Branching capacity of a random walk range.

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Abstract

We consider the branching capacity of the range of a simple random walk on $\mathbb{Z}^d$, with $d \geq 5$, and show that it falls in the same universality class as the volume and the capacity of the range of simple random walks and branching random walks. To be more precise we prove a law of large numbers in dimension $d \geq 6$, with a logarithmic correction in dimension 6, and identify the correct order of growth in dimension 5. The main original part is the law of large numbers in dimension 6, for which one needs a precise asymptotic of the non-intersection probability of an infinite invariant critical tree-indexed walk with a two-sided simple random walk. The result is analogous to the estimate proved by Lawler for the non-intersection probability of an infinite random walk with a two-sided walk in dimension four. While the general strategy of Lawler's proof still applies in this new setting, many steps require new ingredients.

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1 Introduction

We start by recalling some important definitions and we will then state our main results. The branching capacity is defined here in terms of an offspring distribution μ on $\mathbb{N}$, which is fixed in the whole paper and assumed to be critical, in the sense that $\sum_i i\mu(i) = 1$. We further assume that it has a finite and positive variance σ^2. We write the size biased distribution of μ as $\mu_{\rm sb}$, which we recall is defined by $\mu_{\rm sb}(i) = i\mu(i)$, for all $i \geq 0$.

We then consider $\mathcal{T}$ an infinite planar tree, introduced independently in [19] and [4], which generalizes the one-sided version of Le Gall and Lin [16], and which is defined as follows (here the offspring of every vertex are ordered from left to right, and the root is at the bottom of the tree):

- The root produces i offspring with probability $\mu(i-1)$ for every $i \geq 1$. The first offspring of the root is *special*, while the others if they exist are *normal*.
- Special vertices produce offspring independently according to $\mu_{\rm sb}$, while normal vertices produce offspring independently according to μ.
- One of the offspring of a special vertex is chosen at random to be a special vertex, while the rest are normal ones.

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Since μ-trees are almost surely finite, because μ is critical, and since special vertices are guaranteed to have at least one offspring by definition of $\mu_{\rm sb}$, $\mathcal{T}$ has a unique infinite path emanating from the root that we call *spine*: it contains the root together with all the special vertices. We assign label 0 to the root. We assign positive labels to the vertices to the right of the spine according to depth first search from the root and we assign negative labels to the vertices to the left of the spine and the spine vertices as well according to depth first search from infinity, see Figure 1. We call the vertices with negative labels (including the spine vertices) the *past* of $\mathcal{T}$ and denote them $\mathcal{T}_-$, while the vertices with non-negative labels are in the *future* of $\mathcal{T}$ and we denote them $\mathcal{T}_+$. Note that the root does not have any offspring in the past of $\mathcal{T}$.

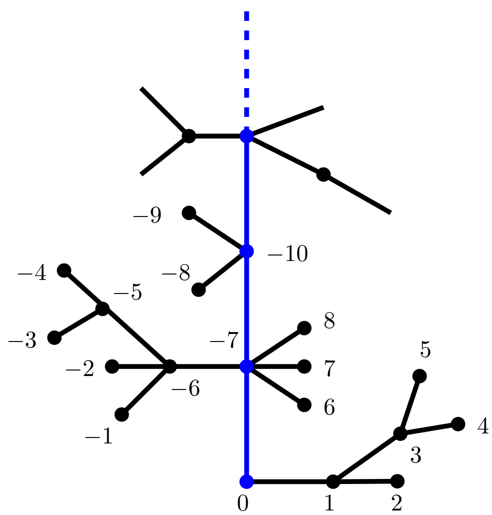

Figure 1: An infinite tree $\mathcal{T}$, with the spine in blue.

Given $x \in \mathbb{Z}^d$, we denote by $(S_u^x)_{u\in\mathcal{T}}$ the random walk indexed by $\mathcal{T}$, starting from x, whose jump distribution is the uniform measure on the neighbors of the origin, and denote its range in the past by

$$\mathcal{T}_-^x = \{S_u^x : u \in \mathcal{T}_-\}.$$

The equilibrium measure e_A of a finite set $A \subset \mathbb{Z}^d$, with $d \geq 5$, has been introduced by Zhu [18], and is defined by,

$$e_A(x) = \mathbb{P}(\mathcal{T}_-^x \cap A = \emptyset).$$

Then the branching capacity of a finite set A is defined similarly as the usual Newtonian capacity, namely

$$\mathrm{BCap}(A) = \sum_{x\in A} e_A(x).$$

Consider now $(X_n)_{n\geq 0}$ an independent simple random walk on $\mathbb{Z}^d$ (i.e. a random walk whose law of increments is the uniform measure on the neighbors of the origin), and define its range at time n as

$$\mathcal{R}_n = \{X_0, \dots, X_n\}.$$

Our main object of study in this paper is the branching capacity of the range $\mathrm{BCap}(\mathcal{R}_n)$, in dimension $d \geq 5$, and our goal is to show that it satisfies the same universal asymptotic behavior as the volume [9] and the capacity [1, 2, 8, 10] of the range, with only a shift of the critical dimension of respectively two and four units, which is here the dimension 6. Interestingly, the same universal results have also been proved recently for the volume [15, 16] and the capacity [4, 5] of a critical branching random walk, and of course it would be of interest to see if they can as well be extended to the branching capacity of a branching random walk, but we leave this for a future work.

Our first result is a strong law of large numbers. The proof is entirely similar to the one for the usual Newtonian capacity, which dates back to Jain and Orey [10], and is reproduced at the end of this paper for reader's convenience (to be more precise the fact that the limiting constant is positive requires a specific argument).

Theorem 1.1. *Assume $d \geq 7$. There exists a constant $c_d > 0$, such that almost surely,*

$$\lim_{n \to \infty} \frac{\mathrm{BCap}(\mathcal{R}_n)}{n} = c_d. \tag{1.1}$$

It is very likely that a central limit theorem, with the usual renormalization in $\sqrt{n}$, could be proved in dimension $d \geq 8$, following the same lines as in [1]. In dimension 7 it is expected that a logarithmic correction should appear in the normalization, but this might be a much more challenging problem, as the corresponding results in the simpler cases of the volume and the capacity of the range of a random walk are already quite involved, see [11, 17] respectively.

The main contribution of this paper is the law of large numbers in dimension 6, which requires some more original work. We only present here a detailed proof of the weak law (with a convergence in probability), but a strong law (with an almost sure convergence) could be proved as well without much additional work, see Remark 3.6 for more details. The main step is to obtain the asymptotic of the expected branching capacity of the range. The general strategy for this is the same as for the capacity of the range, in which case the corresponding result follows from the estimates proved by Lawler [12] for the non-intersection probability between one walk and another independent two-sided walk in dimension four, see [1, 8]. However, one serious issue that arises when working with the tree-indexed walk is the lack of Markov property, which in particular has the damaging consequence that there is no simple last exit formula as one has for a simple random walk. This leads to some non-trivial complications, which fortunately can be overtaken.

Theorem 1.2. *Assume $d = 6$, and that μ has a finite third moment. Then one has the convergence in probability and in L^2,*

$$\lim_{n \to \infty} \frac{\log n}{n} \cdot \mathrm{BCap}(\mathcal{R}_n) = \frac{2\pi^3}{27\sigma^2}.$$

Of course a natural question now would be to prove a central limit theorem, as it was done in [14, 2] respectively for the volume and the capacity of the range. We leave this for future work, as it would require some really new ingredients, in particular one major issue would be to identify a simple expression for the term

$$\chi(A, B) = \mathrm{BCap}(A \cup B) - \mathrm{BCap}(A) - \mathrm{BCap}(B),$$

where A and B are arbitrary finite subsets of $\mathbb{Z}^d$, and improve the bounds that we have on the variance of $\mathrm{BCap}(\mathcal{R}_n)$.

To conclude we provide bounds identifying the correct order of growth of the expected branching capacity of the range in dimension five. The upper bound is easily obtained by using monotonicity of the branching capacity, and known bounds on the branching capacity of balls. The lower bound is more difficult, and we rely here on a recent result of [3] showing a variational characterization of the branching capacity. We also mention that an invariance principle is in progress [6, 7].

Proposition 1.3. *For $d = 5$, there exist positive constants c_5^- and c_5^+, such that for all $n \geq 1$,*

$$c_5^- \cdot \sqrt{n} \leq \mathbb{E}\left[\mathrm{BCap}(\mathcal{R}_n)\right] \leq c_5^+ \cdot \sqrt{n}.$$

Remark 1.4. We note that the proofs of all our results would extend immediately to any symmetric finite range jump distribution both for the tree-indexed walk, and the walk $(X_n)_{n\geq 0}$. It is even likely that one would only need a moment assumption, e.g. as in [6]. Concerning the random walk $(X_n)_{n\geq 0}$, the hypothesis of symmetric jump distribution could be relaxed to a centered jump distribution.

The paper is organized as follows. In Section 2 we prove some preliminary results which could be of general interest. In particular we prove an analogous version in the setting of branching random walks of a key equation discovered by Lawler, relating some non-intersection events and a sum of Green's function along the positions of a random walk, see Lemma 2.7 and Corollary 2.8. We also prove there some quantitative bounds on the speed of convergence toward the branching capacity of a set A, of the (conveniently normalized) probability to hit A for a tree-indexed walk, as the starting point goes to infinity, see Proposition 2.6. Then Section 3 focuses on the case of dimension 6, and we prove there Theorem 1.2, while short proofs of Theorem 1.1 and Proposition 1.3 are given in Section 4.

2 Preliminaries

2.1 Some additional notation

We let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{Z}^d$. For $m \geq 0$, we denote by $B(0,m)$ the closed Euclidean ball of radius m centered at the origin (intersected with $\mathbb{Z}^d$), and for a set $\Lambda \subset \mathbb{Z}^d$, we let $\partial\Lambda$ be the inner boundary of Λ consisting of points of Λ having at least one neighbor outside Λ. We denote by $|A|$ the size of a finite set $A \subset \mathbb{Z}^d$, and we define the diameter of a finite set A as $\mathrm{diam}(A) = 1 + \max\{\|x-y\| : x, y \in A\}$.

We use here the convention that a Geometric random variable X with parameter $p \in (0,1)$ takes values in $\mathbb{N}$, and is such that for any $k \geq 0$, $\mathbb{P}(X = k) = p(1-p)^k$.

We denote by $(\widetilde{X}_k)_{k\geq 0}$ the simple random walk indexed by the vertices of the spine, equipped with its intrinsic labelling (i.e. the vertex on the spine at graph distance k from the root has intrinsic label k). The law of a simple random walk starting from x is denoted by $\mathbb{P}_x$ while the corresponding expectation is denoted by $\mathbb{E}_x$, and we abbreviate them in $\mathbb{P}$ and $\mathbb{E}$ respectively when the walk starts from the origin. For $n \leq m$, we write the range of a random walk $(X_k)_{k\in\mathbb{N}}$ between times n and m as $\mathcal{R}[n,m] = \{X_n, \dots, X_m\}$.

The root of the tree $\mathcal{T}$ is denoted by $\emptyset$.

Given two positive functions f and g, we write $f \lesssim g$, or sometimes also $f = \mathcal{O}(g)$, if there exists a constant $C > 0$, such that $f(x) \leq Cg(x)$ for all x, and likewise write $f \gtrsim g$, if $g \lesssim f$. We write $f = o(g)$ if $f(x)/g(x)$ goes to 0 as x goes to infinity, and $f \sim g$, when $|f - g| = o(g)$.

2.2 Hitting probabilities

We recall here some results from [18] on hitting probabilities for a walk indexed by $\mathcal{T}_-$, or by a critical random tree. We denote by $\mathcal{T}_c$ a μ-Bienaymé-Galton-Watson tree, and by $\mathcal{T}_c^x$ the range of a walk indexed by $\mathcal{T}_c$ starting from x.

