SHARP ESTIMATES FOR DISTINGUISHED RANDOM WALKS ON AFFINE BUILDINGS OF TYPE $\widetilde{A}_r$

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Abstract. We study a distinguished random walk on affine buildings of type $\widetilde{A}_r$, which was already considered by Cartwright, Saloff-Coste and Woess. In rank $r=2$, it is the simple random walk and we obtain optimal global bounds for its transition density (same upper and lower bound, up to multiplicative constants). In the higher rank case, we obtain sharp uniform bounds in fairly large space-time regions which are sufficient for most applications.

In memory of Gerrit van Dijk (1939-2022)

1. Introduction

Heat diffusion in the continuous setting, respectively random walks in the discrete setting are extensively studied. A main issue consists in obtaining sharp upper and/or lower bounds for the heat kernel in the continuous setting, respectively for transition densities of random walks in the discrete setting. Actually there are few cases where the same global bound (up to multiplicative constants) have been obtained. Apart from the Euclidean setting, this was achieved for instance for Riemannian symmetric spaces of noncompact type [2, 3].

This work started as an attempt to obtain similar results for isotropic random walks on affine buildings; it remained unpublished [4] but we believe that the techniques as well as the result may be interesting for the mathematical community. More precisely, we study a distinguished random walk on affine buildings of type $\widetilde{A}_r$. This walk was already considered in [23, 7]; it is the simple random walk in rank $r\le 2$ but not in rank $r>2$. In rank $r=1$, affine buildings are homogeneous trees and the simple random walk was already studied in [8, 9, 27]. In rank $r=2$, we obtain the same global upper and lower bound, up to multiplicative constants. In higher rank $r>2$, we obtain the same result in a fairly large space-time region, as well as a global upper bound, which are sufficient for applications, for instance to deduce a global upper and lower bound for the Green function (see [26]). We recover in particular the spectral gap computed in [23] and the local limit theorem established in [7, 19].

In the meantime, the third author was able to generalize the latter results to finitely supported isotropic random walks on any affine building [26]. Nonetheless we consider that this work is worth publishing, since our results are global for buildings of type $\widetilde{A}_2$ and since our present approach is much simpler than the delicate analysis

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carried out in [26]. Our initial aim consisted in analyzing the behavior of isotropic random walks on affine buildings, in order to describe the Martin compactification of affine buildings. This was eventually achieved in [21].

Our paper is organized as follows. Basics are recalled in Section 2. Section 3 is devoted to our main result, namely a global upper and lower bound for the simple random walk on affine buildings of type $\widetilde{A}_2$. At the end of Section 3 and in Section 4, we extend in part our results to isotropic nearest neighbor random walks in rank $r=2$ and to to the distinguished random walk in rank $r>2$. Appendix A is devoted to some remarkable formulae for the Fourier transform of any nearest neighbor random walk in rank $r=2$.

Our method relies on a careful analysis of the transition density using the inverse Fourier transform. Fourier analysis on p-adic like buildings goes back to Macdonald [10] in the early seventies. There was no other reference available in the literature for a long time, until Cartwright [6] and Parkinson [17, 18] on the one hand, Mantero and Zappa [14, 15] on the other hand, resumed Fourier analysis on affine buildings. Our work, as well as all aforementioned ones about buildings, deals with vertices and it is natural to consider higher dimension simplices in affine buildings. A first study of random walks on chambers in affine buildings of type $\widetilde{A}_2$ is carried out in [20], relying on the theory developed in [16].

2. Preliminaries

2.1. **Notation.** Throughtout the paper, $\mathbb{N}$ will denote the set of nonnegative integers $\{0,1,2,\dots\}$ and $\mathbb{N}^*$ the set of positive integers $\{1,2,\dots\}$. The maximum of two real numbers a and b will be denoted by $a\vee b$, and the minimum by $a\wedge b$. Let us finally specify the meaning of the following binary symbols between two nonnegative functions f and g:

- $f\lesssim g$, resp. $f\gtrsim g$ means that there exists a constant $C>0$ such that $f\leq C g$, resp. $f\geq C g$,
- $f\approx g$ stands for $f\lesssim g$ and $f\gtrsim g$,
- in the case of positive functions, $f\sim g$ means that the ratio $\frac{f}{g}$ tends to 1.

