, we get

$$\begin{gathered} A_{i,j}^{(n)} \le n^{3\gamma}, \qquad \mathcal{A}_{i,j} \le n^{3\gamma} \quad \text{and} \\ |A_{i,j}^{(n)} - \mathcal{A}_{i,j}| \le 2\sqrt{2} t_i^{-1/4} n^{-1/4+3\gamma} \le 4 n^{-\theta/4+3\gamma}. \end{gathered} \tag{29}$$

We use next that

$$\left(\prod_{i=1}^{k} n_i^{-3/2} \right) D_{n_1,\dots,n_k} = \det\bigl((A_{i,j}^{(n)})_{i,j} \bigr) \quad \text{and} \quad \det \overline{M}_{t_1,\dots,t_k} = \det\bigl((\mathcal{A}_{i,j})_{i,j} \bigr).$$

Furthermore, for any matrix M,

$$\det\bigl((M_{i,j})_{i,j} \bigr) = \sum_{\sigma \in \mathcal{S}_k} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{k} M_{i,\sigma(i)},$$

where $\mathcal{S}_k$ is the group of permutations of $\{1, \dots, k\}$ and $\mathrm{sgn}(\sigma)$ is the signature of σ. Therefore, using (29), on $F_{n,n_1,\dots,n_k}$, when $n_i \le n$, for all $i \le k$, we get for n large enough,

$$\begin{aligned} \left| \left(\prod_{i=1}^{k} n_i^{-3/2} \right) D_{n_1,\dots,n_k} - \det \overline{M}_{t_1,\dots,t_k} \right| &\le \sum_{\sigma \in \mathcal{S}_k} \sum_{i=1}^{k} n^{3\gamma(k-1)} |A_{i,\sigma(i)}^{(n)} - \mathcal{A}_{i,\sigma(i)}| \\ &\le 4(k+1)! n^{3\gamma k} n^{-\theta/4} \le n^{-\theta_0}, \end{aligned}$$

according to our assumption on γ.

Thus, for n large enough and every $\varepsilon \ge n^{-\theta_0}$, we get by using Proposition 8

$$\begin{aligned} \sup_{n^\theta \le n_1,\dots,n_k \le n} \mathbb{P}\bigl(\widetilde{\Omega}_{n_1,\dots,n_k}^c(\varepsilon) \bigr) &\le \sup_{n^\theta \le n_1,\dots,n_k \le n} \mathbb{P}\bigl(F_{n,n_1,\dots,n_k}^c \bigr) + \mathbb{P}(\det \overline{M}_{t_1,\dots,t_k} \le 2\varepsilon) \\ &\le \mathcal{O}\bigl(n^{-\theta_0 L} \bigr) + \mathbb{P}\bigl(\overline{\lambda}_{t_1,\dots,t_k} \le (2\varepsilon)^{1/k} \bigr) \\ &= \mathcal{O}\bigl(\varepsilon^L \bigr) \end{aligned}$$

with $\overline{\lambda}_{t_1,\dots,t_k}$ as in Proposition 8. So we just proved that for any $L > 0$, the constant

$$C_L := \sup_{n\ge 1} \sup_{\varepsilon \ge n^{-\theta_0}} \varepsilon^{-L} \sup_{n^\theta \le n_1,\dots,n_k \le n} \mathbb{P}(\widetilde{\Omega}^c_{n_1,\dots,n_k}(\varepsilon))$$

is finite, which gives the first part of the lemma. Then we get for any $p > 0$,

$$\begin{aligned}
&\sup_{n^\theta \le n_1,\dots,n_k \le n} \mathbb{E}\left[\left(\prod_{i=1}^k n_i^{3p/2}\right) D^{-p}_{n_1,\dots,n_k}, \widetilde{\Omega}_{n_1,\dots,n_k}(n^{-\theta_0})\right] \\
&\quad = \sup_{n^\theta \le n_1,\dots,n_k \le n} \int_0^\infty \mathbb{P}\left(n^{-\theta_0} \le \left(\prod_{i=1}^k n_i^{-3/2}\right) D_{n_1,\dots,n_k} \le t^{-1/p}\right) dt \\
&\quad = \sup_{n^\theta \le n_1,\dots,n_k \le n} p \int_{n^{-\theta_0}}^{+\infty} \mathbb{P}\left(n^{-\theta_0} \le \left(\prod_{i=1}^k n_i^{-3/2}\right) D_{n_1,\dots,n_k} \le \varepsilon\right) \frac{d\varepsilon}{\varepsilon^{p+1}} \\
&\quad \le p \int_{n^{-\theta_0}}^1 C_{p+1}\, d\varepsilon + p \int_1^{+\infty} \frac{d\varepsilon}{\varepsilon^{p+1}} < \infty,
\end{aligned}$$

where for the third line we have used the change of variables $t = \varepsilon^{-p}$. This concludes the proof of the lemma. $\square$

The next step is the following:

LEMMA 22. *Let $\eta \in (0, 1/4)$ and $\theta \in (0, 1)$ be given. Then*

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{n^\theta \le n_1,\dots,n_k \le n} \left(\prod_i n_i^{3/4}\right) \\
&\qquad \times \int_{U(\eta)} \left|\varphi_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k) - \mathbb{E}\left[e^{-\sigma^2 Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)/2}\right]\right| d\theta_1 \cdots d\theta_k = 0,
\end{aligned}$$

where

$$Q_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k) := \sum_y \left(\theta_1 N^{(1)}_{n_1}(y) + \dots + \theta_k N^{(k)}_{n_k}(y)\right)^2.$$

