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CAPACITY OF THE RANGE IN DIMENSION 5: ROUGH VARIANCE BOUNDS

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This is a companion paper to [Sch], where we prove some technical estimates. In particular we obtain an upper bound for the variance of the capacity of the range.

1. Introduction.

We prove here technical estimates needed for the companion paper [Sch]. In particular the estimates gathered here show the following rough variance bound:

$$\mathrm{Var}(\mathrm{Cap}(\mathcal{R}_n)) = \mathcal{O}(n \log n),$$

where $\mathcal{R}_n = \{S_0, \dots, S_n\}$ is the range of a random walk on $\mathbb{Z}^5$.

2. Preliminaries.

2.1. *Notation.* We recall here some of the main notation of [Sch]. We consider $(X_i)_{i\geq 1}$ a sequence of independent and identically distributed random variables, whose law is a symmetric and irreducible probability measure having a finite d-th moment. The associated random walk is the process $(S_n)_{n\geq 0}$ defined by $S_n = S_0 + X_1 + \dots + X_n$, for all $n \geq 0$. The walk is called aperiodic if the probability to be at the origin at time n is nonzero for all n large enough, and it is called bipartite if this probability is nonzero only when n is even.

For $x \in \mathbb{Z}^d$, we denote by $\mathbb{P}_x$ the law of the walk starting from $S_0 = x$. When $x = 0$, we simply write it as $\mathbb{P}$. We denote its total range as $\mathcal{R}_\infty := \{S_k\}_{k\geq 0}$, and for $0 \leq k \leq n \leq +\infty$, set $\mathcal{R}[k, n] := \{S_k, \dots, S_n\}$.

For an integer $k \geq 2$, the law of k independent random walks (with the same step distribution) starting from some $x_1, \dots, x_k \in \mathbb{Z}^5$, is denoted by $\mathbb{P}_{x_1,\dots,x_k}$, or simply by $\mathbb{P}$ when they all start from the origin.

We define

$$H_A := \inf\{n \geq 0 \ : \ S_n \in A\}, \quad \text{and} \quad H_A^+ := \inf\{n \geq 1 \ : \ S_n \in A\}, \tag{2.1}$$

respectively for the hitting time and first return time to a subset $A \subset \mathbb{Z}^d$, that we abbreviate respectively as H_x and H_x^+ when A is a singleton $\{x\}$.

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We let $\|x\|$ be the Euclidean norm of $x \in \mathbb{Z}^d$. If X_1 has covariance matrix $\Gamma = \Lambda\Lambda^t$, we define its associated norm as

$$\mathcal{J}^*(x) := |x \cdot \Gamma^{-1} x|^{1/2} = \|\Lambda^{-1} x\|,$$

and set $\mathcal{J}(x) = d^{-1/2}\mathcal{J}^*(x)$ (see [LL10] p.4 for more details).

For a and b some nonnegative reals, we let $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$. We use the letters c and C to denote constants (which could depend on the covariance matrix of the walk), whose values might change from line to line. We also use standard notation for the comparison of functions: we write $f = \mathcal{O}(g)$, or sometimes $f \lesssim g$, if there exists a constant $C > 0$, such that $f(x) \le Cg(x)$, for all x. Likewise, $f = o(g)$ means that $f/g \to 0$, and $f \sim g$ means that f and g are equivalent, that is if $|f - g| = o(f)$. Finally we write $f \asymp g$, when both $f = \mathcal{O}(g)$, and $g = \mathcal{O}(f)$.

2.2. *Transition kernel and Green's function.* We denote by $p_n(x)$ the probability that a random walk starting from the origin ends up at position $x \in \mathbb{Z}^d$ after n steps, that is $p_n(x) := \mathbb{P}[S_n = x]$, and note that for any $x, y \in \mathbb{Z}^d$, one has $\mathbb{P}_x[S_n = y] = p_n(y - x)$. Recall the definitions of Γ and $\mathcal{J}^*$ from the previous subsection, and define

$$\overline{p}_n(x) := \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Gamma}} \cdot e^{-\frac{\mathcal{J}^*(x)^2}{2n}}. \tag{2.2}$$

Theorem 2.1 (**Local Central Limit Theorem**). *There exists a constant $C > 0$, such that for all $n \ge 1$, and all $x \in \mathbb{Z}^d$,*

$$|p_n(x) - \overline{p}_n(x)| \le \frac{C}{n^{(d+2)/2}},$$

in the case of an aperiodic walk, and for bipartite walks,

$$|p_n(x) + p_{n+1}(x) - 2\overline{p}_n(x)| \le \frac{C}{n^{(d+2)/2}}.$$

In addition, under our hypotheses (in particular assuming $\mathbb{E}[\|X_1\|^d] < \infty$), there exists a constant $C > 0$, such that for any $n \ge 1$ and any $x \in \mathbb{Z}^d$ (see Proposition 2.4.6 in [LL10]),

$$p_n(x) \le C \cdot \begin{cases} n^{-d/2} & \text{if } \|x\| \le \sqrt{n}, \\ \|x\|^{-d} & \text{if } \|x\| > \sqrt{n}. \end{cases} \tag{2.3}$$

It is also known (see the proof of Proposition 2.4.6 in [LL10]) that

$$\mathbb{E}[\|S_n\|^d] = \mathcal{O}(n^{d/2}). \tag{2.4}$$

Together with the reflection principle (see Proposition 1.6.2 in [LL10]), and Markov's inequality, this gives that for any $n \geq 1$ and $r \geq 1$,

$$\mathbb{P}\left[\max_{0\leq k\leq n} \|S_k\| \geq r\right] \leq C \cdot \left(\frac{\sqrt{n}}{r}\right)^d. \tag{2.5}$$

Now we define for $\ell \geq 0$, $G_\ell(x) := \sum_{n\geq \ell} p_n(x)$. The **Green's function** is the function $G := G_0$. A union bound gives

$$\mathbb{P}[x \in \mathcal{R}[\ell, \infty)] \leq G_\ell(x). \tag{2.6}$$

By (2.3) there exists a constant $C > 0$, such that for any $x \in \mathbb{Z}^d$, and $\ell \geq 0$,

$$G_\ell(x) \leq \frac{C}{\|x\|^{d-2} + \ell^{\frac{d-2}{2}} + 1}. \tag{2.7}$$

It follows from this bound (together with the corresponding lower bound $G(x) \geq c\|x\|^{2-d}$, which can be deduced from Theorem 2.1), and the fact that G is harmonic on $\mathbb{Z}^d \setminus \{0\}$, that the hitting probability of a ball is bounded as follows (see the proof of [LL10, Proposition 6.4.2]):

$$\mathbb{P}_x[\eta_r < \infty] = \mathcal{O}\left(\frac{r^{d-2}}{1 + \|x\|^{d-2}}\right), \quad \text{with} \quad \eta_r := \inf\{n \geq 0 \ : \ \|S_n\| \leq r\}. \tag{2.8}$$

We shall need as well some control on the overshoot. We state the result we need as a lemma and provide a short proof for the sake of completeness.

Lemma 2.2 (**Overshoot Lemma**). *There exists a constant $C > 0$, such that for all $r \geq 1$, and all $x \in \mathbb{Z}^d$, with $\|x\| \geq r$,*

$$\mathbb{P}_x[\eta_r < \infty, \|S_{\eta_r}\| \leq r/2] \leq \frac{C}{1 + \|x\|^{d-2}}.$$

Proof. We closely follow the proof of Lemma 5.1.9 in [LL10]. Note first that one can alway assume that r is large enough, for otherwise the result follows from (2.8). Then define for $k \geq 0$,

$$Y_k := \sum_{n=0}^{\eta_r} \mathbf{1}\{r + k \leq \|S_n\| < r + (k+1)\}.$$

Let

$$g(x,k) = \mathbb{E}_x[Y_k] = \sum_{n=0}^{\infty} \mathbb{P}_x[r + k \leq \|S_n\| \leq r + k + 1, \, n < \eta_r].$$

One has

$$\mathbb{P}_x[\eta_r < \infty,\, \|S_{\eta_r}\| \le r/2] = \sum_{n=0}^{\infty} \mathbb{P}_x[\eta_r = n+1,\, \|S_{\eta_r}\| \le r/2]$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}_x[\eta_r = n+1,\, \|S_{\eta_r}\| \le r/2,\, r+k \le \|S_n\| < r+k+1]$$
$$\le \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}_x\left[\eta_r > n,\, r+k \le \|S_n\| \le r+k+1,\, \|S_{n+1}-S_n\| \ge \frac{r}{2}+k\right]$$
$$= \sum_{k=0}^{\infty} g(x,k)\mathbb{P}\left[\|X_1\| \ge \frac{r}{2}+k\right] = \sum_{k=0}^{\infty} g(x,k)\sum_{\ell=k}^{\infty}\mathbb{P}\left[\frac{r}{2}+\ell \le \|X_1\| < \frac{r}{2}+\ell+1\right]$$
$$= \sum_{\ell=0}^{\infty}\mathbb{P}\left[\frac{r}{2}+\ell \le \|X_1\| < \frac{r}{2}+\ell+1\right]\sum_{k=0}^{\ell} g(x,k).$$

Now Theorem 2.1 shows that one has $\mathbb{P}_z[\|S_{\ell^2}\| \le r] \ge \rho$, for some constant $\rho > 0$, uniformly in r (large enough), $\ell \ge 1$, and $r \le \|z\| \le r+\ell$. It follows, exactly as in the proof of Lemma 5.1.9 from [LL10], that for any $\ell \ge 1$,

$$\max_{\|z\|\le r+\ell} \sum_{0\le k<\ell} g(z,k) \le \frac{\ell^2}{\rho}.$$

Using in addition (2.8), we get with the Markov property,

$$\sum_{0\le k<\ell} g(x,k) \lesssim \frac{(r+\ell)^{d-2}}{1+\|x\|^{d-2}}\cdot \ell^2,$$

for some constant $C > 0$. As a consequence one has

$$\mathbb{P}_x[\eta_r < \infty,\, \|S_{\eta_r}\| \le r/2]$$
$$\lesssim \frac{1}{1+\|x\|^{d-2}}\sum_{\ell=0}^{\infty}\mathbb{P}\left[\frac{r}{2}+\ell \le \|X_1\| < \frac{r}{2}+\ell+1\right](r+\ell)^{d-2}(\ell+1)^2$$
$$\lesssim \frac{1}{1+\|x\|^{d-2}}\mathbb{E}\left[\|X_1\|^{d-2}(\|X_1\|-r/2)^2\mathbf{1}\{\|X_1\|\ge r/2\}\right] \lesssim \frac{1}{1+\|x\|^{d-2}},$$

since by hypothesis, the d-th moment of X_1 is finite. □

2.3. *Basic tools.* We state here some basic results, which are (for the most part) proved in [Sch].

Lemma 2.3. *There exists $C>0$, such that for all $x\in\mathbb{Z}^d$, and $\ell\ge 0$,*

$$\sum_{z\in\mathbb{Z}^d} G_\ell(z)G(z-x)\le \frac{C}{\|x\|^{d-4}+\ell^{\frac{d-4}{2}}+1}.$$

Lemma 2.4. *One has,*

$$\sup_{x\in\mathbb{Z}^d}\mathbb{E}[G(S_n-x)]=\mathcal{O}\left(\frac{1}{n^{\frac{d-2}{2}}}\right), \tag{2.9}$$

and for any $\alpha\in[0,d)$,

$$\sup_{n\ge 0}\mathbb{E}\left[\frac{1}{1+\|S_n-x\|^\alpha}\right]=\mathcal{O}\left(\frac{1}{1+\|x\|^\alpha}\right). \tag{2.10}$$

Moreover, when $d=5$,

$$\mathbb{E}\left[\Big(\sum_{n\ge k}G(S_n)\Big)^2\right]=\mathcal{O}\left(\frac{1}{k}\right). \tag{2.11}$$

Proof. Only the last statement needs a proof, the others are proved in [Sch]. One simply write, using the Markov property at the second line,

$$\begin{aligned}
\mathbb{E}\left[\Big(\sum_{n\ge k}G(S_n)\Big)^2\right] &= \sum_{x,y}G(x)G(y)\mathbb{E}\left[\sum_{n,m\ge k}\mathbf{1}\{S_n=x,S_m=y\}\right]\\
&\le 2\sum_{x,y}G(x)G(y)\sum_{n\ge k}\sum_{\ell\ge 0}p_n(x)p_\ell(y-x)=2\sum_{x,y}G(x)G(y)G_k(x)G(y-x)\\
&\overset{\text{Lemma } 2.3}{\lesssim}\sum_x\frac{1}{\|x\|^4}G_k(x)\overset{(2.7)}{\lesssim}\frac{1}{k}.
\end{aligned}$$

□

Lemma 2.5. *Let S and $\widetilde{S}$ be two independent walks starting respectively from the origin and some $x\in\mathbb{Z}^d$. Let also ℓ and m be two given nonnegative integers (possibly infinite for m). Define*

$$\tau:=\inf\{n\ge 0\ :\ \widetilde{S}_n\in\mathcal{R}[\ell,\ell+m]\}.$$

Then, for any function $F:\mathbb{Z}^d\to\mathbb{R}_+$,

$$\mathbb{E}_{0,x}[\mathbf{1}\{\tau<\infty\}F(\widetilde{S}_\tau)]\le\sum_{i=\ell}^{\ell+m}\mathbb{E}[G(S_i-x)F(S_i)]. \tag{2.12}$$

In particular, uniformly in ℓ and m,

$$\mathbb{P}_{0,x}[\tau < \infty] = \mathcal{O}\left(\frac{1}{1+\|x\|^{d-4}}\right). \tag{2.13}$$

Moreover, uniformly in $x \in \mathbb{Z}^d$,

$$\mathbb{P}_{0,x}[\tau < \infty] = \begin{cases} \mathcal{O}\left(m \cdot \ell^{\frac{2-d}{2}}\right) & \text{if } m < \infty \\ \mathcal{O}\left(\ell^{\frac{4-d}{2}}\right) & \text{if } m = \infty. \end{cases} \tag{2.14}$$

3. Statement of the main results. Set

$$\varphi_k^n := \mathbb{P}_{S_k}[H^+_{\mathcal{R}_n} = \infty \mid \mathcal{R}_n], \text{ and } Z_k^n := \mathbf{1}\{S_\ell \neq S_k, \text{ for all } \ell = k+1, \dots, n\},$$

for all $0 \le k \le n$. By definition of the capacity one has

$$\mathrm{Cap}(\mathcal{R}_n) = \sum_{k=0}^n Z_k^n \cdot \varphi_k^n.$$

Consider now $(S_n)_{n\in\mathbb{Z}}$ a two-sided random walk starting from the origin (that is $(S_n)_{n\ge 0}$ and $(S_{-n})_{n\ge 0}$ are two independent walks starting from the origin), and denote its total range by $\overline{\mathcal{R}}_\infty := \{S_n\}_{n\in\mathbb{Z}}$. Then for $k \ge 0$, let

$$\varphi(k) := \mathbb{P}_{S_k}[H^+_{\overline{\mathcal{R}}_\infty} = \infty \mid (S_n)_{n\in\mathbb{Z}}], \text{ and } Z(k) := \mathbf{1}\{S_\ell \neq S_k, \text{ for all } \ell \ge k+1\},$$

and define

$$\mathcal{C}_n := \sum_{k=0}^n Z(k)\varphi(k), \quad \text{and} \quad W_n := \mathrm{Cap}(\mathcal{R}_n) - \mathcal{C}_n.$$

We first prove the following result.