2.2. **Root system.** We recall some standard notation and refer e.g. to [5] for more details. In the Euclidean space $\mathbb{R}^{r+1}$, consider the subspace

$$\mathfrak{a}=\{\,z\in\mathbb{R}^{r+1}\,|\,z_1+\ldots+z_{r+1}=0\,\}$$

and the root system of type A_r

$$R=\{\,e_j-e_k\,|\,1\leq j\neq k\leq r+1\,\}.$$

Notice that

- the inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{r+1}$ extends to a $\mathbb{C}$-bilinear form on $\mathbb{C}^{r+1}$ and in particular on the complexification $\mathfrak{a}_{\mathbb{C}}=\mathfrak{a}+i\,\mathfrak{a}$,
- each root $\alpha\in R$ coincides with its coroot $\alpha^\vee=\frac{2}{\|\alpha\|^2}\,\alpha$.

Consider the positive subsystem

$$R^+=\{\,e_j-e_k\,|\,1\leq j<k\leq r+1\,\}$$

in R, the corresponding positive Weyl sector

$$\mathfrak{a}^+=\{\,z\in\mathfrak{a}\,|\,z_1>\ldots>z_{r+1}\,\}$$

in $\mathfrak{a}$ and its closure

$$\mathrm{cl}(\mathfrak{a}^+)=\{\,z\in\mathfrak{a}\,|\,z_1\geq\ldots\geq z_{r+1}\,\},$$

the basis of simple roots

$$\alpha_j = e_j - e_{j+1} \qquad (1 \le j \le r)$$

and the dual basis of fundamental weights

$$\lambda_j = \sum\nolimits_{1\le k\le j} e_k - \tfrac{j}{r+1} \sum\nolimits_{1\le k\le r+1} e_k \qquad (1 \le j \le r)\,.$$

Notice that

$$\sum\nolimits_{j=1}^{r} \lambda_j \quad \text{is equal to} \quad \rho \,=\, \tfrac{1}{2}\sum\nolimits_{\alpha\in R^+} \alpha\,.$$

Let $P = \sum_{j=1}^{r} \mathbb{Z}\lambda_j$ be the weight lattice and $Q = \sum_{j=1}^{r} \mathbb{Z}\alpha_j$ the root sublattice. Let $P^+ = P \cap \mathrm{cl}(\mathfrak{a}^+) = \sum_{j=1}^{r} \mathbb{N}\lambda_j$, resp. $P^{++} = P \cap \mathfrak{a}^+ = \sum_{j=1}^{r} \mathbb{N}^*\lambda_j$, be the subset of dominant weights, resp. strictly dominant weights. The length of $\lambda \in P^+$ is $|\lambda| = \sum_{j=1}^{r} \langle \alpha_j, \lambda\rangle$. The Coxeter complex is a simplicial complex in $\mathfrak{a}$ whose set of vertices is P. Its maximal simplices are called chambers. The fundamental chamber C_0 has vertices $\lambda_0 = 0, \lambda_1, \dots, \lambda_r$. Define the label function $\tau : P \longrightarrow \{0, \dots, r\}$ by the following two conditions:

- For the fundamental chamber, $\tau(\lambda_j) = j$.
- For each chamber, the labels of vertices are exactly $0, \dots, r$.