PROOF. Recall that $U(\eta) = \{|\theta_i| \le n_i^{-1/2-\eta}\ \forall i \le k\}$. Set

$$\begin{aligned}
E_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k) := &\left(\prod_y \varphi_\xi\left(\theta_1 N^{(1)}_{n_1}(y) + \dots + \theta_k N^{(k)}_{n_k}(y)\right)\right) \\
&- e^{-\sigma^2 Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)/2}.
\end{aligned}$$

We have to prove that

$$\int_{U(\eta)} \mathbb{E}\left[\left|E_{n_1,\dots,n_k}(\theta_1, \dots, \theta_k)\right|\right] d\theta_1 \cdots d\theta_k = o\left(\prod_{i=1}^k n_i^{-3/4}\right).$$

Observe that

$$
\begin{aligned}
&E_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)\\
&\quad=\sum_y\Big(\prod_{z>y}\exp\Big(-\frac{\sigma^2}{2}\big(\theta_1N_{n_1}^{(1)}(z)+\cdots+\theta_kN_{n_k}^{(k)}(z)\big)^2\Big)\Big)\\
&\qquad\times\big(\varphi_\xi\big(\theta_1N_{n_1}^{(1)}(y)+\cdots+\theta_kN_{n_k}^{(k)}(y)\big)-e^{-\sigma^2(\theta_1N_{n_1}^{(1)}(y)+\cdots+\theta_kN_{n_k}^{(k)}(y))^2/2}\big)\\
&\qquad\times\Big(\prod_{z<y}\varphi_\xi\big(\theta_1N_{n_1}^{(1)}(z)+\cdots+\theta_kN_{n_k}^{(k)}(z)\big)\Big).
\end{aligned}
$$

Recall now that, since ξ is square integrable, we have $1-\varphi_\xi(u)\sim\sigma^2|u|^2/2$, as $u\to0$. It follows that

$$\big|\varphi_\xi(u)-e^{-\sigma^2u^2/2}\big|\le|u|^2h\big(|u|\big)\qquad\text{for all }u\in\mathbb{R}$$

with h a continuous and monotone function on $[0,+\infty)$ vanishing in 0. In particular, there exists a constant $\varepsilon_0>0$, such that

$$\big|\varphi_\xi(u)\big|\le\exp\big(-\sigma^2|u|^2/4\big)\qquad\text{for all }u\in[-\varepsilon_0,\varepsilon_0].\tag{30}$$

Fix now $\gamma\in(0,\eta)$ and $\theta_0\in(0,\theta/4)$. Next recall (27) and observe that on

$$\Omega(\gamma,\theta_0):=\Omega_{n_1,\dots,n_k}\cap\widetilde{\Omega}_{n_1,\dots,n_k}\big(n^{-\theta_0}\big),$$

if $|\theta_i|\le n_i^{-1/2-\eta}$ for all $i\le k$, then (see the remark following Lemma 16) for all $z\in\mathbb{Z}$,

$$\big|\theta_1N_{n_1}^{(1)}(z)+\cdots+\theta_kN_{n_k}^{(k)}(z)\big|\le kn^{\gamma-\eta},$$

which is smaller than ε_0 for n large enough. Then we get

$$
\begin{aligned}
&\big|E_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)\big|\mathbf{1}_{\Omega(\gamma,\theta_0)}\\
&\quad\le h\big(kn^{\gamma-\eta}\big)e^{-\sigma^2Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)/4}\\
&\qquad\times\sum_y e^{\sigma^2(\theta_1N_{n_1}^{(1)}(y)+\cdots+\theta_kN_{n_k}^{(k)}(y))^2/4}\big(\theta_1N_{n_1}^{(1)}(y)+\cdots+\theta_kN_{n_k}^{(k)}(y)\big)^2\\
&\quad=o(1)\times e^{(\sigma\varepsilon_0)^2}e^{-\sigma^2Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)/4}Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)\\
&\quad=o(1)\times e^{-\sigma^2Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)/8}.
\end{aligned}
$$

Therefore, a change of variables gives

$$\int_{U(\eta)}\big|E_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k)\big|\mathbf{1}_{\Omega(\gamma,\theta_0)}\,d\theta_1\cdots d\theta_k=o(1)\times D_{n_1,\dots,n_k}^{-1/2}\int_{\mathbb{R}^k}e^{-\sigma^2|r|_2^2/8}\,dr,$$

at least when $D_{n_1,\dots,n_k}>0$. The result now follows from Lemmas 16 and 21. $\square$

Finally, Proposition 17 is deduced from the following lemma.