Proposition 2.1 ([18])**.** *Assume $d \geq 5$. There exists positive constants c and C, such that for any finite set A containing the origin, and any x, with $\|x\| \geq 2 \cdot diam(A)$,*

$$c \cdot \frac{\mathrm{BCap}(A)}{\|x\|^{d-4}} \leq \mathbb{P}(\mathcal{T}_-^x \cap A \neq \emptyset) \leq C \cdot \frac{\mathrm{BCap}(A)}{\|x\|^{d-4}}, \tag{2.1}$$

$$c \cdot \frac{\mathrm{BCap}(A)}{\|x\|^{d-2}} \leq \mathbb{P}(\mathcal{T}_c^x \cap A \neq \emptyset) \leq C \cdot \frac{\mathrm{BCap}(A)}{\|x\|^{d-2}}. \tag{2.2}$$

In fact, only the upper bounds will be used, but we also mention the lower bounds for completeness.

2.3 An exact last passage formula

Our tree-indexed random walks are not Markovian, but nevertheless they satisfy a certain last passage formula, which takes the following form.

Lemma 2.2 (Last passage formula)**.** *For any $x \in \mathbb{Z}^d$, $d \geq 5$, and any finite set $A \subseteq \mathbb{Z}^d$, one has*

$$\mathbb{P}\Big((\{x\} \cup \mathcal{T}_-^x) \cap A \neq \emptyset\Big) = \sum_{y \in A} \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+(y,x)\Big],$$

where

$$\mathcal{L}_+(y,x) = \sum_{u \in \mathcal{T}_+} \mathbf{1}\{S_u^y = x\}.$$

Proof. The proof is an immediate application of the shift invariance of the tree $\mathcal{T}$, first identified by Le Gall and Lin for the one-sided version $\mathcal{T}_+$ of the tree [15, 16], and generalized in [4, 19] to the full tree $\mathcal{T}$. More precisely denote by τ the last time in the past when the walk visits A (which is almost surely finite when $d \geq 5$ and A is finite, if the walk ever hits A), and for $n \in \mathbb{Z}$, denote with a slight abuse of notation by S_n^x the position of the walk S^x at the vertex with label n. Then by shift invariance,

$$\begin{aligned}\mathbb{P}\Big((\{x\} \cup \mathcal{T}_-^x) \cap A \neq \emptyset\Big) &= \sum_{n=0}^{\infty} \sum_{y \in A} \mathbb{P}(\tau = -n, S_{-n}^x = y) = \sum_{n=0}^{\infty} \sum_{y \in A} \mathbb{P}(S_{-n}^x = y,\ S_m^x \in A^c,\ \text{for all } m < -n) \\ &= \sum_{n=0}^{\infty} \sum_{y \in A} \mathbb{P}(S_n^y = x, \mathcal{T}_-^y \cap A = \emptyset) = \sum_{y \in A} \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+(y,x)\Big].\end{aligned}$$

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2.4 Green's functions

Recall that the random walk Green's function is defined by

$$g(x,y) = \mathbb{E}_x\Big[\sum_{n \geq 0} \mathbf{1}\{X_n = y\}\Big] = g(0, y-x),$$

with $(X_n)_{n \geq 0}$ a simple random walk. We also let $g(z) = g(0,z)$, and recall that for $d \geq 3$, as $\|z\| \to \infty$ (see [13]),

$$g(z) \sim \frac{a_d}{\|z\|^{d-2}}, \tag{2.3}$$

where

$$a_d = \frac{d}{2}\Gamma(\frac{d}{2}-1)\pi^{-d/2}.$$

We now define

$$G(z) = \sum_{x\in\mathbb{Z}^d} g(x-z)g(x).$$

We shall need a few facts about this function. First, for any $x, z \in \mathbb{Z}^d$ (see e.g. [3]),

$$\mathbb{P}(z \in \mathcal{T}_-^x) \lesssim G(z-x). \tag{2.4}$$

The next result gives the leading order term in the asymptotic behavior of G at infinity.

Lemma 2.3. *Assume $d \geq 5$. Then as $\|z\| \to \infty$,*

$$G(z) \sim \frac{c_d}{\|z\|^{d-4}},$$

with $c_d = \frac{d^2}{2(d-4)} \cdot \pi^{-d/2} \cdot \Gamma(\frac{d}{2}-1)$.

Proof. One has using (2.3) and rotational invariance, $G(z) \sim c_d \cdot \|z\|^{4-d}$, with

$$c_d = a_d^2 \cdot \int_{\mathbb{R}^d} \frac{1}{\|y-u\|^{d-2}} \cdot \frac{1}{\|y\|^{d-2}}\, dy,$$

for any u with $\|u\| = 1$. Note that by integrating over the unit sphere $\mathcal{S}(0,1)$, we find

$$c_d = \frac{a_d^2}{|\mathcal{S}(0,1)|} \cdot \int_{\mathbb{R}^d} \frac{1}{\|y\|^{d-2}} \left(\int_{\mathcal{S}(0,1)} \frac{1}{\|y-u\|^{d-2}}\, du \right) dy.$$

Using next that $z \mapsto \|y\|^{2-d}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$, we find that

$$\frac{1}{|\mathcal{S}(0,1)|} \int_{\mathcal{S}(0,1)} \frac{1}{\|y-u\|^{d-2}}\, du = \begin{cases} \|y\|^{2-d} & \text{if } \|y\| > 1 \\ 1 & \text{if } \|y\| < 1, \end{cases}$$

and a change of variables in polar coordinates then yields

$$c_d = a_d^2 \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\int_0^1 r\, dr + \int_1^\infty r^{3-d}\, dr \right) = a_d^2 \cdot \frac{2\pi^{d/2}}{\Gamma(d/2)} \cdot \frac{d-2}{2(d-4)},$$

which after simplifying gives the desired result. ∎

Finally one should need the following gradient bound.

Lemma 2.4. *Assume $d \geq 5$. One has for any $z, h \in \mathbb{Z}^d$, with $\|h\| \leq \|z\|/2$,*

$$G(z+h) = G(z) \cdot \Big(1 + \mathcal{O}(\frac{\|h\|}{\|z\|})\Big).$$

Proof. The result for the function g is already known, see e.g. [13, Theorem 4.3.1], even for all h satisfying $\|h\| \le \frac{2}{3}\|z\|$. Injecting this in the definition of G, we get for $\|h\| \le \|z\|/2$,

$$\begin{aligned}
G(z+h) &= \sum_{u\in\mathbb{Z}^d} g(z+h+u)g(u) = \sum_{v\in\mathbb{Z}^d} g(z-v)g(h+v) \\
&= \sum_{\|v\|\ge(3/2)\|h\|} g(z-v)g(v)\big(1+\mathcal{O}(\frac{\|h\|}{\|v\|})\big) + \sum_{\|v\|<(3/2)\|h\|} g(z-v)g(h+v) \\
&= G(z)\cdot\Big(1+\mathcal{O}(\frac{\|h\|}{\|z\|})\Big) + \sum_{\|v\|<(3/2)\|h\|} g(z-v)g(h+v) - \mathcal{O}(1)\cdot \sum_{\|v\|<(3/2)\|h\|} g(z-v)g(v) \\
&= G(z)\cdot\Big(1+\mathcal{O}(\frac{\|h\|}{\|z\|})\Big) + \mathcal{O}(\|h\|^2\cdot g(z)) = G(z)\cdot\Big(1+\mathcal{O}(\frac{\|h\|}{\|z\|})\Big).
\end{aligned}$$

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2.5 Variational characterization of the branching capacity

We state here a result from [3] that we shall use only in dimension 5 for proving the lower bound in Proposition 1.3. It shows that the branching capacity is of the same order as the inverse of an energy.

Theorem 2.5 ([3])**.** *Assume $d \ge 5$. There exist positive constants c and C, such that for any nonempty finite set $A \subset \mathbb{Z}^d$,*

$$\frac{c}{\mathrm{BCap}(A)} \le \inf\Big\{\sum_{x,y\in A} G(x-y)\,\nu(x)\nu(y) : \nu \text{ probability measure on } A\Big\} \le \frac{C}{\mathrm{BCap}(A)}.$$

In particular the inverse of the middle term in the above display could provide an alternative definition of the branching capacity, which would be more intrinsic, in that it would not depend on a particular choice of critical probability measure μ. However, it is not clear if with this definition, the law of large numbers would still hold in dimension 6; at least the proof given here would break completely.

2.6 Quantitative bounds on hitting probability

Our goal here is to prove some quantitative bounds, given a finite set $A \subset \mathbb{Z}^d$, on the speed of convergence toward $\mathrm{BCap}(A)$ of the probability that an infinite tree-indexed random walk starting from z hits A, as $\|z\| \to \infty$, when conveniently normalized. We only state the result in dimension 6 for convenience, as we shall only need it in this case, but analogous bounds could be proved in any dimension $d \ge 5$, with the same arguments.

Proposition 2.6. *Assume that μ has a finite third moment. There exists $C > 0$, such that for any finite set $A \subset \mathbb{Z}^6$, containing the origin, and any x, satisfying $\|x\| \ge 8\cdot \mathrm{diam}(A)$,*

$$\left|\frac{\mathbb{P}(\mathcal{T}_-^x \cap A \neq \emptyset)}{G(x)} - \frac{\sigma^2}{2}\cdot \mathrm{BCap}(A)\right| \le C\cdot\Big(\mathrm{BCap}(A)\cdot\frac{\mathrm{diam}(A)^{2/3}}{\|x\|^{2/3}} + |A|\cdot\frac{\mathrm{diam}(A)^{4/3}}{\|x\|^{4/3}}\Big).$$

We mention that since a first version of this paper appeared on arxiv, a result of the same flavor has been derived in [6] using another proof, as well as similar bounds for critical tree-indexed walks.

Proof. By Lemma 2.2 one has with the notation thereof,

$$\mathbb{P}(\mathcal{T}_-^x \cap A \neq \emptyset) = \sum_{y \in A} \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+(y,x)\Big].$$

If the two terms in the expectation above were independent, we would be done, because one can observe that (see e.g. [3]),

$$\mathbb{E}[\mathcal{L}_+(y,x)] = \frac{\sigma^2}{2} G(x-y) + \mathcal{O}(g(x-y)), \tag{2.5}$$

and thus the result would follow directly from Lemma 2.4. The problem is of course that they are not independent, thus our goal will be to decorrelate them as much as possible.

To this end, fix some $y \in A$, as well as a tree indexed walk starting from y, and define

$$\tau_r^y = \inf\big\{k \geq 0 : \widetilde{X}_k \in \partial B(0,r)\big\},$$

with $2\,\mathrm{diam}(A) \leq r \leq \|x\|/2$, to be fixed later, and where we recall that $\widetilde{X}$ refers to the walk indexed by the spine of the tree $\mathcal{T}$. For $0 \leq a \leq b \leq \infty$, we let $\mathcal{F}_+^y[a,b]$ and $\mathcal{F}_-^y[a,b]$ denote the forests of trees respectively in the future and the past of $\mathcal{T}^y$ hanging off the spine at vertices with intrinsic label between a and b. Then let

$$\mathcal{L}_+^1(y,x) = \sum_{u \in \mathcal{F}_+^y[0,\tau_r^y]} \mathbf{1}\{S_u^y = x\}, \quad \text{and} \quad \mathcal{L}_+^2(y,x) = \sum_{u \in \mathcal{F}_+^y[\tau_r^y+1,\infty)} \mathbf{1}\{S_u^y = x\}.$$

One has for any $y \in A$,

$$\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+(y,x)\Big] = \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+^1(y,x)\Big] + \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+^2(y,x)\Big]. \tag{2.6}$$

We upper bound the first term using Lemma 2.4 as follows

$$\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+^1(y,x)\Big] \leq \mathbb{E}\Big[\mathcal{L}_+^1(y,x)\Big] \lesssim g(x) \cdot \mathbb{E}[\tau_r^y] \lesssim g(x) \cdot r^2,$$

using for the last inequality the well-known fact that for a simple random walk, the expected time needed to reach $\partial B(0,r)$ is of order at most r^2. The second term in the right hand side of (2.6), which is the dominant part, will be evaluated using the independence of the forests before and after time τ_r^y, conditionally on the position of $\widetilde{X}$ at this time. More precisely, we first note that

$$\begin{aligned} &\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^y \cap A = \emptyset\} \cdot \mathcal{L}_+^2(y,x)\Big] \qquad (2.7)\\ =&\mathbb{E}\Big[\mathbf{1}\{\mathcal{F}_-^y[0,\tau_r^y] \cap A = \emptyset\} \cdot \mathcal{L}_+^2(y,x)\Big] - \mathbb{E}\Big[\mathbf{1}\{\mathcal{F}_-^y[0,\tau_r^y] \cap A = \emptyset, \mathcal{F}_-^y[\tau_r^y+1,\infty) \cap A \neq \emptyset\} \cdot \mathcal{L}_+^2(y,x)\Big]. \end{aligned}$$