Let $W_0 = \mathfrak{S}_{r+1}$ be the Weyl group, generated by the reflections $r_\alpha(z) = z - \langle \alpha, z\rangle \alpha$ along roots $\alpha \in R$, and let $W = W_0 \ltimes Q$, resp. $\widetilde{W} = W_0 \ltimes P$ be the affine Weyl group, resp. the extended Weyl group, generated by W_0 and by the translations τ_λ along $\lambda \in Q$, resp. $\lambda \in P$. Stabilizers will be denoted by subscripts, for instance $W_0 = \widetilde{W}_0$ is the stabilizer of 0 in W and in $\widetilde{W}$. Let $w \longmapsto q_w$ be an extended parameter function on $\widetilde{W}$. For the type $\widetilde{A}_r$ considered in this work, recall that there is an integer $q > 1$ such that

$$q_w = q^{\,\ell(w)} \qquad \forall\, w \in W_0\,,$$

where $\ell(w)$ denotes the lenght of w in W_0, and

$$q_{t_\lambda} = q^{\,2\langle \rho, \lambda\rangle} \qquad \forall\, \lambda \in P^+.$$

Eventually the following Poincaré polynomial

$$V(q^{-1}) = \sum\nolimits_{w\in V} q_w^{-1}$$

is attached to every subset V of W_0.

2.3. **Affine building.** In this subsection, we follow mostly [22] and refer for more details to [6, 17]. An affine building of type $\widetilde{A}_r$ is a nonempty simplicial complex containing subcomplexes called apartments such that:

- Each apartment is isomorphic to the Coxeter complex associated to $\widetilde{A}_r$.
- Any pair of simplices is contained in an apartment.
- Given two apartments which share at least one chamber (simplex of maximal dimension), there exists a unique isomorphism between them, which fixes pointwise their intersection.

The building will be assumed to be regular, thick and locally finite. By definition this means that, given any chamber C and any face F (simplex of codimension 1) of C, the cardinality of the set of chambers different from C and containing F is independent of C and F, and is equal to $2 \le q < \infty$. We denote by $\mathcal{X}$ the set of vertices (simplices of dimension 0) of the building. In rank $r = 1$, buildings are homogeneous trees with $q+1$ edges. Fix a base point $0 \in \mathcal{X}$. It follows from the above definition that one can define a label function $\tau : \mathcal{X} \to \{0, \dots, r\}$ such that

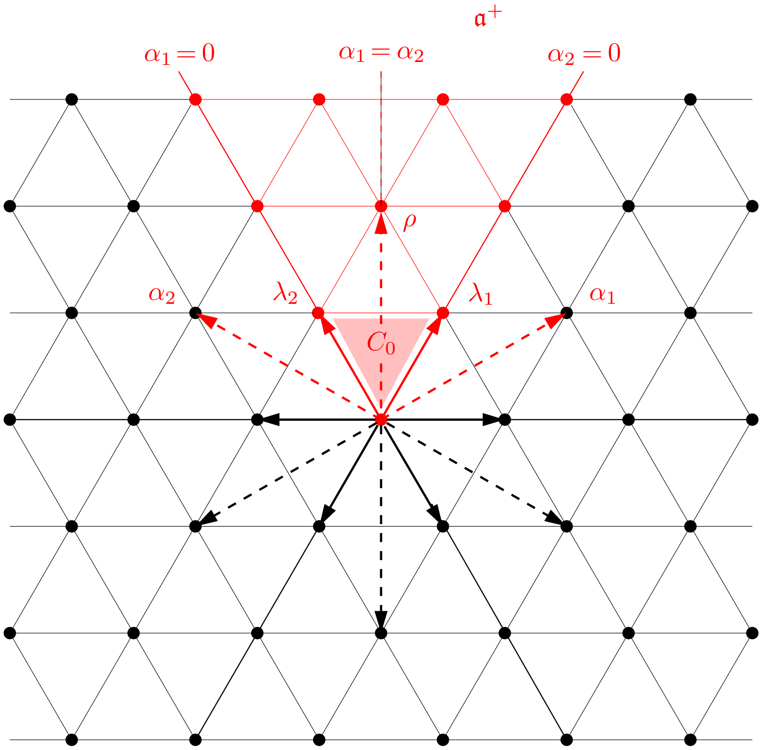

FIGURE 1. Apartment in an affine building of type $\widetilde{A}_2$

$\tau(0) = 0$ and all isomorphisms in the definition preserve labels. Now given $x \in \mathcal{X}$, there exists an apartment $\widetilde{P}$ containing 0 and x and a label preserving isomorphism between $\widetilde{P}$ and P, sending 0 to 0 and x to an element of P^+. The image of x by such an isomorphism is called the radial part of x and will be denoted by x^+. Let