Lemma 3.1. *One has*

$$\mathbb{E}[W_n^2] = \mathcal{O}(n).$$

Proof. Note that $W_n = W_{n,1} + W_{n,2}$, with

$$W_{n,1} = \sum_{k=0}^n (Z_k^n - Z(k))\varphi_k^n, \quad \text{and} \quad W_{n,2} = \sum_{k=0}^n (\varphi_k^n - \varphi(k))Z(k).$$

Consider first the term $W_{n,1}$ which is easier. Observe that $Z_k^n - Z(k)$ is nonnegative and bounded by the indicator function of the event $\{S_k \in \mathcal{R}[n+1,\infty)\}$. Bounding also φ_k^n by one, we get

$$\begin{aligned}\mathbb{E}[W_{n,1}^2] &\le \sum_{\ell=0}^{n}\sum_{k=0}^{n}\mathbb{E}[(Z_\ell^n - Z(\ell))(Z_k^n - Z(k))]\\ &\le \sum_{\ell=0}^{n}\sum_{k=0}^{n}\mathbb{P}\left[S_\ell \in \mathcal{R}[n+1,\infty),\, S_k \in \mathcal{R}[n+1,\infty)\right].\end{aligned}$$

Then noting that $(S_{n+1-k} - S_{n+1})_{k\ge 0}$ and $(S_{n+1+k} - S_{n+1})_{k\ge 0}$ are two independent random walks starting from the origin, we obtain

$$\begin{aligned}\mathbb{E}[W_{n,1}^2] \le \sum_{\ell=1}^{n+1}\sum_{k=1}^{n+1}\mathbb{P}[H_{S_\ell}<\infty,\, H_{S_k}<\infty] &\le 2\sum_{\ell=1}^{n+1}\sum_{k=1}^{n+1}\mathbb{P}[H_{S_\ell}\le H_{S_k}<\infty]\\ \le 2\sum_{1\le \ell\le k\le n+1}\mathbb{P}[H_{S_\ell}\le H_{S_k}<\infty] &+ \mathbb{P}[H_{S_k}\le H_{S_\ell}<\infty].\end{aligned}$$

Using next the Markov property and (2.6), we get with S and $\widetilde{S}$ two independent random walks starting from the origin,

$$\begin{aligned}\mathbb{E}[W_{n,1}^2] &\le 2\sum_{1\le \ell\le k\le n+1}\mathbb{E}[G(S_\ell)G(S_k - S_\ell)] + \mathbb{E}[G(S_k)G(S_k - S_\ell)]\\ &\le 2\sum_{\ell=1}^{n+1}\sum_{k=0}^{n}\mathbb{E}[G(S_\ell)]\cdot\mathbb{E}[G(S_k)] + \mathbb{E}[G(S_\ell+\widetilde{S}_k)G(\widetilde{S}_k)]\\ &\le 4\left(\sup_{x\in\mathbb{Z}^5}\sum_{\ell\ge 0}\mathbb{E}[G(x+S_\ell)]\right)^2 \overset{(2.9)}{=} \mathcal{O}(1).\end{aligned}$$

We proceed similarly with $W_{n,2}$. Observe first that for any $k\ge 0$,

$$0\le \varphi_k^n - \varphi(k) \le \mathbb{P}_{S_k}[H_{\mathcal{R}(-\infty,0]}<\infty \mid S] + \mathbb{P}_{S_k}[H_{\mathcal{R}[n,\infty)}<\infty \mid S].$$

Furthermore, for any $0\le \ell\le k\le n$, the two terms $\mathbb{P}_{S_\ell}[H_{\mathcal{R}(-\infty,0]}<\infty \mid S]$ and $\mathbb{P}_{S_k}[H_{\mathcal{R}[n,\infty)}<\infty \mid S]$ are independent. Therefore,

$$\begin{aligned}\mathbb{E}[W_{n,2}^2] \le \sum_{\ell=0}^{n}\sum_{k=0}^{n}\mathbb{E}[(\varphi_\ell^n - \varphi(\ell))(\varphi_k^n - \varphi(k))] &\le 2\left(\sum_{\ell=0}^{n}\mathbb{P}\left[H_{\mathcal{R}[\ell,\infty)}<\infty\right]\right)^2\\ + 4\sum_{0\le \ell\le k\le n}\mathbb{P}\left[\mathcal{R}_\infty^3\cap(S_\ell+\mathcal{R}_\infty^1)\neq\varnothing,\, \mathcal{R}_\infty^3\cap(S_k+\mathcal{R}_\infty^2)\neq\varnothing\right],\end{aligned} \tag{3.1}$$

where in the last term $\mathcal{R}^1_\infty$, $\mathcal{R}^2_\infty$ and $\mathcal{R}^3_\infty$ are the ranges of three (one-sided) independent walks, independent of $(S_n)_{n\geq 0}$, starting from the origin (denoting here $(S_{-n})_{n\geq 0}$ as another walk $(S^3_n)_{n\geq 0}$). Now (2.14) already shows that the first term on the right hand side of (3.1) is $\mathcal{O}(n)$. For the second one, note that for any $0\leq \ell\leq k\leq n$, one has

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{R}^3_\infty\cap(S_\ell+\mathcal{R}^1_\infty)\neq\varnothing,\, \mathcal{R}^3_\infty\cap(S_k+\mathcal{R}^2_\infty)\neq\varnothing\right]\\
&\leq \mathbb{E}\left[|\mathcal{R}^3_\infty\cap(S_\ell+\mathcal{R}^1_\infty)|\cdot|\mathcal{R}^3_\infty\cap(S_k+\mathcal{R}^2_\infty)|\right]\\
&= \mathbb{E}\left[\mathbb{E}[|\mathcal{R}^3_\infty\cap(S_\ell+\mathcal{R}^1_\infty)|\mid S,\, S^3]\cdot\mathbb{E}[|\mathcal{R}^3_\infty\cap(S_k+\mathcal{R}^2_\infty)|\mid S,\, S^3]\right]\\
&\overset{(2.6)}{\leq} \mathbb{E}\left[\Big(\sum_{m\geq 0} G(S^3_m-S_\ell)\Big)\Big(\sum_{m\geq 0} G(S^3_m-S_k)\Big)\right]\\
&\quad = \mathbb{E}\left[\Big(\sum_{m\geq k} G(S_m-S_{k-\ell})\Big)\Big(\sum_{m\geq k} G(S_m)\Big)\right]\\
&\leq \mathbb{E}\left[\Big(\sum_{m\geq \ell} G(S_m)\Big)^2\right]^{1/2}\cdot\mathbb{E}\left[\Big(\sum_{m\geq k} G(S_m)\Big)^2\right]^{1/2} \overset{(2.11)}{=} \mathcal{O}\left(\frac{1}{1+\sqrt{k\ell}}\right),
\end{aligned}$$

using invariance by time reversal at the penultimate line, and Cauchy-Schwarz at the last one. This concludes the proof of the lemma. □

Now as noticed in [Sch], one has

$$\mathrm{Var}(\mathcal{C}_n)=2\sum_{\ell=1}^n\sum_{k=1}^{\ell}\mathrm{Cov}(Z(0)\varphi(0),Z(k)\varphi(k))+\mathcal{O}(n).$$

We write now $\varphi(0)$ and $\varphi(k)$ as a sum of terms involving intersection and non-intersection probabilities of different parts of the path $(S_n)_{n\in\mathbb{Z}}$. For this, we consider some sequence of integers $(\varepsilon_k)_{k\geq 1}$ satisfying $k>2\varepsilon_k$, for all $k\geq 3$, and whose value will be fixed later. One first step is to reduce the influence of the random variables $Z(0)$ and $Z(k)$, which play a very minor role in the whole proof. Thus we define

$$Z_0:=\mathbf{1}\{S_\ell\neq 0,\,\forall \ell=1,\dots,\varepsilon_k\},\ \text{ and } Z_k:=\mathbf{1}\{S_\ell\neq S_k,\,\forall\ell=k+1,\dots,k+\varepsilon_k\}.$$

One has

$$\mathbb{E}[|Z(0)-Z_0|]=\mathbb{P}[0\in\mathcal{R}[\varepsilon_k+1,\infty)]\overset{(2.6)}{\leq} G_{\varepsilon_k}(0)\overset{(2.7)}{=}\mathcal{O}(\varepsilon_k^{-3/2}),$$

and the same estimate holds for $\mathbb{E}[|Z(k) - Z_k|]$, by the Markov property. Therefore,

$$\mathrm{Cov}(Z(0)\varphi(0), Z(k)\varphi(k)) = \mathrm{Cov}(Z_0\varphi(0), Z_k\varphi(k)) + \mathcal{O}(\varepsilon_k^{-3/2}).$$

Then recall that we consider a two-sided walk $(S_n)_{n\in\mathbb{Z}}$, and that $\varphi(0) = \mathbb{P}[H^+_{\mathcal{R}(-\infty,\infty)} = \infty \mid S]$. Thus one can decompose $\varphi(0)$ as follows:

$$\varphi(0) = \varphi_0 - \varphi_1 - \varphi_2 - \varphi_3 + \varphi_{1,2} + \varphi_{1,3} + \varphi_{2,3} - \varphi_{1,2,3},$$

with

$$\varphi_0 := \mathbb{P}[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S], \quad \varphi_1 := \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty,\ H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],$$

$$\varphi_2 := \mathbb{P}[H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],\ \varphi_3 := \mathbb{P}[H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],$$

$$\varphi_{1,2} := \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty,\ H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty,\ H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],$$

$$\varphi_{1,3} := \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty,\ H^+_{\mathcal{R}[k+1,\infty)} < \infty,\ H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],$$

$$\varphi_{2,3} := \mathbb{P}[H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty,\ H^+_{\mathcal{R}[k+1,\infty)} < \infty,\ H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S],$$

$$\varphi_{1,2,3} := \mathbb{P}[H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]} < \infty, H^+_{\mathcal{R}[\varepsilon_k+1,k]} < \infty, H^+_{\mathcal{R}[k+1,\infty)} < \infty, H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty \mid S].$$

We decompose similarly

$$\varphi(k) = \psi_0 - \psi_1 - \psi_2 - \psi_3 + \psi_{1,2} + \psi_{1,3} + \psi_{2,3} - \psi_{1,2,3},$$

where index 0 refers to the event of avoiding $\mathcal{R}[k-\varepsilon_k, k+\varepsilon_k]$, index 1 to the event of hitting $\mathcal{R}(-\infty,-1]$, index 2 to the event of hitting $\mathcal{R}[0, k-\varepsilon_k-1]$ and index 3 to the event of hitting $\mathcal{R}[k+\varepsilon_k+1,\infty)$ (for a walk starting from S_k this time). Note that φ_0 and ψ_0 are independent. Then write

(3.2)

$$\mathrm{Cov}(Z_0\varphi(0), Z_k\varphi(k)) = -\sum_{i=1}^{3}\left(\mathrm{Cov}(Z_0\varphi_i, Z_k\psi_0) + \mathrm{Cov}(Z_0\varphi_0, Z_k\psi_i)\right)$$
$$+ \sum_{i,j=1}^{3}\mathrm{Cov}(Z_0\varphi_i, Z_k\psi_j) + \sum_{1\le i<j\le 3}\left(\mathrm{Cov}(Z_0\varphi_{i,j}, Z_k\psi_0) + \mathrm{Cov}(Z_0\varphi_0, Z_k\psi_{i,j})\right) + R_{0,k},$$

where $R_{0,k}$ is an error term. The main purpose of this paper is to prove the following estimates.

Proposition 3.2. *One has* $|R_{0,k}| = \mathcal{O}\left(\varepsilon_k^{-3/2}\right)$.

Proposition 3.3. *One has*

(i) $|\operatorname{Cov}(Z_0\varphi_{1,2}, Z_k\psi_0)| + |\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_{2,3})| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\right)$,

(ii) $|\operatorname{Cov}(Z_0\varphi_{1,3}, Z_k\psi_0)|+|\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_{1,3})| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}} \cdot \log(\frac{k}{\varepsilon_k}) + \frac{1}{\varepsilon_k^{3/4}\sqrt{k}}\right)$,

(iii) $|\operatorname{Cov}(Z_0\varphi_{2,3}, Z_k\psi_0)|+|\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_{1,2})| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}} \cdot \log(\frac{k}{\varepsilon_k}) + \frac{1}{\varepsilon_k^{3/4}\sqrt{k}}\right)$.

In the same fashion as Part (i) of the previous proposition, we show:

Proposition 3.4. *For any* $1 \le i < j \le 3$,

$$|\operatorname{Cov}(Z_0\varphi_i, Z_k\psi_j)| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\right), \quad |\operatorname{Cov}(Z_0\varphi_j, Z_k\psi_i)| = \mathcal{O}\left(\frac{1}{\varepsilon_k}\right).$$

Our last result deals with the first sum in the right-hand side of (3.2).

Proposition 3.5. *There exists a constant* $\alpha \in (0,1)$, *such that*

$$\operatorname{Cov}(Z_0\varphi_1, Z_k\psi_0) = \operatorname{Cov}(Z_0\varphi_0, Z_k\psi_3) = 0,$$

$$|\operatorname{Cov}(Z_0\varphi_2, Z_k\psi_0)| + |\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_2)| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\right),$$

$$|\operatorname{Cov}(Z_0\varphi_3, Z_k\psi_0)| + |\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_1)| = \mathcal{O}\left(\frac{\varepsilon_k^{\alpha}}{k^{1+\alpha}}\right).$$

Altogether these propositions show that $\operatorname{Var}(\operatorname{Cap}(\mathcal{R}_n)) = \mathcal{O}(n\log n)$, just by taking $\varepsilon_k := \lfloor k/4 \rfloor$.

4. Proof of Proposition 3.2.

We divide the proof into two lemmas.

Lemma 4.1. *One has*

$$\mathbb{E}[\varphi_{1,2,3}] = \mathcal{O}\left(\frac{1}{\varepsilon_k\sqrt{k}}\right), \quad \textit{and} \quad \mathbb{E}[\psi_{1,2,3}] = \mathcal{O}\left(\frac{1}{\varepsilon_k\sqrt{k}}\right).$$

Lemma 4.2. *For any* $1 \le i < j \le 3$, *and any* $1 \le \ell \le 3$,

$$\mathbb{E}[\varphi_{i,j}\psi_\ell] = \mathcal{O}\left(\varepsilon_k^{-3/2}\right), \quad \textit{and} \quad \mathbb{E}[\varphi_{i,j}]\cdot\mathbb{E}[\psi_\ell] = \mathcal{O}\left(\varepsilon_k^{-3/2}\right).$$

Observe that the $(\varphi_{i,j})_{i,j}$ and $(\psi_{i,j})_{i,j}$ have the same law (up to reordering), and similarly for the $(\varphi_i)_i$ and $(\psi_i)_i$. Furthermore, $\varphi_{i,j} \le \varphi_i$ for any i,j. Therefore by definition of $R_{0,k}$ the proof of Proposition 3.2 readily follows from these two lemmas. For their proofs, we will use the following fact.

LEMMA 4.3. *There exists $C > 0$, such that for any $x, y \in \mathbb{Z}^5$, $0 \le \ell \le m$,*

$$\sum_{i=\ell}^{m} \sum_{z \in \mathbb{Z}^5} p_i(z) G(z-y) p_{m-i}(z-x) \le \frac{C}{(1+\|x\|+\sqrt{m})^5} \left(\frac{1}{1+\|y-x\|} + \frac{1}{1+\sqrt{\ell}+\|y\|} \right).$$

PROOF. Consider first the case $\|x\| \le \sqrt{m}$. By (2.3) and Lemma 2.3,

$$\sum_{i=\ell}^{\lfloor m/2 \rfloor} \sum_{z \in \mathbb{Z}^5} p_i(z) G(z-y) p_{m-i}(z-x) \lesssim \frac{1}{1+m^{5/2}} \sum_{z \in \mathbb{Z}^5} G_\ell(z) G(z-y) \lesssim \frac{(1+m)^{-5/2}}{1+\sqrt{\ell}+\|y\|},$$

with the convention that the first sum is zero when $m < 2\ell$, and

$$\sum_{i=\lfloor m/2 \rfloor}^{m} \sum_{z \in \mathbb{Z}^5} p_i(z) G(z-y) p_{m-i}(z-x) \lesssim \frac{1}{1+m^{5/2}} \sum_{z \in \mathbb{Z}^5} G(z-y) G(z-x) \lesssim \frac{(1+m)^{-5/2}}{1+\|y-x\|}.$$

Likewise, when $\|x\| > \sqrt{m}$, applying again (2.3) and Lemma 2.3, we get

$$\sum_{i=\ell}^{m} \sum_{\|z-x\| \ge \frac{\|x\|}{2}} p_i(z) G(z-y) p_{m-i}(z-x) \lesssim \frac{1}{\|x\|^5} \sum_{z \in \mathbb{Z}^5} G_\ell(z) G(z-y) \lesssim \frac{\|x\|^{-5}}{1+\sqrt{\ell}+\|y\|},$$

$$\sum_{i=\ell}^{m} \sum_{\|z-x\| \le \frac{\|x\|}{2}} p_i(z) G(z-y) p_{m-i}(z-x) \lesssim \frac{1}{\|x\|^5} \sum_{z \in \mathbb{Z}^5} G(z-y) G(z-x) \lesssim \frac{\|x\|^{-5}}{1+\|y-x\|},$$

which concludes the proof of the lemma. □

One can now give the proof of Lemma 4.1.

