

Deviations for the Capacity of the Range of a Random Walk

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Abstract

We obtain estimates for large and moderate deviations for the capacity of the range of a random walk on \mathbb{Z}^d , in dimension $d \geq 5$, both in the upward and downward directions. The results are analogous to those we obtained for the volume of the range in two companion papers [AS17a, AS19]. Interestingly the main lines of the proofs we used for the latter apply as well in this (seemingly different) setting, yet the details of the analysis are different.

Keywords and phrases. Random Walk, Capacity, Range, Large deviations, Moderate deviations.
MSC 2010 subject classifications. Primary 60F05, 60G50.

1 Introduction

We consider a simple random walk $(S_n)_{n \in \mathbb{N}}$ on \mathbb{Z}^d starting from the origin. The range of the walk between two times $k \leq n$, is denoted as $\mathcal{R}[k, n] := \{S_k, \dots, S_n\}$ with the shortcut $\mathcal{R}_n = \mathcal{R}[0, n]$. Its Newtonian capacity, denoted $\text{Cap}(\mathcal{R}_n)$, can be seen as the hitting probability of \mathcal{R}_n by an independent random walk starting from *far away* and properly normalized. Equivalently, using reversibility, it can be expressed as the sum of escape probabilities from \mathcal{R}_n by an independent random walk starting along the range. In other words, $\text{Cap}(\mathcal{R}_n)$ is random and has the following representations:

$$\text{Cap}(\mathcal{R}_n) = \lim_{z \rightarrow \infty} \frac{\mathbb{P}_{0,z}(\tilde{H}_{\mathcal{R}_n} < \infty \mid S)}{G(z)} = \sum_{x \in \mathcal{R}_n} \mathbb{P}_{0,x}(\tilde{H}_{\mathcal{R}_n}^+ = \infty \mid S), \quad (1.1)$$

where $\mathbb{P}_{0,z}$ is the law of two independent walks S and \tilde{S} starting at 0 and z respectively, $G(\cdot)$ is the Green's function, and \tilde{H}_Λ (resp. \tilde{H}_Λ^+) stands for the hitting (resp. return) time of Λ by the walk \tilde{S} .

In view of (1.1), the study of the capacity of the range is intimately related to the question of estimating probabilities of intersection of random walks. This chapter has grown quite large, with several motivations from statistical mechanics keeping the interest alive (see Lawler's celebrated monograph [Law91]). The last decade has witnessed revival interests both after a link between uniform spanning trees and loop erased random walks was discovered (see [LawSW18], [Hut18] for recent results) and after the introduction of random interacements by Sznitman in [S10] which mimic a random walk confined in a region of volume comparable to its time span.

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The study of the capacity of the range of a random walk has a long history. Jain and Orey [JO69] show that in any dimension $d \geq 3$, there exists a constant $\gamma_d \in [0, \infty)$, such that almost surely

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{Cap}(\mathcal{R}_n) = \gamma_d, \quad \text{and} \quad \gamma_d > 0, \quad \text{if and only if, } d \geq 5. \quad (1.2)$$

The first order asymptotics is obtained in dimension 3 in [C17], where $\text{Cap}(\mathcal{R}_n)$ scales like \sqrt{n} . Dimension 4 is *the critical dimension*, and a central limit theorem with a non-gaussian limit is established in [ASS19b]. In higher dimension a central limit theorem is proved in [Sch19] for $d = 5$, and in [ASS18] for $d \geq 6$.

Here, we mainly study the downward deviations for the capacity of the range in dimension $d \geq 5$, in the moderate and large deviations regimes. We also establish a large deviations principle in the upward direction. Our analysis is, as in our previous works [AS17a, AS19], related to the celebrated large deviation analysis of the volume of the Wiener sausage by van den Berg, Bolthausen and den Hollander [BBH01]. The folding of the Wiener sausage, under squeezing its volume, became a paradigm of *folding*, with localization in a domain with holes of order one (the picture of a Swiss Cheese popularized in [BBH01]). The variational formula for the rate function was shown to have minimizers of different nature in $d = 3$ and in $d \geq 5$ suggesting dimension-dependent optimal scenarii to achieve the deviation. For the discrete analogue of the Wiener sausage, we established in [AS17a, AS20] some path properties confirming some observations of [BBH01].

Main results Our first result concerns the large and moderate deviations in dimension 7 and higher. In this case we obtain upper and lower bounds for the moderate deviations which are of the same order (on a logarithmic scale), and we cover almost the whole non-Gaussian regime (see Remark 1.2 below).

Theorem 1.1. *Assume $d \geq 7$. There exist positive constants ε , $\underline{\kappa}$ and $\bar{\kappa}$ (only depending on the dimension), such that for any $n^{\frac{d-2}{d}} \cdot \log n \leq \zeta \leq \varepsilon n$, and for n large enough,*

$$\exp\left(-\underline{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right) \leq \mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \leq \exp\left(-\bar{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right).$$

Remark 1.2. In dimension 7 and higher it is known [ASS18] that the variance of $\text{Cap}(\mathcal{R}_n)$ is of order n , so the cost of deviations should jump from a Gaussian regime with speed ζ^2/n , to a regime with speed of order $\zeta^{1-2/(d-2)}$, with a transition occurring when both speed are equal, that is for ζ of order $n^{(d-2)/d}$. So our result covers (almost) the whole non-Gaussian regime.

In dimension 5, we obtain similar estimates, but we do not reach the Gaussian regime:

Theorem 1.3. *Assume $d = 5$. There exist positive constants ε , $\underline{\kappa}$ and $\bar{\kappa}$, such that for any $n^{5/7} \cdot \log n \leq \zeta \leq \varepsilon n$, and n large enough,*

$$\exp\left(-\underline{\kappa} \cdot \left(\frac{\zeta^2}{n}\right)^{1/3}\right) \leq \mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \leq \exp\left(-\bar{\kappa} \cdot \left(\frac{\zeta^2}{n}\right)^{1/3}\right).$$

Remark 1.4. In $d = 5$, the variance of $\text{Cap}(\mathcal{R}_n)$ is of order $n \log n$, [Sch19]. Thus, the moderate deviations should pass from a gaussian regime with a speed of order $\zeta^2/(n \log n)$, to a large deviation regime with a speed of order $(\zeta^2/n)^{1/3}$, with a transition occurring for ζ of order $\sqrt{n}(\log n)^{3/4}$. Note that in the case of the volume of the range a similar transition has been established by Chen [Chen10] in dimension 3 (in fact he obtained even stronger asymptotics in this case), see also [AS19].

Remark 1.5. In dimension 6 our result is less precise. One can only show that

$$\exp\left(-\underline{\kappa} \cdot \zeta^{1/2}\right) \leq \mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \leq \exp\left(-\frac{\bar{\kappa}}{\log(n/\zeta)} \cdot \zeta^{1/2}\right).$$

Our next result provide path properties of the trajectory under the constraint of moderate deviations. To state it, one needs some more notation. Define for $r > 0$, and $x \in \mathbb{Z}^d$, the cube of side-length r , as

$$Q(x, r) := \left[x - \frac{r}{2}, x + \frac{r}{2}\right)^d \cap \mathbb{Z}^d.$$

Given $\Lambda \subseteq \mathbb{Z}^d$, and $n \geq 0$, let $\ell_n(\Lambda)$ be the time spent in Λ before time n . For $\rho \in (0, 1]$, and r, n positive integers, we let

$$\mathcal{C}_n(r, \rho) := \{x \in r\mathbb{Z}^d : \ell_n(Q(x, r)) \geq \rho r^d\}, \quad \text{and} \quad \mathcal{V}_n(r, \rho) := \bigcup_{x \in \mathcal{C}_n(r, \rho)} Q(x, r). \quad (1.3)$$

Define also for a sequence of values of deviation $(\zeta_n)_{n \geq 1}$,

$$\rho_{\text{typ}} := \begin{cases} \zeta_n^{5/3}/n^{7/3} & \text{if } d = 5 \\ \zeta_n^{-2/(d-2)} & \text{if } d \geq 7, \end{cases} \quad \tau_{\text{typ}} := \begin{cases} n & \text{if } d = 5 \\ \zeta_n & \text{if } d \geq 7, \end{cases} \quad \text{and} \quad \chi_d := \begin{cases} 5/7 & \text{if } d = 5 \\ \frac{d-2}{d} & \text{if } d \geq 7. \end{cases}$$

Theorem 1.6. *Assume $d = 5$, or $d \geq 7$. There are positive constants $\alpha, \beta, \varepsilon$ and C , such that for any sequences $(r_n)_{n \geq 1}$, and $(\zeta_n)_{n \geq 1}$, satisfying*

$$r_n^{d-2} \rho_{\text{typ}} \geq C \log n, \quad \text{and} \quad n^{\chi_d} \cdot \log n \leq \zeta_n \leq \varepsilon n,$$

one has

$$\lim_{n \rightarrow \infty} \mathbb{P}(\ell_n(\mathcal{V}_n(r_n, \beta \rho_{\text{typ}})) \geq \alpha \tau_{\text{typ}} \mid \text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta_n) = 1.$$

The final result concerns the upward deviations. Our decomposition (1.4) allows us to adapt a beautiful argument of Hamana and Kesten, [HK], written for the volume of the range of a random walk.