$$V_\lambda(0) = \{\, y \in \mathcal{X} \mid y^+ = \lambda \,\},$$

be the so-called sphere of radius $\lambda \in P^+$ centered at 0. For every $y \in V_\lambda(0)$, we set $|y| = |\lambda|$ and $y_j = \langle \alpha_j, \lambda \rangle$. More generally, the sphere $V_\lambda(x)$ of radius $\lambda \in P^+$ and center $x \in \mathcal{X}$ consists of all $y \in \mathcal{X}$ such that there exist an apartment $\widetilde{P}$ containing $\{x, y\}$ and an isomorphism between $\widetilde{P}$ and P which preserves labels up to translation and which sends x to 0 and y to λ. The cardinality of $V_\lambda(x)$ is independent of x and is given by

$$N_\lambda = \tfrac{W_0(q^{-1})}{(W_0 \cap W_\lambda)(q^{-1})}\, q^{2\langle \rho, \lambda \rangle}.$$

Finally $(x, y)^+ = \lambda \in P^+$ and $d(x, y) = |\lambda| \in [0, +\infty)$ are the vectorial and scalar distances between x and y. Both are invariant under isomorphisms of $\mathcal{X}$.

2.4. **Special functions.** Consider the fundamental skew invariant polynomial

$$\boldsymbol{\pi}(z) = \prod\nolimits_{\alpha \in R^+} \langle \alpha, z \rangle\,,$$

the Weyl denominator

$$\Delta(z)=\sum\nolimits_{w\in W_0}(\det w)\,e^{\langle w.\rho,z\rangle}=\prod\nolimits_{\alpha\in R^+}\big(e^{\frac{\langle\alpha,z\rangle}{2}}-e^{-\frac{\langle\alpha,z\rangle}{2}}\big)\,,$$

and the following functions on $\mathfrak{a}_{\mathbb{C}}$:

$$\begin{aligned}\mathbf{b}(z)&=\prod\nolimits_{\alpha\in R^+}\big(1-q^{-1}e^{-\langle\alpha,z\rangle}\big)\,,\\ \mathbf{c}(z)=e^{\langle\rho,z\rangle}\,\Delta(z)^{-1}\,\mathbf{b}(z)&=\prod\nolimits_{\alpha\in R^+}\tfrac{1-q^{-1}e^{-\langle\alpha,z\rangle}}{1-e^{-\langle\alpha,z\rangle}}\,,\\ h(z)&=\sum\nolimits_{j=1}^{r}\sum\nolimits_{\lambda\in W_0.\lambda_j}e^{\langle\lambda,z\rangle}\,.\end{aligned}\tag{2.1}$$

Notice that $\mathbf{b}(z)$ is bounded on $\mathrm{cl}(\mathfrak{a}^+)+i\,\mathfrak{a}$, as well as all its derivatives, and moreover that $|\mathbf{b}(z)|$ is bounded there from below. The function $h(z)$ is a linear combination of symmetric Macdonald functions

$$P_\lambda(z)=W_0(q^{-1})^{-1}\,q_{t_\lambda}^{-1/2}\sum\nolimits_{w\in W_0}\mathbf{c}(w.z)\,e^{\langle\lambda,w.z\rangle}\tag{2.2}$$