PROOF OF LEMMA 4.1. Since $\varphi_{1,2,3}$ and $\psi_{1,2,3}$ have the same law, it suffices to prove the result for $\varphi_{1,2,3}$. Let $(S_n)_{n \in \mathbb{Z}}$ and $(\widetilde{S}_n)_{n \ge 0}$ be two independent random walks starting from the origin. Define

$$\tau_1 := \inf\{n \ge 1 \, : \, \widetilde{S}_n \in \mathcal{R}(-\infty, -\varepsilon_k - 1]\}, \ \tau_2 := \inf\{n \ge 1 \, : \, \widetilde{S}_n \in \mathcal{R}[\varepsilon_k + 1, k]\},$$

and

$$\tau_3 := \inf\{n \ge 1 \, : \, \widetilde{S}_n \in \mathcal{R}[k+1, \infty)\}.$$

One has

$$\mathbb{E}[\varphi_{1,2,3}] \le \sum_{i_1 \ne i_2 \ne i_3} \mathbb{P}[\tau_{i_1} \le \tau_{i_2} \le \tau_{i_3}]. \tag{4.1}$$

We first consider the term corresponding to $i_1 = 1$, $i_2 = 2$, and $i_3 = 3$. One has by the Markov property,

$$\mathbb{P}[\tau_1 \le \tau_2 \le \tau_3 < \infty] \overset{(2.13)}{\lesssim} \mathbb{E}\left[\frac{\mathbf{1}\{\tau_1 \le \tau_2 < \infty\}}{1 + \|\widetilde{S}_{\tau_2} - S_k\|}\right] \overset{(2.12)}{\lesssim} \sum_{i=\varepsilon_k}^{k} \mathbb{E}\left[\frac{G(S_i - \widetilde{S}_{\tau_1})\mathbf{1}\{\tau_1 < \infty\}}{1 + \|S_i - S_k\|}\right].$$

Now define $\mathcal{G}_i := \sigma((S_j)_{j\le i}) \vee \sigma((\widetilde{S}_n)_{n\ge 0})$, and note that τ_1 is $\mathcal{G}_i$-measurable for any $i \ge 0$. Moreover, the Markov property and (2.3) show that

$$\mathbb{E}\left[\frac{1}{1 + \|S_i - S_k\|} \mid \mathcal{G}_i\right] \lesssim \frac{1}{\sqrt{k-i}}.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}[\tau_1 \le \tau_2 \le \tau_3 < \infty] \lesssim \sum_{i=\varepsilon_k}^{k} \mathbb{E}\left[\mathbf{1}\{\tau_1 < \infty\} \cdot \frac{G(S_i - \widetilde{S}_{\tau_1})}{1 + \sqrt{k-i}}\right] \\
&\lesssim \sum_{z\in\mathbb{Z}^5} \mathbb{P}[\tau_1 < \infty,\, \widetilde{S}_{\tau_1} = z] \cdot \left(\sum_{i=\varepsilon_k}^{k/2} \frac{\mathbb{E}[G(S_i - z)]}{\sqrt{k}} + \sum_{i=k/2}^{k} \frac{\mathbb{E}[G(S_i - z)]}{1 + \sqrt{k-i}}\right) \\
&\overset{(2.9)}{\lesssim} \frac{1}{\sqrt{k\varepsilon_k}} \cdot \mathbb{P}[\tau_1 < \infty] \overset{(2.13)}{\lesssim} \frac{1}{\varepsilon_k\sqrt{k}}.
\end{aligned}$$

We consider next the term corresponding to $i_1 = 1$, $i_2 = 3$ and $i_3 = 2$, whose analysis slightly differs from the previous one. First Lemma 4.3 gives

(4.2)

$$\begin{aligned}
&\mathbb{P}[\tau_1 \le \tau_3 \le \tau_2 < \infty] = \sum_{x,y\in\mathbb{Z}^5} \mathbb{E}\left[\mathbf{1}\{\tau_1 \le \tau_3 < \infty, \widetilde{S}_{\tau_3} = y, S_k = x\} \sum_{i=\varepsilon_k}^{k} G(S_i - y)\right] \\
&= \sum_{x,y\in\mathbb{Z}^5} \left(\sum_{i=\varepsilon_k}^{k} \sum_{z\in\mathbb{Z}^5} p_i(z) G(z-y) p_{k-i}(x-z)\right) \mathbb{P}\left[\tau_1 \le \tau_3 < \infty, \widetilde{S}_{\tau_3} = y \mid S_k = x\right] \\
&\lesssim \sum_{x\in\mathbb{Z}^5} \frac{1}{(\|x\| + \sqrt{k})^5} \left(\frac{\mathbb{P}[\tau_1 \le \tau_3 < \infty \mid S_k = x]}{\sqrt{\varepsilon_k}} + \mathbb{E}\left[\frac{\mathbf{1}\{\tau_1 \le \tau_3 < \infty\}}{1 + \|\widetilde{S}_{\tau_3} - x\|} \;\Big|\; S_k = x\right]\right).
\end{aligned}$$

We then have

$$\begin{aligned}
&\mathbb{P}[\tau_1 \le \tau_3 < \infty \mid S_k = x] \overset{(2.13)}{\lesssim} \mathbb{E}\left[\frac{\mathbf{1}\{\tau_1 < \infty\}}{1 + \|\widetilde{S}_{\tau_1} - x\|}\right] \\
&\overset{(2.12)}{\lesssim} \sum_{y\in\mathbb{Z}^5} \frac{G_{\varepsilon_k}(y)G(y)}{1 + \|y - x\|} \overset{\text{Lemma 2.3}}{\lesssim} \frac{1}{(1 + \|x\|)\sqrt{\varepsilon_k}} + \sum_{\|y-x\|\le \frac{\|x\|}{2}} \frac{G_{\varepsilon_k}(y)G(y)}{1 + \|y-x\|}.
\end{aligned}$$

Moreover, when $\|x\| \geq \sqrt{\varepsilon_k}$, one has

$$\sum_{\|y-x\|\leq \frac{\|x\|}{2}} \frac{G_{\varepsilon_k}(y)G(y)}{1+\|y-x\|} \overset{(2.7)}{\lesssim} \frac{1}{\|x\|^6} \sum_{\|y-x\|\leq \frac{\|x\|}{2}} \frac{1}{1+\|y-x\|} \lesssim \frac{1}{\|x\|^2},$$

while, when $\|x\| \leq \sqrt{\varepsilon_k}$,

$$\sum_{\|y-x\|\leq \frac{\|x\|}{2}} \frac{G_{\varepsilon_k}(y)G(y)}{1+\|y-x\|} \overset{(2.7)}{\lesssim} (1+\|x\|)\varepsilon_k^{-3/2} \lesssim \frac{1}{\varepsilon_k}.$$

Therefore, it holds for any x,

$$\mathbb{P}[\tau_1 \leq \tau_3 < \infty \mid S_k = x] \lesssim \frac{1}{(1+\|x\|)\sqrt{\varepsilon_k}}. \tag{4.3}$$

Similarly, one has

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_1 \leq \tau_3 < \infty\}}{1+\|\widetilde{S}_{\tau_3}-x\|} \;\Big|\; S_k = x\right] \leq \mathbb{E}\left[\sum_{y\in\mathbb{Z}^5} \frac{G(y-\widetilde{S}_{\tau_1})G(y-x)}{1+\|y-x\|}\mathbf{1}\{\tau_1<\infty\}\right]$$

(4.4)

$$\leq \mathbb{E}\left[\frac{\mathbf{1}\{\tau_1<\infty\}}{1+\|\widetilde{S}_{\tau_1}-x\|^2}\right] \leq \sum_{y\in\mathbb{Z}^5} \frac{G_{\varepsilon_k}(y)G(y)}{1+\|y-x\|^2} \lesssim \frac{1}{(1+\|x\|^2)\sqrt{\varepsilon_k}}.$$

Injecting (4.3) and (4.4) into (4.2) finally gives

$$\mathbb{P}[\tau_1 \leq \tau_2 \leq \tau_3 < \infty] \lesssim \frac{1}{\varepsilon_k\sqrt{k}}.$$

The other terms in (4.1) are entirely similar, so this concludes the proof of the lemma. □

For the proof of Lemma 4.2, one needs some additional estimates that we state as two separate lemmas.

Lemma 4.4. *There exists a constant $C>0$, such that for any $x,y\in\mathbb{Z}^5$,*

$$\sum_{i=\varepsilon_k}^{k-\varepsilon_k}\sum_{z\in\mathbb{Z}^5} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5}\left(\frac{1}{1+\|z-x\|}+\frac{1}{\sqrt{k-i}}\right)$$
$$\leq C\cdot\begin{cases} \frac{1}{k^{5/2}}\left(\frac{1}{1+\|x\|^2}+\frac{1}{\varepsilon_k}\right)+\frac{1}{k^{3/2}\varepsilon_k^{3/2}(1+\|y-x\|)} & \text{if } \|x\|\leq\sqrt{k} \\ \frac{1}{\|x\|^5\varepsilon_k}\left(1+\frac{k}{\sqrt{\varepsilon_k}(1+\|y-x\|)}\right) & \text{if } \|x\|>\sqrt{k}. \end{cases}$$

Proof. We proceed similarly as for the proof of Lemma 4.3. Assume first that $\|x\| \leq \sqrt{k}$. On one hand, using Lemma 2.3, we get

$$\sum_{i=\varepsilon_k}^{k/2} \frac{1}{\sqrt{k-i}} \sum_{z\in\mathbb{Z}^5} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \lesssim \frac{1}{k^3} \sum_{z\in\mathbb{Z}^5} G_{\varepsilon_k}(z)G(z-y) \lesssim \frac{1}{k^{5/2}\sqrt{k\varepsilon_k}},$$

and,

$$\begin{aligned}
&\sum_{i=\varepsilon_k}^{k/2} \sum_{z\in\mathbb{Z}^5} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5(1+\|z-x\|)} \lesssim \frac{1}{k^{5/2}} \sum_{z\in\mathbb{Z}^5} \frac{G_{\varepsilon_k}(z)G(z-y)}{1+\|z-x\|} \\
&\lesssim \frac{1}{k^{5/2}} \left(\sum_{\|z-x\|\geq \frac{\|x\|}{2}} \frac{G_{\varepsilon_k}(z)G(z-y)}{1+\|z-x\|} + \sum_{\|z-x\|\leq \frac{\|x\|}{2}} \frac{G_{\varepsilon_k}(z)G(z-y)}{1+\|z-x\|} \right) \\
&\lesssim \frac{1}{k^{5/2}} \left(\frac{1}{(1+\|x\|)\sqrt{\varepsilon_k}} + \frac{1}{1+\|x\|^2} \right) \lesssim \frac{1}{k^{5/2}} \left(\frac{1}{1+\|x\|^2} + \frac{1}{\varepsilon_k} \right).
\end{aligned}$$

On the other hand, by (2.3)

$$\sum_{i=k/2}^{k-\varepsilon_k} \sum_{\|z\|>2\sqrt{k}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \left(\frac{1}{1+\|z-x\|} + \frac{1}{\sqrt{k-i}} \right) \lesssim \frac{1}{k^2} \sum_{\|z\|>2\sqrt{k}} \frac{G(z-y)}{\|z\|^5} \lesssim k^{-\frac{7}{2}}.$$

Furthermore,

$$\sum_{i=k/2}^{k-\varepsilon_k} \frac{1}{\sqrt{k-i}} \sum_{\|z\|\leq 2\sqrt{k}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \lesssim \frac{1}{k^2\varepsilon_k} \sum_{\|z\|\leq 2\sqrt{k}} \frac{G(z-y)}{1+\|z-x\|^3} \lesssim \frac{(k^2\varepsilon_k)^{-1}}{1+\|y-x\|},$$

and

$$\begin{aligned}
\sum_{i=\frac{k}{2}}^{k-\varepsilon_k} \sum_{\|z\|\leq 2\sqrt{k}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \frac{1}{1+\|z-x\|} &\lesssim \frac{1}{k^{3/2}\varepsilon_k^{3/2}} \sum_{\|z\|\leq 2\sqrt{k}} \frac{G(z-y)}{1+\|z-x\|^3} \\
&\lesssim \frac{1}{k^{3/2}\varepsilon_k^{3/2}} \frac{1}{1+\|y-x\|}.
\end{aligned}$$

Assume now that $\|x\| > \sqrt{k}$. One has on one hand, using Lemma 2.3,

$$\sum_{i=\varepsilon_k}^{k-\varepsilon_k} \sum_{\|z-x\|\geq \frac{\|x\|}{2}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \left(\frac{1}{1+\|z-x\|} + \frac{1}{\sqrt{k-i}} \right) \lesssim \frac{1}{\|x\|^5\varepsilon_k}.$$

On the other hand,

$$\sum_{i=\varepsilon_k}^{k-\varepsilon_k} \sum_{\|z-x\|\le \frac{\|x\|}{2}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \frac{1}{1+\|z-x\|} \lesssim \frac{k}{\|x\|^5 \varepsilon_k^{3/2}} \sum_{z\in\mathbb{Z}^5} \frac{G(z-y)}{1+\|z-x\|^3} \lesssim \frac{k}{\|x\|^5 \varepsilon_k^{3/2}(1+\|y-x\|)},$$

and

$$\sum_{i=\varepsilon_k}^{k-\varepsilon_k} \frac{1}{\sqrt{k-i}} \sum_{\|z-x\|\le \frac{\|x\|}{2}} \frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5} \lesssim \frac{\sqrt{k}}{\|x\|^5\varepsilon_k} \sum_{z\in\mathbb{Z}^5} \frac{G(z-y)}{1+\|y-x\|^3} \lesssim \frac{\sqrt{k}}{\|x\|^5\varepsilon_k(1+\|y-x\|)},$$

concluding the proof of the lemma. □

Lemma 4.5. *There exists a constant $C>0$, such that for any $x,y\in\mathbb{Z}^5$,*

$$\sum_{v\in\mathbb{Z}^5} \frac{1}{(\|v\|+\sqrt{k})^5}\left(\frac{1}{1+\|x-v\|}+\frac{1}{1+\|x\|}\right)\frac{1}{(\|x-v\|+\sqrt{\varepsilon_k})^5}\left(\frac{1}{1+\|y-x\|}+\frac{1}{1+\|y-v\|}\right)$$

$$\le C\cdot \begin{cases} \frac{1}{k^2\varepsilon_k}\left(\frac{1}{\sqrt{\varepsilon_k}}+\frac{1}{1+\|x\|}+\frac{1}{1+\|y-x\|}+\frac{\sqrt{\varepsilon_k}}{(1+\|x\|)(1+\|y-x\|)}\right) & \text{if } \|x\|\le\sqrt{k} \\ \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^5\sqrt{\varepsilon_k}}\left(\frac{1}{1+\|y-x\|}+\frac{1}{\sqrt{k}}\right) & \text{if } \|x\|>\sqrt{k}. \end{cases}$$

Proof. Assume first that $\|x\|\le\sqrt{k}$. In this case it suffices to notice that on one hand, for any $\alpha\in\{3,4\}$, one has

$$\sum_{\|v\|\le 2\sqrt{k}} \frac{1}{(1+\|x-v\|^\alpha)(1+\|y-v\|^{4-\alpha})} = \mathcal{O}(\sqrt{k}),$$

and on the other hand, for any $\alpha,\beta\in\{0,1\}$,

$$\sum_{\|v\|>2\sqrt{k}} \frac{1}{\|v\|^{10+\alpha}(1+\|y-v\|)^\beta} = \mathcal{O}(k^{-5/2-\alpha-\beta}).$$

Assume next that $\|x\|>\sqrt{k}$. In this case it is enough to observe that

$$\sum_{\|v\|\le\frac{\sqrt{k}}{2}} \left(\frac{1}{1+\|x-v\|}+\frac{1}{\|x\|}\right)\left(\frac{1}{1+\|y-x\|}+\frac{1}{1+\|y-v\|}\right) \lesssim \frac{k^2}{(1+\|y-x\|)},$$

$$\sum_{\|v\|\geq \frac{\sqrt{k}}{2}} \frac{1}{\|v\|^5(\sqrt{\varepsilon_k}+\|x-v\|)^5} \lesssim \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^5},$$

$$\sum_{\|v\|\geq \frac{\sqrt{k}}{2}} \frac{1}{\|v\|^5(\sqrt{\varepsilon_k}+\|x-v\|)^5(1+\|y-v\|)} \lesssim \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^5}\left(\frac{1}{\sqrt{k}}+\frac{1}{1+\|y-x\|}\right).$$

$\square$

Proof of Lemma 4.2. First note that for any ℓ, one has $\mathbb{E}[\psi_\ell]=\mathcal{O}(\varepsilon_k^{-1/2})$, by (2.14). Using also similar arguments as in the proof of Lemma 4.1, that we will not reproduce here, one can see that $\mathbb{E}[\varphi_{i,j}]=\mathcal{O}(\varepsilon_k^{-1})$, for any $i\neq j$. Thus only the terms of the form $\mathbb{E}[\varphi_{i,j}\psi_\ell]$ are at stake.