Considering the first term on the right hand side, using (2.5) and Lemma 2.4, we already get

$$\mathbb{E}\Big[\mathbf{1}\{\mathcal{F}_-^y[0,\tau_r^y] \cap A = \emptyset\} \cdot \mathcal{L}_+^2(y,x)\Big] = \frac{\sigma^2}{2} G(x) \cdot \Big(1 + \mathcal{O}(\frac{r}{\|x\|})\Big) \cdot \mathbb{P}(\mathcal{F}_-^y[0,\tau_r^y] \cap A = \emptyset).$$

Moreover, by (2.1) one has (recall that the dimension is equal to 6 here),

$$\mathbb{P}(\mathcal{F}_-^y[\tau_r^y, \infty) \cap A \neq \emptyset) \lesssim \frac{\mathrm{BCap}(A)}{r^2} \lesssim \frac{\mathrm{diam}(A)^2}{r^2}.$$

In particular by choosing r large enough, one can always ensure that the probability on the left hand side is smaller than $1/2$. As a consequence,

$$\mathbb{P}(\mathcal{F}_-^y[0,\tau_r^y]\cap A=\emptyset)=e_A(y)\cdot\Big(1+\mathcal{O}(\frac{\operatorname{diam}(A)^2}{r^2})\Big). \tag{2.8}$$

Altogether this gives

$$\mathbb{E}\Big[\mathbf{1}\{\mathcal{F}_-^y[0,\tau_r^y]\cap A=\emptyset\}\cdot\mathcal{L}_+^2(y,x)\Big]=\frac{\sigma^2}{2}G(x)\cdot e_A(y)\cdot\Big\{1+\mathcal{O}\Big(\frac{r}{\|x\|}+\frac{\operatorname{diam}(A)^2}{r^2}\Big)\Big\}.$$

Now it remains to consider the second term in (2.7). By (2.8), one has

$$\mathbb{E}\Big[\mathbf{1}\{\mathcal{F}_-^y[0,\tau_r^y]\cap A=\emptyset,\mathcal{F}_-^y[\tau_r^y,\infty)\cap A\neq\emptyset\}\cdot\mathcal{L}_+^2(y,x)\Big]\lesssim e_A(y)\cdot\sup_{z\in\partial B(0,r)}\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^z\cap A\neq\emptyset\}\cdot\mathcal{L}_+(z,x)\Big]. \tag{2.9}$$

We let $r_0=2\operatorname{diam}(A)$, and first upper bound the time spent at x after the spine hits $\partial B(0,r_0)$ if it ever happens. Using some independence and (2.5), we get that for any $z\in\partial B(0,r)$,

$$\mathbb{E}\Big[\mathbf{1}\{\tau_{r_0}^z<\infty\}\cdot\mathcal{L}_+(\widetilde{X}_{\tau_{r_0}^z},x)\Big]\lesssim\frac{g(z)}{\overline{g}(r_0)}\cdot G(x), \tag{2.10}$$

with the notation $\overline{g}(s)=s^{-4}$, for $s>0$. It amounts next to upper bound the time spent at x in the future before the spine hits $\partial B(0,r_0)$ under the event that the past hits A. Denote by $\ell_c(u,x)$ the time spent at x by a walk indexed by a μ-Bienaymé-Galton-Watson tree with the root conditioned to have only one child, and starting from u. Let also $\ell_c(u,A)=\sum_{y\in A}\ell_c(u,y)$ be the time spent in A by this walk. Consider now $(\ell_c^{(i)}(u,x))_{i\in\mathbb{Z}}$ and $(\ell_c^{(i)}(u,A))_{i\in\mathbb{Z}}$ sequences of independent copies of these random variables. Finally let (d_-,d_+) be two random variables with joint distribution $\mathbb{P}(d_-=i,d_+=j)=\mu(i+j+1)$, representing the number of offspring respectively in the past and the future of a vertex on the spine, and set

$$\ell_c^-(u,x)=\sum_{i=1}^{d_-}\ell_c^{(-i)}(u,A),\quad\text{and}\quad\ell_c^+(u,x)=\sum_{i=1}^{d_+}\ell_c^{(i)}(u,x).$$

We upper bound,

$$\mathcal{L}_+(z,x)\le\mathbf{1}\{\tau_{r_0}^z<\infty\}\cdot\mathcal{L}_+(\widetilde{X}_{\tau_{r_0}^z},x)+\sum_{0\le n<\tau_{r_0}^z}\ell_c^+(\widetilde{X}_n,x), \tag{2.11}$$

where with a slight abuse of notation we assume that for different indices n and k, $\ell_c^+(\widetilde{X}_n,x)$ and $\ell_c^+(\widetilde{X}_k,x)$ are independent. Similarly, and using the same convention, a union bound gives

$$\mathbf{1}\{\mathcal{T}_-^z\cap A\neq\emptyset\}\le\mathbf{1}\{\tau_{r_0}^z<\infty\}+\sum_{1\le m<\tau_{r_0}^z}\mathbf{1}\{\ell_c^-(\widetilde{X}_m,A)>0\}. \tag{2.12}$$

Since μ has a finite third moment, one has $\mathbb{E}[d_-d_+]<\infty$, and using further that μ has mean one, we can see that $\mathbb{E}[\ell_c(v,x)]\lesssim g(x-v)$, for any $v\in\mathbb{Z}^6$. Together with (2.2) and a union bound, we deduce that for any $n,m<\tau_{r_0}^z$,

$$\mathbb{E}\Big[\mathbf{1}\{\ell_c^-(\widetilde{X}_m,A)>0\}\cdot\ell_c^+(\widetilde{X}_n,x)\mid\widetilde{X}\Big]\lesssim g(x-\widetilde{X}_n)\cdot\frac{\operatorname{BCap}(A)}{\|\widetilde{X}_m\|^4}.$$

On the other hand, applying the Markov property and standard estimates for hitting probabilities for the simple random walk, entail

$$\mathbb{E}\Big[\mathbf{1}\{n < \tau_{r_0}^z < \infty\} \cdot \ell_c^+(\widetilde{X}_n, x) \mid \widetilde{X}_0, \dots, \widetilde{X}_n\Big] \lesssim g(x - \widetilde{X}_n) \cdot \frac{g(\widetilde{X}_n)}{\overline{g}(r_0)}.$$

Using now (2.10), (2.11) and (2.12), together with the last two displays, and denoting by $\ell(u)$ the time spent at u by the walk $\widetilde{X}$, we get

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^z \cap A \neq \emptyset\} \cdot \mathcal{L}_+(z,x)\Big] \\
&\lesssim \mathbb{E}\Big[\Big(\sum_{1 \le m < \tau_{r_0}^z} \mathbf{1}\{\ell_c^-(\widetilde{X}_m, A) > 0\} + \mathbf{1}\{\tau_{r_0}^z < \infty\}\Big) \cdot \sum_{0 \le n < \tau_{r_0}^z} \ell_c^+(\widetilde{X}_n, x)\Big] + \frac{g(z)}{\overline{g}(r_0)} \cdot G(x) \\
&\lesssim \sum_{\|u\|, \|v\| > r_0} \mathbb{E}_z[\ell(u)\ell(v)] \cdot g(x-u) \cdot \frac{\mathrm{BCap}(A)}{\|v\|^4} + \sum_{\|u\| > r_0} \mathbb{E}_z[\ell(u)] \cdot g(x-u) \cdot \frac{g(u)}{\overline{g}(r_0)} + \frac{g(z)}{\overline{g}(r_0)} \cdot G(x).
\end{aligned}$$

On the other hand, the Markov property gives that

$$\mathbb{E}_z[\ell(u)\ell(v)] \lesssim g(u-v) \cdot (g(z-u) + g(z-v)).$$

and elementary computation then show that for any $z \in \partial B(0,r)$,

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^z \cap A \neq \emptyset\} \cdot \mathcal{L}_+(z,x)\Big] &\lesssim \Big(\frac{G(x)}{r^2} + \frac{1}{r_0 \cdot \|x\|^3}\Big) \cdot \mathrm{BCap}(A) + \frac{r_0^4}{r^4} \cdot G(x) \\
&\lesssim \mathrm{diam}(A)^2 \cdot \Big(\frac{G(x)}{r^2} + \frac{1}{r_0 \cdot \|x\|^3}\Big),
\end{aligned}$$

using that $\mathrm{BCap}(A) \lesssim \mathrm{diam}(A)^2$ for the last inequality. Injecting this in (2.9), summing over $y \in A$, and gathering all the estimates obtained so far yields

$$\frac{\mathbb{P}(\mathcal{T}_-^x \cap A \neq \emptyset)}{G(x)} = \frac{\sigma^2}{2} \cdot \mathrm{BCap}(A) \cdot \Big(1 + \mathcal{O}(\frac{r}{\|x\|} + \frac{\mathrm{diam}(A)^2}{r^2})\Big) + \mathcal{O}\Big(|A| \cdot \frac{r^2}{\|x\|^2}\Big).$$

Then taking $r = (\|x\| \cdot \mathrm{diam}(A)^2)^{1/3}$, concludes the proof of the proposition. ■

2.7 Lawler's identity and first consequences

We give here an analogous version for the branching capacity of a wonderful identity discovered by Lawler in the setting of Newtonian capacity, see e.g. [12], and which has also been used successfully by Bai and Wan when studying the capacity of a branching random walk in the recent work [4].

For $n \geq 1$, let ξ_n^l and ξ_n^r be two independent Geometric random variables with parameter $1/n$. Let $(X_k)_{k \in \mathbb{Z}}$ be a two-sided simple random walk starting from the origin at time 0. Then for $a \leq b \in \mathbb{Z}$, define $\mathcal{R}[a,b] = \{X_a, \dots, X_b\}$, and let

$$e_n := \mathbf{1}\big\{0 \notin \mathcal{R}[1, \xi_n^r]\big\}.$$

Now consider an infinite invariant tree $\mathcal{T}$ independent of the walk X, and let $\mathcal{A}_n$ be the event

$$\mathcal{A}_n = \big\{\mathcal{T}_-^0 \cap \mathcal{R}[-\xi_n^l, \xi_n^r] = \emptyset\big\}.$$

Write also, with the notation of Lemma 2.2,

$$\mathcal{L}_n = \sum_{j=-\xi_n^l}^{\xi_n^r} \mathcal{L}_+(0, X_j).$$

Lemma 2.7. *Assume $d \geq 5$. Then for any $n \geq 1$,*

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot \mathcal{L}_n\Big] = 1.$$

Proof. For $m \geq 0$, and a nearest neighbor path $(x_1, \dots, x_m)$, define the event

$$B(m, x_1, \dots, x_m) = \{\xi_n^l + \xi_n^r = m, X_{k-\xi_n^l} = X_{-\xi_n^l} + x_k, \text{ for } 1 \leq k \leq m\}.$$

Let also for $0 \leq j \leq m$,

$$B(m, j, x_1, \dots, x_m) = B(m, x_1, \dots, x_m) \cap \{\xi_n^l = j, \xi_n^r = m - j\}.$$

Note that all these events have the same probability, and thus for any $0 \leq j \leq m$,

$$\mathbb{P}\big(B(m, j, x_1, \dots, x_m)\big) = \frac{\mathbb{P}\big(B(m, x_1, \dots, x_m)\big)}{m+1}.$$