LEMMA 23. *Let $t_1, \ldots, t_k \in (0,1)$ and $\eta \in (0, 1/8)$ be given. Then*

$$\int_{U(\eta)} \mathbb{E}\big[e^{-\sigma^2 Q_{n_1,\ldots,n_k}(\theta_1,\ldots,\theta_k)/2}\big]\,d\theta_1\cdots d\theta_k$$
$$= \left(\frac{\sqrt{2\pi}}{\sigma}\right)^k \mathcal{C}_{t_1,\ldots,t_k} n^{-3k/4} + o_k\big(n^{-3k/4}\big).$$

PROOF. First write

$$\int_{U(\eta)} e^{-\sigma^2 Q_{n_1,\ldots,n_k}(\theta_1,\ldots,\theta_k)/2}\,d\theta_1\cdots d\theta_k = I_{n_1,\ldots,n_k} - J_{n_1,\ldots,n_k},$$

where $I_{n_1,\ldots,n_k}$ is the integral over $\mathbb{R}^k$ and $J_{n_1,\ldots,n_k}$ is the integral over $\{\exists j : |\theta_j| > n_j^{-1/2-\eta}\}$. A change of variables gives

$$I_{n_1,\ldots,n_k} = \sigma^{-k} D_{n_1,\ldots,n_k}^{-1/2} \int_{\mathbb{R}^k} e^{-|r|_2^2/2}\,dr.$$

According to Proposition 7, we know that

$$n^{-3k/4} D_{n_1,\ldots,n_k} \overset{(\mathcal{L})}{\Longrightarrow} \widetilde{\mathcal{D}}_{T_1,\ldots,T_k}$$

as $n \to \infty$ and $n_i/n \to t_i$, for $i = 1, \ldots, k$. This, combined with Lemma 21, shows that

$$\mathbb{E}[I_{n_1,\ldots,n_k}] = \left(\frac{\sqrt{2\pi}}{\sigma}\right)^k \mathcal{C}_{t_1,\ldots,t_k} n^{-3k/4} + o_k\big(n^{-3k/4}\big),$$

and it just remains to estimate $\mathbb{E}[J_{n_1,\ldots,n_k}]$.

First consider the matrix $A_{n_1,\ldots,n_k} := (\langle N_{n_i}^{(i)}, N_{n_j}^{(j)}\rangle)_{i,j\le k}$, and denote by $\mu_{n_1,\ldots,n_k}$ its smallest eigenvalue.

Let now $\theta \in (0,1)$, $0 < \theta_0 < \frac{\theta}{4}$ and $\gamma > 0$ be such that $2\eta + \theta_0 + 3\gamma(k-1) < 1/4$. We know that on $\Omega_{n_1,\ldots,n_k}$,

$$\operatorname{tr}(A_{n_1,\ldots,n_k}) = \sum_{i=1}^k \sum_y N_{n_i}^{(i)}(y)^2 \le k n^{3/2+3\gamma}\big(1 + o_k(1)\big).$$

We deduce that all eigenvalues of $A_{n_1,\ldots,n_k}$ are smaller than the right-hand side of the above inequality. In particular, on $\widetilde{\Omega}_{n_1,\ldots,n_k}(n^{-\theta_0})$, there exists a constant $c > 0$ (depending only on k and the t_i's), such that

$$\mu_{n_1,\ldots,n_k} \ge \frac{D_{n_1,\ldots,n_k}}{(kn^{3/2+3\gamma}(1+o_k(1)))^{k-1}}$$
$$\ge c n^{3/2-\theta_0-3\gamma(k-1)}\big(1 - o_k(1)\big).$$

Then we get

$$\mu_{n_1,\ldots,n_k} n^{-1-2\eta} \ge n^{1/2-2\eta-\theta_0-3\gamma(k-1)}\big(1 - o_k(1)\big)$$
$$\ge n^{1/4}\big(1 - o_k(1)\big),$$

since $2\eta + \theta_0 + 3\gamma(k-1) < 1/4$ by hypothesis. Note, moreover, that

$$Q_{n_1,\dots,n_k}(\theta_1,\dots,\theta_k) \ge \mu_{n_1,\dots,n_k}\big|(\theta_1,\dots,\theta_k)\big|_2^2.$$

Therefore, a change of variables gives

$$J_{n_1,\dots,n_k} \le \mu_{n_1,\dots,n_k}^{-k/2} \int_{\{|r|_2 \ge \mu_{n_1,\dots,n_k}^{1/2} n^{-1/2-\eta}\}} e^{-\sigma^2 |r|_2^2/2}\,dr$$

and it then follows from the first part of Lemma 21 that $\mathbb{E}[J_{n_1,\dots,n_k}] = o_k(n^{-3k/4})$. This concludes the proof of the lemma. □

5.5. *Proof of Proposition* 18. Let $\theta_0 \in (0, 1/4)$ be fixed. Consider the events

$$H(\theta_1,\dots,\theta_k) := \big\{\big|\theta_1 N_{n_1}^{(1)}(y) + \cdots + \theta_k N_{n_k}^{(k)}(y)\big| \le \varepsilon_0 \text{ for all } y \in \mathbb{Z}\big\},$$

where ε_0 is as in (30) and

$$\widetilde{H}(\theta_1,\dots,\theta_k) := H(\theta_1,\dots,\theta_k) \cap \widetilde{\Omega}_{n_1,\dots,n_k}\big(n^{-\theta_0}\big).$$