Let $(S_n)_{n\in\mathbb{Z}}$, $(\widetilde{S}_n)_{n\geq 0}$ and $(\widehat{S}_n)_{n\geq 0}$ be three independent walks starting from the origin. Recall the definition of τ_1, τ_2 and τ_3 from the proof of Lemma 4.1, and define analogously

$$\widehat{\tau}_1 := \inf\{n\geq 1 : S_k+\widehat{S}_n\in\mathcal{R}(-\infty,-1]\},\ \widehat{\tau}_2 := \inf\{n\geq 1 : S_k+\widehat{S}_n\in\mathcal{R}[0,k-\varepsilon_k-1]\},$$

and

$$\widehat{\tau}_3 := \inf\{n\geq 1 : S_k+\widehat{S}_n\in\mathcal{R}[k+\varepsilon_k+1,\infty)\}.$$

When $\ell\neq i,j$, one can take advantage of the independence between the different parts of the range of S, at least once we condition on the value of S_k. This allows for instance to write

$$\mathbb{E}[\varphi_{1,2}\psi_3]\leq \mathbb{P}[\tau_1<\infty,\ \tau_2<\infty,\ \widehat{\tau}_3<\infty] = \mathbb{P}[\tau_1<\infty,\ \tau_2<\infty]\mathbb{P}[\widehat{\tau}_3<\infty]\lesssim \varepsilon_k^{-3/2},$$

using independence for the second equality and our previous estimates for the last one. Similarly,

$$\begin{aligned}\mathbb{E}[\varphi_{1,3}\psi_2] &\leq \sum_{x\in\mathbb{Z}} \mathbb{P}[\tau_1<\infty,\ \tau_3<\infty \mid S_k=x]\times \mathbb{P}[\widehat{\tau}_2<\infty,\ S_k=x]\\ &\lesssim \sum_{x\in\mathbb{Z}^5} \frac{1}{(1+\|x\|)\sqrt{\varepsilon_k}}\cdot\frac{1}{(1+\|x\|+\sqrt{k})^5}\left(\frac{1}{1+\|x\|}+\frac{1}{\sqrt{\varepsilon_k}}\right)\lesssim \frac{1}{\varepsilon_k\sqrt{k}},\end{aligned}$$

using (4.3) and Lemma 4.3 for the second inequality. The term $\mathbb{E}[\varphi_{2,3}\psi_1]$ is handled similarly. We consider now the other cases. One has

$$\mathbb{E}[\varphi_{2,3}\psi_3]\leq \mathbb{P}[\tau_2\leq\tau_3<\infty,\ \widehat{\tau}_3<\infty]+\mathbb{P}[\tau_3\leq\tau_2<\infty,\ \widehat{\tau}_3<\infty]. \tag{4.5}$$

By using the Markov property at time τ_2, one can write

$$\mathbb{P}[\tau_2 \le \tau_3 < \infty,\, \widehat{\tau}_3 < \infty] \le \sum_{x,y\in\mathbb{Z}^5} \mathbb{E}\left[\left(\sum_{i=0}^{\infty} G(S_i - y + x)\right)\left(\sum_{j=\varepsilon_k}^{\infty} G(S_j)\right)\right] \mathbb{P}[\tau_2 < \infty, \widetilde{S}_{\tau_2} = y, S_k = x].$$

Then applying Lemmas 2.3 and 4.3, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=0}^{\varepsilon_k} G(S_i - y + x)\right)\left(\sum_{j=\varepsilon_k}^{\infty} G(S_j)\right)\right]\\
&= \sum_{v\in\mathbb{Z}^5} \mathbb{E}\left[\left(\sum_{i=0}^{\varepsilon_k} G(S_i - y + x)\right)\mathbf{1}\{S_{\varepsilon_k} = v\}\right] \mathbb{E}\left[\left(\sum_{j=0}^{\infty} G(S_j + v)\right)\right]\\
&\lesssim \sum_{v\in\mathbb{Z}^5} \frac{1}{1+\|v\|}\cdot\left(\sum_{i=0}^{\varepsilon_k} p_i(z) G(z-y+x) p_{\varepsilon_k - i}(v-z)\right)\\
&\lesssim \sum_{v\in\mathbb{Z}^5} \frac{1}{1+\|v\|}\frac{1}{(\|v\|+\sqrt{\varepsilon_k})^5}\left(\frac{1}{1+\|v-y+x\|} + \frac{1}{1+\|y-x\|}\right) \lesssim \frac{\varepsilon_k^{-1/2}}{1+\|y-x\|}.
\end{aligned} \tag{4.6}$$

Likewise,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{i=\varepsilon_k}^{\infty} G(S_i - y + x)\right)\left(\sum_{j=\varepsilon_k}^{\infty} G(S_j)\right)\right] &\le \sum_{z\in\mathbb{Z}^5} G_{\varepsilon_k}(z)\left(\frac{G(z-y+x)}{1+\|z\|} + \frac{G(z)}{1+\|z-y+x\|}\right)\\
&\lesssim \frac{1}{\sqrt{\varepsilon_k}(1+\|y-x\|)}.
\end{aligned} \tag{4.7}$$

Recall now that by (2.14), one has $\mathbb{P}[\tau_2 < \infty] \lesssim \varepsilon_k^{-1/2}$. Moreover, from the proof of Lemma 4.1, one can deduce that

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_2 < \infty\}}{\|\widetilde{S}_{\tau_2} - S_k\|}\right] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}.$$

Combining all these estimates we conclude that

$$\mathbb{P}[\tau_2 \le \tau_3 < \infty,\, \widehat{\tau}_3 < \infty] \lesssim \frac{1}{\varepsilon_k\sqrt{k}}.$$

We deal next with the second term in the right-hand side of (4.5). Applying the Markov property at time τ_3, and then Lemma 4.3, we obtain

$$\begin{aligned}
&\mathbb{P}[\tau_3 \le \tau_2 < \infty,\, \widehat{\tau}_3 < \infty] \\
&\le \sum_{x,y\in\mathbb{Z}^5} \left(\sum_{i=\varepsilon_k}^{k} \mathbb{E}[G(S_i - y)\mathbf{1}\{S_k = x\}] \right) \mathbb{P}[\tau_3 < \infty, \widehat{\tau}_3 < \infty, \widetilde{S}_{\tau_3} = y \mid S_k = x] \\
&\lesssim \sum_{x,y\in\mathbb{Z}^5} \frac{1}{(\|x\| + \sqrt{k})^5} \left(\frac{1}{1 + \|y - x\|} + \frac{1}{\sqrt{\varepsilon_k}} \right) \mathbb{P}[\tau_3 < \infty, \widehat{\tau}_3 < \infty, \widetilde{S}_{\tau_3} = y \mid S_k = x] \\
&\lesssim \sum_{x\in\mathbb{Z}^5} \frac{1}{(\|x\| + \sqrt{k})^5} \left(\frac{\mathbb{P}[\tau_3 < \infty, \widehat{\tau}_3 < \infty \mid S_k = x]}{\sqrt{\varepsilon_k}} + \mathbb{E}\left[\frac{\mathbf{1}\{\tau_3 < \infty, \widehat{\tau}_3 < \infty\}}{1 + \|\widetilde{S}_{\tau_3} - x\|} \;\Big|\; S_k = x \right] \right) \\
&\lesssim \sum_{x\in\mathbb{Z}^5} \frac{1}{(\|x\| + \sqrt{k})^5} \left(\frac{1}{\varepsilon_k(1 + \|x\|)} + \mathbb{E}\left[\frac{\mathbf{1}\{\tau_3 < \infty, \widehat{\tau}_3 < \infty\}}{1 + \|\widetilde{S}_{\tau_3} - x\|} \;\Big|\; S_k = x \right] \right), \qquad (4.8)
\end{aligned}$$

using also (4.6) and (4.7) (with $y = 0$) for the last inequality. We use now (2.8) and Lemma 2.2 to remove the denominator in the last expectation above. Define for $r \ge 0$, and $x \in \mathbb{Z}^5$,

$$\eta_r(x) := \inf\{n \ge 0 \;:\; \|\widetilde{S}_n - x\| \le r\}.$$

On the event when $r/2 \le \|\widetilde{S}_{\eta_r(x)} - x\| \le r$, one applies the Markov property at time $\eta_r(x)$, and we deduce from (2.8) and Lemma 2.2 that

$$\begin{aligned}
&\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3 < \infty,\, \widehat{\tau}_3 < \infty\}}{1 + \|\widetilde{S}_{\tau_3} - x\|} \;\Big|\; S_k = x \right] \le \frac{\mathbb{P}[\tau_3 < \infty,\, \widehat{\tau}_3 < \infty \mid S_k = x]}{1 + \|x\|} \\
&\qquad + \sum_{i=0}^{\log_2 \|x\|} \frac{\mathbb{P}\left[\tau_3 < \infty,\, \widehat{\tau}_3 < \infty,\, 2^i \le \|\widetilde{S}_{\tau_3} - x\| \le 2^{i+1} \mid S_k = x\right]}{2^i} \\
&\lesssim \frac{1}{\sqrt{\varepsilon_k}(1 + \|x\|^2)} + \sum_{i=0}^{\log_2 \|x\|} \frac{\mathbb{P}\left[\eta_{2^{i+1}}(x) \le \tau_3 < \infty,\, \widehat{\tau}_3 < \infty \mid S_k = x\right]}{2^i} \\
&\lesssim \frac{\varepsilon_k^{-1/2}}{1 + \|x\|^2} + \frac{\mathbb{P}[\widehat{\tau}_3 < \infty]}{1 + \|x\|^3} + \sum_{i=0}^{\log_2 \|x\|} \frac{2^{2i}}{1 + \|x\|^3} \max_{\|z\| \ge 2^i} \mathbb{P}_{0,0,z}\left[H_{\mathcal{R}[\varepsilon_k,\infty)} < \infty, \widetilde{H}_{\mathcal{R}_\infty} < \infty \right],
\end{aligned}$$

where in the last probability, H and $\widetilde{H}$ refer to hitting times by two independent walks, independent of S, starting respectively from the origin and from z. Then it follows from (4.6) and (4.7) that

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3<\infty,\widehat{\tau}_3<\infty\}}{1+\|\widetilde{S}_{\tau_3}-x\|}\;\Big|\;S_k=x\right]\lesssim\frac{1}{\sqrt{\varepsilon_k}(1+\|x\|^2)}. \tag{4.9}$$

Combining this with (4.8), it yields that

$$\mathbb{P}[\tau_2\le\tau_3<\infty,\widehat{\tau}_3<\infty]\lesssim\frac{1}{\varepsilon_k\sqrt{k}}.$$

The terms $\mathbb{E}[\varphi_{1,3}\psi_3]$ and $\mathbb{E}[\varphi_{1,3}\psi_1]$ are entirely similar, and we omit repeating the proof. Thus it only remains to consider the terms $\mathbb{E}[\varphi_{2,3}\psi_2]$ and $\mathbb{E}[\varphi_{1,2}\psi_2]$. Since they are also similar we only give the details for the former. We start again by writing

$$\mathbb{E}[\varphi_{2,3}\psi_2]\le\mathbb{P}[\tau_2\le\tau_3<\infty,\,\widehat{\tau}_2<\infty]+\mathbb{P}[\tau_3\le\tau_2<\infty,\,\widehat{\tau}_2<\infty]. \tag{4.10}$$

Then one has

(4.11)

$$\begin{aligned}
&\mathbb{P}[\tau_3\le\tau_2<\infty,\,\widehat{\tau}_2<\infty]\\
&\le\sum_{x,y\in\mathbb{Z}^5}\mathbb{E}\left[\left(\sum_{i=\varepsilon_k}^{k}G(S_i-y)\right)\left(\sum_{j=0}^{k-\varepsilon_k}G(S_j-x)\right)\mathbf{1}\{S_k=x\}\right]\mathbb{P}[\tau_3<\infty,\widetilde{S}_{\tau_3}=y\mid S_k=x]\\
&\le\sum_{x,y\in\mathbb{Z}^5}\left(\sum_{i=\varepsilon_k}^{k}\sum_{j=0}^{k-\varepsilon_k}\sum_{z,w\in\mathbb{Z}^5}\mathbb{P}[S_i=z,S_j=w,S_k=x]G(z-y)G(w-x)\right)\\
&\qquad\times\mathbb{P}[\tau_3<\infty,\widetilde{S}_{\tau_3}=y\mid S_k=x].
\end{aligned}$$

Now for any $x,y\in\mathbb{Z}^5$,

$$\Sigma_1(x,y):=\sum_{i=\varepsilon_k}^{k-\varepsilon_k}\sum_{j=\varepsilon_k}^{k-\varepsilon_k}\sum_{z,w\in\mathbb{Z}^5}\mathbb{P}[S_i=z,\ S_j=w,\ S_k=x]G(z-y)G(w-x)$$

$$\le 2\sum_{i=\varepsilon_k}^{k-\varepsilon_k}\sum_{z\in\mathbb{Z}^5}p_i(z)G(z-y)\left(\sum_{j=i}^{k-\varepsilon_k}\sum_{w\in\mathbb{Z}^5}p_{j-i}(w-z)G(w-x)p_{k-j}(x-w)\right)$$

$$=2\sum_{i=\varepsilon_k}^{k-\varepsilon_k}\sum_{z\in\mathbb{Z}^5}p_i(z)G(z-y)\left(\sum_{j=\varepsilon_k}^{k}\sum_{w\in\mathbb{Z}^5}p_j(w)G(w)p_{k-i-j}(w+x-z)\right)$$

$$\overset{\text{Lemma 4.3}}{\lesssim}\sum_{i=\varepsilon_k}^{k-\varepsilon_k}\sum_{z\in\mathbb{Z}^5}\frac{p_i(z)G(z-y)}{(\|z-x\|+\sqrt{k-i})^5}\left(\frac{1}{1+\|z-x\|}+\frac{1}{\sqrt{k-i}}\right)$$

$$\overset{\text{Lemma 4.4}}{\lesssim}\begin{cases}\frac{1}{k^{5/2}}\left(\frac{1}{1+\|x\|^2}+\frac{1}{\varepsilon_k}\right)+\frac{1}{k^{3/2}\varepsilon_k^{3/2}(1+\|y-x\|)} & \text{if } \|x\|\le\sqrt{k}\\ \frac{1}{\|x\|^5\varepsilon_k}\left(1+\frac{k}{\sqrt{\varepsilon_k}(1+\|y-x\|)}\right) & \text{if } \|x\|>\sqrt{k}.\end{cases}$$