Theorem 1.7. *Assume $d \geq 5$. The following limit exists for all $x > 0$:*

$$\psi_d(x) := - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\text{Cap}(\mathcal{R}_n) \geq n \cdot x).$$

Furthermore, there exists a constant $\gamma_d^* > \gamma_d$ (defined in (1.2)), such that the function ψ_d is continuous and convex on $[0, \gamma_d^*]$, increasing on $[\gamma_d, \gamma_d^*]$, and satisfies

$$\psi_d(x) \begin{cases} = 0 & \text{if } x \leq \gamma_d \\ \in (0, \infty) & \text{if } x \in (\gamma_d, \gamma_d^*] \\ = \infty & \text{if } x > \gamma_d^*. \end{cases}$$

We also obtain Gaussian upper bounds (up to a logarithmic factor) in the regime of moderate deviations, see Proposition 2.4.

Our approach to downward deviations. The cornerstone of our approach is a decomposition formula obtained in [ASS19a]:

$$\forall A, B \text{ finite sets of } \mathbb{Z}^d, \quad \text{Cap}(A \cup B) = \text{Cap}(A) + \text{Cap}(B) - \chi_C(A, B), \quad (1.4)$$

where $\chi_C(A, B)$ called *the cross-term* has a nice expression. In this work, the decomposition (1.4) allows us to follow a simple approach devised in [AS17a], and later improved in [AS19], to study downward deviations for the volume of the range in dimensions $d \geq 3$. We partition the time-period of length n into intervals of length $T \leq n$, and by iterating (1.4) appropriately one can write our functional of the range, $\text{Cap}(\mathcal{R}_n)$, as a sum of i.i.d. terms minus a certain sum of cross-terms of the form $\chi_C(\mathcal{R}_{iT}, \mathcal{R}[iT, (i+1)T])$, with i going from 1 to $\lfloor n/T \rfloor$. The so-called corrector, is the sum of these cross-terms that we integrate over $\mathcal{R}[iT, (i+1)T]$. We then show that for some appropriate time-scale T it is this corrector which is responsible for (most of) the deviations. The final step is to estimate the cost for such deviations. This analysis is similar to the corresponding one for the volume of the range that we performed in [AS19], but it also requires some new ingredients, in particular Lemmas 4.1, 4.2 and 4.3.

Heuristics. We use the sign \approx to express that two quantities are *of the same order*. As already mentioned, the first step in this work is a simple decomposition for the capacity of a union of sets in term of a cross-term

$$\chi_C(A, B) \approx 2 \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x(H_A^+ = \infty) \cdot G(x - y) \cdot \mathbb{P}_y(H_B^+ = \infty), \quad (1.5)$$

see (2.9) and (2.12) for a precise expression. The key phenomenon responsible for producing a small capacity for the range of a random walk is *an increase of the cross-term on an appropriate scale*. In other words, the walk *folds* into a ball-like domain in order to increase some *self-interaction* captured by the cross-term. Now to be more concrete, let us divide the range $\mathcal{R}[0, 2n]$ into two subsets $\mathcal{R}[0, n]$ and $\mathcal{R}[n, 2n]$. Let us call, for simplicity $\mathcal{R}_n^1 = \mathcal{R}[0, n] - S_n$, and $\mathcal{R}_n^2 = \mathcal{R}[n, 2n] - S_n$ the two subranges translated by S_n so that they become independent. By translation invariance of the capacity, we obtain

$$\text{Cap}(\mathcal{R}[0, 2n]) = \text{Cap}(\mathcal{R}_n^1) + \text{Cap}(\mathcal{R}_n^2) - \chi_C(\mathcal{R}_n^1, \mathcal{R}_n^2).$$

Now, assume that both walks stay inside a ball of radius R a time of order $\tau \leq n$, and are unconstrained afterward. Thus, under the strategy we mentioned,

$$\begin{aligned} \chi_C(\mathcal{R}_n^1, \mathcal{R}_n^2) &\approx G(R) \times \text{Cap}(\mathcal{R}_\tau^1) \times \text{Cap}(\mathcal{R}_\tau^2) + \mathcal{O}(G(\sqrt{n})n^2) \\ &\approx G(R)(\min(\tau, R^{d-2}))^2 + \mathcal{O}(n^{\frac{6-d}{2}}). \end{aligned} \quad (1.6)$$

The term $\mathcal{O}(G(\sqrt{n})n^2)$ appears if τ is smaller than n , and accounts for the unconstrained contribution to the cross-term. In obtaining (1.6), we have used that if $\mathcal{R}_\tau^1, \mathcal{R}_\tau^2$ are inside a ball of radius R , their capacity is bounded by the capacity of the ball, which is of order R^{d-2} , as well as by their volume bounded by τ . Thus, it is useless to consider τ larger than R^{d-2} , since then τ no more affects the cross-term and increasing τ (or decreasing R below τ) only makes the cost of the strategy larger. Now, to reach a deviation of order ζ , we have

$$\frac{1}{R^{d-2}} \tau^2 \approx \zeta. \quad (1.7)$$

Recall that the cost of being localized a time τ in a ball of radius R is of order $\exp(-\tau/R^2)$ (up to a constant in the exponential). So we need to find a choice of (τ, R) which minimizes this cost under the constraint (1.7). In other words one needs to maximize $\sqrt{\zeta} \cdot R^{(d-6)/2}$. This leads to two regimes.

- When $d = 5$, R (and then τ) is as large as possible. So, $\tau = n$ and $R^{d-2} = n^2/\zeta$ by (1.7). The strategy is time homogeneous for any ζ !
- When $d \geq 7$, then τ is as small as possible, that is $\tau = R^{d-2} = \zeta$. The strategy is time-inhomogeneous.

When $d = 6$, the strategy remains unknown, but the cost should be of order $\exp(-\sqrt{\zeta})$.

Application to a polymer melt. The model of random interlacements, introduced by Sznitman [S10], is roughly speaking the union of the ranges of trajectories obtained by a Poisson point process on the space of doubly infinite trajectories, and is such that the probability of avoiding a set K is $\exp(-u \cdot \text{Cap}(K))$, where $u > 0$ is a fixed parameter. With this in mind, let us consider the following model of polymer among a polymer melt interacting by exclusion. We distinguish one polymer, a simple random walk, interacting with a cloud of other random walk trajectories modeled by random interlacements which we call for short *the melt*. The interaction is through exclusion: the walk and the melt do not intersect. When integrating over the interlacements law, the measure on the walk with the effective interaction has a density proportional to $\exp(-u \cdot \text{Cap}(\mathcal{R}_n))$, with respect to the law of a simple random walk.

As a corollary of our deviation estimates, one can address key issues on this polymer. Since this follows in the same way as the study of the Gibbs measure tilted by the volume of the range was a corollary of [AS17a], we repeat neither the statements, nor the proofs here. The simplest and most notable difference is that the proper scaling of the temperature which provides a phase transition is when it is of order $n^{-2/(d-2)}$ in dimension $d \geq 5$. Moreover, our polymer measure in dimension d behaves similarly as the polymer measure of [AS17a] in dimension $d - 2$. Theorem 1.8 of [AS17a] is true here also after the drop in dimension is performed.

Organization. Let us explain how the rest of the paper is organized while sketching the skeleton of our approach. The key decomposition relation (1.4) is given for arbitrary sets in (2.9) and for the range in (3.1). Section 3, and more precisely Proposition 3.2 makes the link between downward deviations for the capacity and upward deviations of a corrector. The corrector itself is studied in Section 4. Propositions 4.5 and 4.6 imply the upper bounds respectively in Theorems 1.1 and 1.3, as well as Theorem 1.6. In Section 2, we recall basic facts on Green's function, and on the Newtonian capacity. In Section 5, we prove the lower bounds in Theorems 1.1 and 1.3. Finally, we prove Theorem 1.7 concerning the upward deviations in Section 6.

2 Preliminaries

2.1 Further notation

For $z \in \mathbb{Z}^d$, $d \geq 5$, we denote by \mathbb{P}_z the law of the simple random walk starting from z , and let

$$G(z) := \mathbb{E} \left[\sum_{n=0}^{\infty} \mathbf{1}\{S_n = z\} \right],$$

be the Green's function. It is known (see [Law91]) that for some positive constant C ,

$$G(z) \leq \frac{C}{\|z\|^{d-2} + 1}, \quad \text{for all } z \in \mathbb{Z}^d, \quad (2.1)$$

with $\|\cdot\|$ the Euclidean norm. We also consider for $T > 0$, and $z \in \mathbb{Z}^d$,

$$G_T(z) := \mathbb{E} \left[\sum_{n=0}^T \mathbf{1}\{S_n = z\} \right].$$

In particular for any $z \in \mathbb{Z}^d$, and $T \geq 1$,

$$\mathbb{P}(z \in \mathcal{R}_T) \leq G_T(z). \quad (2.2)$$

For $A \subset \mathbb{Z}^d$, we denote by $|A|$ the cardinality of A , and by

$$H_A := \inf\{n \geq 0 : S_n \in A\}, \quad \text{and} \quad H_A^+ := \inf\{n \geq 1 : S_n \in A\},$$

respectively the hitting time of A and the first return time to A .