Then by using (30) and the argument at the end of the proof of Lemma 23, we get

$$\begin{aligned}\int_{V(\eta)} &\mathbb{E}\Big[\prod_y \big|\varphi_\xi\big(\theta_1 N_{n_1}^{(1)}(y) + \cdots + \theta_k N_{n_k}^{(k)}(y)\big)\big|, \widetilde{H}(\theta_1,\dots,\theta_k)\Big]\,d\theta_1\cdots d\theta_k \\ &= o_k\big(n^{-3k/4}\big)\end{aligned}$$

and, thanks to Lemma 21,

$$\begin{aligned}&\sup_{n\ge 1}\sup_{n^\theta \le n_1,\dots,n_k \le n}\Big(\prod_{i=1}^k n_i^{3/4}\Big) \\ &\quad\times \int_{V(\eta)} \mathbb{E}\Big[\prod_y \big|\varphi_\xi\big(\theta_1 N_{n_1}^{(1)}(y) + \cdots + \theta_k N_{n_k}^{(k)}(y)\big)\big|, \widetilde{H}(\theta_1,\dots,\theta_k)\Big]\,d\theta_1\cdots d\theta_k \\ &\quad< \infty.\end{aligned}$$

On the other hand, by using the Hölder continuity of the local time (see Lemma 16), we get

$$\begin{aligned}&\mathbb{P}\big[H(\theta_1,\dots,\theta_k)^c, \#\big\{y \in \mathbb{Z} : \big|\theta_1 N_{n_1}^{(1)}(y) + \cdots + \theta_k N_{n_k}^{(k)}(y)\big| \in [\varepsilon_0/2, \varepsilon_0]\big\} \le n^{1/4}\big] \\ &\quad= o_k\big(n^{-3k/4}\big),\end{aligned}$$

uniformly in $(\theta_1,\dots,\theta_k) \in V(\eta)$ and

$$\begin{aligned}&\sup_{(\theta_1,\dots,\theta_k)\in V(\eta)}\sup_{n\ge 1}\sup_{n^\theta \le n_1,\dots,n_k \le n}\Big(\prod_{i=1}^k n_i^{3/4}\Big) \\ &\quad\times \mathbb{P}\big[H(\theta_1,\dots,\theta_k)^c, \\ &\qquad\quad \#\big\{y \in \mathbb{Z} : \big|\theta_1 N_{n_1}^{(1)}(y) + \cdots + \theta_k N_{n_k}^{(k)}(y)\big| \in [\varepsilon_0/2, \varepsilon_0]\big\} \le n^{1/4}\big] < \infty.\end{aligned}$$

Finally, by using again (30), we obtain

$$\mathbb{P}\Big[H(\theta_1,\dots,\theta_k)^c, \Big|\prod_y \varphi_\xi\big(\theta_1 N_{n_1}^{(1)}(y)+\cdots+\theta_k N_{n_k}^{(k)}(y)\big)\Big| > e^{-(\sigma\varepsilon_0/2)^2 n^{1/4}/4}\Big]$$
$$= o_k\big(n^{-3k/4}\big)$$

and

$$\sup_{n\ge 1}\sup_{n^\theta\le n_1,\dots,n_k\le n}\Bigg(\prod_{i=1}^k n_i^{3/4}\Bigg)$$
$$\times \mathbb{P}\Big[H(\theta_1,\dots,\theta_k)^c,$$
$$\Big|\prod_y \varphi_\xi\big(\theta_1 N_{n_1}^{(1)}(y)+\cdots+\theta_k N_{n_k}^{(k)}(y)\big)\Big| > e^{-(\sigma\varepsilon_0/2)^2 n^{1/4}/4}\Big] < \infty.$$

The proposition now follows with Lemma 21.

6. Proof of Corollary 6. We first observe that for $k=1$, the result follows from (3), since we can write

$$\begin{aligned}\mathbb{E}[\mathcal{N}_n(0)] &= \sum_{i=0}^n \mathbb{P}(Z_i=0) = \sum_{i=0}^{\lfloor n/d_0\rfloor}\mathbb{P}(Z_{id_0}=0)\\ &\underset{n\to\infty}{\sim} \frac{d}{\sigma}\sum_{i=0}^{\lfloor n/d_0\rfloor} p_{1,1}(0)(id_0)^{-3/4}\\ &\underset{n\to\infty}{\sim} \frac{4d}{\sigma d_0}p_{1,1}(0)n^{1/4} = \frac{d}{\sigma d_0}\mathcal{M}_{1,1}(0)n^{1/4}\end{aligned}$$

and

$$\mathcal{M}_{1,1}(0) = \int_0^1 p_{1,t}(0)\,dt = \int_0^1 p_{1,1}(0)t^{-3/4}\,dt = 4p_{1,1}(0).$$

We prove now the result for some general $k\ge 1$. Fix some $\theta\in(0,1/4)$ and write

$$\begin{aligned}&n^{-k/4}\mathbb{E}[\mathcal{N}_n(0)^k]\\ &\quad= n^{-k/4}\sum_{n_1,\dots,n_k\le n}\mathbb{P}(Z_{n_1}=\cdots=Z_{n_k}=0)\\ &\quad= n^{-k/4}\sum_{n_1,\dots,n_k\le\lfloor n/d_0\rfloor}\mathbb{P}(Z_{n_1d_0}=\cdots=Z_{n_kd_0}=0)\\ &\quad= k!n^{3k/4}\int_{0\le u_1\le\cdots\le u_k\le 1/d_0}\mathbb{P}(Z_{\lfloor nu_1\rfloor d_0}=\cdots=Z_{\lfloor nu_k\rfloor d_0}=0)\,du_1\cdots du_k\end{aligned}$$

$$= k!n^{3k/4} \int_{n^{\theta-1} \le u_1 \le \cdots \le u_k \le 1/d_0} \mathbb{P}(Z_{\lfloor nu_1 \rfloor d_0} = \cdots = Z_{\lfloor nu_k \rfloor d_0} = 0)\, du_1 \cdots du_k + o(1).$$