Thus one can write, with $x_0 = 0$,

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot \mathcal{L}_n\Big] &= \sum_{m=0}^{\infty} \sum_{(x_1,\dots,x_m)} \frac{\mathbb{P}(B(m, x_1, \dots, x_m))}{m+1} \sum_{j=0}^{m} \mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot \mathcal{L}_n \;\Big|\; B(m, j, x_1, \dots, x_m)\Big] \\
&= \sum_{m=0}^{\infty} \sum_{(x_1,\dots,x_m)} \frac{\mathbb{P}(B(m, x_1, \dots, x_m))}{m+1} \sum_{j=0}^{m} \mathbf{1}\{x_k \neq x_j, \text{ for all } k > j\} \\
&\qquad \times \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^{x_j} \cap \{0, x_1, \dots, x_m\} = \emptyset\} \cdot \big(\sum_{\ell=0}^{m} \mathcal{L}_+(x_j, x_\ell)\big)\Big] \\
&= \sum_{m=0}^{\infty} \sum_{(x_1,\dots,x_m)} \frac{\mathbb{P}(B(m, x_1, \dots, x_m))}{m+1} \sum_{\ell=0}^{m} \sum_{j=0}^{m} \mathbf{1}\{x_k \neq x_j, \text{ for all } k > j\} \cdot \\
&\qquad \times \mathbb{E}\Big[\mathbf{1}\{\mathcal{T}_-^{x_j} \cap \{0, x_1, \dots, x_m\} = \emptyset\} \cdot \mathcal{L}_+(x_j, x_\ell)\Big] \\
&= \sum_{m=0}^{\infty} \sum_{(x_1,\dots,x_m)} \mathbb{P}\big(B(m, x_1, \dots, x_m)\big) = 1,
\end{aligned}$$

using Lemma 2.2 for the penultimate equality. ■

We now provide some first consequences of this lemma. Define for $n \in \mathbb{N} \cup \{\infty\}$,

$$U_n = \sum_{j=-\xi_n^l}^{\xi_n^r} \sum_{i \geq 0} d_i \cdot g(X_j, \widetilde{X}_i), \quad \text{and} \quad Z_n = \sum_{j=-\xi_n^l}^{\xi_n^r} \sum_{i \geq 0} d_i \cdot \mathbf{1}\{X_j = \widetilde{X}_i\}, \tag{2.13}$$

with d_i the number of offspring in the future of the i-th vertex on the spine, and where we recall $(\widetilde{X}_i)_{i \geq 0}$ is the walk indexed by the spine of $\mathcal{T}$. Let also $\ell_n = \sum_{j=-\xi_n^l}^{0} \mathbf{1}\{X_j = 0\}$.

Corollary 2.8. *Assume $d \geq 5$. Then for all $n \geq 1$,*

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot (U_n + \ell_n - Z_n)\Big] = 1.$$

Moreover, if $d \geq 7$, then

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_\infty} \cdot e_\infty \cdot (U_\infty + \ell_\infty - Z_\infty)\Big] = 1.$$

Proof. The idea for the first identity is to start from the equation given by Lemma 2.7, and then condition with respect to the sigma-field

$$\mathcal{G}_n = \sigma\Big((d_i)_{i\geq 0}, (\widetilde{X}_i)_{i\geq 0}, (X_j)_{-\xi_n^l \leq j \leq \xi_n^r}\Big).$$

The main observation is that conditionally on $\mathcal{G}_n$, the random variables $\mathcal{L}_n$ and $\mathbf{1}_{\mathcal{A}_n} \cdot e_n$ are independent. Moreover, for each i, if we denote by $\mathcal{T}_i$ the tree hanging off the spine in the future at its i-th vertex, to which we remove the root, then for any $y \in \mathbb{Z}^d$,

$$\mathbb{E}\Big[\sum_{u\in\mathcal{T}_i} \mathbf{1}\{S_u = y\} \mid \mathcal{G}_n\Big] = d_i \cdot \big(g(\widetilde{X}_i, y) - \mathbf{1}\{\widetilde{X}_i = y\}\big),$$

since for a random walk indexed by a critical tree, starting from x, and conditioned by the fact that the root of the tree has exactly one offspring, the mean number of visits to a site y is equal to $g(x, y) - \mathbf{1}\{x = y\}$, if we do not count the starting point. Therefore, summing over $i \geq 0$, we get

$$\mathbb{E}\big[\mathcal{L}_n \mid \mathcal{G}_n\big] = \ell_n + \sum_{i=0}^{\infty} \sum_{-\xi_n^l \leq j \leq \xi_n^r} d_i \cdot \big(g(\widetilde{X}_i, X_j) - \mathbf{1}\{\widetilde{X}_i = X_j\}\big), \tag{2.14}$$

which proves already the first claim of the corollary.

For the second claim note that by definition almost surely the sequence $\mathbf{1}_{\mathcal{A}_n} \cdot e_n$ converges toward $\mathbf{1}_{\mathcal{A}_\infty} \cdot e_\infty$, while $(U_n + \ell_n - Z_n)_{n\geq 0}$ converges toward $U_\infty + \ell_\infty - Z_\infty$. Moreover, for each $n \geq 1$, one has $0 \leq \mathbf{1}_{\mathcal{A}_\infty} \cdot e_\infty \cdot (U_n + \ell_n - Z_n) \leq U_\infty + \ell_\infty$, and if $d \geq 7$, by (2.3) and Lemma 2.3,

$$\mathbb{E}[\ell_\infty + U_\infty] \lesssim g(0) + \sum_{u,v\in\mathbb{Z}^d} g(u - v)g(u)g(v) = g(0) + \sum_{u\in\mathbb{Z}^d} G(u)g(u) < \infty.$$

Thus the second claim follows from the first one and the dominated convergence theorem. ∎

3 Proof of Theorem 1.2

We assume in the whole section that $d = 6$, and that μ has a finite third moment.

3.1 Concentration of the variable U_n

We just state here our main estimates concerning the mean and variance of the variable U_n. The proof is postponed to Section 3.6, as it is a bit long and technical.

Proposition 3.1. *One has as* $n \to \infty$,

$$\mathbb{E}[U_n] \sim \frac{27\sigma^2}{2\pi^3} \cdot \log n, \quad \textit{and} \quad \mathrm{Var}(U_n) = \mathcal{O}(\log n).$$

3.2 Rough bounds on the probability of the event $\mathcal{A}_n$

We prove here some rough upper bound on the probability of the event $\mathcal{A}_n$, as well as on the event

$$\mathcal{B}_n = \big\{\mathcal{T}_-^0 \cap \mathcal{R}[0, \xi_n^r] = \emptyset\big\}.$$

Lemma 3.2. *One has*

$$\mathbb{P}(\mathcal{A}_n) \lesssim \frac{1}{\log n}, \quad \text{and} \quad \mathbb{P}(\mathcal{B}_n) \lesssim \frac{1}{\sqrt{\log n}}.$$

Proof. Recall the definitions of U_n and Z_n given in (2.13), and let

$$\mathcal{E}_n = \Big\{ |(U_n - Z_n) - \mathbb{E}[(U_n - Z_n)]| \ge \frac{1}{2}\mathbb{E}[U_n] \Big\}.$$

Note that in any dimension $d \ge 5$, and using that μ has a finite third moment,

$$\begin{aligned}\mathbb{E}[(Z_\infty)^2] &= \sum_{x,y\in\mathbb{Z}^d} \sum_{\substack{i_1,i_2\ge 0\\ j_1,j_2\in\mathbb{Z}}} \mathbb{E}[d_{i_1}d_{i_2}] \cdot \mathbb{P}(\widetilde{X}_{i_1} = x, \widetilde{X}_{i_2} = y) \cdot \mathbb{P}(X_{j_1} = x, X_{j_2} = y) \\ &\lesssim \sum_{x,y\in\mathbb{Z}^d} g(x)^2 g(y-x)^2 \lesssim 1, \end{aligned} \tag{3.1}$$

and hence by Proposition 3.1 and Chebyshev's inequality,

$$\mathbb{P}(\mathcal{E}_n) \lesssim \frac{1}{\log n}.$$

Then using in addition Corollary 2.8, we get

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n\big] \lesssim \mathbb{P}(\mathcal{E}_n) + \mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot \mathbf{1}_{\mathcal{E}_n^c}\big] \lesssim \frac{1}{\log n} + \frac{\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n \cdot (U_n - Z_n)\big]}{\mathbb{E}[U_n]} \lesssim \frac{1}{\log n}. \tag{3.2}$$

We want now to remove e_n from the expectation in the left-hand side. Denote by σ the last visiting time of the origin by the walk $(X_k)_{k\ge 0}$. Let

$$\mathcal{A}_n^\sigma = \big\{\mathcal{T}_-^0 \cap \big(\mathcal{R}[-\xi^l_{\sqrt{n}}, 0] \cup \mathcal{R}[\sigma, \sigma + \xi^r_{\sqrt{n}}]\big) = \emptyset\big\}.$$

Since the law of the walk X after time σ, is the law of a walk conditioned on not returning to the origin after time 0, one has

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n) &\le \mathbb{P}(\mathcal{A}_n^\sigma) + \mathbb{P}(\sigma + \xi^r_{\sqrt{n}} > \xi^r_n) \lesssim \mathbb{E}\big[\mathbf{1}_{\mathcal{A}_{\sqrt{n}}} \cdot e_{\sqrt{n}}\big] + \mathbb{P}(\sigma + \xi^r_{\sqrt{n}} > \xi^r_n) \\ &\overset{(3.2)}{\lesssim} \frac{1}{\log n} + \mathbb{P}(\sigma + \xi^r_{\sqrt{n}} > \xi^r_n) \lesssim \frac{1}{\log n},\end{aligned}$$

where the last bound follows from basic estimates. Indeed on one hand,

$$\mathbb{P}(\sigma \ge \sqrt{n}) \le \sum_{k\ge\sqrt{n}} \mathbb{P}(X_k = 0) \lesssim \sum_{k\ge\sqrt{n}} k^{-3} \lesssim \frac{1}{n},$$

and on the other hand, by standard properties of geometric random variables,

$$\mathbb{P}(\xi^r_{\sqrt{n}} \ge \xi^r_n - \sqrt{n}) \le \mathbb{P}(\xi^r_{\sqrt{n}} \ge n^{3/4}) + \mathbb{P}(\xi^r_n \le 2n^{3/4}) \lesssim n^{-1/4}.$$

Thus so far we have proved the first inequality of the corollary. The second one follows by Cauchy-Schwarz inequality. Indeed, using also the independence between $\mathcal{R}[0, \xi^r_n]$ and $\mathcal{R}[-\xi^l_n, 0]$, one deduces that

$$\mathbb{P}(\mathcal{B}_n)^2 \le \mathbb{E}[\mathbb{P}(\mathcal{B}_n \mid \mathcal{T}_-^0)^2] = \mathbb{P}(\mathcal{A}_n).$$

■

3.3 Probability estimates of some non-intersection events

Our main goal in this section is to prove estimates on some non-intersection events, which are simple consequences of Lemma 3.2. Denote by ξ_n a Geometric random variable with parameter $1/n$, independent of everything else, and for $\varepsilon > 0$, we denote by x_ε the hitting point of $\partial B(0, 1/\varepsilon)$ by the walk $\widetilde{X}$ indexed by the spine of $\mathcal{T}$, starting from the origin. We start with the following estimate.