We also need the following well known fact, see [Law91]. There exists a constant $C > 0$, such that for any $R > 0$ and $z \in \mathbb{Z}^d$,

$$\mathbb{P}_z \left(\inf_{k \geq 0} \|S_k\| \leq R \right) \leq C \cdot \left(\frac{R}{\|z\|} \right)^{d-2}. \quad (2.3)$$

2.2 On the capacity

The capacity of a finite subset $A \subset \mathbb{Z}^d$, with $d \geq 3$, is defined by

$$\text{Cap}(A) := \lim_{\|z\| \rightarrow \infty} \frac{1}{G(z)} \mathbb{P}_z(H_A < \infty). \quad (2.4)$$

It is well known, see Proposition 2.2.1 of [Law91], that the capacity is monotone for inclusion:

$$\text{Cap}(A) \leq \text{Cap}(B), \quad \text{for any } A \subset B, \quad (2.5)$$

and satisfies the sub-additivity relation

$$\text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B) - \text{Cap}(A \cap B), \quad \text{for all } A, B \subset \mathbb{Z}^d. \quad (2.6)$$

Another equivalent definition of the capacity is the following (see (2.12) of [Law91]).

$$\text{Cap}(A) = \sum_{x \in A} \mathbb{P}_x(H_{A^+} = \infty). \quad (2.7)$$

In particular it implies that

$$\text{Cap}(A) \leq |A|, \quad \text{for all } A \subset \mathbb{Z}^d. \quad (2.8)$$

The starting point for our decomposition is the definition (2.4) of the capacity in terms of a hitting time. It implies that for any two finite subsets $A, B \subset \mathbb{Z}^d$,

$$\text{Cap}(A \cup B) = \text{Cap}(A) + \text{Cap}(B) - \chi_C(A, B), \quad (2.9)$$

with

$$\chi_C(A, B) := \lim_{z \rightarrow \infty} \frac{1}{G(z)} \mathbb{P}_z(\{H_A < \infty\} \cap \{H_B < \infty\}). \quad (2.10)$$

Note that by (2.4) and the latter formula, one has

$$0 \leq \chi_C(A, B) \leq \min(\text{Cap}(A), \text{Cap}(B)). \quad (2.11)$$

Now, we have shown in [ASS19b] that $\chi_C(A, B) = \chi(A, B) + \chi(B, A) - \varepsilon(A, B)$, with

$$\chi(A, B) = \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x(H_{A \cup B}^+ = \infty) \cdot G(x - y) \cdot \mathbb{P}_y(H_B^+ = \infty), \quad (2.12)$$

and,

$$0 \leq \varepsilon(A, B) \leq \text{Cap}(A \cap B) \leq |A \cap B|,$$

where the last inequality follows from (2.8).

We will need some control on the speed of convergence in (1.2).

Lemma 2.1. *Assume $d \geq 5$. One has*

$$|\mathbb{E}[\text{Cap}(\mathcal{R}_n)] - \gamma_d n| = \mathcal{O}(\psi_d(n)), \quad (2.13)$$

with

$$\psi_d(n) = \begin{cases} \sqrt{n} & \text{if } d = 5 \\ \log n & \text{if } d = 6 \\ 1 & \text{if } d \geq 7. \end{cases} \quad (2.14)$$

Proof. By Proposition 1.2 in [ASS18], one has the rough lower bound:

$$\text{Cap}(\mathcal{R}_{n+m}) \geq \text{Cap}(\mathcal{R}_n) + \text{Cap}(\mathcal{R}[n, n+m]) - 2 \sum_{k=0}^n \sum_{\ell=n}^{n+m} G(S_k - S_\ell),$$

for any integers $n, m \geq 1$. Even though a better inequality is used in (2.9) below, this result together with the subadditivity relation (2.6) are enough to conclude the proof, using Hammersley's lemma and Lemma 3.2 in [ASS18], which controls the moments of the error term in the lower bound. For the details, we refer to the proof of (1.13) in [AS17b], which is entirely similar. \square

The next result provides some useful bounds on the variance of the capacity of the range, which were obtained in [Sch19] in case of dimension 5, and in [ASS18] in higher dimension.

Proposition 2.2. *One has,*

$$\text{var}(\text{Cap}(\mathcal{R}_n)) = \begin{cases} \mathcal{O}(n \log n) & \text{if } d = 5 \\ \mathcal{O}(n) & \text{if } d \geq 6. \end{cases}$$

Remark 2.3. Actually sharp asymptotics are known: in dimension 5, one has $\text{var}(\text{Cap}(\mathcal{R}_n)) \sim \sigma_5 n \log n$, and in higher dimension $\text{var}(\text{Cap}(\mathcal{R}_n)) \sim \sigma_d n$, for some positive constant $(\sigma_d)_{d \geq 5}$, see respectively [Sch19] and [ASS18].

As a consequence of the previous results one can obtain Gaussian type upper bounds for the moderate deviations in the upward deviations.

Proposition 2.4. *There exist positive constants $(c_d)_{d \geq 5}$, such that for any $n \geq 2$, and $\lambda > 0$,*

$$\mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \geq \lambda) \leq \begin{cases} \exp\left(-c_5 \cdot \frac{\lambda^2}{n(\log n)^3}\right) & \text{if } d = 5 \\ \exp\left(-c_6 \cdot \frac{\lambda^2}{n(\log \log n)^2}\right) & \text{if } d = 6 \\ \exp\left(-c_d \cdot \frac{\lambda^2}{n}\right) & \text{if } d \geq 7. \end{cases}$$

Proof. For simplicity let us concentrate on the proof when $d = 5$. We will explain at the end the necessary modifications to the proof when $d \geq 6$. Note first that one can always assume that ζ is smaller than $n/2$. We use now (1.4) repeatedly along a dyadic decomposition of $\{0, \dots, n\}$. This gives for $L \geq 1$, with $m_L := \lfloor n/2^L \rfloor$,

$$\text{Cap}(\mathcal{R}_n) = \sum_{i=1}^{2^L} \text{Cap}(\mathcal{R}_{m_L}^{(i)}) - \sum_{\ell=1}^L \Sigma_\ell,$$

where the $\mathcal{R}_{m_L}^{(i)}$ are consecutive pieces of the range of length either m_L or $m_L + 1$, and

$$\Sigma_\ell := \sum_{j=1}^{2^{\ell-1}} \chi_C(\mathcal{R}_{m_\ell}^{(2^{j-1})}, \mathcal{R}_{m_\ell}^{(2^j)}),$$

with similar notation as above, in particular $m_\ell = \lfloor n/2^\ell \rfloor$. Thus,

$$\mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] > \zeta) \leq \mathbb{P}\left(\sum_{i=1}^{2^L} \text{Cap}(\mathcal{R}_{m_L}^{(i)}) - \mathbb{E}[\text{Cap}(\mathcal{R}_{m_L}^{(i)})] > \frac{\zeta}{2}\right) + \sum_{\ell=1}^L \mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{2L}\right) \quad (2.15)$$

We fix now L , such that $n/\zeta \leq m_L \leq 2n/\zeta$. The first term in (2.15) is ruled out using Bernstein's inequality, and Lemma 2.2, which give

$$\mathbb{P}\left(\sum_{i=1}^{2^L} \text{Cap}(\mathcal{R}_{m_L}^{(i)}) - \mathbb{E}[\text{Cap}(\mathcal{R}_{m_L}^{(i)})] > \zeta/2\right) \leq \exp\left(-c \frac{\zeta^2}{n \log m_L}\right), \quad (2.16)$$

for some constant $c > 0$. Concerning the sum in (2.15), note first that by (2.13), one has

$$\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^{\ell/2} \sqrt{n}),$$

for any $\ell \geq 1$. Therefore, one can assume that ℓ is such that $2^{\ell/2} \sqrt{n} > c\zeta/L$, for some constant $c > 0$, for otherwise the corresponding probability is zero. For such ℓ one has by using standard concentration results (see Theorem 4.4. in [CL06]):

$$\mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{2L}\right) \leq \exp\left(-c \frac{(\zeta/L)}{\sqrt{m_\ell} + Ln(\log m_\ell)/\zeta}\right) \leq \exp\left(-\frac{c\zeta^2}{n(\log n)^3}\right),$$

which completes the proof in case $d = 5$. In case $d \geq 6$, the variance is linear. So first, the term $\log m_L$ can be removed in (2.16). Moreover, in case $d = 6$, one has $\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^\ell \log n)$, and thus only the ℓ 's such that $\zeta \geq 2^\ell \geq c\zeta/\log n$ need to be considered. There are order $\log \log n$ such integers, and for each of them one has by the same argument as above,

$$\mathbb{P}\left(\mathbb{E}[\Sigma_\ell] - \Sigma_\ell > \frac{\zeta}{C \log \log n}\right) \leq \exp(-c\zeta^2/(\log \log n)^2),$$

which concludes the proof in case $d = 6$. The case $d \geq 7$ is similar, since this time $\mathbb{E}[\Sigma_\ell] = \mathcal{O}(2^\ell)$, and thus there are only a bounded number of integers ℓ 's that need to be considered. \square

3 Transfer of downward deviations to the corrector

The possibility of establishing the heuristic picture described in the introduction stems from writing the capacity of a union of sets as a sum of capacities and a cross-term. The latter though typically small is nonetheless responsible for the fluctuations. Iterating this decomposition leads to an expression of the capacity of the range as a sum of i.i.d. terms minus a sum of cross-terms. The so-called corrector is obtained by summing appropriate conditional expectations of these cross-terms.

Our first result in this section, Lemma 3.1, provides an explicit expression for (what turns out to be an upper bound for) this corrector in terms of a sum of convoluted Green's functions taken along the trajectory and weighted by escape probability terms. We then recall a result from [AS19], which relates the deviations of the capacity to those of the corrector, which we state here as Proposition 3.2.