Indeed, for the last equality, we use Theorem 5 which implies that for any $\ell \ge 1$,

$$n^{3k/4} \int_{0 \le u_1 \le \cdots \le u_\ell \le n^{\theta-1} \le u_{\ell+1} \le \cdots \le u_k \le 1/d_0} \mathbb{P}(Z_{\lfloor nu_1 \rfloor d_0} = \cdots = Z_{\lfloor nu_k \rfloor d_0} = 0)\, du_1 \cdots du_k$$
$$\le C n^{3\ell/4 + (\theta-1)\ell} \int_{0 \le u_{\ell+1} \le \cdots \le u_k \le 1/d_0} (u_{\ell+1} \cdots u_k)^{-3/4} = o(1),$$

since $\theta < 1/4$ and where C is the constant appearing in Theorem 5. Then, by using again Theorem 5, we can apply the Lebesgue dominated convergence theorem, and we get

$$\begin{aligned} n^{-k/4} &\sum_{n_1, \dots, n_k} \mathbb{P}(Z_{n_1} = \cdots = Z_{n_k} = 0) \\ &= k! \left(\frac{d}{\sigma}\right)^k \int_{0 \le u_1 \le u_2 \le \cdots \le u_k \le 1/d_0} p_{k, u_1 d_0, u_2 d_0, \dots, u_k d_0}(0, \dots, 0)\, du + o(1) \\ &= \left(\frac{d}{\sigma}\right)^k \int_{[0, 1/d_0]^k} p_{k, u_1 d_0, u_2 d_0, \dots, u_k d_0}(0, \dots, 0)\, du + o(1) \\ &= \left(\frac{d}{d_0 \sigma}\right)^k \int_{[0,1]^k} p_{k, u_1, u_2, \dots, u_k}(0, \dots, 0)\, du + o(1) \\ &= \left(\frac{d}{d_0 \sigma}\right)^k \mathcal{M}_{k,1}(0) + o(1). \end{aligned}$$

This concludes the proof of the corollary.

We notice that similar calculations show that for any $r \ge 1$, any $k_1, \dots, k_r \ge 1$ and any $0 < t_1 < \cdots < t_r$,

$$\begin{aligned} &\mathbb{E}[\mathcal{N}_{[nt_1]}(0)^{k_1} \cdots \mathcal{N}_{[nt_r]}(0)^{k_r}] \\ &\quad \sim \left(\frac{d}{\sigma d_0}\right)^{k_1 + \cdots + k_r} n^{(k_1 + \cdots + k_r)/4} \mathbb{E}[\mathcal{L}_{t_1}(0)^{k_1} \cdots \mathcal{L}_{t_r}(0)^{k_r}] \end{aligned} \tag{31}$$

as $n \to \infty$.

At this point, it is also not difficult to see that the sequence $(\mathcal{N}_{[nt]}(0)/n^{1/4}, t \ge 0)$ is tight in the Skorokhod space $\mathbb{D}(\mathbb{R})$. For this, notice that for any $T > 0$ and $p \ge 1$, there exists a constant $C = C(T, p)$, such that for all $t \in [0, T]$, $h > 0$ and $\eta > 0$,

$$\begin{aligned} \mathbb{P}\big(\mathcal{N}_{[n(t+h)]}(0) - \mathcal{N}_{[nt]}(0) \ge \eta n^{1/4}\big) &\le \eta^{-p} n^{-p/4} \mathbb{E}\big[\big(\mathcal{N}_{[n(t+h)]}(0) - \mathcal{N}_{[nt]}(0)\big)^p\big] \\ &\le C \eta^{-p} h^{p/4}. \end{aligned}$$

Indeed, the second inequality follows from the proof of Corollary 6. Since $\mathcal{N}_{[nt]}(0)$ is a nondecreasing process, the tightness follows, for instance, from Lemma (1.7), page 517, in [18].

7. Proof of Proposition 7. It was proved by Kesten and Spitzer [10] that the normalized self-intersection local time of the random walk converges in distribution to its continuous counterpart. A similar convergence is proved for the mutual intersection local time in Chen's book [6]. We prove Proposition 7 by following carefully their proof.