Lemma 3.3. *For every $\varepsilon \in (0,1)$, there exists a constant $C(\varepsilon) > 0$, such that for all $n \geq 2$,*

$$\mathbb{P}\Big(\mathcal{T}_-^{x_\varepsilon} \cap \mathcal{R}[0, \xi_n] = \emptyset\Big) \leq \frac{C(\varepsilon)}{\sqrt{\log n}}.$$

Proof. Define for $\varepsilon \in (0,1)$,

$$\widetilde{\tau}_\varepsilon = \inf\{k \geq 0 : \widetilde{X}_k \in \partial B(0, 1/\varepsilon)\},$$

where we recall that $\widetilde{X}$ is the random walk indexed by the spine of $\mathcal{T}^0$. In particular, $x_\varepsilon = \widetilde{X}_{\widetilde{\tau}_\varepsilon}$, by definition. Now we let $\mathcal{D}_\varepsilon$ be the event that the path $\widetilde{X}$ up to time $\widetilde{\tau}_\varepsilon$ avoids $\mathcal{R}[0, \xi_n]$ and that none of the vertices on the spine up to time $\widetilde{\tau}_\varepsilon$ has any normal offspring. Note that there exists a constant $c(\varepsilon) > 0$, such that for any $x \in \partial B(0, 1/\varepsilon)$ and any path γ starting from the origin, for which

$$\mathbb{P}(\widetilde{X}_{\widetilde{\tau}_\varepsilon} = x, \mathcal{R}[0, \xi_n] = \gamma, \mathcal{T}_-^0 \cap \mathcal{R}[0, \xi_n] = \emptyset) > 0,$$

one also has

$$\mathbb{P}(\mathcal{D}_\varepsilon, \widetilde{X}_{\widetilde{\tau}_\varepsilon} = x, \mathcal{R}[0, \xi_n] = \gamma) \geq c(\varepsilon) \cdot \mathbb{P}(\widetilde{X}_{\widetilde{\tau}_\varepsilon} = x, \mathcal{R}[0, \xi_n] = \gamma),$$

since in particular in this case x cannot be disconnected from the origin within $B(0, 1/\varepsilon)$ by the path γ. Then one has

$$\begin{aligned}
\mathbb{P}(\mathcal{T}_-^0 \cap \mathcal{R}[0, \xi_n] = \emptyset) &\geq \mathbb{P}(\mathcal{T}_-^{x_\varepsilon} \cap \mathcal{R}[0, \xi_n] = \emptyset, \mathcal{D}_\varepsilon) \\
&= \sum_{x \in \partial B(0,1/\varepsilon)} \sum_\gamma \mathbb{P}(\mathcal{T}_-^x \cap \mathcal{R}[0, \xi_n] = \emptyset, \widetilde{X}_{\widetilde{\tau}_\varepsilon} = x, \mathcal{R}[0, \xi_n] = \gamma, \mathcal{D}_\varepsilon) \\
&\geq c(\varepsilon) \cdot \sum_{x \in \partial B(0,1/\varepsilon)} \sum_\gamma \mathbb{P}(\mathcal{T}_-^x \cap \mathcal{R}[0, \xi_n] = \emptyset, \widetilde{X}_{\widetilde{\tau}_\varepsilon} = x, \mathcal{R}[0, \xi_n] = \gamma) \\
&= c(\varepsilon) \cdot \mathbb{P}(\mathcal{T}_-^{x_\varepsilon} \cap \mathcal{R}[0, \xi_n] = \emptyset),
\end{aligned}$$

using for the last equality the fact that after time $\widetilde{\tau}_\varepsilon$ the tree-indexed walk is independent of what it has done before this time. Then we conclude the proof using Lemma 3.2. ∎

We prove now a second estimate. Recall the definitions of $\widetilde{\tau}_\varepsilon$ given in the proof of the previous lemma. Then denote by $\mathcal{F}_-^0[0, \widetilde{\tau}_\varepsilon]$ the forest consisting of all the subtrees in the past of $\mathcal{T}^0$ hanging of the spine from vertices with intrinsic label between 0 and $\widetilde{\tau}_\varepsilon$.

Lemma 3.4. *There exists a constant $C > 0$, such that for every $\varepsilon \in (0,1)$,*

$$\limsup_{n \to \infty} \mathbb{P}(\mathcal{F}_-^0[0, \widetilde{\tau}_\varepsilon] \cap \mathcal{R}[0, \xi_n] = \emptyset) \leq \frac{C}{\sqrt{\log(1/\varepsilon)}}.$$

Proof. Let ξ_ε be a Geometric random variable with parameter ε, independent of everything else. Note that for any $n \geq \varepsilon^{-3}$,

$$\mathbb{P}(\xi_\varepsilon > \xi_n) \leq \mathbb{P}(\xi_\varepsilon \geq \varepsilon^{-2}) + \mathbb{P}(\xi_n \leq \varepsilon^{-2}) \lesssim \varepsilon,$$

and thus one can always replace ξ_n by ξ_ε in the statement of the proposition. Moreover, for any $x \in \partial B(0, 1/\varepsilon)$,

$$\mathbb{P}(\mathcal{T}_-^x \cap \mathcal{R}[0,\xi_\varepsilon] \neq \emptyset) \le \mathbb{E}\Big[|\mathcal{T}_-^x \cap \mathcal{R}[0,\xi_\varepsilon]|\Big] \lesssim \sum_{u\in\mathbb{Z}^6} G(x-u)\cdot \mathbb{P}(u \in \mathcal{R}[0,\xi_\varepsilon]) \lesssim \varepsilon.$$

Therefore, for n large enough,

$$\mathbb{P}(\mathcal{F}_-^0[0,\widetilde{\tau}_\varepsilon] \cap \mathcal{R}[0,\xi_n] = \emptyset) \lesssim \mathbb{P}(\mathcal{T}_-^0 \cap \mathcal{R}[0,\xi_\varepsilon] = \emptyset) + \varepsilon,$$

and then the result follows from Lemma 3.2. ■

3.4 Asymptotic of the mean

Here we compute the leading order term in the asymptotic of the expectation of the branching capacity of the range.

Proposition 3.5. *One has*

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\big] \sim \frac{2\pi^3}{27\sigma^2}\cdot\frac{1}{\log n},$$

and

$$\mathbb{E}\left[\mathrm{BCap}(\mathcal{R}_n)\right] \sim \frac{2\pi^3}{27\sigma^2}\cdot\frac{n}{\log n}.$$

Proof. Let us start with the first claim of the proposition. By Lemma 3.2 and (3.1), one has using also Cauchy-Schwarz inequality,

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\cdot Z_n] \le \mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot e_n]^{1/2}\cdot\mathbb{E}[Z_\infty^2]^{1/2} \lesssim \frac{1}{\sqrt{\log n}}.$$

Therefore Corollary 2.8 gives

$$\mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\cdot U_n\Big] = 1 + \mathcal{O}\big(\frac{1}{\sqrt{\log n}}\big). \tag{3.3}$$

Then letting $\overline{U}_n = U_n - \mathbb{E}[U_n]$, and using Proposition 3.1, we get

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\big] = \frac{1}{\mathbb{E}[U_n]} - \frac{\mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\cdot\overline{U}_n]}{\mathbb{E}[U_n]} + \mathcal{O}\big(\frac{1}{(\log n)^{3/2}}\big),$$

and it amounts now to bound the second term on the right hand side. For $\varepsilon > 0$, let

$$Y_\varepsilon = \mathbf{1}\{|U_n - \mathbb{E}[U_n]| > \varepsilon\cdot\mathbb{E}[U_n]\}.$$

One has

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot e_n\cdot|\overline{U}_n|] \le \varepsilon\cdot\mathbb{E}[U_n]\cdot\mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot e_n] + \mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot Y_\varepsilon\cdot|\overline{U}_n|],$$

and again it suffices to bound the second term on the right-hand side. By Cauchy-Schwarz inequality and Proposition 3.1, we have

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot Y_\varepsilon\cdot|\overline{U}_n|] \lesssim \mathbb{E}[\mathbf{1}_{\mathcal{A}_n}\cdot Y_\varepsilon]^{1/2}\cdot\sqrt{\log n}.$$

Now define

$$U_n^+ = \sum_{j=0}^{\xi_n^r} \sum_{i\geq 0} d_i \cdot g(X_j, \widetilde{X}_i), \quad \text{and} \quad U_n^- = \sum_{j=-\xi_n^\ell}^{-1} \sum_{i\geq 0} d_i \cdot g(X_j, \widetilde{X}_i).$$

Let also

$$Y_\varepsilon^+ = \mathbf{1}\{|U_n^+ - \mathbb{E}[U_n^+]| > \varepsilon \cdot \mathbb{E}[U_n^+]\}, \quad \text{and} \quad Y_\varepsilon^- = \mathbf{1}\{|U_n^- - \mathbb{E}[U_n^-]| > \varepsilon \cdot \mathbb{E}[U_n^-]\}.$$

One has

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_{2\varepsilon}] \leq \mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^+] + \mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^-],$$

and it suffices to bound the term $\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^+]$, since the other term can be handled by a similar argument. We further decompose it into two terms as follows. Recall the definition of $\widetilde{\tau}_\varepsilon$ from the proof of Lemma 3.3, and let

$$U_n^\varepsilon = \sum_{j=0}^{\xi_n^r} \sum_{i=0}^{\widetilde{\tau}_\varepsilon} d_i \cdot g(X_j, \widetilde{X}_i), \quad \text{and} \quad \widetilde{U}_n^\varepsilon = \sum_{j=0}^{\xi_n^r} \sum_{i=\widetilde{\tau}_\varepsilon+1}^{\infty} d_i \cdot g(X_j, \widetilde{X}_i).$$

Now set $\varepsilon' = \exp(-\varepsilon^{-6})$, and define

$$Y_\varepsilon^1 = \mathbf{1}\Big\{\big|U_n^{\varepsilon'} - \mathbb{E}[U_n^{\varepsilon'}]\big| > \frac{\varepsilon}{2} \cdot \mathbb{E}[U_n^+]\Big\}, \quad \text{and} \quad Y_\varepsilon^2 = \mathbf{1}\Big\{\big|\widetilde{U}_n^{\varepsilon'} - \mathbb{E}[\widetilde{U}_n^{\varepsilon'}]\big| > \frac{\varepsilon}{2} \cdot \mathbb{E}[U_n^+]\Big\}.$$

One has

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^+] \leq \mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^1] + \mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^2].$$

Moreover, letting $y_\varepsilon = \widetilde{X}_{\widetilde{\tau}_{\varepsilon'/4}}$, we can write using independence,

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^1] &\leq \mathbb{E}\Big[Y_\varepsilon^1 \cdot \mathbf{1}\big\{\mathcal{T}_-^{y_\varepsilon} \cap \mathcal{R}[-\xi_n^l, 0] = \emptyset\big\}\Big] \\
&= \sum_{x\in\partial B(0,1/\varepsilon')} \sum_{y\in\partial B(0,4/\varepsilon')} \mathbb{E}[Y_\varepsilon^1 \cdot \mathbf{1}\{\widetilde{X}_{\widetilde{\tau}_{\varepsilon'}} = x\}] \cdot \mathbb{P}_x(\widetilde{X}_{\widetilde{\tau}_{\varepsilon'/4}} = y) \cdot \mathbb{P}(\mathcal{T}_-^y \cap \mathcal{R}[-\xi_n^l, 0] = \emptyset) \\
&\lesssim \sum_{x\in\partial B(0,1/\varepsilon')} \sum_{y\in\partial B(0,4/\varepsilon')} \mathbb{E}[Y_\varepsilon^1 \cdot \mathbf{1}\{\widetilde{X}_{\widetilde{\tau}_{\varepsilon'}} = x\}] \cdot \mathbb{P}(\widetilde{X}_{\widetilde{\tau}_{\varepsilon'/4}} = y) \cdot \mathbb{P}(\mathcal{T}_-^y \cap \mathcal{R}[-\xi_n^l, 0] = \emptyset) \\
&= \mathbb{E}[Y_\varepsilon^1] \cdot \mathbb{P}(\mathcal{T}_-^{y_\varepsilon} \cap \mathcal{R}[-\xi_n^l, 0] = \emptyset),
\end{aligned}$$

using also Harnack's inequality at the third line, see e.g. [13, Lemma 6.3.7]. Now the same argument as the one used for proving Proposition 3.1, see Remark 3.8, shows that $\mathbb{E}[Y_\varepsilon^1] \lesssim \frac{1}{\varepsilon^2 \log n}$. Using additionally Lemma 3.3, then yields

$$\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^1] \lesssim \frac{C(\varepsilon'/4)}{\varepsilon^2 (\log n)^{3/2}}.$$

Similarly one has, with the notation from Lemma 3.4, and using this result,

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\mathcal{A}_n} \cdot Y_\varepsilon^2] &\leq \mathbb{E}\Big[Y_\varepsilon^2 \cdot \mathbf{1}\{\mathcal{F}_-^0[0, \widetilde{\tau}_{4\varepsilon'}] \cap \mathcal{R}[-\xi_n^\ell, 0] = \emptyset\}\Big] \\
&\lesssim \mathbb{E}[Y_\varepsilon^2] \cdot \mathbb{P}\big(\mathcal{F}_-^0[0, \widetilde{\tau}_{4\varepsilon'}] \cap \mathcal{R}[-\xi_n^\ell, 0] = \emptyset\big) \\
&\lesssim \frac{1}{\sqrt{\log(1/\varepsilon')}} \cdot \frac{1}{\varepsilon^2 \log n} = \frac{\varepsilon}{\log n}.
\end{aligned}$$

Since this holds for all $\varepsilon \in (0,1)$, combining all the previous estimates proves that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n} \cdot e_n\big] = \frac{1}{\mathbb{E}[U_n]} + o\big(\frac{1}{\log n}\big),$$

concluding the proof of the first part of the proposition, thanks again to Proposition 3.1.