Thus, the strategy is similar to the one used to treat downward deviations for the range developed in [AS19]. However the form of the corrector is slightly different. Roughly it involves a convolution of the Green's function with itself together multiplied by escape probability terms, where in [AS17a] only the Green's function appeared.

A detailed analysis of this corrector is carried out in Sections 4.4 and 4.3. Before we can state precisely the result, some preliminary notation is required.

For $I \subset \mathbb{N}$, we write $\mathcal{R}(I) := \{S_k, k \in I\}$, for the set of visited sites during times $k \in I$. Since for any two intervals $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I \cup J) = \mathcal{R}(I) \cup \mathcal{R}(J)$, (2.9) gives

$$\text{Cap}(\mathcal{R}(I \cup J)) = \text{Cap}(\mathcal{R}(I)) + \text{Cap}(\mathcal{R}(J)) - \chi_{\mathcal{C}}(\mathcal{R}(I), \mathcal{R}(J)). \quad (3.1)$$

Next, given two sets A and B , their symmetric difference is defined as $A \Delta B := (A \cap B^c) \cup (B \cap A^c)$. Note in particular that for any $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I) \Delta \mathcal{R}(J) \subset \mathcal{R}(I \Delta J)$. Moreover, it follows from (2.5), (2.6) and (2.8) that for any $A, B \subset \mathbb{Z}^d$,

$$|\text{Cap}(A) - \text{Cap}(B)| \leq \text{Cap}(A \Delta B) \leq |A \Delta B|.$$

Applying this inequality to ranges on some intervals I and J , we get

$$|\text{Cap}(\mathcal{R}(I)) - \text{Cap}(\mathcal{R}(J))| \leq |I \Delta J|. \quad (3.2)$$

Now given some integer $T \leq n$, we define for $j \geq 0$ and $k \geq 1$,

$$I_{j,k} := [j + (k-1)T, j + kT], \quad \text{and} \quad \tilde{I}_{j,k} := I_{j,1} \cup \dots \cup I_{j,k}.$$

It follows from (3.2) that almost surely

$$|\text{Cap}(\mathcal{R}_n) - \frac{1}{T} \sum_{j=0}^{T-1} \text{Cap}(\mathcal{R}(\tilde{I}_{j, \lfloor n/T \rfloor}))| \leq T. \quad (3.3)$$

On the other hand, applying (3.1) recursively we obtain for any $j = 0, \dots, T-1$,

$$\text{Cap}(\mathcal{R}(\tilde{I}_{j, \lfloor n/T \rfloor})) = \sum_{k=1}^{\lfloor n/T \rfloor} \text{Cap}(\mathcal{R}(I_{j,k})) - \sum_{k=1}^{\lfloor n/T \rfloor - 1} \chi_{\mathcal{C}}(\mathcal{R}(\tilde{I}_{j,k}), \mathcal{R}(I_{j,k+1})). \quad (3.4)$$

Define now

$$\xi_n^*(T) := \frac{1}{T} \sum_{j=0}^{T-1} \sum_{\ell=1}^{\lfloor n/T \rfloor} \mathbb{E} \left[\chi_{\mathcal{C}}(\mathcal{R}(\tilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})) \mid \mathcal{F}_{j+\ell T} \right],$$

and

$$\xi_n(T) := \sum_{k=0}^n \sum_{x \in \mathcal{R}_k} \mathbb{P}_x(H_{\mathcal{R}_k}^+ = \infty) \cdot \frac{G \star G_T(x - S_k)}{T}. \quad (3.5)$$

Lemma 3.1. *One has, for any $n \geq 1$ and $1 \leq T \leq n$,*

$$\xi_n^*(T) \leq 2\xi_n(T).$$

Proof. By (2.12), for any sets A and B ,

$$\chi(A, B) \leq \tilde{\chi}(A, B) := \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x(H_A^+ = \infty) \cdot G(x - y) \cdot \mathbb{P}_y(H_B^+ = \infty).$$

Note that $\tilde{\chi}$ is symmetric in the sense that $\tilde{\chi}(A, B) = \tilde{\chi}(B, A)$, for any A, B . Bounding the last probability term by one, we get

$$\chi(A, B) + \chi(B, A) \leq 2\bar{\chi}(A, B), \quad \text{with} \quad \bar{\chi}(A, B) := \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x(H_A^+ = \infty) \cdot G(x - y).$$

Now for any j, k , the Markov property and translation invariance of the simple random walk give

$$\begin{aligned} \mathbb{E} \left[\bar{\chi}(\mathcal{R}(\tilde{I}_{j,k}), \mathcal{R}(I_{j,k+1})) \mid \mathcal{F}_{j+kT} \right] &= \sum_{x \in \mathcal{R}(\tilde{I}_{j,k})} \mathbb{P}_x(H_{\mathcal{R}(\tilde{I}_{j,k})}^+ = \infty) \sum_{y \in \mathbb{Z}^d} G(x - y) \cdot \mathbb{P}(y \in \mathcal{R}(I_{j,k+1}) \mid \mathcal{F}_{j+kT}) \\ &\stackrel{(2.2)}{\leq} \sum_{x \in \mathcal{R}(\tilde{I}_{j,k})} \mathbb{P}_x(H_{\mathcal{R}(\tilde{I}_{j,k})}^+ = \infty) \cdot G \star G_T(x - S_{j+kT}), \end{aligned}$$

and the lemma follows using (3.4). □

We then obtain the main result of this section. The proof is exactly the same as for Corollary 4.2 in [AS19], and is thus omitted.

Proposition 3.2. *There exists a positive constant c , such that for any $n \geq 2$ any $1 \leq T \leq n$, and any $\zeta \geq \frac{n \log n}{T}$,*

$$\mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \leq 2T \exp(-c\frac{\zeta}{T}) + \mathbb{P}(\xi_n(T) \geq c\zeta).$$

Remark 3.3. In dimension 5, the mean of $\xi_n(T)$ is of order n/\sqrt{T} . So the upper deviations for $\xi_n(T)$ start to decay only for $\zeta > n/\sqrt{T}$, and since on the other hand one needs to take T at most of order $(\zeta n)^{1/3}$, to ensure the term $\exp(-c\zeta/T)$ to have the right decay, this imposes the condition $\zeta > n^{5/7}$. In particular the approach we have here has no chance to work up to the Gaussian regime. On the other hand in dimension 7 and higher, the mean of $\xi_n(T)$ is of order n/T , and T can be chosen of order $\zeta^{2/(d-2)}$, which only imposes the a priori condition $\zeta > n^{(d-2)/d}$, leaving a chance to cover entirely the non-Gaussian regime.

4 Upper Bounds

We prove here the upper bounds in Theorems 1.1 and 1.3, as well as Theorem 1.6, thanks to Proposition 3.1 in [AS19].

4.1 Basic estimates

For $r > 0$, and $x \in \mathbb{R}^d$, we recall that

$$Q(x, r) := [x - r/2, x + r/2]^d \cap \mathbb{Z}^d,$$

and for simplicity let also $Q(r) := Q(0, r)$.

Lemma 4.1. *Assume that $d \geq 5$. There exists a constant $C_1 > 0$, such that for any $r \geq 1$, and any $\Lambda \subset Q(r)$,*

$$\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \leq C_1 r^2. \quad (4.1)$$

Proof. Without loss of generality, one can assume $r \geq 2$. For $i \geq 0$, write

$$\Lambda_i := \Lambda \cap (Q(r2^{-i}) \setminus Q(r2^{-i-1})),$$

and define $L := \lfloor \log r \rfloor$. Then, for some positive constants C_0 and C_1 ,

$$\begin{aligned} \sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) &\leq \sum_{i=0}^L \sum_{x \in \Lambda_i} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \\ &\leq \sum_{i=0}^L \left(\frac{2^{i+1}}{r}\right)^{d-4} \text{Cap}(\Lambda_i) \leq \sum_{i=0}^L \left(\frac{2^{i+1}}{r}\right)^{d-4} \text{Cap}\left(Q\left(\frac{r}{2^i}\right)\right) \\ &\leq C_0 \sum_{i=0}^L \left(\frac{2^{i+1}}{r}\right)^{d-4} \cdot \left(\frac{r}{2^i}\right)^{d-2} \leq C_1 r^2. \end{aligned}$$

□

The second result we need is the following.

Lemma 4.2. *Assume $d \geq 5$. There exists a constant $C_2 > 0$, such that for any $x \in \mathbb{Z}^d$, and any $T \geq 1$,*

$$\varphi_T(x) := \frac{G \star G_T(x)}{T} \leq C_2 \cdot \min\left(\frac{1}{1 + \|x\|^{d-2}}, \frac{1}{T(1 + \|x\|^{d-4})}\right).$$

Proof. First $G_T \leq G$, so that $G \star G_T \leq G \star G$, and an elementary computation gives that $G \star G(x) \leq C_2/(1 + \|x\|^{d-4})$, for all $x \in \mathbb{Z}^d$, and some $C_2 > 0$. This already proves one of the two desired bounds.