For $j = 1, \dots, k$ and $a < b$, let

$$T_{n_j}^{(j)}(a,b) := \frac{1}{n_j} \sum_{a \le n^{-1/2} y < b} N_{n_j}^{(j)}(y),$$

denote the time spent by $S_{\lfloor n_j \cdot \rfloor}^{(j)}/\sqrt{n}$ in $[a,b)$ before time n_j. The mutual intersection local time of $S_{[n_j \cdot]}^{(j)}/\sqrt{n}$ and $S_{[n_{j'} \cdot]}^{(j')}/\sqrt{n}$ before time 1 is defined by

$$\begin{aligned} T_{n_j,n_{j'}}^{(j,j')} &:= \frac{\sqrt{n}}{n_j n_{j'}} \langle N_{n_j}^{(j)}, N_{n_{j'}}^{(j')} \rangle \\ &= \frac{\sqrt{n}}{n_j n_{j'}} \sum_{x \in \mathbb{Z}} \sum_{k=1}^{n_j} \sum_{\ell=1}^{n_{j'}} \mathbf{1}_{\{S_k^{(j)} = x\}} \mathbf{1}_{\{S_\ell^{(j')} = x\}}. \end{aligned}$$

For any $\varepsilon > 0$, consider the regularizing functions $p_\varepsilon(x) := e^{-x^2/2\varepsilon}/\sqrt{2\pi\varepsilon}$ and set

$$T_{\varepsilon,n_j,n_{j'}}^{(j,j')} := \frac{1}{\sqrt{n} n_j n_{j'}} \sum_{x \in \mathbb{Z}} \sum_{k=1}^{n_j} \sum_{\ell=1}^{n_{j'}} p_\varepsilon\left(\frac{S_k^{(j)} - x}{\sqrt{n}}\right) p_\varepsilon\left(\frac{S_\ell^{(j')} - x}{\sqrt{n}}\right).$$

Similarly, let

$$\Lambda_j(a,b) := \frac{1}{T_j} \int_a^b L_{T_j}^{(j)}(x)\, dx$$

denote the time spent by $B^{(j)}$ in $[a,b)$ before time T_j, and let

$$\Lambda_{j,j'} := \frac{1}{T_j T_{j'}} \int_{\mathbb{R}} L_{T_j}^{(j)}(x) L_{T_{j'}}^{(j')}(x)\, dx$$

denote the mutual intersection local time of $B_{T_j \cdot}^{(j)}$ and $B_{T_{j'} \cdot}^{(j')}$. Finally, set for every $\varepsilon > 0$,

$$\Lambda_{\varepsilon,j,j'} := \int_{\mathbb{R}} \left(\int_{[0,1]^2} p_\varepsilon(B_{sT_j} - x) p_\varepsilon(B_{tT_{j'}} - x)\, ds\, dt \right) dx.$$

We will use the following lemmas:

LEMMA 24 (Lemma 5.3.1 in Chen). *For all $j \neq j'$,*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \limsup_{n_j/n\to T_j, n_{j'}/n\to T_{j'}} \mathbb{E}\big[\big|T^{(j,j')}_{n_j,n_{j'}} - T^{(j,j')}_{\varepsilon,n_j,n_{j'}}\big|^2\big] = 0.$$

LEMMA 25 (Theorem 2.2.3 in Chen). *For all $j \neq j'$, the sequence $(\Lambda_{\varepsilon,j,j'}, \varepsilon > 0)$ converges in L^2 to $\Lambda_{j,j'}$, as ε goes to* 0.

We can then already deduce the following:

LEMMA 26. *For any $m_1, \dots, m_k \geq 1$ and any $-\infty < a_{j,\ell} < b_{j,\ell} < \infty$ ($j = 1, \dots, k$ and $\ell = 1, \dots, m_j$),*

$$\big(\big(T^{(j)}_{n_j}(a_{j,\ell}, b_{j,\ell})\big)_{j=1,\dots,k,\ell=1,\dots,m_j}, \big(T^{(j,j')}_{n_j,n_{j'}}\big)_{1\leq j<j'\leq k}\big)$$

converges in distribution to

$$\big(\big(\Lambda_j(a_{j,\ell}, b_{j,\ell})\big)_{j=1,\dots,k,\ell=1,\dots,m_j}, (\Lambda_{j,j'})_{1\leq j<j'\leq k}\big)$$

as $n \to +\infty$, and $n_j/n \to T_j$ for all $j \leq k$.

PROOF OF LEMMA 26. Let $\theta_{j,\ell}$ (for $j = 1, \dots, k$ and $\ell = 1, \dots, m_j$) and $\overline{\theta}_{j,j'}$ (for $1 \leq j < j' \leq k$) be some fixed real numbers. It suffices to prove that

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^k\sum_{\ell=1}^{m_j}\theta_{j,\ell}T^{(j)}_{n_j}(a_{j,\ell}, b_{j,\ell}) + i\sum_{1\leq j<j'\leq k}\overline{\theta}_{j,j'}T^{(j,j')}_{n_j,n_{j'}}\right)\right)$$

converges to

$$\mathbb{E}\left(\exp\left(i\sum_{j=1}^k\sum_{\ell=1}^{m_j}\theta_{j,\ell}\Lambda_j(a_{j,\ell}, b_{j,\ell}) + i\sum_{1\leq j<j'\leq k}\overline{\theta}_{j,j'}\Lambda_{j,j'}\right)\right).$$

Lemmas 24 and 25 show that we can replace the $T^{(j,j')}_{n_j,n_{j'}}$ and $\Lambda_{j,j'}$, respectively, by $T^{(j,j')}_{\varepsilon,n_j,n_{j'}}$ and $\Lambda_{\varepsilon,j,j'}$.