(see [13], [18]). For the type $\widetilde{A}_r$ considered in this work, there is indeed a close connection, namely

$$P_\lambda(z)=\overbrace{\tfrac{(W_0\cap W_\lambda)(q^{-1})}{W_0(q^{-1})}\,q^{-\langle\rho,\lambda\rangle}}^{N_\lambda^{-1}\,q^{\langle\rho,\lambda\rangle}}\,P_\lambda(e^z;q^{-1})$$

between symmetric Macdonald polynomials P_λ and Hall-Littlewood polynomials $P_\lambda(\,.\,;q^{-1})$, which boil down to symmetric monomials when λ is a fundamental weight (see for instance [11, pp. 209 & 299], [12, § 10], [6, pp. 99–100]). Thus

$$P_{\lambda_j}(z)=N_{\lambda_j}^{-1}\,q^{\langle\rho,\lambda_j\rangle}\sum\nolimits_{\lambda\in W_0.\lambda_j}e^{\langle\lambda,z\rangle}$$

and

$$h(z)=\sum\nolimits_{j=1}^{r}N_{\lambda_j}\,q^{-\langle\rho,\lambda_j\rangle}\,P_{\lambda_j}(z)\,.$$

Now the *fundamental spherical function* is defined on $\mathcal{X}$ by

$$F_0(x)\;=\;P_\lambda(0)\quad\text{with}\quad\lambda=x^+.$$

It is a positive eigenfunction of the Hecke algebra $\mathcal{A}$ described in Subsection 2.5 below. Specifically, for every $\mu\in P^+$, there exists a constant $c_\mu>0$ such that

$$\sum\nolimits_{y\in V_\mu(x)}F_0(y)=c_\mu\,F_0(x)\,,\tag{2.3}$$

for all $x\in\mathcal{X}$ (see [18, Theorem 3.22]). Its behavior is given in the following proposition. The statement and the proof are similar to the symmetric space case [1], which was generalized to the hypergeometric setting in [24].

Proposition 2.1. *We have*

$$F_0(x)\;\approx\;q^{-\langle\rho,\lambda\rangle}\prod\nolimits_{\alpha\in R^+}\big(1+\langle\alpha,\lambda\rangle\big).\tag{2.4}$$

Moreover,

$$F_0(x)\sim\text{const.}\;\boldsymbol{\pi}(\lambda)\,q^{-\langle\rho,\lambda\rangle},\tag{2.5}$$

as $\langle\alpha,\lambda\rangle\to+\infty$ *for all* $\alpha\in R^+$.

Proof. Let us multiply (2.2) by $\boldsymbol{\pi}(z)$, in order to remove the singularity of the **c**-function at the origin, and apply the operator $\boldsymbol{\pi}\big(\frac{\partial}{\partial z}\big)\big|_{z=0}$. We obtain in this way

$$P_\lambda(0) = q^{-\langle\rho,\lambda\rangle}\, p(\lambda)\,,$$

where p is a polynomial of the form

$$p \,=\, \text{const.}\,\boldsymbol{\pi} \,+\, \text{linear combination of subproducts of }\, \boldsymbol{\pi}\,.$$

This proves (2.5) and (2.4) far away from the walls. Eventually (2.3) enables us to extend (2.4) up to the walls, since it implies that $F_0(x) \approx F_0(y)$ for all neighbors $x,y\in\mathcal{X}$ and more generally for all $x,y\in\mathcal{X}$ at any fixed distance. □

2.5. **Averaging operators and Fourier-Gelfand transform.** We denote by $\mathcal{A}$ the linear span of mean operators on $\mathcal{X}$:

$$A_\lambda f(x) \,=\, \tfrac{1}{N_\lambda} \sum\nolimits_{y\in V_\lambda(x)} f(y) \qquad \forall\, x\in\mathcal{X},\ \forall\,\lambda\in P^+.$$