For the second part we use first that

$$\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] = \sum_{k=0}^{n} \mathbb{P}\big(\mathcal{T}_-^0 \cap \mathcal{R}[-k, n-k] = \emptyset, 0 \notin \mathcal{R}[1, n-k]\big).$$

Then the lower bound follows from the first part, by writing

$$\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] \geq n \cdot \mathbb{P}\big(\mathcal{T}_-^0 \cap \mathcal{R}[-n, n] = \emptyset, 0 \notin \mathcal{R}[1, n]\big) \geq n \cdot \mathbb{E}[\mathbf{1}_{\mathcal{A}_{n(\log n)^2}} \cdot e_{n(\log n)^2}] - o\big(\frac{n}{\log n}\big),$$

using for the last inequality that for a Geometric random variable ξ with parameter p, one has $\mathbb{P}(\xi \leq n) \leq p(n+1)$. For the upper bound we write similarly, with $n' = n/(\log n)^2$,

$$\begin{aligned}\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] &\leq 2n' + (n - 2n') \cdot \mathbb{P}\big(\mathcal{T}_-^0 \cap \mathcal{R}[-n', n'] = \emptyset, 0 \notin \mathcal{R}[1, n']\big) \\ &\leq 2n' + (n - 2n') \cdot \mathbb{E}[\mathbf{1}_{\mathcal{A}_{n/(\log n)^4}} \cdot e_{n/(\log n)^4}] - o\big(\frac{n}{\log n}\big),\end{aligned}$$

using this time that for a Geometric random variable ξ with parameter p, one has $\mathbb{P}(\xi > n') \leq (1-p)^{n'}$. This concludes the proof of the proposition. ■

3.5 Conclusion

We may now conclude the proof of the weak law of large numbers in dimension 6.

Proof of Theorem 1.2. To conclude the proof it suffices to show that

$$\mathrm{Var}(\mathrm{BCap}(\mathcal{R}_n)) = o(\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)]^2). \tag{3.4}$$

For this one can follow verbatim the proof of Chang [8], which we briefly recall for reader's convenience. Note first that by Proposition 2.6, one has for any z with $\|z\| \geq 8n$,

$$\frac{\sigma^2}{2} \cdot \mathrm{BCap}(\mathcal{R}_n) = \frac{\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_n \neq \emptyset \mid \mathcal{R}_n)}{G(z)} + \mathcal{O}\big(\frac{1}{n}\big).$$

As a consequence,

$$\frac{\sigma^2}{2} \cdot \mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] = \frac{\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_n \neq \emptyset)}{G(z)} + \mathcal{O}\big(\frac{1}{n}\big), \tag{3.5}$$

and with $\widetilde{\mathcal{T}}_-^z$ an independent copy of $\mathcal{T}_-^z$,

$$\frac{\sigma^4}{4} \cdot \mathbb{E}\big[\mathrm{BCap}(\mathcal{R}_n)^2\big] = \frac{\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_n \neq \emptyset, \widetilde{\mathcal{T}}_-^z \cap \mathcal{R}_n \neq \emptyset)}{G(z)^2} + \mathcal{O}(1), \tag{3.6}$$

where we use $\mathrm{BCap}(\mathcal{R}_n) \leq n+1$ to show that the error term is well $\mathcal{O}(1)$. Next, we define

$$\tau_1 = \inf\{k : X_k \in \mathcal{T}_-^z\}, \quad \text{and} \quad \tau_2 = \inf\{k : X_k \in \widetilde{\mathcal{T}}_-^z\}.$$

One has by symmetry,

$$\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_n \neq \emptyset, \widetilde{\mathcal{T}}_-^z \cap \mathcal{R}_n \neq \emptyset) \leq 2\mathbb{P}(\tau_1 \leq \tau_2 \leq n). \tag{3.7}$$

Then by using the Markov property for the walk X, we get

$$\mathbb{P}(\tau_1 \leq \tau_2 \leq n) = \mathbb{E}\left[\mathbf{1}\{\tau_1 \leq n\} \cdot \mathbb{P}_{X_{\tau_1}}(\widetilde{\mathcal{T}}_-^z \cap \mathcal{R}[0, n-\tau_1] \neq \emptyset \mid \tau_1)\right]. \tag{3.8}$$

Letting $k(n) = (\sigma^2/2) \cdot \mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)]$, one has using also Lemma 2.4 and (3.5),

$$\mathbb{P}_{X_{\tau_1}}(\widetilde{\mathcal{T}}_-^z \cap \mathcal{R}[0, n-\tau_1] \neq \emptyset \mid \tau_1) = G(z) \cdot k(n-\tau_1) + \mathcal{O}\big(\frac{G(z)}{n}\big).$$

Injecting this in (3.8) and using (2.1) gives

$$\mathbb{P}(\tau_1 \leq \tau_2 \leq n) = G(z) \cdot \mathbb{E}\left[\mathbf{1}\{\tau_1 \leq n\} \cdot k(n-\tau_1)\right] + \mathcal{O}\big(G(z)^2\big). \tag{3.9}$$

The final step is to show that conditionally on the event $\{\tau_1 \leq n\}$, the random variable τ_1/n converges in law to a uniform random variable in $[0,1]$, as $n \to \infty$ (uniformly in z with $\|z\| \geq n^3$). For this one can write using (3.5) and Proposition 3.5, that for any $s \in (0,1)$,

$$\mathbb{P}(\tau_1 \leq ns \mid \tau_1 \leq n) = \frac{\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_{ns} \neq \emptyset)}{\mathbb{P}(\mathcal{T}_-^z \cap \mathcal{R}_n \neq \emptyset)} = \frac{k(ns) + \mathcal{O}(1)}{k(n) + \mathcal{O}(1)} = s \cdot (1 + o(1)).$$

Using this and (3.9), as well as (3.5) and Proposition 3.5, yields

$$\mathbb{P}(\tau_1 \leq \tau_2 \leq n) = G(z) \cdot \mathbb{E}\left[\mathbf{1}\{\tau_1 \leq n\} \cdot k(n-\tau_1)\right] + \mathcal{O}\big(G(z)^2\big) = G(z)^2 \cdot k(n)^2 \cdot (1+o(1)) + \mathcal{O}(G(z)^2),$$

and plugging this into (3.7) and (3.6) concludes the proof of (3.4), and thus the proof of Theorem 1.2 as well. ■

Remark 3.6 (Sketch of proof of the strong law of large numbers)**.** We now briefly explain how one can strengthen the weak law into a strong law of large numbers. The main point is to obtain a quantitative bound on the second order term in the asymptotic expansion of the expected branching capacity of the range. More precisely one needs a bound of the form

$$\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] = \frac{2\pi^3}{27\sigma^2} \cdot \frac{n}{\log n} + \mathcal{O}\big(\frac{n}{(\log n)^{1+\delta}}\big), \tag{3.10}$$

for some $\delta > 0$. Indeed, once this is obtained, then a careful look at the previous proof above reveals that this would yield a better bound on the variance, namely

$$\frac{\mathrm{Var}(\mathrm{BCap}(\mathcal{R}_n))}{\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)]^2} = \mathcal{O}\big(\frac{1}{(\log n)^\delta}\big).$$

In turn, once such bound is known, then one can follow exactly the same proof as in [2] to deduce almost sure convergence. Roughly, using a dyadic decomposition scheme, one can express the branching capacity of the range as a sum of independent and (almost) identically distributed terms, plus a sum of so-called crossed terms, whose mean is controlled. Hence, one has for any $L \geq 1$ a decomposition of the form

$$\mathrm{BCap}(\mathcal{R}_n) = \sum_{i=0}^{2^L-1} \mathrm{BCap}(\mathcal{R}_n^{(i,L)}) + \sum_{\ell=1}^{L}\sum_{j=0}^{2^{\ell-2}} \chi(\mathcal{R}_n^{(2j,\ell)}, \mathcal{R}_n^{(2j+1,\ell)}),$$

where $\chi(A,B)$ is defined in the introduction, and $\mathcal{R}_n^{(j,\ell)} = \mathcal{R}[j\frac{n}{2^\ell},(j+1)\frac{n}{2^\ell}]$. Here, as we take L of order $\log\log n$, the main contribution comes from the first sum, the second sum is shown to have a small mean, thanks to (3.10). As a consequence one can deduce almost sure convergence of $\frac{\log n}{n}\cdot \mathrm{BCap}(\mathcal{R}_n)$ along a subsequence of the form $a_n = \exp(n^{1-\delta'})$, for some $\delta'\in(0,1)$, just using Chebyshev's inequality and the Borel-Cantelli lemma. Finally using that a_{n+1}/a_n converges to one, and monotonicity of the branching capacity, one easily extends this convergence along a subsequence into an almost sure convergence for the initial sequence.

Thus the whole proof boils down to proving (3.10), for some $\delta>0$. For this one can follow roughly the same strategy as in the proof of Proposition 3.5, but with a different truncation of the variable U_n^+. In fact reproducing the same first steps, one can see that the main problem is to prove a bound of the form

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^+\big] = \mathcal{O}\big(\frac{1}{(\log n)^\delta}\big), \tag{3.11}$$

where $Y_n^+ = \mathbf{1}\big\{|U_n^+ - \mathbb{E}[U_n^+]| > (\log n)^{\frac{9}{10}}\big\}$. To this end, fix some $L\geq 0$ and define

$$V_n^L = \sum_{j=0}^{\xi_n^r}\sum_{i=0}^{\tau_L} d_i\cdot g(X_j,\widetilde X_i), \quad \text{and} \quad W_n^L = \sum_{j=0}^{\xi_n^r}\sum_{i=\tau_L}^{\infty} d_i\cdot g(X_j,\widetilde X_i),$$

where

$$\tau_L = \inf\{k\geq 0 : \widetilde X_k \in \partial B(0,2^L)\}.$$

Write also $Y_n^1 = \mathbf{1}\{|V_n^L - \mathbb{E}[V_n^L]| > \frac12 (\log n)^{\frac{9}{10}}\}$ and $Y_n^2 = \mathbf{1}\{|W_n^L - \mathbb{E}[W_n^L]| > \frac12(\log n)^{\frac{9}{10}}\}$, so that

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^+\big] \leq \mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^1\big] + \mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^2\big].$$

Now a similar proof as the one of Proposition 3.1 can show that

$$\mathrm{Var}(V_n^L)\lesssim L, \quad \text{and} \quad \mathrm{Var}(W_n^L)\lesssim \log n,$$

and thus $\mathbb{E}[Y_n^1]\lesssim L/(\log n)^{\frac{18}{10}}$, while $\mathbb{E}[Y_n^2]\lesssim (\log n)^{-4/5}$. Therefore a similar argument as in the original proof can show that if $L=\sqrt{\log n}$, then

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^2\big] \leq \mathbb{E}[\mathbf{1}\big\{\mathcal{F}_-^0[0,\tau_{\sqrt{\log n}}]\cap \mathcal{R}[-\xi_n^l,0]=\emptyset\big\}\cdot Y_n^2\big] \lesssim (\log n)^{-\frac{21}{20}},$$

and on the other hand, one can simply write

$$\mathbb{E}\big[\mathbf{1}_{\mathcal{A}_n}\cdot Y_n^1\big] \leq \mathbb{P}(\mathcal{A}_n)^{1/2}\cdot \mathbb{E}\big[Y_n^1\big]^{1/2} \lesssim (\log n)^{-\frac{23}{20}},$$

which give (3.11) as wanted. Actually the last step is to show that one also has

$$\mathbb{E}[U_n] = \frac{27\sigma^2}{2\pi^3}\cdot \log n + \mathcal{O}(\sqrt{\log n}),$$

but this is more routine (yet slightly tedious) computation. For this one can follow the same argument as the one given in the next section, and use a finer asymptotic of the function G, itself following from finer asymptotic of the function g, which is well-known, see e.g. [13, Theorem 4.3.1].