Observe now that the map

$$\begin{aligned}\big(x^{(j)}\big)_{j\leq k} \mapsto{}& \sum_{j=1}^k\sum_{\ell=1}^{m_j}\theta_{j,\ell}\int_0^1 \mathbf{1}_{[a_{j,\ell}\leq x^{(j)}_s<b_{j,\ell}]}\,ds \\ &+ \sum_{1\leq j<j'\leq k}\overline{\theta}_{j,j'}\int_{\mathbb{R}}\int_{[0,1]^2} p_\varepsilon\big(x^{(j)}_s - x\big)p_\varepsilon\big(x^{(j')}_t - x\big)\,ds\,dt\,dx\end{aligned}$$

is continuous on $\mathbb{D}([0,1], \mathbb{R}^k)$ for the Skorokhod topology. Observe, moreover, that for all fixed $\varepsilon > 0$,

$$T^{(j,j')}_{\varepsilon,n_j,n_{j'}} = \int_{\mathbb{R}}\int_{[0,1]^2} p_\varepsilon\left(\frac{S^{(j)}_{[n_js]}}{\sqrt{n}} - x\right)p_\varepsilon\left(\frac{\widetilde{S}_{[n_{j'}t]}}{\sqrt{n}} - x\right)ds\,dt\,dx + o(1).$$

Therefore, the weak convergence of $(S^{(j)}_{[n_j\cdot]}/\sqrt{n}, j \le k)$ toward $(B^{(j)}_{T_j\cdot}, j \le k)$ implies that

$$\sum_{j=1}^{k}\sum_{\ell=1}^{m_j}\theta_{j,\ell}T^{(j)}_{n_j}(a_{j,\ell}, b_{j,\ell}) + \sum_{1\le j<j'\le k}\overline{\theta}_{j,j'}T^{(j,j')}_{\varepsilon,n_j,n_{j'}}$$

converges in distribution to

$$\sum_{j=1}^{k}\sum_{\ell=1}^{m_j}\theta_{j,\ell}\Lambda_j(a_{j,\ell}, b_{j,\ell}) + \sum_{1\le j<j'\le k}\overline{\theta}_{j,j'}\Lambda_{\varepsilon,j,j'}.$$

The result follows. $\square$

We finish now the proof of Proposition 7. Let θ_j (for $j = 1,\dots,k$) and $\theta_{j,j'}$ (for $1 \le j < j' \le k$) be some fixed real numbers. We proceed like in [10] by decomposing the set of all possible indices into small slices where sharp estimates can be made. Define, in the slice $[\tau\ell\sqrt{n}, \tau(\ell+1)\sqrt{n})$, an average occupation time by

$$T_j(\tau,\ell,n) := \frac{n_j}{n}T^{(j)}_{n_j}\big(\tau\ell, \tau(\ell+1)\big).$$

Set also

$$U(\tau,M,n) := \sum_{j=1}^{k}\theta_j n^{-3/2}\sum_{|x|\ge M\tau\sqrt{n}} N^{(j)}_{n_j}(x)^2,$$

$$V(\tau,M,n) := \sum_{j=1}^{k}\frac{\theta_j}{\tau}\sum_{-M\le\ell<M}\big(T_j(\tau,\ell,n)\big)^2 + \sum_{1\le j<j'\le k}\theta_{j,j'}\frac{n_jn_{j'}}{n^2}T^{(j,j')}_{n_j,n_{j'}}$$

and

$$\begin{aligned} A(\tau,M,n) &:= \sum_{j=1}^{k}\theta_j n^{-3/2}\sum_{x\in\mathbb{Z}} N^{(j)}_{n_j}(x)^2 \\ &\quad + \sum_{1\le j<j'\le k}\theta_{j,j'}\frac{n_jn_{j'}}{n^2}T^{(j,j')}_{n_j,n_{j'}} - U(\tau,M,n) - V(\tau,M,n) \\ &= \sum_{j=1}^{k}\theta_j n^{-3/2}\sum_{-M\le\ell<M}\ \sum_{a(\ell,n)\le x<a(\ell+1,n)}\bigg(N^{(j)}_{n_j}(x)^2 \\ &\qquad - \frac{n^2\times(T_j(\tau,\ell,n))^2}{(\tau\sqrt{n})^2}\bigg) \\ &\quad + o(1). \end{aligned}$$

It follows from computations in [10] (see, in particular, Lemmas 1, 2 and 3) that $A(\tau, M, n)$ converges in probability to zero as $M\tau^{3/2} \to 0$. Moreover,

$$\mathbb{P}\big(U(\tau, M, n) \neq 0\big) \leq \mathbb{P}\Big(\exists j \leq k : \sup_{m \leq n_j} |S_m^{(j)}| > M\tau\sqrt{n}\Big),$$

and it is well known that the right-hand term goes to 0, as $M\tau \to \infty$, and $n_j/n \to T_j$, for all $j \leq k$.