Then $\mathcal{A}$ is a (commutative) polynomial $*$-algebra with generators $A_{\lambda_1},\dots,A_{\lambda_r}$. Consider $\mathcal{A}$ as a subalgebra of the algebra $\mathcal{L}(\ell^2(\mathcal{X}))$ of bounded linear operators on $\ell^2(\mathcal{X})$. Then the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathcal{L}(\ell^2(\mathcal{X}))$ is a commutative C^*-algebra and the Fourier-Gelfand transform defines an isomorphism between $\overline{\mathcal{A}}$ and the algebra of continuous functions on $i\mathfrak{a}$, which are invariant under $W_0\ltimes i2\pi Q$. Such functions can be viewed as W_0-invariant continuous functions on iU, where

$$U \,=\, \{\,\theta\in\mathfrak{a} \mid |\langle\alpha,\theta\rangle|\le 2\pi \ \ \forall\,\alpha\in R\,\},$$

is a W_0-invariant fundamental domain for the action of the lattice $2\pi Q$ on $\mathfrak{a}$. Specifically, the image of A_λ is the Macdonald polynomial P_λ.

Finally the following inversion formula holds, for every $A\in\overline{\mathcal{A}}$ and $x,y\in\mathcal{X}$ (see [10, Theorem 5.1.5] or [18, Theorem 5.2]):

$$(A\delta_x)(y) = C_0 \int_U \widehat{A}(i\theta)\, P_\lambda(-i\theta)\, \tfrac{d\theta}{|\mathbf{c}(i\theta)|^2}\,, \tag{2.6}$$

where $C_0 = \frac{W_0(q^{-1})}{(2\pi)^r\,|W_0|}$ and $\lambda = (x,y)^+$.

2.6. **Distinguished random walk.** In this paper, we consider mainly the Markov chain on $\mathcal{X}$ with transition probability

$$p(x,y) = \begin{cases} \sigma\, q_{t_{\lambda_j}}^{-1/2} & \text{if } y\in V_{\lambda_j}(x),\\ \ \ 0 & \text{otherwise}, \end{cases}$$

where

$$\sigma^{-1} \,=\, \sum\nolimits_{j=1}^{r} N_{\lambda_j}\, q_{t_{\lambda_j}}^{-1/2}.$$

This distinguished random walk, which was already considered in [23] and [7], is the *simple* random walk in rank one and two, but not in higher rank. Its generator

$$Af(x) = \sum_{y\in\mathcal{X}} p(x,y)\, f(y) = \sigma \sum_{j=1}^{r} q_{t_{\lambda_j}}^{-1/2} N_{\lambda_j} A_{\lambda_j} f(x), \tag{2.7}$$

corresponds to $\widehat{A} = \sigma h$ via the Fourier-Gelfand transform and its spectral radius is equal to $\boldsymbol{\sigma} = \sigma h(0)$, where $h(0) = \sum_{j=1}^{r} |W_0.\lambda_j| = 2(2^r-1)$.

3. Global heat kernel estimates in rank two

For every integer $n\geq 2$ and for every $x\in\mathcal{X}$ with $|x|<n$, set

$$\delta=\tfrac{x^+ +\rho}{n+2}=\underbrace{\tfrac{x_1+1}{n+2}}_{\delta_1}\lambda_1+\underbrace{\tfrac{x_2+1}{n+2}}_{\delta_2}\lambda_2\,.$$

Notice that the coordinates δ_j belong to the interval $(0,1)$. Thus $\delta\in\mathfrak{a}^+$ and $|\delta|=\frac{|x|+2}{n+2}<1$. This section is mainly devoted to the proof of the following global estimate for the transition probabilities $p_n(x)=p_n(0,x)$ of the random walk (2.7).