We also have

$$\Sigma_2(x,y):=\sum_{i=k-\varepsilon_k}^{k}\sum_{j=0}^{k-\varepsilon_k}\sum_{z,w\in\mathbb{Z}^5}\mathbb{P}[S_i=z,S_j=w,S_k=x]G(z-y)G(w-x)$$

$$=\sum_{i=k-\varepsilon_k}^{k}\sum_{j=0}^{k-\varepsilon_k}\sum_{z,v,w\in\mathbb{Z}^5}\mathbb{P}[S_j=w,S_{k-\varepsilon_k}=v,S_i=z,S_k=x]G(z-y)G(w-x)$$

$$=\sum_{v\in\mathbb{Z}^5}\left(\sum_{j=0}^{k-\varepsilon_k}\sum_{w\in\mathbb{Z}^5}p_j(w)p_{k-\varepsilon_k-j}(v-w)G(w-x)\right)\left(\sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5}p_i(z-v)p_{\varepsilon_k-i}(x-z)G(z-y)\right),$$

and applying then Lemmas 4.3 and 4.5, gives

$$\Sigma_2(x,y)$$
$$\lesssim\sum_{v\in\mathbb{Z}^5}\frac{1}{(\|v\|+\sqrt{k})^5}\left(\frac{1}{1+\|x-v\|}+\frac{1}{1+\|x\|}\right)\frac{1}{(\|x-v\|+\sqrt{\varepsilon_k})^5}\left(\frac{1}{1+\|y-x\|}+\frac{1}{1+\|y-v\|}\right)$$

$$\lesssim\begin{cases}\frac{1}{k^2\varepsilon_k}\left(\frac{1}{\sqrt{\varepsilon_k}}+\frac{1}{1+\|x\|}+\frac{1}{1+\|y-x\|}+\frac{\sqrt{\varepsilon_k}}{(1+\|x\|)(1+\|y-x\|)}\right) & \text{if } \|x\|\le\sqrt{k}\\ \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^5\sqrt{\varepsilon_k}}\left(\frac{1}{1+\|y-x\|}+\frac{1}{\sqrt{k}}\right) & \text{if } \|x\|>\sqrt{k}.\end{cases}$$

Likewise, by reversing time, one has

$$\Sigma_3(x,y) := \sum_{i=\varepsilon_k}^{k} \sum_{j=0}^{\varepsilon_k} \sum_{z,w\in\mathbb{Z}^5} \mathbb{P}[S_i = z, S_j = w, S_k = x] G(z-y) G(w-x)$$

$$= \sum_{i=0}^{k-\varepsilon_k} \sum_{j=k-\varepsilon_k}^{k} \sum_{z,v,w\in\mathbb{Z}^5} \mathbb{P}[S_i = z-x, S_{k-\varepsilon_k} = v-x, S_j = w-x, S_k = -x] G(z-y) G(w-x)$$

$$= \sum_{v\in\mathbb{Z}^5} \left(\sum_{i=0}^{k-\varepsilon_k} \sum_{z\in\mathbb{Z}^5} p_i(z-x) p_{k-\varepsilon_k-i}(v-z) G(z-y) \right) \left(\sum_{j=0}^{\varepsilon_k} \sum_{w\in\mathbb{Z}^5} p_j(w-v) p_{\varepsilon_k-j}(w) G(w-x) \right)$$

$$\lesssim \sum_{v\in\mathbb{Z}^5} \frac{1}{(\|v-x\|+\sqrt{k})^5} \left(\frac{1}{1+\|y-v\|} + \frac{1}{1+\|y-x\|} \right) \frac{1}{(\|v\|+\sqrt{\varepsilon_k})^5} \left(\frac{1}{1+\|x\|} + \frac{1}{1+\|x-v\|} \right),$$

and then a similar argument as in the proof of Lemma 4.5 gives the same bound for $\Sigma_3(x,y)$ as for $\Sigma_2(x,y)$. Now recall that (4.11) yields

$$\mathbb{P}[\tau_3 \le \tau_2 < \infty, \widehat{\tau}_2 < \infty] \le \sum_{x,y\in\mathbb{Z}^5} \left(\Sigma_1(x,y) + \Sigma_2(x,y) + \Sigma_3(x,y)\right) \mathbb{P}[\tau_3 < \infty, \widetilde{S}_{\tau_3} = y \mid S_k = x].$$

Recall also that by (2.13),

$$\mathbb{P}[\tau_3 < \infty \mid S_k = x] \lesssim \frac{1}{1+\|x\|},$$

and

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3 < \infty\}}{1+\|\widetilde{S}_{\tau_3} - x\|} \,\Big|\, S_k = x \right] \le \sum_{y\in\mathbb{Z}^5} \frac{G(y)G(y-x)}{1+\|y-x\|} \lesssim \frac{1}{1+\|x\|^2}.$$

Furthermore, for any $\alpha \in \{1,2,3\}$, and any $\beta \ge 6$,

$$\sum_{\|x\|\le\sqrt{k}} \frac{1}{1+\|x\|^\alpha} \lesssim k^{\frac{5-\alpha}{2}}, \qquad \sum_{\|x\|\ge\sqrt{k}} \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^\beta} \le \sum_{\|x\|\ge\sqrt{\varepsilon_k}} \frac{\log(\frac{\|x\|}{\sqrt{\varepsilon_k}})}{\|x\|^\beta} \lesssim \varepsilon_k^{\frac{5-\beta}{2}}.$$

Putting all these pieces together we conclude that

$$\mathbb{P}[\tau_3 \le \tau_2 < \infty,\, \widehat{\tau}_2 < \infty] \lesssim \varepsilon_k^{-3/2}.$$

We deal now with the other term in (4.10). As previously, we first write using the Markov property, and then using (2.12) and Lemma 2.3,

$$\mathbb{P}[\tau_2 \le \tau_3 < \infty,\, \widehat{\tau}_2 < \infty] \le \mathbb{E}\left[\frac{\mathbf{1}\{\tau_2 < \infty,\, \widehat{\tau}_2 < \infty\}}{1+\|\widetilde{S}_{\tau_2} - S_k\|} \right].$$

Then using (2.8) and Lemma 2.2 one can handle the denominator in the last expectation, the same way as for (4.9), and we conclude similarly that

$$\mathbb{P}[\tau_2 \leq \tau_3 < \infty, \, \widehat{\tau}_2 < \infty] \lesssim \varepsilon_k^{-3/2}.$$

This finishes the proof of Lemma 4.2. $\square$

5. Proof of Propositions 3.3 and 3.4. For the proof of these propositions we shall need the following estimate.

Lemma 5.1. *One has for all $x, y \in \mathbb{Z}^5$,*

$$\sum_{i=k-\varepsilon_k}^{k} \mathbb{E}\left[G(S_i - y)\mathbf{1}\{S_k = x\}\right]$$
$$\lesssim \varepsilon_k \left(\frac{\log(2 + \frac{\|y-x\|}{\sqrt{\varepsilon_k}})}{(\|x\| + \sqrt{k})^5(\|y-x\| + \sqrt{\varepsilon_k})^3} + \frac{\log(2 + \frac{\|y\|}{\sqrt{k}})}{(\|x\| + \sqrt{\varepsilon_k})^5(\|y\| + \sqrt{k})^3} \right).$$

Proof. One has using (2.3) and (2.7),

$$\sum_{i=k-\varepsilon_k}^{k} \mathbb{E}\left[G(S_i - y)\mathbf{1}\{S_k = x\}\right] = \sum_{i=k-\varepsilon_k}^{k} \sum_{z\in\mathbb{Z}^5} p_i(z)G(z-y)p_{k-i}(x-z)$$
$$\lesssim \sum_{z\in\mathbb{Z}^5} \frac{\varepsilon_k}{(\|z\| + \sqrt{k})^5(1 + \|z-y\|^3)(\|x-z\| + \sqrt{\varepsilon_k})^5}$$
$$\lesssim \frac{1}{\varepsilon_k^{3/2}(\|x\| + \sqrt{k})^5} \sum_{\|z-x\|\leq\sqrt{\varepsilon_k}} \frac{1}{1 + \|z-y\|^3}$$
$$+ \frac{\varepsilon_k}{(\|x\| + \sqrt{k})^5} \sum_{\sqrt{\varepsilon_k}\leq\|z-x\|\leq\frac{\|x\|}{2}} \frac{1}{(1 + \|z-y\|^3)(1 + \|z-x\|^5)}$$
$$+ \frac{\varepsilon_k}{(\|x\| + \sqrt{\varepsilon_k})^5} \sum_{\|z-x\|\geq\frac{\|x\|}{2}} \frac{1}{(\|z\| + \sqrt{k})^5(1 + \|z-y\|^3)}.$$

Then it suffices to observe that

$$\sum_{\|z-x\|\leq\sqrt{\varepsilon_k}} \frac{1}{1 + \|z-y\|^3} \lesssim \frac{\varepsilon_k^{5/2}}{(\|y-x\| + \sqrt{\varepsilon_k})^3},$$

$$\sum_{\sqrt{\varepsilon_k}\leq\|z-x\|\leq\frac{\|x\|}{2}} \frac{1}{(1 + \|z-y\|^3)(1 + \|z-x\|^5)} \lesssim \frac{\log(2 + \frac{\|y-x\|}{\sqrt{\varepsilon_k}})}{(\|y-x\| + \sqrt{\varepsilon_k})^3},$$

$$\sum_{z\in\mathbb{Z}^5} \frac{1}{(\|z\|+\sqrt{k})^5(1+\|z-y\|^3)} \lesssim \frac{\log(2+\frac{\|y\|}{\sqrt{k}})}{(\|y\|+\sqrt{k})^3}.$$

□

Proof of Proposition 3.3 (i). This part is the easiest: it suffices to observe that $\varphi_{1,2}$ is a sum of one term which is independent of $Z_k\psi_0$ and another one, whose expectation is negligible. To be more precise, define

$$\varphi_{1,2}^0 := \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]}=\infty,\ H^+_{\mathcal{R}(-\infty,-\varepsilon_k-1]}<\infty,\ H^+_{\mathcal{R}[\varepsilon_k+1,k-\varepsilon_k-1]}<\infty \mid S\right],$$

and note that $Z_0\varphi^0_{1,2}$ is independent of $Z_k\psi_0$. It follows that

$$|\operatorname{Cov}(Z_0\varphi_{1,2},Z_k\psi_0)| = |\operatorname{Cov}(Z_0(\varphi_{1,2}-\varphi^0_{1,2}),Z_k\psi_0)| \le \mathbb{P}\left[\tau_1<\infty,\ \tau_*<\infty\right],$$

with τ_1 and τ_* the hitting times respectively of $\mathcal{R}(-\infty,-\varepsilon_k]$ and $\mathcal{R}[k-\varepsilon_k,k]$ by another walk $\widetilde{S}$ starting from the origin, independent of S. Now, using (2.3), we get

$$\begin{aligned}
\mathbb{P}[\tau_1\le\tau_*<\infty] &\le \mathbb{E}\left[\mathbf{1}\{\tau_1<\infty\}\left(\sum_{i=k-\varepsilon_k}^{k} G(S_i-\widetilde{S}_{\tau_1})\right)\right]\\
&\le \sum_{y\in\mathbb{Z}^5}\left(\sum_{z\in\mathbb{Z}^5}\sum_{i=k-\varepsilon_k}^{k} p_i(z)G(z-y)\right)\mathbb{P}[\tau_1<\infty,\ \widetilde{S}_{\tau_1}=y]\\
&\lesssim \frac{\varepsilon_k}{k^{3/2}}\,\mathbb{P}[\tau_1<\infty] \overset{(2.14)}{\lesssim} \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.
\end{aligned}$$

Likewise, using now Lemma 2.3,

$$\begin{aligned}
\mathbb{P}[\tau_*\le\tau_1<\infty] &\le \mathbb{E}\left[\mathbf{1}\{\tau_*<\infty\}\left(\sum_{i=\varepsilon_k}^{\infty} G(S_{-i}-\widetilde{S}_{\tau_*})\right)\right]\\
&\le \sum_{y\in\mathbb{Z}^5}\left(\sum_{z\in\mathbb{Z}^5} G_{\varepsilon_k}(z)G(z-y)\right)\mathbb{P}[\tau_*<\infty,\ \widetilde{S}_{\tau_*}=y]\\
&\lesssim \frac{1}{\sqrt{\varepsilon_k}}\,\mathbb{P}[\tau_*<\infty] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}},
\end{aligned}$$

and the first part of (i) follows. But since Z_0 and Z_k have played no role here, the same computation gives the result for the covariance between $Z_0\varphi_0$ and $Z_k\psi_{2,3}$ as well. □

Proof of Proposition 3.3 (ii)-(iii). These parts are more involved. Since they are entirely similar, we only prove (iii), and as for (i) we only give the details for the covariance between $Z_0\varphi_{2,3}$ and $Z_k\psi_0$, since Z_0 and Z_k will not play any role here. We define similarly as in the proof of (i),

$$\varphi_{2,3}^0 := \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty,\, H^+_{\mathcal{R}[\varepsilon_k,k-\varepsilon_k]} < \infty,\, H^+_{\mathcal{R}[k+\varepsilon_k,\infty)} < \infty \mid S\right],$$

but observe that this time, the term $\varphi_{2,3}^0$ is no more independent of ψ_0. This entails some additional difficulty, on which we shall come back later, but first we show that one can indeed replace $\varphi_{2,3}$ by $\varphi_{2,3}^0$ in the computation of the covariance. For this, denote respectively by τ_2, τ_3, τ_* and τ_{**} the hitting times of $\mathcal{R}[\varepsilon_k,k]$, $\mathcal{R}[k,\infty)$, $\mathcal{R}[k-\varepsilon_k,k]$, and $\mathcal{R}[k,k+\varepsilon_k]$ by $\widetilde{S}$. One has

$$\mathbb{E}[|\varphi_{2,3}-\varphi_{2,3}^0|] \le \mathbb{P}[\tau_2<\infty,\, \tau_{**}<\infty] + \mathbb{P}[\tau_3<\infty,\, \tau_*<\infty].$$

Using (2.3), (2.12) and Lemma 2.3, we get

$$\begin{aligned}\mathbb{P}[\tau_*\le\tau_3<\infty] &\le \mathbb{E}\left[\frac{\mathbf{1}\{\tau_*<\infty\}}{1+\|\widetilde{S}_{\tau_*}-S_k\|}\right] \le \sum_{i=k-\varepsilon_k}^{k} \mathbb{E}\left[\frac{G(S_i)}{1+\|S_i-S_k\|}\right]\\ &\lesssim \sum_{i=k-\varepsilon_k}^{k} \mathbb{E}\left[\frac{G(S_i)}{1+\sqrt{k-i}}\right] \lesssim \sum_{z\in\mathbb{Z}^5}\sum_{i=k-\varepsilon_k}^{k}\frac{p_i(z)G(z)}{1+\sqrt{k-i}}\\ &\lesssim \sqrt{\varepsilon_k}\sum_{z\in\mathbb{Z}^5}\frac{1}{(\|z\|+\sqrt{k})^5}G(z)\lesssim\frac{\sqrt{\varepsilon_k}}{k^{3/2}}.\end{aligned}$$

Next, applying Lemma 5.1, we get

$$\begin{aligned}&\mathbb{P}[\tau_3\le\tau_*<\infty]\\ &\le \sum_{x,y\in\mathbb{Z}^5}\mathbb{E}\left[\left(\sum_{i=k-\varepsilon_k}^{k}G(S_i-y)\right)\mathbf{1}\{S_k=x\}\right]\mathbb{P}[\tau_3<\infty, \widetilde{S}_{\tau_3}=y\mid S_k=x]\\ &\lesssim \varepsilon_k\sum_{x\in\mathbb{Z}^5}\left(\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3<\infty\}\log(2+\frac{\|\widetilde{S}_{\tau_3}-x\|}{\sqrt{\varepsilon_k}})}{(\|x\|+\sqrt{k})^5(\sqrt{\varepsilon_k}+\|\widetilde{S}_{\tau_3}-x\|)^3}\,\Big|\, S_k=x\right]\right.\\ &\qquad\left.+\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3<\infty\}\log(2+\frac{\|\widetilde{S}_{\tau_3}\|}{\sqrt{k}})}{(\|x\|+\sqrt{\varepsilon_k})^5(\sqrt{k}+\|\widetilde{S}_{\tau_3}\|)^3}\,\Big|\, S_k=x\right]\right).\end{aligned}$$