Now, Lemma 26 shows that $V(\tau, M, n)$ converges in law to

$$\sum_{j=1}^{k} \frac{\theta_j}{\tau} \sum_{-M \leq \ell < M} \bigg(\int_{\ell\tau}^{(\ell+1)\tau} L_{T_j}^{(j)}(x)\,dx\bigg)^2 + \sum_{1 \leq j < j' \leq k} \theta_{j,j'} \int_{\mathbb{R}} L_{T_j}^{(j)}(x) L_{T_{j'}}^{(j')}(x)\,dx.$$

But the map $x \mapsto L_t^{(j)}(x)$ being a.s. continuous with compact support, this last sum converges, as $\tau \to 0$ and $M\tau \to \infty$, to

$$\sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} L_{T_j}^{(j)}(x)^2\,dx + \sum_{1 \leq j < j' \leq k} \theta_{j,j'} \int_{\mathbb{R}} L_{T_j}^{(j)}(x) L_{T_{j'}}^{(j')}(x)\,dx.$$

The proposition follows.

REFERENCES

[1] BILLINGSLEY, P. (1995). *Probability and Measure*, 3rd ed. Wiley, New York. MR1324786

[2] BORODIN, A. N. (1979). A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR* **246** 786–787. MR0543530

[3] BORODIN, A. N. (1986). On the character of convergence to Brownian local time. II. *Probab. Theory Related Fields* **72** 251–277.

[4] CASTELL, F., GUILLOTIN-PLANTARD, N. and PÈNE, F. (2013). Limit theorems for one and two-dimensional random walks in random scenery. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 506–528. MR3088379

[5] CASTELL, F., GUILLOTIN-PLANTARD, N., PÈNE, F. and SCHAPIRA, B. (2011). A local limit theorem for random walks in random scenery and on randomly oriented lattices. *Ann. Probab.* **39** 2079–2118. MR2932665

[6] CHEN, X. (2010). *Random Walk Intersections: Large Deviations and Related Topics. Mathematical Surveys and Monographs* **157**. Amer. Math. Soc., Providence, RI. MR2584458

[7] CHEN, X., LI, W. V., ROSIŃSKI, J. and SHAO, Q.-M. (2011). Large deviations for local times and intersection local times of fractional Brownian motions and Riemann–Liouville processes. *Ann. Probab.* **39** 729–778. MR2789511

[8] DOMBRY, C. and GUILLOTIN-PLANTARD, N. (2009). Discrete approximation of a stable self-similar stationary increments process. *Bernoulli* **15** 195–222. MR2546804

[9] GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67. MR0556414

[10] KESTEN, H. and SPITZER, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50** 5–25. MR0550121

[11] KHOSHNEVISAN, D. (2003). The codimension of the zeros of a stable process in random scenery. In *Séminaire de Probabilités XXXVII. Lecture Notes in Math.* **1832** 236–245. Springer, Berlin. MR2053048

[12] KHOSHNEVISAN, D. and LEWIS, T. M. (1999). Iterated Brownian motion and its intrinsic skeletal structure. In *Seminar on Stochastic Analysis*, *Random Fields and Applications* (*Ascona*, 1996). *Progress in Probability* **45** 201–210. Birkhäuser, Basel. MR1712242

[13] LEBEDEV, N. N. (1972). *Special Functions and Their Applications*. Dover, New York. Revised edition, translated from the Russian and edited by Richard A. Silverman, unabridged and corrected republication. MR0350075

[14] LE DOUSSAL, P. (1992). Diffusion in layered random flows, polymers, electrons in random potentials, and spin depolarization in random fields. *J. Stat. Phys.* **69** 917–954. MR1192029

[15] LE GALL, J. F. Mouvement Brownien, processus de branchement et superprocessus. Master course. Available at http://www.math.u-psud.fr/~jflegall.

[16] MARCUS, M. B. and ROSEN, J. (2006). *Markov Processes*, *Gaussian Processes*, *and Local Times*. *Cambridge Studies in Advanced Mathematics* **100**. Cambridge Univ. Press, Cambridge. MR2250510

[17] PORT, S. C. and STONE, C. J. (1978). *Brownian Motion and Classical Potential Theory*: *Probability and Mathematical Statistics*. Academic Press, New York. MR0492329

[18] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Springer, Berlin. MR1725357

[19] XIAO, Y. (1999). The Hausdorff dimension of the level sets of stable processes in random scenery. *Acta Sci. Math.* (*Szeged*) **65** 385–395. MR1702175

F. CASTELL
LATP, UMR CNRS 6632
CENTRE DE MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ AIX-MARSEILLE I
39, RUE JOLIOT CURIE
13 453 MARSEILLE CEDEX 13
FRANCE
E-MAIL: Fabienne.Castell@cmi.univ-mrs.fr

N. GUILLOTIN-PLANTARD
INSTITUT CAMILLE JORDAN
UNIVERSITÉ DE LYON
CNRS UMR 5208
UNIVERSITÉ LYON 1, 43
BOULEVARD DU 11 NOVEMBRE 1918
69622 VILLEURBANNE
FRANCE
E-MAIL: nadine.guillotin@univ-lyon1.fr

F. PÈNE
LABORATOIRE DE MATHÉMATIQUES
UNIVERSITÉ EUROPÉENNE DE BRETAGNE
UNIVERSITÉ DE BREST
UMR CNRS 6205, 29238 BREST CEDEX
FRANCE
E-MAIL: francoise.pene@univ-brest.fr

B. SCHAPIRA
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS-SUD 11
CNRS UMR 8628, BÂT. 425
F-91405 ORSAY, CEDEX
FRANCE
E-MAIL: bruno.schapira@math.u-psud.fr