Theorem 3.1. *Let $\phi(\delta)=\min_{z\in\mathfrak{a}}\Phi_\delta(z)$, where $\Phi_\delta(z)=\log\frac{h(z)}{6}-\langle\delta,z\rangle$. Then*

$$p_n(x)\approx\frac{(1+|x|)\,(1+x_1)\,(1+x_2)}{n^3\sqrt{n-|x|}\,\sqrt{n-x_1\vee x_2}}\,\boldsymbol{\sigma}^n\,q^{-\langle\rho,x^+\rangle}\,e^{n\,\phi(\delta)}\tag{3.1}$$

uniformly in the range $|x|<n$. In the limit case $|x|=n$,

$$p_n(x)\approx\sigma^n\,q^{-n}\,n^n\,(x_1\vee x_2)^{-(x_1\vee x_2)}\,(x_1\wedge x_2+1)^{-(x_1\wedge x_2)-\frac12}\,.$$

Remark 3.2. *Let us comment on some factors occurring on the right hand side of (3.1). The exponential decay $\boldsymbol{\sigma}^n$ is produced by the spectral radius $\boldsymbol{\sigma}=\frac{3}{q+1+q^{-1}}<1$ of A, $F_0(x)\approx(1+|x|)(1+x_1)(1+x_2)\,q^{-\langle\rho,x^+\rangle}$ is the W_0-invariant ground state of A, and $e^{n\,\phi(\delta)}$ is a Gaussian type factor.*

Remark 3.3. *Recall ([8], see also [27]) the corresponding result in rank one i.e. for homogeneous trees:*

$$p_n(x)\approx\frac{1+|x|}{n\sqrt{1+n-|x|}}\,\boldsymbol{\sigma}^n\,q^{-\frac{|x|}{2}}\,e^{n\,\phi(\delta)}\quad\forall\;|x|\leq n\text{ with same parities.}\tag{3.2}$$

Here $\boldsymbol{\sigma}=\frac{2}{q^{1/2}+q^{-1/2}}$, $\delta=\frac{|x|+1}{n+1}$ and

$$\phi(\delta)\,=\,-\tfrac12\,\{(1+\delta)\log(1+\delta)+(1-\delta)\log(1-\delta)\}\,.$$

As $\lim_{\delta\nearrow1}\phi(\delta)=-\log 2$, notice that (3.2) agrees with the obvious expression $p_n(x)=(q+1)^{-n}$ in the limit case $|x|=n$.

Theorem 3.1 is proved by combining different arguments, depending on the relative sizes of $|x|$ and n and on the position of $\lambda=x^+$ in $\mathrm{cl}(\mathfrak{a}^+)$. In most cases, the proof relies on suitable versions of the inversion formula (2.6), which yields

$$p_n(x)=C_0\,\sigma^n\int_U h(i\theta)^n\,P_\lambda(-i\theta)\,\frac{d\theta}{|\mathbf{c}(i\theta)|^2}\,,\tag{3.3}$$

and thus boils down to estimating integrals of the form

$$\int_U e^{(n+2)\,\Psi(\theta)}\,a(\theta)\,d\theta\,,\tag{3.4}$$

where Ψ is a complex phase involving the function h and a is an amplitude involving the functions $\mathbf{b}$ or $\mathbf{c}$. Let us start the proof of Theorem 3.1 with a series of auxiliary results.

3.1. **Local Harnack inequalities.**

Lemma 3.4. *There exists a constant $C>0$ such that*

$$p_{n+1}(x)\ge C\,p_n(y)$$

for every $n\in\mathbb{N}^*$ and for all neighbors $x,y\in\mathcal{X}$.

This inequality is an immediate consequence of the very definition

$$p_{n+1}(x)=A\,p_n(x)=\tfrac{1}{2(q^2+q+1)}\sum_{\substack{y\in\mathcal{X}\\ d(y,x)=1}}p_n(y)\qquad\forall\,n\in\mathbb{N}^*,\;\forall\,x\in\mathcal{X}.$$

Next result follows by iteration and by using the fact that the random walk between two neighbors in $\mathcal{X}$ is *aperiodic*, meaning that

$$\gcd\{\,n\in\mathbb{N}^*\,|\,p_n(x,y)>0\,\}=1\qquad\forall\,x\neq y\,.$$

Corollary 3.5. *For every $m\in\mathbb{N}^*$, there exists $C>0$ such that*

$$p_{n+m}(x)\ge C\,p_n(y)\tag{3.5}$$

for every $n\in\mathbb{N}^*$ and for all $x,y\in\mathcal{X}$ such that $0<d(x,y)\le m$.