Moreover,

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3<\infty\}\log(2+\frac{\|\widetilde{S}_{\tau_3}-x\|}{\sqrt{\varepsilon_k}})}{(\sqrt{\varepsilon_k}+\|\widetilde{S}_{\tau_3}-x\|)^3}\,\Big|\,S_k=x\right]\overset{(2.12)}{\leq}\sum_{y\in\mathbb{Z}^5}\frac{G(y)G(y-x)\log(2+\frac{\|y-x\|}{\sqrt{\varepsilon_k}})}{(\sqrt{\varepsilon_k}+\|y-x\|)^3}$$
$$\lesssim\frac{1}{\sqrt{\varepsilon_k}(1+\|x\|)^3},$$

and

$$\mathbb{E}\left[\frac{\mathbf{1}\{\tau_3<\infty\}\log(2+\frac{\|\widetilde{S}_{\tau_3}\|}{\sqrt{k}})}{(\sqrt{k}+\|\widetilde{S}_{\tau_3}\|)^3}\,\Big|\,S_k=x\right]\overset{(2.12)}{\leq}\sum_{y\in\mathbb{Z}^5}\frac{G(y)G(y-x)\log(2+\frac{\|y\|}{\sqrt{k}})}{(\sqrt{k}+\|y\|)^3}$$
$$\lesssim\frac{1}{\sqrt{k}(1+\|x\|)(\sqrt{k}+\|x\|)^2}.$$

Furthermore, it holds

$$\sum_{x\in\mathbb{Z}^5}\frac{1}{(\|x\|+\sqrt{k})^5(1+\|x\|)^3}\lesssim\frac{1}{k^{3/2}},$$
$$\sum_{x\in\mathbb{Z}^5}\frac{1}{(\|x\|+\sqrt{\varepsilon_k})^5(1+\|x\|)(\sqrt{k}+\|x\|)^2}\lesssim\frac{1}{\sqrt{k\varepsilon_k}},$$

which altogether proves that

$$\mathbb{P}[\tau_3\leq\tau_*<\infty]\lesssim\frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

Likewise,

$$\mathbb{P}[\tau_2\leq\tau_{**}<\infty]\leq\sum_{x,y\in\mathbb{Z}^5}\mathbb{E}\left[\sum_{i=0}^{\varepsilon_k}G(S_i-y+x)\right]\mathbb{P}[\tau_2<\infty,\widetilde{S}_{\tau_2}=y,S_k=x],$$

and using (2.7), we get

$$\mathbb{E}\left[\sum_{i=0}^{\varepsilon_k}G(S_i-y+x)\right]=\sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5}p_i(z)G(z-y+x)$$
$$\lesssim\sum_{\|z\|\leq\sqrt{\varepsilon_k}}G(z)G(z-y+x)+\varepsilon_k\sum_{\|z\|\geq\sqrt{\varepsilon_k}}\frac{G(z-y+x)}{\|z\|^5}$$
$$\lesssim\frac{\varepsilon_k}{(\|y-x\|+\sqrt{\varepsilon_k})^2(1+\|y-x\|)}+\varepsilon_k\,\frac{\log\left(2+\frac{\|y-x\|}{\sqrt{\varepsilon_k}}\right)}{(\|y-x\|+\sqrt{\varepsilon_k})^3}$$
$$\lesssim\varepsilon_k\,\frac{\log\left(2+\frac{\|y-x\|}{\sqrt{\varepsilon_k}}\right)}{(\|y-x\|+\sqrt{\varepsilon_k})^2(1+\|y-x\|)}.$$

Therefore, using the Markov property,

$$\mathbb{P}[\tau_2 \le \tau_{**} < \infty] \lesssim \varepsilon_k \cdot \mathbb{E}\left[\frac{\log\left(2 + \frac{\|\widetilde{S}_{\tau_2} - S_k\|}{\sqrt{\varepsilon_k}}\right) \cdot \mathbf{1}\{\tau_2 < \infty\}}{(\|\widetilde{S}_{\tau_2} - S_k\| + \sqrt{\varepsilon_k})^2(1 + \|\widetilde{S}_{\tau_2} - S_k\|)}\right]$$

$$\lesssim \varepsilon_k \sum_{i=\varepsilon_k}^{k} \mathbb{E}[G(S_i)] \cdot \mathbb{E}\left[\frac{\log\left(2 + \frac{\|S_{k-i}\|}{\sqrt{\varepsilon_k}}\right)}{(\|S_{k-i}\| + \sqrt{\varepsilon_k})^2(1 + \|S_{k-i}\|)}\right].$$

Furthermore, using (2.3) we obtain after straightforward computations,

$$\mathbb{E}\left[\frac{\log\left(2 + \frac{\|S_{k-i}\|}{\sqrt{\varepsilon_k}}\right)}{(\|S_{k-i}\| + \sqrt{\varepsilon_k})^2(1 + \|S_{k-i}\|)}\right] \lesssim \frac{\log\left(2 + \frac{k-i}{\varepsilon_k}\right)}{\sqrt{k-i}(\varepsilon_k + k - i)},$$

and using in addition (2.9), we conclude that

$$\mathbb{P}[\tau_2 \le \tau_{**} < \infty] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}} \cdot \log(\frac{k}{\varepsilon_k}).$$

Similarly, using Lemma 4.3 we get

$$\mathbb{P}[\tau_{**} \le \tau_2 < \infty]$$

$$= \sum_{x,y\in\mathbb{Z}^5} \mathbb{P}[\tau_{**} < \infty,\, \widetilde{S}_{\tau_{**}} = y \mid S_k = x] \cdot \mathbb{E}\left[\sum_{i=\varepsilon_k}^{k} G(S_i - y)\mathbf{1}\{S_k = x\}\right]$$

$$\lesssim \sum_{x\in\mathbb{Z}^5} \frac{1}{(\|x\| + \sqrt{k})^5}\left(\mathbb{E}\left[\frac{1\{\tau_{**} < \infty\}}{1 + \|\widetilde{S}_{\tau_{**}} - x\|}\,\Big|\, S_k = x\right] + \frac{\mathbb{P}[\tau_{**} < \infty \mid S_k = x]}{\sqrt{\varepsilon_k}}\right).$$

Moreover, one has

$$\mathbb{P}[\tau_{**} < \infty \mid S_k = x] \le \sum_{i=0}^{\varepsilon_k} \mathbb{E}[G(S_i + x)] \lesssim \sum_{i=0}^{\varepsilon_k} \sum_{z\in\mathbb{Z}^5} \frac{1}{(1 + \|z\| + \sqrt{i})^5(1 + \|z + x\|^3)}$$

$$\lesssim \sum_{\|z\|\le\sqrt{\varepsilon_k}} \frac{1}{(1 + \|z\|^3)(1 + \|z + x\|^3)} + \sum_{\|z\|\ge\sqrt{\varepsilon_k}} \frac{\varepsilon_k}{\|z\|^5(1 + \|z + x\|^3)}$$

$$\lesssim \frac{\varepsilon_k \log(2 + \frac{\|x\|}{\sqrt{\varepsilon_k}})}{(\sqrt{\varepsilon_k} + \|x\|)^2(1 + \|x\|)},$$

and likewise

$$\mathbb{E}\left[\frac{1\{\tau_{**}<\infty\}}{1+\|\widetilde{S}_{\tau_{**}}-x\|}\,\Big|\,S_k=x\right]\le\sum_{i=0}^{\varepsilon_k}\sum_{z\in\mathbb{Z}^5}\frac{1}{(1+\|z\|+\sqrt{i})^5(1+\|z-x\|^3)(1+\|z\|)}$$
$$\lesssim\sum_{\|z\|\le\sqrt{\varepsilon_k}}\frac{1}{(1+\|z\|^4)(1+\|z-x\|^3)}+\sum_{\|z\|\ge\sqrt{\varepsilon_k}}\frac{\varepsilon_k}{\|z\|^6(1+\|z-x\|^3)}$$
$$\lesssim\frac{\sqrt{\varepsilon_k}}{(\|x\|+\sqrt{\varepsilon_k})(1+\|x\|^2)}.$$

Then it follows as above that

$$\mathbb{P}[\tau_{**}\le\tau_2<\infty]\lesssim\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\cdot\log(\frac{k}{\varepsilon_k}).$$

In other words we have proved that

$$\mathbb{E}[|\varphi_{2,3}-\varphi_{2,3}^0|]\lesssim\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\cdot\log(\frac{k}{\varepsilon_k}).$$

We then have to deal with the fact that $Z_0\varphi_{2,3}^0$ is not really independent of $Z_k\psi_0$. Therefore, we introduce the new random variables

$$\widetilde{Z}_k:=\mathbf{1}\{S_i\ne S_k\ \forall i=k+1,\dots,\varepsilon_k'\},\ \widetilde{\psi}_0:=\mathbb{P}_{S_k}\left[H^+_{\mathcal{R}[k-\varepsilon_k',k+\varepsilon_k']}=\infty\mid S\right],$$

where $(\varepsilon_k')_{k\ge0}$ is another sequence of integers, whose value will be fixed later. For the moment we only assume that it satisfies $\varepsilon_k'\le\varepsilon_k/4$, for all k. One has by (2.7) and (2.14),

$$\mathbb{E}[|Z_k\psi_0-\widetilde{Z}_k\widetilde{\psi}_0|]\lesssim\frac{1}{\sqrt{\varepsilon_k'}}.\tag{5.1}$$

Furthermore, for any $y\in\mathbb{Z}^5$,

(5.2)
$$\mathbb{E}\left[\varphi_{2,3}^0\mid S_{k+\varepsilon_k}-S_{k-\varepsilon_k}=y\right]=\sum_{x\in\mathbb{Z}^5}\mathbb{E}\left[\varphi_{2,3}^0\mathbf{1}\{S_{k-\varepsilon_k}=x\}\mid S_{k+\varepsilon_k}-S_{k-\varepsilon_k}=y\right]$$
$$\le\sum_{x\in\mathbb{Z}^5}\mathbb{P}\left[\widetilde{\mathcal{R}}_\infty\cap\mathcal{R}[\varepsilon_k,k-\varepsilon_k]\ne\varnothing,\,\widetilde{\mathcal{R}}_\infty\cap(x+y+\widehat{\mathcal{R}}_\infty)\ne\varnothing,\,S_{k-\varepsilon_k}=x\right],$$

where in the last probability, $\widetilde{\mathcal{R}}_\infty$ and $\widehat{\mathcal{R}}_\infty$ are the ranges of two independent walks, independent of S, starting from the origin. Now x and y being fixed, define

$$\tau_1:=\inf\{n\ge0:\widetilde{S}_n\in\mathcal{R}[\varepsilon_k,k-\varepsilon_k]\},\ \tau_2:=\inf\{n\ge0:\widetilde{S}_n\in(x+y+\widehat{\mathcal{R}}_\infty)\}.$$

Applying (2.12) and the Markov property we get

$$\begin{aligned}\mathbb{P}[\tau_1 \le \tau_2 < \infty,\, S_{k-\varepsilon_k} = x] &\le \mathbb{E}\left[\frac{\mathbf{1}\{\tau_1 < \infty,\, S_{k-\varepsilon_k} = x\}}{1 + \|\widetilde{S}_{\tau_1} - (x+y)\|}\right] \\ &\le \sum_{i=\varepsilon_k}^{k-\varepsilon_k} \sum_{z\in\mathbb{Z}^5} \frac{p_i(z)G(z)p_{k-\varepsilon_k-i}(x-z)}{1+\|z-(x+y)\|} \\ &\lesssim \frac{1}{(\|x\|+\sqrt{k})^5}\left(\frac{1}{\sqrt{\varepsilon_k}(1+\|x+y\|)} + \frac{1}{1+\|x\|^2}\right),\end{aligned}$$

using also similar computations as in the proof of Lemma 4.3 for the last inequality. It follows that for some constant $C > 0$, independent of y,

$$\sum_{x\in\mathbb{Z}^5} \mathbb{P}[\tau_1 \le \tau_2 < \infty,\, S_{k-\varepsilon_k} = x] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}.$$

On the other hand, by Lemmas 4.3 and 2.5,

$$\begin{aligned}\mathbb{P}[\tau_2 \le \tau_1 < \infty,\, S_{k-\varepsilon_k} = x] &\lesssim \frac{1}{(\|x\|+\sqrt{k})^5}\left(\mathbb{E}\left[\frac{\mathbf{1}\{\tau_2<\infty\}}{1+\|\widetilde{S}_{\tau_2} - x\|}\right] + \frac{\mathbb{P}[\tau_2<\infty]}{\sqrt{\varepsilon_k}}\right) \\ &\lesssim \frac{1}{(\|x\|+\sqrt{k})^5}\left(\frac{1}{\sqrt{\varepsilon_k}(1+\|x+y\|)} + \frac{1}{1+\|x\|^2}\right),\end{aligned}$$

and it follows as well that

$$\sum_{x\in\mathbb{Z}^5} \mathbb{P}[\tau_2 \le \tau_1 < \infty, S_{k-\varepsilon_k} = x] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}.$$

Coming back to (5.2), we deduce that

$$\mathbb{E}\left[\varphi_{2,3}^0 \mid S_{k+\varepsilon_k} - S_{k-\varepsilon_k} = y\right] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}, \tag{5.3}$$

with an implicit constant independent of y. Together with (5.1), this gives

$$\begin{aligned}&\mathbb{E}\left[\varphi_{2,3}^0 |Z_k\psi_0 - \widetilde{Z}_k\widetilde{\psi}_0|\right] \\ &= \sum_{y\in\mathbb{Z}^5} \mathbb{E}\left[\varphi_{2,3}^0 \mid S_{k+\varepsilon_k} - S_{k-\varepsilon_k} = y\right]\cdot \mathbb{E}\left[|Z_k\psi_0 - \widetilde{Z}_k\widetilde{\psi}_0|\mathbf{1}\{S_{k+\varepsilon_k} - S_{k-\varepsilon_k} = y\}\right] \\ &\lesssim \frac{1}{\sqrt{k\varepsilon_k\varepsilon_k'}}.\end{aligned}$$

Thus at this point we have shown that

$$\mathrm{Cov}(Z_0\varphi_{2,3}, Z_k\psi_0) = \mathrm{Cov}(Z_0\varphi_{2,3}^0, \widetilde{Z}_k\widetilde{\psi}_0) + \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\cdot \log(\frac{k}{\varepsilon_k}) + \frac{1}{\sqrt{k\varepsilon_k\varepsilon_k'}}\right).$$

Note next that

$$\begin{aligned}\mathrm{Cov}(Z_0\varphi_{2,3}^0, \widetilde{Z}_k\widetilde{\psi}_0) = \sum_{y,z\in\mathbb{Z}^5} &\mathbb{E}\left[Z_0\varphi_{2,3}^0 \mid S_{k+\varepsilon_k} - S_{k-\varepsilon_k'} = y\right] \\ &\times \mathbb{E}\left[\widetilde{Z}_k\widetilde{\psi}_0 \mathbf{1}\{S_{k+\varepsilon_k'} - S_{k-\varepsilon_k'} = z\}\right]\left(p_{\varepsilon_k-\varepsilon_k'}(y-z) - p_{\varepsilon_k+\varepsilon_k'}(y)\right).\end{aligned}$$

Moreover, one can show exactly as (5.3) that uniformly in y,

$$\mathbb{E}\left[\varphi_{2,3}^0 \mid S_{k+\varepsilon_k} - S_{k-\varepsilon_k'} = y\right] \lesssim \frac{1}{\sqrt{k\varepsilon_k}}.$$

Therefore by using also (2.5) and Theorem 2.1, we see that

$$\begin{aligned}&|\mathrm{Cov}(Z_0\varphi_{2,3}^0, \widetilde{Z}_k\widetilde{\psi}_0)| \\ &\lesssim \frac{1}{\sqrt{k\varepsilon_k}} \sum_{\|y\|\le\varepsilon_k^{\frac{6}{10}}} \sum_{\|z\|\le\varepsilon_k^{\frac{1}{10}}\cdot\sqrt{\varepsilon_k'}} p_{2\varepsilon_k'}(z)\,|\overline{p}_{\varepsilon_k-\varepsilon_k'}(y-z) - \overline{p}_{\varepsilon_k+\varepsilon_k'}(y)| + \frac{1}{\varepsilon_k\sqrt{k}}.\end{aligned}$$

Now straightforward computations show that for y and z as in the two sums above, one has for some constant $c>0$,

$$|\overline{p}_{\varepsilon_k-\varepsilon_k'}(y-z) - \overline{p}_{\varepsilon_k+\varepsilon_k'}(y)| \lesssim \left(\frac{\|z\|}{\sqrt{\varepsilon_k}} + \frac{\varepsilon_k'}{\varepsilon_k}\right)\overline{p}_{\varepsilon_k-\varepsilon_k'}(cy),$$

at least when $\varepsilon_k' \le \sqrt{\varepsilon_k}$, as will be assumed in a moment. Using also that $\sum_z \|z\| p_{2\varepsilon_k'}(z) \lesssim \sqrt{\varepsilon_k'}$, we deduce that

$$|\mathrm{Cov}(Z_0\varphi_{2,3}^0, \widetilde{Z}_k\widetilde{\psi}_0)| = \mathcal{O}\left(\frac{\sqrt{\varepsilon_k'}}{\varepsilon_k\sqrt{k}}\right).$$

This concludes the proof as we choose $\varepsilon_k' = \lfloor\sqrt{\varepsilon_k}\rfloor$. □

We can now quickly give the proof of Proposition 3.4.