For the other one write, by definition of G_T ,

$$G \star G_T(x) = \sum_{y \in \mathbb{Z}^d} G(x - y)G_T(y) = \sum_{k=1}^T \mathbb{E}[G(x - S_k)]. \quad (4.2)$$

Let τ be the hitting time of the cube $Q(x, 2)$ for the walk starting at 0, and note that one can assume $\|x\| \geq 4$. Since G is harmonic on $\mathbb{Z}^d \setminus \{0\}$, we have for any $k \geq 0$, $\mathbb{E}[G(x - S_{k \wedge \tau})] = G(x)$. This entails

$$G(x) = \mathbb{E}[\mathbf{1}\{\tau \geq k\}G(x - S_k)] + \mathbb{E}[\mathbf{1}\{\tau < k\}G(x - S_\tau)] \geq \mathbb{E}[G(x - S_k)] - \mathbb{E}[\mathbf{1}\{\tau < k\}G(x - S_k)].$$

Now, we use that $G(x)$ is bounded by $G(0)$, so that the previous inequality gives

$$\mathbb{E}[G(x - S_k)] \leq G(x) + G(0)\mathbb{P}(\tau < \infty) \stackrel{(2.3)}{\leq} (1 + CG(0)) \cdot G(x),$$

for some constant $C > 0$. Injecting this in (4.2) and using (2.1), proves the second inequality. \square

Our last estimate requires some new notation. For a (deterministic) function $S : \mathbb{N} \rightarrow \mathbb{Z}^d$ (not necessarily to the nearest neighbor), and for any $\mathcal{K} \subset \mathbb{N}$, we define for any $\Lambda \subset \mathbb{Z}^d$,

$$\ell_{\mathcal{K}}(\Lambda) := \sum_{k \in \mathcal{K}} \mathbf{1}\{S(k) \in \Lambda\}.$$

Lemma 4.3. *Assume $d \geq 3$. Let $S : \mathbb{N} \rightarrow \mathbb{Z}^d$, and $\mathcal{K} \subset \mathbb{N}$, be such that for some $\rho \in (0, 1)$ and $r \geq 1$,*

$$\ell_{\mathcal{K}}(Q(x, r)) \leq \rho|Q(r)|, \quad \text{for all } x \in r\mathbb{Z}^d.$$

There exists a constant $C_3 > 0$ (independent of ρ, r, S , and \mathcal{K}), such that for any $z \in \mathbb{Z}^d$,

$$\sum_{k \in \mathcal{K}} \frac{\mathbf{1}(\|S(k) - z\| \geq 2r)}{\|S(k) - z\|^{d-2}} \leq C_3 \rho^{1-\frac{2}{d}} |\mathcal{K}|^{2/d}. \quad (4.3)$$

Proof. We start by proving that for any $R \geq 2r$, and any $z \in \mathbb{Z}^d$,

$$\sum_{k \in \mathcal{K}} \frac{\mathbf{1}(2r \leq \|S(k) - z\| \leq R)}{\|S(k) - z\|^{d-2}} \leq C_3 \rho R^2. \quad (4.4)$$

Consider a covering of the cube $Q(z, R)$ by a partition made of smaller cubes homeomorphic to $Q(r)$, with centers in the set $z + r\mathbb{Z}^d$. For each $x \in z + r\mathbb{Z}^d$, with $x \neq z$, the contribution of the points $S(k)$ lying in $Q(x, r)$ to the sum we need to bound, is upper bounded (up to some constant) by $\rho r^d \cdot \|x - z\|^{2-d}$, and (4.4) follows as we observe that, for some constant $C > 0$,

$$\sum_{x \in z + r\mathbb{Z}^d} \frac{\mathbf{1}\{r \leq \|z - x\| \leq R\}}{\|z - x\|^{d-2}} \leq C \frac{R^2}{r^d}.$$

We then deduce (4.3), by observing that by rearranging the points $(S(k))_{k \in \mathcal{K}}$, one can only increase the sum (at least up to a multiplicative constant) by assuming they are all in $Q(z, 2(\frac{|\mathcal{K}|}{\rho})^{1/d})$, and still satisfy the hypothesis of the lemma. \square

4.2 The sets \mathcal{K}_n

We recall here our main tool from [AS19], which requires some new notation: for $r > 0$, we define

$$\mathcal{A}(r) := \{\mathcal{C} \subset \mathbb{Z}^d : |\mathcal{C}| < \infty, \|x - y\| > 2dr \text{ for all } x \neq y \in \mathcal{C}\}.$$

Then for $n \geq 0$, and $\Lambda \subseteq \mathbb{Z}^d$, define the time spent in Λ by the walk up to time n as

$$\ell_n(\Lambda) := \sum_{k=0}^n \mathbf{1}\{S_k \in \Lambda\}.$$

Finally given $\rho > 0$, $r \geq 1$, and $n \geq 1$, set

$$\mathcal{K}_n(r, \rho) := \{k \leq n : \ell_n(Q(S_k, r)) \geq \rho|Q(r)|\}. \quad (4.5)$$

Theorem 4.4 ([AS19]). *There exist positive constants C_0 and κ , such that for any $\rho > 0$, $r \geq 1$, and $n \geq 1$, satisfying*

$$\rho r^{d-2} \geq C_0 \log n, \quad (4.6)$$

one has for any $L \geq 1$,

$$\mathbb{P}(|\mathcal{K}_n(r, \rho)| \geq L) \leq C_0 \exp\left(-\kappa \rho^{\frac{2}{d}} L^{1-\frac{2}{d}}\right).$$

4.3 Dimension seven and larger

We assume here that $d \geq 7$, and fix in this case the value of T as

$$T := \left\lceil \frac{2\kappa}{c} \cdot \zeta^{\frac{2}{d-2}} \right\rceil.$$

Under the event of moderate deviations considered here (when the capacity of the range up to time n is reduced by an amount ζ from its mean value), the walk typically folds its trajectory a time of order ζ , in a region of volume $\zeta^{d/(d-2)}$. Thus the typical density of the range in the folding region is

$$\bar{\rho} := \zeta^{-\frac{2}{d-2}}.$$

Define ρ_i , r_i , and L_i , for $i \in \mathbb{Z}$, by

$$\rho_i := 2^{-i} \cdot \bar{\rho}, \quad r_i^{d-2} \cdot \rho_i = C_0 \log n, \quad \text{and} \quad L_i := \zeta \cdot 2^{\frac{2i}{d-2}},$$

with C_0 as in Theorem 4.4. Let then $\widehat{\mathcal{K}}_0 := \mathcal{K}_n(r_0, \bar{\rho})$, and for $i \geq 1$,

$$\widehat{\mathcal{K}}_i := \mathcal{K}_n(r_i, \rho_i) \setminus \left(\bigcup_{0 \leq j < i} \mathcal{K}_n(r_j, \rho_j) \right), \quad \text{and} \quad \widetilde{\mathcal{K}}_i := \mathcal{K}_n(r_{-i}, \rho_{-i}) \setminus \left(\bigcup_{-i < j \leq 0} \mathcal{K}_n(r_j, \rho_j) \right).$$

Define for $A > 0$, $\delta > 0$, and $I \in \mathbb{N}$,

$$\mathcal{E}(A, \delta, I) := \left(\bigcap_{0 \leq i \leq I} \{|\widehat{\mathcal{K}}_i| \leq \delta L_i\} \right) \cap \left(\bigcap_{i > I} \{|\widehat{\mathcal{K}}_i| \leq A L_i\} \right) \cap \left(\bigcap_{i < 0} \{|\widetilde{\mathcal{K}}_i| \leq A L_i\} \right).$$

Proposition 4.5. *For any $A > 0$, there exist $\delta > 0$ and $I \geq 0$, such that for any $n \geq 2$, and $n^{\frac{d-2}{d}} \cdot \log n \leq \zeta \leq n$,*

$$\mathcal{E}(A, \delta, I) \subseteq \{\xi_n(T) \leq \zeta\}.$$

Proof. We note that

$$\xi_n(T) \leq \Sigma_1 + \Sigma_2 + 2\Sigma_3 + 2\Sigma_4 + 2\Sigma_5 + 2\Sigma_6,$$

where

$$\begin{aligned} \Sigma_1 &:= \sum_{k \in \widehat{\mathcal{K}}_0} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \mathbb{P}_x(H_{\mathcal{R}_k}^+ = \infty) \cdot \mathbf{1}\{x \in Q(S_k, \sqrt{T})\}, \\ \Sigma_2 &:= \sum_{k \in \widehat{\mathcal{K}}_0} \sum_{k' \in \widehat{\mathcal{K}}_0} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \notin Q(S_k, \sqrt{T})\}, \\ \Sigma_3 &:= \sum_{i \geq 1} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \in Q(S_k, r_{i-1}) \setminus Q(S_k, r_0)\}, \end{aligned}$$

$$\begin{aligned}\Sigma_4 &:= \sum_{i \geq 1} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \in Q(S_k, r_0)\}, \\ \Sigma_5 &:= \sum_{i \geq 1} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{j \geq i} \sum_{k' \in \widehat{\mathcal{K}}_j} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \notin Q(S_k, r_{j-1})\}, \\ \Sigma_6 &:= \sum_{k \in \widehat{\mathcal{K}}_0} \sum_{j \geq 1} \sum_{k' \in \widehat{\mathcal{K}}_j} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \notin Q(S_k, r_{j-1})\}.\end{aligned}$$

Now assume that $\mathcal{E}(A, \delta, I)$ holds. By Lemmas 4.1 and 4.2, one has

$$\Sigma_1 \leq C_1 C_2 |\widehat{\mathcal{K}}_0| \leq \frac{\zeta}{6},$$

provided $\delta < 1/(6C_1 C_2)$. Furthermore,

$$\Sigma_2 \leq C_2 \frac{|\widehat{\mathcal{K}}_0|^2}{T^{\frac{d-2}{2}}} \leq \frac{\zeta}{6},$$

provided again that δ is chosen small enough.