3.6 Proof of Proposition 3.1

We start by a preliminary result. Let

$$G_n = \sum_{-\xi_n^l \leq j \leq \xi_n^r} G(X_j). \tag{3.12}$$

Lemma 3.7. *One has*

$$\mathbb{E}[G_n] \sim \frac{27}{\pi^3} \cdot \log n, \quad \text{and} \quad \mathrm{Var}(G_n) \lesssim \log n.$$

Proof. Let $\lambda = (1 - 1/n)$. One has by (2.3) and Lemma 2.3,

$$\begin{aligned}\mathbb{E}[G_n] &= G(0) + 2\sum_{j=1}^{\infty} \lambda^j \cdot \mathbb{E}[G(X_j)] \sim 2\sum_{j=1}^{n} \mathbb{E}[G(X_j)] \sim 2 \sum_{\|u\| \leq \sqrt{n}} G(u)g(u) \sim 2c_6 a_6 \pi^3 \int_1^{\sqrt{n}} \frac{1}{r}\, dr \\ &= c_6 a_6 \pi^3 \log n = 27\pi^{-3} \log n,\end{aligned}$$

and we now deal with the variance. For this, unfortunately it does not seem possible to use an explicit computation as it was done in dimension four by Lawler, see the proof of [12, Proposition 3.4.1], since the function G is no longer harmonic when $d = 6$. However, the heuristic argument given there still holds, and we shall use it here. More precisely, the idea is that parts of the trajectories of X between times 2^i and 2^{i+1} are almost independent for different i's. In order to formalize it, we introduce some more notation. First notice that by symmetry it suffices to bound the variance of $G_n^+ = \sum_{k=0}^{\xi_n} G(X_k)$, where ξ_n is a Geometric random variable with parameter $1/n$, independent of the walk X. Then for $\ell \geq 0$, define $B_\ell = B(0, 2^{\ell-1})$, and

$$\tau_\ell = \inf\{k \geq 0 : X_k \in \partial B_\ell\}.$$

Let also

$$Y_\ell^{(n)} = \sum_{\tau_\ell \wedge \xi_n \leq k < \tau_{\ell+1} \wedge \xi_n} G(X_k), \quad \text{and} \quad Y_\ell = \sum_{\tau_\ell \leq k < \tau_{\ell+1}} G(X_k),$$

so that in particular,

$$G_n^+ = \sum_{\ell=0}^{\infty} Y_\ell^{(n)}.$$

Recall that for a simple random walk, starting from ∂B_ℓ, the probability to hit $B(0, r)$, for some $r < 2^\ell$, is of order $g(2^\ell)/g(r)$, where we use the convention $g(r) = r^{2-d}$. It follows that

$$\sup_{\ell \geq 0} \sup_{x \in \partial B_\ell} \mathbb{E}_x[Y_\ell^2] \lesssim \sup_{\ell \geq 0} \sum_{\|u\|, \|v\| \leq 2^\ell} G(u)G(u+v)g(2^\ell)^2 \lesssim 1. \tag{3.13}$$

Thus for any $\ell \geq 0$, by the Markov property,

$$\mathbb{E}[Y_\ell^{(n)}] \leq \mathbb{P}(\xi_n \geq \tau_\ell) \cdot \sup_{x \in \partial B_\ell} \mathbb{E}_x[Y_\ell] \lesssim \mathbb{P}(\xi_n \geq \tau_\ell),$$

and likewise

$$\mathbb{E}[(Y_\ell^{(n)})^2] \leq \mathbb{P}(\xi_n \geq \tau_\ell) \cdot \sup_{x \in \partial B_\ell} \mathbb{E}_x[Y_\ell^2] \lesssim \mathbb{P}(\xi_n \geq \tau_\ell). \tag{3.14}$$

Moreover,

$$\mathrm{Var}(G_n^+) = \sum_{\ell \geq 0} \mathrm{Var}(Y_\ell^{(n)}) + 2 \sum_{0 \leq \ell < m} \Big(\mathbb{E}[Y_\ell^{(n)} \cdot Y_m^{(n)}] - \mathbb{E}[Y_\ell^{(n)}] \cdot \mathbb{E}[Y_m^{(n)}] \Big). \tag{3.15}$$

The first sum above is handled using (3.14), which shows that it is bounded by

$$\sum_{\ell\geq 0}\mathbb{E}[(Y_\ell^{(n)})^2]\lesssim\sum_{\ell\geq 0}\mathbb{P}(\xi_n\geq\tau_\ell)\lesssim\log n.$$

It amounts now to bound the second sum in (3.15). Define for $y\in B_m$ and $x\in\partial B_m$,

$$H_m(y,x)=\mathbb{P}_y(X_{\tau_m}=x).$$

It is known, see e.g. [13, Proposition 6.4.4], that uniformly in $y\in B_\ell$, with $\ell\leq m-1$,

$$H_m(y,x)=H_m(0,x)(1+\mathcal{O}(2^{\ell-m})). \tag{3.16}$$

Now define

$$L=\frac{1}{2}\log_2(n)-2\log_2(\log n),\quad\text{and}\quad M=\frac{1}{2}\log_2(n)-\log_2(\log n),$$

where $\log_2(a)=\frac{\log(a)}{\log 2}$, for $a>0$. Write for $\ell,m\geq 0$,

$$\Delta_{\ell,m}^{(n)}=\mathbb{E}[Y_\ell^{(n)}\cdot Y_m^{(n)}]-\mathbb{E}[Y_\ell^{(n)}]\cdot\mathbb{E}[Y_m^{(n)}].$$

We first bound for $\ell\leq L$ and $m\geq M$, using the memoryless property of geometric random variables,

$$\begin{aligned}\mathbb{E}[Y_\ell^{(n)}\cdot Y_m^{(n)}]&\leq\sum_{y\in\partial B_{\ell+1}}\sum_{x\in\partial B_M}\mathbb{E}\left[Y_\ell^{(n)}\mathbf{1}\{X_{\tau_{\ell+1}}=y\}\right]\cdot H_M(y,x)\cdot\mathbb{E}_x[Y_m^{(n)}]\\&\leq\mathbb{E}[Y_\ell^{(n)}]\cdot\sum_{x\in\partial B_M}H_M(0,x)(1+\mathcal{O}(2^{L-M}))\cdot\mathbb{E}_x[Y_m^{(n)}],\end{aligned}$$

and for $m\geq M$,

$$\begin{aligned}\mathbb{E}[Y_m^{(n)}]&=\sum_{x\in\partial B_M}\mathbb{P}(\xi_n>\tau_M,X_{\tau_M}=x)\cdot\mathbb{E}_x[Y_m^{(n)}]\\&=\sum_{x\in\partial B_M}H_M(0,x)\cdot\mathbb{E}_x[Y_m^{(n)}]-\sum_{x\in\partial B_M}\mathbb{P}(\xi_n\leq\tau_M,X_{\tau_M}=x)\cdot\mathbb{E}_x[Y_m^{(n)}]\\&\geq\sum_{x\in\partial B_M}H_M(0,x)\cdot\mathbb{E}_x[Y_m^{(n)}]-\mathbb{P}(\xi_n\leq\tau_M)\cdot\sup_{x\in\partial B_M}\mathbb{P}_x[\xi_n\leq\tau_m].\end{aligned}$$

Note also that

$$\mathbb{P}(\xi_n\leq\tau_M)=\sum_{k\geq 0}\mathbb{P}(\xi_n=k)\cdot\mathbb{P}(\tau_M\geq k)\leq\frac{\mathbb{E}[\tau_M]}{n}\lesssim\frac{2^{2M}}{n}\lesssim\frac{1}{(\log n)^2}.$$

Altogether, this gives

$$\sum_{\ell=0}^{L}\sum_{m\geq N}\Delta_{\ell,m}^{(n)}\lesssim\log n.$$

Moreover,

$$\sum_{L\leq\ell<m}\mathbb{E}[Y_\ell^{(n)}\cdot Y_m^{(n)}]\leq\sum_{L\leq\ell<m}\mathbb{P}(\xi_n\geq\tau_\ell)\cdot\sup_{x\in\partial B_{\ell+1}}\mathbb{P}_x(\xi_n\geq\tau_m)\lesssim(\log\log n)^2,$$

and it just remains to consider the case when $\ell < m \le M$. Note that the case $m = \ell + 1$ can be handled using a similar bound as (3.14). Furthermore, if $\ell + 2 \le m$, then by (3.16),

$$\begin{aligned}\mathbb{E}[Y_\ell^{(n)} \cdot Y_m^{(n)}] \le \mathbb{E}[Y_\ell \cdot Y_m] &= \sum_{y \in \partial B_{\ell+1}} \sum_{x \in \partial B_m} \mathbb{E}[Y_\ell \cdot \mathbf{1}\{X_{\tau_{\ell+1}} = y\}] \cdot H_m(y,x) \cdot \mathbb{E}_x[Y_m] \\ &\le (1 + \mathcal{O}(2^{\ell-m})) \cdot \mathbb{E}[Y_\ell] \cdot \mathbb{E}[Y_m].\end{aligned}$$

Conversely, one can use that by Cauchy-Schwarz inequality,

$$\mathbb{E}[Y_\ell^{(n)}] \ge \mathbb{E}[Y_\ell] - \mathbb{E}[Y_\ell^2]^{1/2} \cdot \mathbb{P}(\xi_n \le \tau_{\ell+1})^{1/2} \overset{(3.13)}{\ge} \mathbb{E}[Y_\ell] - C \cdot \mathbb{P}(\xi_n \le \tau_{\ell+1})^{1/2},$$

for some constant $C > 0$, and that for $\ell \le M$,

$$\mathbb{P}(\xi_n \le \tau_{\ell+1}) \le \frac{\mathbb{E}[\tau_{\ell+1}]}{n} \lesssim \frac{2^{2M}}{n} \lesssim \frac{1}{(\log n)^2},$$

which altogether give as well

$$\sum_{0 \le \ell < m \le N} \Delta_{\ell,m}^{(n)} \lesssim \log n.$$

This conclude the proof of the upper bound for the variance. ■

We now move to proving concentration for U_n. The proof is based on a similar idea.

Proof of Proposition 3.1. First recall, see e.g. [3], that for each $i \ge 1$, and $k \ge 0$, one has $\mathbb{P}(d_i = k) = \sum_{j \ge k+1} \mu(j)$, and hence

$$\mathbb{E}[d_i] = \sum_{k \ge 1} (k-1) \sum_{j \ge k} \mu(j) = \sum_{j \ge 1} \frac{j(j-1)}{2} \mu(j) = \frac{\sigma^2}{2},$$

while $\mathbb{E}[d_0] = 1$. Thus, recalling (2.13) and the definition of G_n from (3.12), we get

$$\mathbb{E}[U_n \mid X, \xi_n^r, \xi_n^l] = \frac{\sigma^2}{2} \cdot G_n + (1 - \frac{\sigma^2}{2}) g_n, \tag{3.17}$$

with

$$g_n = \sum_{j=-\xi_n^l}^{\xi_n^r} g(X_j).$$

Therefore the result for the expectation of U_n follows from Lemma 3.7 together with the fact that $\mathbb{E}[g_n] \le 2G(0)$. We shall now use that

$$\mathrm{Var}(U_n) = \mathbb{E}\Big[\mathrm{Var}(U_n \mid X, \xi_n^l, \xi_n^r)\Big] + \mathrm{Var}\Big(\mathbb{E}[U_n \mid X, \xi_n^l, \xi_n^r]\Big). \tag{3.18}$$

Lemma 3.7 and (3.17) yield

$$\mathrm{Var}\Big(\mathbb{E}[U_n \mid X, \xi_n^l, \xi_n^r]\Big) \lesssim \mathrm{Var}(G_n) + \mathbb{E}[g_n^2] \lesssim \log n + \mathbb{E}[g_n^2].$$

Furthermore,

$$\mathbb{E}[g_n^2] \le 4\mathbb{E}\Big[\Big(\sum_{j \ge 0} g(X_j)\Big)^2\Big] \le 8 \sum_{0 \le j \le k} \mathbb{E}[g(X_j) g(X_k)] = 8 \sum_{u,v \in \mathbb{Z}^d} g(u)^2 g(u+v) g(v) \lesssim 1,$$

and thus it only remains to consider the first term on the right-hand side of (3.18). Here we will use again that μ has a finite third moment, which implies that d_i has a finite second moment for all $i \geq 0$.