Proof of Proposition 3.4. Case $1\le i<j\le 3$. First note that $Z_0\varphi_1$ and $Z_k\psi_3$ are independent, so only the cases $i=1$ and $j=2$, or $i=2$ and $j=3$ are at stake. Let us only consider the case $i=2$ and $j=3$, since the

other one is entirely similar. Define, in the same fashion as in the proof of Proposition 3.3,

$$\varphi_2^0 := \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]} = \infty,\ H^+_{\mathcal{R}[\varepsilon_k+1,k-\varepsilon_k]} < \infty \mid S\right].$$

One has by using independence and translation invariance,

$$\mathbb{E}[|\varphi_2 - \varphi_2^0|\psi_3] \le \mathbb{P}[H_{\mathcal{R}[k-\varepsilon_k,k]} < \infty] \cdot \mathbb{P}[H_{\mathcal{R}[\varepsilon_k,\infty)} < \infty] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}},$$

which entails

$$\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_3) = \mathrm{Cov}(Z_0\varphi_2^0, Z_k\psi_3) + \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\right) \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}},$$

since $Z_0\varphi_2^0$ and $Z_k\psi_3$ are independent.

Case $1 \le j \le i \le 3$. Here one can use entirely similar arguments as those from the proof of Lemma 4.2, and we therefore omit the details. □

6. Proof of Proposition 3.5. We need to estimate here the covariances $\mathrm{Cov}(Z_0\varphi_i, Z_k\psi_0)$ and $\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_{4-i})$, for all $1 \le i \le 3$.

Case $i = 1$. It suffices to observe that $Z_0\varphi_1$ and $Z_k\psi_0$ are independent, as are $Z_0\varphi_0$ and $Z_k\psi_3$. Thus their covariances are equal to zero.

Case $i = 2$. We first consider the covariance between $Z_0\varphi_2$ and $Z_k\psi_0$, which is easier to handle. Define

$$\widetilde{\varphi}_2 := \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,k-\varepsilon_k-1]} = \infty,\ H^+_{\mathcal{R}[k-\varepsilon_k,k]} < \infty \mid S\right],$$

and note that $Z_0(\varphi_2 - \widetilde{\varphi}_2)$ is independent of $Z_k\psi_0$. Therefore

$$\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_0) = \mathrm{Cov}(Z_0\widetilde{\varphi}_2, Z_k\psi_0).$$

Then we decompose ψ_0 as $\psi_0 = \psi_0^1 - \psi_0^2$, where

$$\psi_0^1 := \mathbb{P}_{S_k}[H^+_{\mathcal{R}[k,k+\varepsilon_k]} = \infty \mid S],\ \psi_0^2 := \mathbb{P}_{S_k}[H^+_{\mathcal{R}[k,k+\varepsilon_k]} = \infty, H^+_{\mathcal{R}[k-\varepsilon_k,k-1]} < \infty \mid S].$$

Using now that $Z_k\psi_0^1$ is independent of $Z_0\widetilde{\varphi}_2$ we get

$$\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_0) = -\,\mathrm{Cov}(Z_0\widetilde{\varphi}_2, Z_k\psi_0^2).$$

Let $(\widetilde{S}_n)_{n\ge 0}$ and $(\widehat{S}_n)_{n\ge 0}$ be two independent walks starting from the origin, and define

$$\tau_1 := \inf\{n \ge 0 : S_{k-n} \in \widetilde{\mathcal{R}}[1,\infty)\},\ \tau_2 := \inf\{n \ge 0 : S_{k-n} \in (S_k + \widehat{\mathcal{R}}[1,\infty))\}.$$

We decompose

$$\begin{aligned}&\mathrm{Cov}(Z_0\widetilde{\varphi}_2, Z_k\psi_0^2)\\&= \mathbb{E}\left[Z_0\widetilde{\varphi}_2 Z_k\psi_0^2 \mathbf{1}\{\tau_1 \le \tau_2\}\right] + \mathbb{E}\left[Z_0\widetilde{\varphi}_2 Z_k\psi_0^2 \mathbf{1}\{\tau_1 > \tau_2\}\right] - \mathbb{E}[Z_0\widetilde{\varphi}_2]\mathbb{E}[Z_k\psi_0^2].\end{aligned}$$

We bound the first term on the right-hand side simply by the probability of the event $\{\tau_1 \le \tau_2 \le \varepsilon_k\}$, which we treat later, and for the difference between the last two terms, we use that

$$\left|\mathbf{1}\{\tau_2 < \tau_1 \le \varepsilon_k\} - \sum_{i=0}^{\varepsilon_k} \mathbf{1}\left\{\tau_2 = i,\, H^+_{\mathcal{R}[k-\varepsilon_k, k-i-1]} < \infty\right\}\right| \le \mathbf{1}\{\tau_1 \le \tau_2 \le \varepsilon_k\}.$$

Using also that the event $\{\tau_2 = i\}$ is independent of $(S_n)_{n \le k-i}$, we deduce that

$$\begin{aligned}&|\,\mathrm{Cov}(Z_0\widetilde{\varphi}_2, Z_k\psi_0^2)|\\&\le 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \sum_{i=0}^{\varepsilon_k} \mathbb{P}[\tau_2 = i]\left|\mathbb{P}\left[H^+_{\mathcal{R}[k-\varepsilon_k, k-i]} < \infty\right] - \mathbb{P}\left[H^+_{\mathcal{R}[k-\varepsilon_k, k]} < \infty\right]\right|\\&\le 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \sum_{i=0}^{\varepsilon_k} \mathbb{P}[\tau_2 = i] \cdot \mathbb{P}\left[H^+_{\mathcal{R}[k-i,k]} < \infty\right]\\&\overset{(2.14)}{\le} 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \frac{C}{k^{3/2}} \sum_{i=0}^{\varepsilon_k} i\mathbb{P}[\tau_2 = i]\\&\le 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \frac{C}{k^{3/2}} \sum_{i=0}^{\varepsilon_k} \mathbb{P}[\tau_2 \ge i]\\&\overset{(2.14)}{\le} 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \frac{C}{k^{3/2}} \sum_{i=0}^{\varepsilon_k} \frac{1}{\sqrt{i}} \le 2\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] + \frac{C\sqrt{\varepsilon_k}}{k^{3/2}}.\end{aligned}$$

Then it amounts to bound the probability of τ_1 being smaller than τ_2:

$$\begin{aligned}&\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] = \sum_{x,y\in\mathbb{Z}^5} \sum_{i=0}^{\varepsilon_k} \mathbb{P}\left[\tau_1 = i, i \le \tau_2 \le \varepsilon_k, S_k = x,\, S_{k-i} = x+y\right]\\&\le \sum_{x,y\in\mathbb{Z}^5} \sum_{i=0}^{\varepsilon_k} \mathbb{P}\left[\tau_1 = i, S_{k-i} = x+y, (x+\widehat{\mathcal{R}}_\infty) \cap \mathcal{R}[k-\varepsilon_k, k-i] \neq \varnothing, S_k = x\right]\\&\le \sum_{x,y\in\mathbb{Z}^5} \sum_{i=0}^{\varepsilon_k} \mathbb{P}\left[\widetilde{\mathcal{R}}_\infty \cap (x+\mathcal{R}[0,i-1]) = \varnothing, S_i = y,\, x+y \in \widetilde{\mathcal{R}}_\infty\right]\\&\qquad \times \mathbb{P}\left[\widehat{\mathcal{R}}_\infty \cap (y+\mathcal{R}[0,\varepsilon_k - i]) \neq \varnothing, S_{k-i} = -x-y\right],\end{aligned}$$

using invariance by time reversal of S, and where we stress the fact that in the first probability in the last line, $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are two independent ranges starting from the origin. Now the last probability can be bounded using (2.6) and Lemma 4.3, which give

$$\mathbb{P}\left[\widehat{\mathcal{R}}_\infty \cap (y+\mathcal{R}[0,\varepsilon_k - i]) \neq \varnothing, S_{k-i} = -x-y\right] \le \sum_{j=0}^{\varepsilon_k - i} \mathbb{E}\left[G(S_j+y)\mathbf{1}\{S_{k-i}=-x-y\}\right]$$
$$= \sum_{j=0}^{\varepsilon_k - i}\sum_{z\in\mathbb{Z}^5} p_j(z)G(z+y)p_{k-i-j}(z+x+y) = \sum_{j=k-\varepsilon_k}^{k-i}\sum_{z\in\mathbb{Z}^5} p_j(z)G(z-x)p_{k-i-j}(z-x-y)$$
$$\lesssim \frac{1}{(\|x+y\|+\sqrt{k})^5}\left(\frac{1}{1+\|y\|}+\frac{1}{\sqrt{k}+\|x\|}\right).$$

It follows that

$$\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] \lesssim \sum_{x,y\in\mathbb{Z}^5}\sum_{i=0}^{\varepsilon_k} \frac{G(x+y)p_i(y)}{(\|x+y\|+\sqrt{k})^5}\left(\frac{1}{1+\|y\|}+\frac{1}{\sqrt{k}+\|x\|}\right),$$

and then standard computations show that

$$\mathbb{P}[\tau_1 \le \tau_2 \le \varepsilon_k] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}. \tag{6.1}$$

Taking all these estimates together proves that

$$\mathrm{Cov}(Z_0\varphi_2, Z_k\psi_0) \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

We consider now the covariance between $Z_0\varphi_0$ and $Z_k\psi_2$. Here a new problem arises due to the random variable Z_0, which does not play the same role as Z_k, but one can use similar arguments. In particular the previous proof gives

$$\mathrm{Cov}(Z_0\varphi_0, Z_k\psi_2) = -\,\mathrm{Cov}((1-Z_0)\varphi_0, Z_k\psi_2) + \mathcal{O}\left(\frac{\sqrt{\varepsilon_k}}{k^{3/2}}\right).$$

Then we decompose as well $\varphi_0 = \varphi_0^1 - \varphi_0^2$, with

$$\varphi_0^1 := \mathbb{P}[H^+_{\mathcal{R}[k-\varepsilon_k,k]} = \infty \mid S],\ \varphi_0^2 := \mathbb{P}[H^+_{\mathcal{R}[k-\varepsilon_k,k]} = \infty, H^+_{\mathcal{R}[k+1,k+\varepsilon_k]} < \infty \mid S].$$

Using independence we get

$$\mathrm{Cov}((1-Z_0)\varphi_0^1, Z_k\psi_2) = \mathbb{E}[\varphi_0^1]\cdot \mathrm{Cov}((1-Z_0), Z_k\psi_2).$$

Then we define in the same fashion as above,

$$\widetilde{\tau}_0 := \inf\{n \geq 1 : S_n = 0\},\ \widetilde{\tau}_2 := \inf\{n \geq 0 : S_n \in (S_k + \widehat{\mathcal{R}}[1,\infty))\},$$

with $\widehat{\mathcal{R}}$ the range of an independent walk starting from the origin. Recall that by definition $1 - Z_0 = \mathbf{1}\{\widetilde{\tau}_0 \leq \varepsilon_k\}$. Thus one can write

$$\mathrm{Cov}((1-Z_0), Z_k\psi_2) = \mathbb{E}[Z_k\psi_2\mathbf{1}\{\widetilde{\tau}_2 \leq \widetilde{\tau}_0 \leq \varepsilon_k\}] + \mathbb{E}[Z_k\psi_2\mathbf{1}\{\widetilde{\tau}_0 < \widetilde{\tau}_2\}] - \mathbb{P}[\widetilde{\tau}_0 \leq \varepsilon_k]\mathbb{E}[Z_k\psi_2].$$

On one hand, using (2.6), the Markov property, and (2.9),

$$\begin{aligned}
&\mathbb{E}[Z_k\psi_2\mathbf{1}\{\widetilde{\tau}_2 \leq \widetilde{\tau}_0 \leq \varepsilon_k\}] \leq \mathbb{P}[\widetilde{\tau}_2 \leq \widetilde{\tau}_0 \leq \varepsilon_k] \leq \sum_{y\in\mathbb{Z}^5} \mathbb{P}[\widetilde{\tau}_2 \leq \varepsilon_k,\, S_{\widetilde{\tau}_2} = y] \cdot G(y) \\
&\leq \sum_{i=0}^{\varepsilon_k} \mathbb{E}\left[G(S_i - S_k)G(S_i)\right] \leq \sum_{i=0}^{\varepsilon_k} \mathbb{E}[G(S_{k-i})] \cdot \mathbb{E}[G(S_i)] \lesssim \frac{1}{k^{3/2}} \sum_{i=0}^{\varepsilon_k} \frac{1}{1+i^{3/2}} \lesssim \frac{1}{k^{3/2}}.
\end{aligned}$$

On the other hand, similarly as above,

$$\begin{aligned}
&\mathbb{E}[Z_k\psi_2\mathbf{1}\{\widetilde{\tau}_0 < \widetilde{\tau}_2\}] - \mathbb{P}[\widetilde{\tau}_0 \leq \varepsilon_k] \cdot \mathbb{E}[Z_k\psi_2] \\
&\leq \mathbb{P}[\widetilde{\tau}_2 \leq \widetilde{\tau}_0 \leq \varepsilon_k] + \sum_{i=1}^{\varepsilon_k} \mathbb{P}[\widetilde{\tau}_0 = i] \Big(\mathbb{P}\left[(S_k + \widehat{\mathcal{R}}[1,\infty)) \cap \mathcal{R}[i+1,\varepsilon_k] \neq \varnothing\right] - \mathbb{P}[\widetilde{\tau}_2 \leq \varepsilon_k]\Big) \\
&\lesssim \frac{1}{k^{3/2}} + \sum_{i=1}^{\varepsilon_k} \mathbb{P}[\widetilde{\tau}_0 = i]\mathbb{P}[\widetilde{\tau}_2 \leq i] \overset{(2.14)}{\lesssim} \frac{1}{k^{3/2}} + \frac{1}{k^{3/2}} \sum_{i=1}^{\varepsilon_k} i\mathbb{P}[\widetilde{\tau}_0 = i]
\end{aligned}$$

(6.2)

$$\lesssim \frac{1}{k^{3/2}} + \frac{1}{k^{3/2}} \sum_{i=1}^{\varepsilon_k} \mathbb{P}[\widetilde{\tau}_0 \geq i] \overset{(2.6),(2.7)}{\lesssim} \frac{1}{k^{3/2}} + \frac{1}{k^{3/2}} \sum_{i=1}^{\varepsilon_k} \frac{1}{1+i^{3/2}} \lesssim \frac{1}{k^{3/2}}.$$