Next by definition, when $k \in \widehat{\mathcal{K}}_i$, the density in the shell $Q(S_k, r_j) \setminus Q(S_k, r_{j-1})$ is bounded by ρ_{j-1} , for all $1 \leq j < i$. Therefore, letting J being the smallest integer such that $r_J \geq \sqrt{T}$, and $C > 0$ some constant,

$$\begin{aligned}\Sigma_3 &\leq C_2 \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| \left\{ \sum_{j=1}^{J-1} \frac{\rho_j r_j^d}{T r_j^{d-4}} + \sum_{j=J}^{i-1} \frac{\rho_j r_j^d}{r_j^{d-2}} \right\} \\ &\leq C \log n \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| \left\{ \sum_{j=1}^{J-1} \frac{1}{T r_j^{d-6}} + \sum_{j=J}^{i-1} \frac{1}{r_j^{d-4}} \right\} \\ &\leq C \log n \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| \left\{ \frac{1}{T r_0^{d-6}} + \frac{1}{r_J^{d-4}} \right\} \\ &\leq C n \log n \left\{ \frac{1}{T r_0^{d-6}} + \frac{1}{T^{\frac{d-4}{2}}} \right\} \leq \frac{\zeta}{12},\end{aligned}$$

at least for n large enough, since $\frac{n \log n}{T} \leq \zeta$, by hypothesis. Similarly,

$$\begin{aligned}\Sigma_4 &\leq C_2 \sum_{i \geq 1} |\widetilde{\mathcal{K}}_i| \left\{ \frac{r_{-i}^{4/d} \rho_{-i}^{4/d}}{T} + \sum_{j=-i}^0 \frac{\rho_j r_j^d}{T r_j^{d-4}} \right\} \\ &\leq C \log n \sum_{i \geq 1} |\widetilde{\mathcal{K}}_i| \left\{ \frac{r_{-i}^{4/d} \rho_{-i}^{4/d}}{T} + \sum_{j=-i}^0 \frac{1}{T r_j^{d-6}} \right\} \\ &\leq C \log n \sum_{i \geq 1} |\widetilde{\mathcal{K}}_i| \left\{ \frac{r_{-i}^{4/d} \rho_{-i}^{4/d}}{T} + \frac{1}{T} \right\} \\ &\leq \frac{C n (\log n)^{1 + \frac{4}{d(d-2)}}}{T} \leq \frac{\zeta}{12},\end{aligned}$$

using again the hypothesis on ζ . Eventually by (4.3), we get for some constant $C > 0$,

$$\begin{aligned} \Sigma_5 &\leq C_2 \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| \sum_{j \geq i} |\widehat{\mathcal{K}}_j|^{2/d} \rho_j^{1-\frac{2}{d}} \leq C_2 A^{2/d} \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| \sum_{j \geq i} 2^{\frac{4j}{d(d-2)} - j(1-\frac{2}{d})} \\ &\leq CA^{2/d} \sum_{i \geq 1} |\widehat{\mathcal{K}}_i| 2^{-i(1-\frac{2}{d-2})} \leq CA^{2/d} \zeta \left\{ \delta \sum_{i \leq I} 2^{-i\frac{d-6}{d-2}} + A \sum_{i \geq I} 2^{-i\frac{d-6}{d-2}} \right\} \leq \frac{\zeta}{12}, \end{aligned}$$

choosing first I large enough, and then δ small enough. Exactly the same argument gives as well $\Sigma_6 \leq \zeta/12$, and this concludes the proof of the proposition. \square

4.4 Dimension five

We assume here that $d = 5$ and prove the upper bound in Theorem 1.3. We fix here

$$T := \lceil \frac{2\kappa}{c} \cdot (\zeta n)^{1/3} \rceil,$$

and

$$\bar{\rho} := \zeta^{5/3} n^{-7/3}.$$

Define next ρ_i , r_i , and L_i , for $i \in \mathbb{Z}$, by

$$\rho_i := 2^i \cdot \bar{\rho}, \quad r_i^3 \cdot \rho_i = C_0 \log n, \quad \text{and} \quad L_i := n \cdot 2^{-\frac{2i}{3}},$$

with C_0 as in Theorem 4.4. Let then for $i \in \mathbb{Z}$,

$$\widehat{\mathcal{K}}_i := \mathcal{K}_n(r_i, \rho_i) \setminus \left(\bigcup_{j>i} \mathcal{K}_n(r_j, \rho_j) \right).$$

Finally, for $A > 0$, $\delta > 0$, and $I \in \mathbb{N}$, let

$$\mathcal{E}(A, \delta, I) := \left(\bigcap_{-I \leq i \leq I} \{|\widehat{\mathcal{K}}_i| \leq \delta L_i\} \right) \cap \left(\bigcap_{i>I} \{|\widehat{\mathcal{K}}_i| \leq AL_i\} \right).$$

Proposition 4.6. *For any $A > 0$, there exist $\delta > 0$ and $I \geq 0$, such that for any $n \geq 2$, and $n^{5/7} \cdot \log n \leq \zeta \leq n$,*

$$\mathcal{E}(A, \delta, I) \subseteq \{\xi_n(T) \leq \zeta\}.$$

Proof. Let

$$\widetilde{\mathcal{K}}_0 := \left(\bigcup_{i \geq -I} \mathcal{K}_n(r_i, \rho_i) \right)^c,$$

and then note that

$$\xi_n(T) \leq 2\Sigma_1 + 2\Sigma_2 + \Sigma_3 + \Sigma_4,$$

where,

$$\Sigma_1 := \sum_{i \geq -I} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \in Q(S_k, r_i)\},$$

$$\begin{aligned}\Sigma_2 &:= \sum_{i \geq -I} \sum_{k \in \widehat{\mathcal{K}}_i} \sum_{0 \leq j \leq i} \sum_{k' \in \widehat{\mathcal{K}}_j} \varphi_T(S_{k'} - S_k) \cdot \mathbf{1}\{S_{k'} \notin Q(S_k, r_{j+1})\}, \\ \Sigma_3 &:= \sum_{k \in \widetilde{\mathcal{K}}_0} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \in Q(S_k, r_{-I})\}, \\ \Sigma_4 &:= \sum_{k \in \widetilde{\mathcal{K}}_0} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbf{1}\{x \notin Q(S_k, r_{-I})\},\end{aligned}$$

Assume now that the event $\mathcal{E}(A, \delta, I)$ holds. Let J be the smallest integer, such that $r_J \leq \sqrt{T}$. One has for some constant $C > 0$,

$$\Sigma_1 \leq C \sum_{i \geq -I} |\widehat{\mathcal{K}}_i| \left(\sum_{i \leq j \leq J} \frac{\rho_j r_j^5}{r_j^3} + \sum_{j > \max(i, J)} \frac{\rho_j r_j^5}{T r_j} \right) \leq C \frac{n \log n}{\sqrt{T}} \leq \frac{\zeta}{8},$$

by hypothesis on ζ . The same argument gives as well $\Sigma_3 \leq \zeta/4$.

Using next Lemma 4.3,

$$\begin{aligned}\Sigma_2 &\leq C_1 C_2 \sum_{i \geq -I} |\widehat{\mathcal{K}}_i| \sum_{-I \leq j \leq i} |\widehat{\mathcal{K}}_j|^{2/5} \rho_j^{3/5} \\ &\leq C \frac{\zeta}{n} \cdot \sum_{i \geq -I} |\widehat{\mathcal{K}}_i| \sum_{-I \leq j \leq i} 2^{j(-\frac{4}{15} + \frac{3}{5})} \\ &\leq C \zeta \cdot \left\{ \delta \sum_{-I \leq i \leq I} 2^{-i/3} + A \sum_{i \geq I} 2^{-i/3} \right\} \leq \frac{\zeta}{8},\end{aligned}$$

taking first I large enough, and then δ small enough. The same argument gives as well $\Sigma_4 \leq \zeta/4$, which concludes the proof of the proposition. \square

5 Lower Bounds

We prove here the lower bounds in Theorems 1.1 and 1.3. In fact in dimension 5 the result covers a larger range of possible values for ζ .