For $\ell \geq 0$, define

$$\widetilde{\tau}_\ell = \inf\{k \geq 0 : \widetilde{X}_k \in \partial B_\ell\},$$

and for $x \in \mathbb{Z}^6$,

$$\widetilde{Y}_\ell(x) = \sum_{j=\widetilde{\tau}_\ell}^{\widetilde{\tau}_{\ell+1}-1} d_j \cdot g(x, \widetilde{X}_j).$$

Note that by using (3.16), one has for any $m \geq \ell + 2$, uniformly over $x, y \in \mathbb{Z}^6$,

$$\mathbb{E}[\widetilde{Y}_\ell(x) \cdot \widetilde{Y}_m(y)] \leq (1 + \mathcal{O}(2^{\ell-m})) \cdot \mathbb{E}[\widetilde{Y}_\ell(x)] \cdot \mathbb{E}[\widetilde{Y}_m(y)].$$

Moreover, repeating the argument used for (3.13) yields for $m \geq 0$,

$$\mathbb{E}[\widetilde{Y}_m(y)] \lesssim \sum_{u \in B_{m+1}} g(y, u) g(2^m) \lesssim \rho_m(y),$$

with

$$\rho_m(y) = g(y) 2^{2m} \cdot \mathbf{1}\{y \notin B_m\} + G(2^m) \cdot \mathbf{1}\{y \in B_m\}.$$

As a consequence, letting

$$h_\ell(y) = \frac{2^\ell}{(1 + \|y\|)^3} \cdot \mathbf{1}\{y \notin B_\ell\} + 2^{-2\ell} \cdot \mathbf{1}\{y \in B_\ell\},$$

we get that for any $\ell \geq 0$,

$$\sum_{m \geq \ell+1} 2^{\ell-m} \cdot \mathbb{E}[\widetilde{Y}_m(y)] \lesssim h_\ell(y).$$

It follows that uniformly over $x, y \in \mathbb{Z}^6$,

$$\sum_{\ell \geq 0} \sum_{m \geq \ell+2} \Big(\mathbb{E}[\widetilde{Y}_\ell(x) \cdot \widetilde{Y}_m(y)] - \mathbb{E}[\widetilde{Y}_\ell(x)] \cdot \mathbb{E}[\widetilde{Y}_m(y)] \Big) \lesssim \sum_{\ell \geq 0} \rho_\ell(x) \cdot h_\ell(y) \leq \sum_{\ell \geq 0} h_\ell(x) \cdot h_\ell(y).$$

On the other hand, a similar computation as above yields

$$\mathbb{E}\big[\widetilde{Y}_\ell(x) \cdot \big(\widetilde{Y}_\ell(y) + \widetilde{Y}_{\ell+1}(y)\big)\big] \lesssim \sum_{u,v \in B_{\ell+1}} g(x,u) g(y, u+v) g(2^\ell)^2 \lesssim \rho_\ell(x) \cdot \rho_\ell(y) \leq h_\ell(x) \cdot h_\ell(y).$$

Altogether this gives

$$\sum_{\ell \geq 0} \sum_{m \geq 0} \Big(\mathbb{E}[\widetilde{Y}_\ell(x) \cdot \widetilde{Y}_m(y)] - \mathbb{E}[\widetilde{Y}_\ell(x)] \cdot \mathbb{E}[\widetilde{Y}_m(y)] \Big) \lesssim \sum_{\ell \geq 0} h_\ell(x) \cdot h_\ell(y).$$

From this we infer that

$$\mathrm{Var}(U_n \mid X, \xi_n^\ell, \xi_n^r) \lesssim \sum_{-\xi_n^\ell \leq j,k \leq \xi_n^r} \sum_{\ell \geq 0} h_\ell(X_j) \cdot h_\ell(X_k). \tag{3.19}$$

Now, for any fixed $\ell \geq 0$, one has

$$\mathbb{E}\left[\sum_{0 \leq j \leq k \leq \xi_n^r} h_\ell(X_j) h_\ell(X_k) \right] \lesssim \sum_{\|u\|, \|v\| \leq \sqrt{n}} h_\ell(u) \cdot h_\ell(u+v) \cdot g(u) g(v) \lesssim \frac{n^2}{n^2 + 2^{4\ell}},$$

and likewise,

$$\mathbb{E}\Big[\sum_{0\le j\le \xi_n^r} h_\ell(X_j)\Big]^2 \lesssim \left(\sum_{\|u\|\le\sqrt{n}} h_\ell(u)g(u)\right)^2 \lesssim \frac{n^2}{n^2+2^{4\ell}}.$$

Thus as wanted,

$$\mathbb{E}\Big[\mathrm{Var}(U_n \mid X, \xi_n^\ell, \xi_n^r)\Big] \lesssim \sum_{\ell\ge 0} \frac{n^2}{n^2+2^{4\ell}} \lesssim \log n,$$

concluding the proof of the proposition. ■

Remark 3.8. Let us explain here how one can adapt the proof above to show that both $\mathbb{E}[Y_\varepsilon^1]$ and $\mathbb{E}[Y_\varepsilon^2]$ appearing in the proof of Proposition 3.5 are $\mathcal{O}(\frac{1}{\varepsilon^2 \log n})$. First, since $\mathbb{E}[U_n^+]$ is of order $\log n$, by Chebyshev's inequality it suffices to show that $\mathrm{Var}(U_n^{\varepsilon'})$ and $\mathrm{Var}(\widetilde{U}_n^{\varepsilon'})$ are $\mathcal{O}(\log n)$, with the notation thereof. Note first that $U_n^{\varepsilon'} + \widetilde{U}_n^{\varepsilon'} = U_n^+$, and applying readily the same proof as above shows that $\mathrm{Var}(U_n^+) = \mathcal{O}(\log n)$. Hence, by triangular inequality, it suffices to bound the variance of $U_n^{\varepsilon'}$. The starting point is still to decompose the variance in two terms, as in (3.18). Concerning the term $\mathbb{E}[\mathrm{Var}(U_n^{\varepsilon'} \mid X, \xi_n^r)]$, the same proof as above applies, in particular the bound (3.19) still holds, and one can even restrict the sum on the right-hand side to indices $j,k\ge 0$, and ℓ such that $2^\ell \le \frac{2}{\varepsilon'}$. Concerning the other term $\mathrm{Var}(\mathbb{E}[U_n^{\varepsilon'} \mid X, \xi_n^r])$, the same computation can be done as well, just replacing the function $G(x)$ appearing there by the function $G^{\varepsilon'}(x)$, defined as the expected time spent at x by a simple random walk starting from the origin before it exits the ball centered at the origin with radius $1/\varepsilon'$. In particular $G^{\varepsilon'}(x) = 0$, when $\|x\| > 1/\varepsilon'$, but for $\|x\| \le \frac{1}{2\varepsilon'}$, it is of the same order as $G(x)$. Therefore, again all the proof above goes through with only minor modification.

4 Proofs of Theorem 1.1 and Proposition 1.3

Proof of Proposition 1.3. The proof is the same as in [1], which we recall for completeness. For the lower bound, we let

$$L_n(x) = \sum_{k=0}^n \mathbf{1}\{S_k = x\},$$

and $\nu_n(x) = \frac{L_n(x)}{n+1}$, which defines a probability measure supported on $\mathcal{R}_n$. Thus one can use Theorem 2.5, which gives that

$$\mathrm{BCap}(\mathcal{R}_n) \gtrsim \frac{n^2}{\sum_{x,y\in\mathcal{R}_n} G(x-y)L_n(x)L_n(y)}.$$

Then by using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] &\gtrsim \frac{n^2}{\sum_{x,y\in\mathcal{R}_n} G(x-y)\mathbb{E}[L_n(x)L_n(y)]} \ge \frac{n^2}{\mathbb{E}[\sum_{0\le k\le n}\sum_{0\le \ell\le n} G(X_k - X_\ell)]} \\ &\gtrsim \frac{n}{\mathbb{E}[\sum_{0\le k\le n} G(X_k)]} \gtrsim \frac{n}{\sum_{\|u\|\le\sqrt{n}} G(u)g(u)} \gtrsim \sqrt{n}. \end{aligned} \tag{4.1}$$

The upper bound comes from the fact that the branching capacity is monotone for inclusion, and thus if $R_n = \max_{0\le k\le n}\|X_k\|$, then $\mathrm{BCap}(\mathcal{R}_n) \le \mathrm{BCap}(B(0,R_n)) \lesssim R_n$, as we know from [18] that the branching capacity of a ball of radius R is of order R in dimension 5. Therefore $\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] \lesssim \mathbb{E}[R_n]$, and the desired upper bound follows since it is well known that $\mathbb{E}[R_n] \lesssim \sqrt{n}$. ■

Proof of Theorem 1.1. The fact that the limit exists in (1.1) follows from the ergodic theorem, exactly as in [10]. Let us recall the argument for reader's convenience. First one has

$$\mathrm{BCap}(\mathcal{R}_n) = \sum_{k=0}^{n} e_{\mathcal{R}_n}(X_k) \cdot \mathbf{1}\big\{X_k \notin \{X_{k+1}, \dots, X_n\}\big\}.$$

Thus, letting $\mathcal{R}_\infty$ and $\widetilde{\mathcal{R}}_\infty$ be two independent infinite ranges starting from the origin, one has

$$\frac{\mathrm{BCap}(\mathcal{R}_n)}{n} \geq \frac{1}{n} \sum_{k=0}^{n} e_{\mathcal{R}_\infty \cup \widetilde{\mathcal{R}}_\infty}(X_k) \cdot \mathbf{1}\big\{X_k \notin \{X_{k+1}, \dots, \}\big\},$$

and the ergodic theorem implies that the right hand side converges almost surely as $n \to \infty$, toward (with the notation of Corollary 2.8)

$$c_d = \mathbb{E}\Big[e_{\mathcal{R}_\infty \cup \widetilde{\mathcal{R}}_\infty}(0) \cdot \mathbf{1}\big\{0 \notin \mathcal{R}[1, \infty)\big\}\Big] = \mathbb{E}\Big[\mathbf{1}_{\mathcal{A}_\infty} \cdot e_\infty\Big],$$

which provides already the lower bound

$$\liminf_{n \to \infty} \frac{\mathrm{BCap}(\mathcal{R}_n)}{n} \geq c_d.$$

To get the upper bound, notice that for any $n \geq 1$,

$$\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)] \leq 2\sqrt{n} + (n - 2\sqrt{n}) \cdot \mathbb{E}\Big[e_{\mathcal{R}[0,\sqrt{n}] \cup \widetilde{\mathcal{R}}[0,\sqrt{n}]} \cdot \mathbf{1}\{0 \notin \mathcal{R}[0, \sqrt{n}]\}\Big],$$

and since by monotone convergence the expectation on the right hand side converges to c_d as $n \to \infty$, it follows that

$$\limsup_{n \to \infty} \frac{\mathbb{E}[\mathrm{BCap}(\mathcal{R}_n)]}{n} \leq c_d.$$

Now fix some integer $m \geq 1$, and observe that by subadditivity of the branching capacity, see [18], one has

$$\mathrm{BCap}(\mathcal{R}_n) \leq \sum_{i=0}^{\lfloor n/m \rfloor - 1} \mathrm{BCap}(\mathcal{R}[im, (i+1)m]).$$

Since the right-hand side is a sum of independent and identically distributed terms, one get by Kolmogorov's strong law of large numbers,

$$\limsup_{n \to \infty} \frac{\mathrm{BCap}(\mathcal{R}_n)}{n} \leq \frac{\mathbb{E}[\mathrm{BCap}(\mathcal{R}_m)]}{m}.$$

Since this holds for any m, we obtain the converse inequality,

$$\limsup_{n \to \infty} \frac{\mathrm{BCap}(\mathcal{R}_n)}{n} \leq \limsup_{m \to \infty} \frac{\mathbb{E}[\mathrm{BCap}(\mathcal{R}_m)]}{m} \leq c_d.$$

Finally, to see that c_d is positive when $d \geq 7$, one can use the second statement of Corollary 2.8. It has already been seen in its proof that $\mathbb{E}[U_\infty]$ is finite, which implies that U_∞ is finite almost surely. Together with the second claim of Corollary 2.8, we deduce that $\mathbf{1}_{\mathcal{A}_\infty} \cdot e_\infty$ is not almost surely equal to zero, and thus $c_d > 0$. ∎

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