In other terms, we have already shown that

$$|\,\mathrm{Cov}((1-Z_0)\varphi_0^1, Z_k\psi_2)| \lesssim \frac{1}{k^{3/2}}.$$

The case when φ_0^1 is replaced by φ_0^2 is entirely similar. Indeed, we define

$$\widetilde{\tau}_1 := \inf\{n \geq 0 : S_n \in \widetilde{\mathcal{R}}[1,\infty)\},$$

with $\widetilde{\mathcal{R}}$ the range of a random walk starting from the origin, independent of S and $\widehat{\mathcal{R}}$. Then we set $\widetilde{\tau}_{0,1} := \max(\widetilde{\tau}_0, \widetilde{\tau}_1)$, and exactly as for (6.1) and (6.2), one has

$$\mathbb{P}[\widetilde{\tau}_2 \leq \widetilde{\tau}_{0,1} \leq \varepsilon_k] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}},$$

and

$$\mathbb{E}\left[(1-Z_0)\varphi_0^2 Z_k\psi_2 \mathbf{1}\{\widetilde{\tau}_{0,1}<\widetilde{\tau}_2\}\right] - \mathbb{E}[(1-Z_0)\varphi_0^2]\cdot\mathbb{E}[Z_k\psi_2]$$
$$\le \mathbb{P}[\widetilde{\tau}_2\le \widetilde{\tau}_{0,1}\le \varepsilon_k] + \sum_{i=0}^{\varepsilon_k}\mathbb{P}[\widetilde{\tau}_{0,1}=i]\cdot\mathbb{P}[\widetilde{\tau}_2\le i] \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

Altogether, this gives

$$|\operatorname{Cov}(Z_0\varphi_0, Z_k\psi_2)| \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

Case $i=3$. We only need to treat the case of the covariance between $Z_0\varphi_3$ and $Z_k\psi_0$, as the other one is entirely similar here. Define

$$\widetilde{\varphi}_3 := \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]\cup\mathcal{R}[k+\varepsilon_k+1,\infty)}=\infty,\ H^+_{\mathcal{R}[k,k+\varepsilon_k]}<\infty \mid S\right].$$

The proof of the case $i=2$, already shows that

$$|\operatorname{Cov}(Z_0\widetilde{\varphi}_3, Z_k\psi_0)| \lesssim \frac{\sqrt{\varepsilon_k}}{k^{3/2}}.$$

Define next

$$h_3 := \varphi_3-\widetilde{\varphi}_3 = \mathbb{P}\left[H^+_{\mathcal{R}[-\varepsilon_k,\varepsilon_k]}=\infty,\ H^+_{\mathcal{R}[k+\varepsilon_k+1,\infty)}<\infty \mid S\right].$$

Assume for a moment that $\varepsilon_k \ge k^{\frac{9}{20}}$. We will see later another argument when this condition is not satisfied. Then define $\varepsilon_k' := \lfloor \varepsilon_k^{10/9}/k^{1/9}\rfloor$, and note that one has $\varepsilon_k'\le\varepsilon_k$. Write $\psi_0=\psi_0'+h_0$, with

$$\psi_0' := \mathbb{P}\left[H^+_{\mathcal{R}[k-\varepsilon_k'+1,k+\varepsilon_k'-1]}=\infty \mid S\right],$$

and

$$h_0 := \mathbb{P}\left[H^+_{\mathcal{R}[k-\varepsilon_k'+1,k+\varepsilon_k'-1]}=\infty,\ H^+_{\mathcal{R}[k-\varepsilon_k,k-\varepsilon_k']\cup\mathcal{R}[k+\varepsilon_k',k+\varepsilon_k]}<\infty \mid S\right].$$

Define also

$$Z_k' := \mathbf{1}\{S_\ell\neq S_k,\ \text{for all } \ell=k+1,\dots,k+\varepsilon_k'-1\}.$$

One has

$$\operatorname{Cov}(Z_0h_3,Z_k\psi_0) = \operatorname{Cov}(Z_0h_3,Z_k'\psi_0') + \operatorname{Cov}(Z_0h_3,Z_k'h_0) + \operatorname{Cov}(Z_0h_3,(Z_k-Z_k')\psi_0).$$

For the last of the three terms, one can simply notice that, using the Markov property at the first return time to S_k (for the walk S), and then (2.6), (2.7), and (2.14), we get

$$\begin{aligned}\mathbb{E}[h_3(Z_k - Z_k')] &\le \mathbb{E}[Z_k - Z_k'] \times \mathbb{P}[\widetilde{\mathcal{R}}_\infty \cap \mathcal{R}\,[k,\infty) \neq \varnothing] \\ &\lesssim \frac{1}{(\varepsilon_k')^{3/2}\sqrt{k}} \lesssim \frac{1}{\varepsilon_k^{5/3} k^{1/3}} \lesssim \frac{1}{k^{\frac{13}{12}}},\end{aligned}$$

using our hypothesis on ε_k for the last equality. As a consequence, it also holds

$$|\operatorname{Cov}(Z_0 h_3, (Z_k - Z_k')\psi_0)| \lesssim k^{-\frac{13}{12}}.$$

Next we write

$$\operatorname{Cov}(Z_0 h_3, Z_k' h_0) = \sum_{x,y\in\mathbb{Z}^5} (p_{k-2\varepsilon_k}(x-y) - p_k(x)) H_1(y) H_2(x), \tag{6.3}$$

where

$$H_1(y) := \mathbb{E}\left[Z_k' h_0 \mathbf{1}\{S_{k+\varepsilon_k} - S_{k-\varepsilon_k} = y\}\right],\ H_2(x) := \mathbb{E}\left[Z_0 h_3 \mid S_{k+\varepsilon_k} - S_{\varepsilon_k} = x\right].$$

Define $r_k := (k/\varepsilon_k')^{1/8}$. By using symmetry and translation invariance,

$$\begin{aligned}&\sum_{\|y\|\ge\sqrt{\varepsilon_k} r_k} H_1(y) \le \mathbb{P}\left[H_{\mathcal{R}[-\varepsilon_k,-\varepsilon_k']\cup\mathcal{R}[\varepsilon_k',\varepsilon_k]} < \infty,\ \|S_{\varepsilon_k} - S_{-\varepsilon_k}\| \ge \sqrt{\varepsilon_k} r_k\right] \\ &\le 2\mathbb{P}\left[H_{\mathcal{R}[\varepsilon_k',\varepsilon_k]} < \infty,\ \|S_{\varepsilon_k}\| \ge \sqrt{\varepsilon_k}\frac{r_k}{2}\right] + 2\mathbb{P}\left[H_{\mathcal{R}[\varepsilon_k',\varepsilon_k]} < \infty,\ \|S_{-\varepsilon_k}\| \ge \sqrt{\varepsilon_k}\frac{r_k}{2}\right] \\ &\overset{(2.14),(2.5)}{\le} 2\mathbb{P}\left[H_{\mathcal{R}[\varepsilon_k',\varepsilon_k]} < \infty,\ \|S_{\varepsilon_k}\| \ge \sqrt{\varepsilon_k}\frac{r_k}{2}\right] + \frac{C}{\sqrt{\varepsilon_k'} r_k^5}.\end{aligned}$$

Considering the first probability on the right-hand side, define τ as the first hitting time (for S), after time ε_k', of another independent walk $\widetilde{S}$ (starting from the origin). One has

$$\begin{aligned}&\mathbb{P}\left[H_{\mathcal{R}[\varepsilon_k',\varepsilon_k]} < \infty,\ \|S_{\varepsilon_k}\| \ge \sqrt{\varepsilon_k}\frac{r_k}{2}\right] \\ &\le \mathbb{P}[\|S_\tau\| \ge \sqrt{\varepsilon_k}\frac{r_k}{4},\ \tau \le \varepsilon_k] + \mathbb{P}[\|S_{\varepsilon_k} - S_\tau\| \ge \sqrt{\varepsilon_k}\frac{r_k}{4},\ \tau \le \varepsilon_k].\end{aligned}$$

Using then the Markov property at time τ, we deduce with (2.14) and (2.5),

$$\mathbb{P}[\|S_{\varepsilon_k} - S_\tau\| \ge \sqrt{\varepsilon_k}\frac{r_k}{4},\ \tau \le \varepsilon_k] \lesssim \frac{1}{\sqrt{\varepsilon_k'} r_k^5}.$$

Likewise, using the Markov property at the first time when the walk exit the ball of radius $\sqrt{\varepsilon_k} r_k/4$, and applying then (2.5) and (2.13), we get as well

$$\mathbb{P}[\|S_\tau\| \geq \sqrt{\varepsilon_k}\frac{r_k}{4},\, \tau \leq \varepsilon_k] \lesssim \frac{1}{\sqrt{\varepsilon_k} r_k^6}.$$

Furthermore, for any y, one has

$$\sum_{x\in\mathbb{Z}^5} p_{k-2\varepsilon_k}(x-y)H_2(x) \overset{(2.3),(2.13)}{\lesssim} \sum_{x\in\mathbb{Z}^5} \frac{1}{(1+\|x+y\|)(\|x\|+\sqrt{k})^5} \lesssim \frac{1}{\sqrt{k}},$$

with an implicit constant, which is uniform in y (and the same holds with $p_k(x)$ instead of $p_{k-2\varepsilon_k}(x-y)$). Similarly, define $r_k' := (k/\varepsilon_k')^{\frac{1}{10}}$. One has for any y, with $\|y\| \leq \sqrt{\varepsilon_k} r_k$,

$$\sum_{\|x\|\geq \sqrt{k} r_k'} p_{k-2\varepsilon_k}(x-y)H_2(x) \overset{(2.5),(2.13)}{\lesssim} \frac{1}{\sqrt{k}(r_k')^6}.$$

Therefore coming back to (6.3), and using that by (2.13), $\sum_y H_1(y) \lesssim 1/\sqrt{\varepsilon_k'}$, we get

$$\begin{aligned}
&\mathrm{Cov}(Z_0h_3, Z_k'h_0)\\
&= \sum_{\|x\|\leq\sqrt{k}r_k'} \sum_{\|y\|\leq\sqrt{\varepsilon_k}r_k} (p_{k-2\varepsilon_k}(x-y)-p_k(x))H_1(y)H_2(x) + \mathcal{O}\left(\frac{1}{\sqrt{k\varepsilon_k'}(r_k')^6} + \frac{1}{\sqrt{k\varepsilon_k'}r_k^5}\right)\\
&= \sum_{\|x\|\leq\sqrt{k}r_k'} \sum_{\|y\|\leq\sqrt{\varepsilon_k}r_k} (p_{k-2\varepsilon_k}(x-y)-p_k(x))H_1(y)H_2(x) + \mathcal{O}\left(\frac{(\varepsilon_k')^{\frac{1}{10}}}{k^{\frac{11}{10}}}\right).
\end{aligned}$$

Now we use the fact $H_1(y) = H_1(-y)$. Thus the last sum is equal to half of the following:

$$\begin{aligned}
&\sum_{\|x\|\leq\sqrt{k}r_k'} \sum_{\|y\|\leq\sqrt{\varepsilon_k}r_k} (p_{k-2\varepsilon_k}(x-y)+p_{k-2\varepsilon_k}(x+y)-2p_k(x))H_1(y)H_2(x)\\
&\overset{\text{Theorem 2.1},(2.13)}{\leq} \sum_{\|x\|\leq\sqrt{k}r_k'} \sum_{\|y\|\leq\sqrt{\varepsilon_k}r_k} (\overline{p}_{k-2\varepsilon_k}(x-y)+\overline{p}_{k-2\varepsilon_k}(x+y)-2\overline{p}_k(x))H_1(y)H_2(x)\\
&\qquad + \mathcal{O}\left(\frac{(r_k')^4}{k^{3/2}\sqrt{\varepsilon_k'}}\right),
\end{aligned}$$

(with an additional factor 2 in front in case of a bipartite walk). Note that the error term above is $\mathcal{O}(k^{-11/10})$, by definition of r'_k. Moreover, straightforward computations show that for any x and y as in the sum above,

$$|\overline{p}_{k-2\varepsilon_k}(x-y)+\overline{p}_{k-2\varepsilon_k}(x+y)-2\overline{p}_k(x)|\lesssim\left(\frac{\|y\|^2+\varepsilon_k}{k}\right)\overline{p}_k(cx).$$

In addition one has (with the notation as above for τ),

$$\begin{aligned}&\sum_{y\in\mathbb{Z}^5}\|y\|^2\,H_1(y)\le 2\mathbb{E}\left[\|S_{\varepsilon_k}-S_{-\varepsilon_k}\|^2\mathbf{1}\{\tau\le\varepsilon_k\}\right]\\ &\le 4\mathbb{E}[\|S_{\varepsilon_k}\|^2]\mathbb{P}[\tau\le\varepsilon_k]+4\mathbb{E}\left[\|S_{\varepsilon_k}\|^2\mathbf{1}\{\tau\le\varepsilon_k\}\right]\\ &\overset{(2.5),(2.14)}{\lesssim}\ \frac{\varepsilon_k}{\sqrt{\varepsilon'_k}}+\mathbb{E}\left[\|S_\tau\|^2\mathbf{1}\{\tau\le\varepsilon_k\}\right]+\mathbb{E}\left[\|S_{\varepsilon_k}-S_\tau\|^2\mathbf{1}\{\tau\le\varepsilon_k\}\right]\\ &\overset{(2.5),(2.14)}{\lesssim}\ \frac{\varepsilon_k}{\sqrt{\varepsilon'_k}}+\sum_{r\ge\sqrt{\varepsilon_k}}r\mathbb{P}\left[\|S_\tau\|\ge r,\,\tau\le\varepsilon_k\right]\overset{(2.5),(2.13)}{\lesssim}\ \frac{\varepsilon_k}{\sqrt{\varepsilon'_k}},\end{aligned}$$

using also the Markov property in the last two inequalities (at time τ for the first one, and at the exit time of the ball of radius r for the second one). Altogether, this gives

$$|\operatorname{Cov}(Z_0h_3,Z'_kh_0)|\lesssim\frac{\varepsilon_k}{k^{3/2}\sqrt{\varepsilon'_k}}+\frac{(\varepsilon'_k)^{\frac{1}{10}}}{k^{\frac{11}{10}}}\lesssim\frac{(\varepsilon_k)^{\frac19}}{k^{\frac{10}{9}}}.$$

In other words, for any sequence $(\varepsilon_k)_{k\ge1}$, such that $\varepsilon_k\ge k^{9/20}$, one has

$$\operatorname{Cov}(Z_0h_3,Z_k\psi_0)=\operatorname{Cov}(Z_0h_3,Z'_k\psi'_0)+\mathcal{O}\left(\frac{(\varepsilon_k)^{\frac19}}{k^{\frac{10}{9}}}+\frac{1}{k^{\frac{13}{12}}}\right).$$

One can then iterate the argument with the sequence (ε'_k) in place of (ε_k), and (after at most a logarithmic number of steps), we are left to consider a sequence (ε_k), satisfying $\varepsilon_k\le k^{9/20}$. In this case, we use similar arguments as above. Define $\widetilde{H}_1(y)$ as $H_1(y)$, but with $Z_k\psi_0$ instead of Z'_kh_0 in the expectation, and choose $r_k:=\sqrt{k/\varepsilon_k}$, and $r'_k=k^{\frac{1}{10}}$. Then we obtain exactly as above,

$$
\begin{aligned}
&\mathrm{Cov}(Z_0 h_3, Z_k\psi_0)\\
&= \sum_{\|x\|\le \sqrt{k} r'_k} \sum_{\|y\|\le\sqrt{k}} (p_{k-2\varepsilon_k}(x-y) - p_k(x))\widetilde{H}_1(y)H_2(x) + \mathcal{O}\left(\frac{1}{r_k^5\sqrt{k}} + \frac{1}{(r'_k)^6\sqrt{k}}\right)\\
&= \sum_{\|x\|\le \sqrt{k} r'_k} \sum_{\|y\|\le\sqrt{k}} (\overline{p}_{k-2\varepsilon_k}(x-y) - \overline{p}_k(x))\widetilde{H}_1(y)H_2(x) + \mathcal{O}\left(\frac{1}{k^{\frac{11}{10}}}\right)\\
&\lesssim \frac{\varepsilon_k}{k^{3/2}} + \frac{1}{k^{\frac{11}{10}}} \lesssim \frac{1}{k^{\frac{21}{20}}},
\end{aligned}
$$

which concludes the proof of the proposition.

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