Proposition 5.1. *Assume $d = 5$. There exist positive constants ε_0 and $\underline{\kappa}$, such that for any $n \geq 2$, and any $\sqrt{n}(\log n)^3 \leq \zeta \leq \varepsilon_0 n$, one has*

$$\mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \geq \exp(-\underline{\kappa} \cdot (\frac{\zeta^2}{n})^{1/3}).$$

Proof. The proof of (2.12) in [ASS19b] reveals that for any finite $A, B \subset \mathbb{Z}^d$, one has also

$$\text{Cap}(A \cup B) \leq \text{Cap}(A) + \text{Cap}(B) - \chi_0(A, B), \quad (5.1)$$

with

$$\chi_0(A, B) := \sum_{x \in A \setminus B} \sum_{y \in B} \mathbb{P}_x(H_{A \cup B}^+ = \infty) G(y - x) \mathbb{P}_y(H_B^+ = \infty).$$

Now given $n \geq 1$, set $\ell = \lfloor \frac{n}{10} \rfloor$, and $m = n - \ell$. We apply (5.1) with $A = \mathcal{R}_m$ and $B = \mathcal{R}[m, n]$. Fix $\varepsilon_0 > 0$ (later chosen small enough), and define

$$E := \{\overline{\text{Cap}}(\mathcal{R}_n) \geq -\varepsilon_0 n\},$$

where we use the notation $\overline{\text{Cap}}(\mathcal{R}_n)$ for the centered capacity. Using (5.1), Lemma 2.1, and Proposition 2.4, we deduce that for some constant $c > 0$,

$$\begin{aligned} \mathbb{P}(-\varepsilon_0 n \leq \overline{\text{Cap}}(\mathcal{R}_n) \leq -\zeta) &\geq \mathbb{P}(E, \chi_0(\mathcal{R}_m, \mathcal{R}[m, n]) \geq 4\zeta) - \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}_m) \geq \zeta) \\ &\quad - \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}[m, n]) \geq \zeta) \\ &\geq \mathbb{P}(E, \chi_0(\mathcal{R}_m, \mathcal{R}[m, n]) \geq 4\zeta) - 2 \exp\left(-c \frac{\zeta^2}{n(\log n)^3}\right). \end{aligned} \quad (5.2)$$

Note that when $\zeta \geq \sqrt{n}(\log n)^3$, then $\zeta^2/(n(\log n)^3) \geq (\log n)(\zeta^2/n)^{1/3}$, and therefore the last term above is negligible. Now, let $\rho > 0$ be some small constant (to be fixed later) and consider the event

$$F := \{\|S_k\| \leq \rho \cdot n^{2/3} \zeta^{-1/3}, \quad \text{for all } k \leq n\}.$$

Note that by (2.1) and (2.7), on the event F ,

$$\chi_0(\mathcal{R}_m, \mathcal{R}[m, n]) \geq c_\rho \cdot \frac{\zeta}{n^2} \cdot \text{Cap}(\mathcal{R}[m, n]) \cdot (\text{Cap}(\mathcal{R}_n) - \text{Cap}(\mathcal{R}[m, n])), \quad (5.3)$$

for some constant $c_\rho > 0$, going to infinity as ρ goes to zero. Furthermore, by (2.6), one has

$$\text{Cap}(\mathcal{R}_n) \leq \text{Cap}(\mathcal{R}_m) + \text{Cap}(\mathcal{R}[m, n]),$$

and thus by Lemma 2.1 and Proposition 2.4, by taking ε_0 small enough, we get for n large enough,

$$\begin{aligned} \mathbb{P}\left(\text{Cap}(\mathcal{R}[m, n]) \leq \gamma_5 \frac{\ell}{2}, E\right) &\leq \mathbb{P}(\text{Cap}(\mathcal{R}_m) \geq \gamma_5(m + \ell/3)) \\ &\leq \mathbb{P}\left(\overline{\text{Cap}}(\mathcal{R}_m) \geq \gamma_5 \frac{\ell}{10}\right) \leq \exp\left(-c' \frac{n}{(\log n)^3}\right), \end{aligned}$$

for some constant $c' > 0$, and with γ_5 as in (1.2). Similarly one has for some possibly smaller constant $c' > 0$,

$$\mathbb{P}\left(\text{Cap}(\mathcal{R}_n) - \text{Cap}(\mathcal{R}[m, n]) \leq \gamma_5 \frac{n}{4}, E\right) \leq \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}[m, n]) \geq \gamma_5 \ell) \leq \exp\left(-c' \frac{n}{(\log n)^3}\right).$$

Then (5.3) gives

$$\begin{aligned} \mathbb{P}(\chi_0(\mathcal{R}_m, \mathcal{R}[m, n]) \geq \frac{c_\rho \gamma_5^2}{100} \cdot \zeta, E) &\geq \mathbb{P}(\chi_0(\mathcal{R}_m, \mathcal{R}[m, n]) \geq \frac{c_\rho \gamma_5^2}{100} \cdot \zeta, E \cap F) \\ &\geq \mathbb{P}(E \cap F) - 2 \exp\left(-c' \frac{n}{(\log n)^3}\right) \\ &\geq \mathbb{P}(F) - \mathbb{P}(E^c) - 2 \exp\left(-c' \frac{n}{(\log n)^3}\right). \end{aligned}$$

Coming back to (5.2), and choosing ρ , such that $c_\rho \geq 300/\gamma_5^2$, we deduce that

$$\mathbb{P}(\overline{\text{Cap}}(\mathcal{R}_n) \leq -\zeta) = \mathbb{P}(-\varepsilon_0 n \leq \overline{\text{Cap}}(\mathcal{R}_n) \leq -\zeta) + \mathbb{P}(E^c)$$

$$\geq \mathbb{P}(F) - 2 \exp\left(-c' \frac{n}{(\log n)^3}\right) - 2 \exp(-c(\log n) \cdot \zeta^{2/3} n^{-1/3}).$$

Moreover, it is well known that for any $\rho > 0$, there exists $\kappa > 0$, such that

$$\mathbb{P}(F) \geq \exp(-\kappa \cdot \zeta^{2/3} n^{-1/3}),$$

and this concludes the proof. \square

In dimension 6 and more the result reads as follows.

Proposition 5.2. *Assume $d \geq 7$. There exist positive constants ε_0 , K and $\underline{\kappa}$, such that for any $n \geq 2$ and any $Kn^{\frac{d-2}{d}} \leq \zeta \leq \varepsilon_0 n$, one has*

$$\mathbb{P}(\text{Cap}(\mathcal{R}_n) - \mathbb{E}[\text{Cap}(\mathcal{R}_n)] \leq -\zeta) \geq \exp\left(-\underline{\kappa} \cdot \zeta^{1-\frac{2}{d-2}}\right).$$

In dimension $d = 6$, the same result holds for $n^{\frac{d-2}{d}} (\log \log n)^2 \leq \zeta \leq \varepsilon_0 n$.

Proof. We prove the result for $d \geq 7$ to keep notation simple, but the same argument works as well for $d = 6$. Set $\ell := \lfloor 5\zeta/\gamma_d \rfloor$. Using (2.6), Lemma 2.1, and Proposition 2.4, we obtain that for some constant $c > 0$,

$$\begin{aligned} \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}_n) \leq -\zeta) &\geq \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}_\ell) \leq -3\zeta) - \mathbb{P}(\overline{\text{Cap}}(\mathcal{R}[\ell, n]) \geq \zeta) \\ &\geq \mathbb{P}(\text{Cap}(\mathcal{R}_\ell) \leq \zeta) - \exp\left(-c \cdot \frac{\zeta^2}{n}\right), \end{aligned}$$

at least provided ζ is large enough, which one can always assume. Now the hypothesis on ζ implies that the last term is negligible, provided K is chosen large enough, and by the same argument as in the proof of Proposition 5.1, one can see that the first term on the right-hand side is of the right order (which is of the order of the event F where the walk stays confined in a ball of radius $c'\zeta^{1/(d-2)}$, with $c' > 0$ small enough, during the whole time ℓ). This concludes the proof of the proposition. \square

6 Upward Deviations

We prove here Theorem 1.7. Thanks to our decomposition (2.9), we can adapt the approach of Hamana and Kesten [HK], who proved a similar result for the size of the range.

The approach of Hamana and Kesten is based on first proving an approximate subadditivity relation for the probability of upward deviations, that is the existence of some constants $\chi \in (0, 1)$, $c > 0$, and $C > 0$, such that for any $m, n \geq 1$ integers, and y, z positive reals,

$$\mathbb{P}(|\mathcal{R}_{m+n}| \geq y + z - Ca(m, n)) \geq c \chi^{a(m, n)} \mathbb{P}(|\mathcal{R}_n| \geq y) \mathbb{P}(|\mathcal{R}_m| \geq z), \quad (6.1)$$

with

$$a(m, n) := (n \cdot m)^{\frac{1}{d+1}}.$$

The second step, which is general and only based on (6.1) and the fact that (when $d \geq 2$) one has $\lim_{m, n \rightarrow \infty} \frac{a(m, n)}{n \vee m} = 0$, shows that the following limit exists,

$$\psi(x) := - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(|\mathcal{R}_n| \geq x \cdot n), \quad \text{for all } x > 0,$$

and that ψ is continuous and convex on $[0, 1]$. Here we prove an analogous result as (6.1), and use their general argument to conclude.

Proof of Theorem 1.7. We first prove an analogous result as (6.1), but with $a(m, n)$ replaced by the function:

$$\tilde{a}(m, n) = (n \cdot m)^{\frac{1}{d-1}}.$$

In other words we establish the following inequality. There exists $\chi \in (0, 1)$, and $C > 0$, such that for any m, n integers and y, z positive reals,

$$\mathbb{P}(\text{Cap}(\mathcal{R}_{m+n}) \geq y + z - C\tilde{a}(m, n)) \geq \frac{1}{2}\chi^{\tilde{a}(m, n)}\mathbb{P}(\text{Cap}(\mathcal{R}_n) \geq y)\mathbb{P}(\text{Cap}(\mathcal{R}_m) \geq z). \quad (6.2)$$

The first step is to obtain the analogue of the following simple deterministic bound used in [HK]: if \mathcal{R}_n and $\tilde{\mathcal{R}}_m$ are two independent copies of the range, there is a positive constant C , such that for any $r \geq 1$

$$\frac{1}{|Q(r)|} \sum_{z \in Q(r)} |(z + \mathcal{R}_n) \cap \tilde{\mathcal{R}}_m| \leq C \frac{n \cdot m}{r^d}.$$

The corresponding bound in our context reads as follows:

$$\frac{1}{|Q(r)|} \sum_{z \in Q(r)} \sum_{x \in \mathcal{R}_n} \sum_{y \in \tilde{\mathcal{R}}_m} G(x - y + z) \leq C \frac{n \cdot m}{r^{d-2}}, \quad (6.3)$$

and is a direct consequence of (2.1) and the fact that for any $x \in \mathbb{Z}^d$, and for some constant $C > 0$,

$$\sum_{z \in Q(r)} \frac{1}{1 + \|z - x\|^{d-2}} \leq Cr^2.$$

Now to lighten notation, we simply write $a = \tilde{a}(m, n) = \lfloor (mn)^{\frac{1}{d-1}} \rfloor$. Using that the capacity is translation-invariant, we deduce

$$\begin{aligned} \text{Cap}(\mathcal{R}_{m+n+a}) &\stackrel{(2.5)}{\geq} \text{Cap}(\mathcal{R}_n \cup \mathcal{R}[n+a, n+m+a]) \\ &\stackrel{(2.9)}{=} \text{Cap}(\bar{\mathcal{R}}_n) + \text{Cap}(\tilde{\mathcal{R}}_m) - \chi_C(\bar{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + S'_a), \end{aligned} \quad (6.4)$$

with $\bar{\mathcal{R}}_n := \mathcal{R}_n - S_n$, $S'_a := S_{n+a} - S_n$, and $\tilde{\mathcal{R}}_m := \mathcal{R}[n+a, n+m+a] - S_{n+a}$. The Markov property implies that $\bar{\mathcal{R}}_n$ and $\tilde{\mathcal{R}}_m$ are independent, and distributed as \mathcal{R}_n and \mathcal{R}_m respectively. Furthermore,

$$\chi_C(\bar{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + S'_a) \stackrel{(2.12)}{\leq} \sum_{x \in \bar{\mathcal{R}}_n} \sum_{y \in \tilde{\mathcal{R}}_m} G(x - y - S'_a). \quad (6.5)$$

Now, one idea of Hamana and Kesten [HK] is to bound the law of S'_a by a uniform law on the cube $Q(a/d)$. Indeed for any $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$, one has

$$\mathbb{P}(S'_a = x) \geq \frac{1}{(2d)^a}, \quad (6.6)$$

since there is at least one path of length a going from 0 to x . Write $\bar{Q}(a/d)$ for the set of sites $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$. Then for any $x \in \bar{Q}(a/d)$, and any $\alpha > 0$,

$$\mathbb{P}\left(\text{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - \frac{\alpha}{2}\right) \stackrel{(6.4)}{\geq} \mathbb{P}(S'_a = x) \cdot \mathbb{P}\left(\text{Cap}(\bar{\mathcal{R}}_n) \geq z, \text{Cap}(\tilde{\mathcal{R}}_m) \geq y, \chi_C(\bar{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + x) \leq \frac{\alpha}{2}\right).$$

Integrating with respect to the uniform measure on $\overline{Q}(a/d)$, we get

$$\begin{aligned} \mathbb{P}(\text{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - \frac{\alpha}{2}) &\stackrel{(6.6)}{\geq} \frac{1}{(2d)^a} \\ &\times \frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbb{P}\left(\text{Cap}(\overline{\mathcal{R}}_n) \geq z, \text{Cap}(\tilde{\mathcal{R}}_m) \geq y, \chi_C(\overline{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + x) \leq \frac{\alpha}{2}\right). \end{aligned} \quad (6.7)$$

We need now to estimate the mean of $\chi_C(\overline{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + \cdot)$ with respect to the uniform measure. According to (6.3), there is a positive constant C , such that

$$\frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \chi_C(\overline{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + x) \leq C \frac{m \cdot n}{a^{d-2}} \leq Ca, \quad (6.8)$$

where the last inequality follows from the definition of a . Then by Chebychev's inequality, we obtain

$$\frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbf{1}(\chi_C(\overline{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + x) \leq 2Ca) \geq \frac{1}{2}. \quad (6.9)$$

As a consequence,

$$\begin{aligned} \mathbb{P}(\text{Cap}(\mathcal{R}_{m+n}) \geq z + y - a - 4Ca) &\stackrel{(2.6),(2.8)}{\geq} \mathbb{P}(\text{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - 4Ca) \\ &\stackrel{(6.7)}{\geq} \frac{1}{(2d)^a} \cdot \mathbb{E}\left[\mathbf{1}(\text{Cap}(\overline{\mathcal{R}}_n) \geq z) \cdot \mathbf{1}(\text{Cap}(\tilde{\mathcal{R}}_m) \geq y) \times \frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbf{1}(\chi_C(\overline{\mathcal{R}}_n, \tilde{\mathcal{R}}_m + x) \leq 2Ca)\right] \\ &\stackrel{(6.9)}{\geq} \frac{1}{2(2d)^a} \cdot \mathbb{P}(\text{Cap}(\mathcal{R}_n) \geq z) \mathbb{P}(\text{Cap}(\mathcal{R}_m) \geq y), \end{aligned}$$

proving (6.2), with $\chi = 1/(2d)$.

It then follows from the general arguments of Hamana and Kesten, see Lemma 3 in [HK], that the following limit exists for all $x > 0$:

$$\psi_d(x) := - \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(\text{Cap}(\mathcal{R}_n) \geq nx).$$

We now prove that the range for which $\psi_d(x)$ is finite is not empty. Define for $n \geq 0$,

$$c_n := \max_{\gamma: \{0, \dots, n\} \rightarrow \mathbb{Z}^d} \text{Cap}(\{\gamma(0), \dots, \gamma(n)\}), \quad (6.10)$$

where the max is taken over all nearest neighbor paths of length $n + 1$. By (2.6), it follows that $c_{n+m} \leq c_n + c_m$, for all $n, m \geq 0$, so that by Fekete's lemma, the limit $\lim_{n \rightarrow \infty} c_n/n$ exists. Call γ_d^* this limit. Note that $\psi_d(x)$ is finite on $[\gamma_d, \gamma_d^*]$, since the probability that the simple random walk follows the path realizing the maximum in (6.10) is larger than or equal to $1/(2d)^{n+1}$, so that $\psi_d(x) \leq \log(2d)$, for all $x \leq \gamma_d^*$. Conversely, by definition of c_n , one has $\psi_d(x) = \infty$ for all $x > \gamma_d^*$. Furthermore, it follows from Lemma 3 and Proposition 4 in [HK], that ψ_d is continuous, and convex on $(0, \gamma_d^*]$. Now Proposition 2.4 and Lemma 2.1 show that when $d = 5$, $\psi_d(x) \geq c(x - \gamma_5)^3$, for all $x \geq \gamma_d$. Likewise, when $d \geq 6$, we get $\psi_d(x) \geq c(x - \gamma_d)^3$, for $\gamma_d \leq x \leq 1$. Using convexity, this also shows that ψ_d is increasing on $[\gamma_d, \gamma_d^*]$. In addition one has $\psi_d(x) = 0$ for all $x < \gamma_d$, by definition of γ_d as the limit of the (normalized) expected capacity, and using that by (2.8), $\text{Cap}(\mathcal{R}_n) \leq n$.

Finally we show that $\gamma_d^* > \gamma_d$.

Consider \mathcal{D}_n the set of *no double backtrack at even times* paths of length $n + 1$ that we introduced in [AS17b]. By definition, this is simply the set of paths $\gamma : \{0, \dots, n\} \rightarrow \mathbb{Z}^d$, such that for any even $k \leq n$, one has $\gamma(k + 2) \neq \gamma(k)$. The only important property we need is that from a no-backtrack walk \tilde{S} , and a sum of independent geometric variables $\{\xi_i, i \in \mathbb{N}\}$, with parameter $1/(2d)^2$, we can build a simple random walk S such that

$$\mathcal{R}[0, n + 2 \sum_{i \leq n/2} \xi_i] = \tilde{\mathcal{R}}_n.$$

Thus, for any $\alpha > 0$, we have by (2.6) and (2.8),

$$\text{Cap}(\tilde{\mathcal{R}}_n) \geq \text{Cap}(\mathcal{R}_{(1+\alpha)n}) - \mathbf{1} \left(\sum_{i \leq n/2} \xi_i < \frac{\alpha n}{2} \right) \cdot (1 + \alpha)(n + 1).$$

By taking the maximum over \mathcal{D}_n on the left hand side, and then the expectation on the right hand side, we obtain

$$c_n \geq \max_{\pi \in \mathcal{D}_n} \text{Cap}(\pi) \geq \mathbb{E}[\text{Cap}(\mathcal{R}_{(1+\alpha)n})] - (1 + \alpha)(n + 1) \cdot \mathbb{P} \left(\sum_{i \leq n/2} \xi_i < \frac{\alpha n}{2} \right). \quad (6.11)$$

Now take $\alpha < 1/(2d)^2$, and use Chebyshev's inequality, to see that the last term of (6.11) is $\mathcal{O}(1)$. Together with Lemma 2.1 it implies that

$$c_n \geq \gamma_d(1 + \alpha)n - \mathcal{O}(\sqrt{n}),$$

which indeed proves that $\gamma_d < \gamma_d^*$. □

Acknowledgements: Perla Sousi participated at an early stage of the project and we thank her for stimulating discussions, and her proof of Lemma 4.1. We also thank Quentin Berger and Julien Poisat for the idea of considering the polymer melt. The authors were partly supported by the French Agence Nationale de la Recherche under grants ANR-17-CE40-0032 and ANR-16-CE93-0003.

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