Remark 3.6. *Inequality (3.5) remains true when $x=y$. This is proved by the same arguments if $m>1$ but, if $m=1$, this follows only a posteriori from Theorem 3.1.*

3.2. **Remarkable formulae.**

The function

$$h=e^{\lambda_1}+e^{-\lambda_1}+e^{\lambda_2}+e^{-\lambda_2}+e^{\lambda_1-\lambda_2}+e^{\lambda_2-\lambda_1}$$

enjoys the product formula

$$\begin{aligned}h+2&=(e^{\lambda_1}+1)(e^{-\lambda_2}+1)(e^{\lambda_2-\lambda_1}+1)\\&=(2\cosh\tfrac{\lambda_1}{2})(2\cosh\tfrac{\lambda_2}{2})(2\cosh\tfrac{\lambda_1-\lambda_2}{2})\end{aligned}\tag{3.6}$$

and the differentiation formula

$$\boldsymbol{\pi}(\partial)\,h^{n+3}=(n+3)^2(n+2)\left[\,h+2\,\tfrac{n+1}{n+3}\right]h^n\,\Delta\qquad\forall\,n\ge-2\,.\tag{3.7}$$

These formulae, which are easily checked (see Appendix A for more general results), play key roles in our analysis of (3.3).

3.3. **Real phase $\Phi=\Phi_\delta$.**

Let $\delta\in\mathrm{cl}(\mathfrak{a}^+)$ with $|\delta|=\delta_1+\delta_2<1$, where $\delta_j=\langle\alpha_j,\delta\rangle\ge0$ are the coordinates of δ in the basis $\{\lambda_1,\lambda_2\}$. In this subsection, we study the function

$$\Phi(z)=\Phi_\delta(z)=\log\tfrac{h(z)}{6}-\langle\delta,z\rangle\qquad\forall\,z\in\mathfrak{a},$$

which was introduced in the statement of Theorem 3.1 and whose dependency on δ will no more be indicated.

Lemma 3.7. (a) *The function Φ is strictly convex, tends to $+\infty$ at infinity and reaches its minimum $\phi(\delta)\in(-\log 6,0]$ at a single point $s\in\mathrm{cl}(\mathfrak{a}^+)$, which is the unique solution to the equation*

$$\tfrac{dh(s)}{h(s)}=\delta\,.\tag{3.8}$$

Moreover,

$$d\phi(\delta)=-\,s\,.\tag{3.9}$$

(b) *The map $\delta\longmapsto s$ is real analytic and bijective between $\{\,\delta\in\mathrm{cl}(\mathfrak{a}^+)\,|\,|\delta|<1\,\}$ and* $\mathrm{cl}(\mathfrak{a}^+)$.

(c) *We have* $h(s)\approx e^{s^1\vee s^2}$*, where* $s^j=\langle\lambda_j,s\rangle$ *denote the coordinates of* s *in the basis* $\{\alpha_1,\alpha_2\}$*.*
(d) *The jth coordinates* δ_j *and* $s_j=\langle\alpha_j,s\rangle$ *in the basis* $\{\lambda_1,\lambda_2\}$ *vanish simultaneously.*
(e) $|\delta|\to 1$ *if and only if* $|s|\to+\infty$*. More precisely,* $1-|\delta|\approx e^{-(s^1\wedge s^2)}$*.*
(f) *We have* $1-\delta_1\vee\delta_2\approx e^{-|s^1-s^2|}$*. More precisely,*

$$e^{-|s^1-s^2|}=\tfrac{1-\delta_1\vee\delta_2}{\delta_1\vee\delta_2}+\mathcal{O}(1-|\delta|)\quad \text{as } |\delta|\to 1\,.$$

(g) $\delta_1-\delta_2$ *and* $s_1-s_2=3(s^1-s^2)$ *have the same sign. Moreover*

$$|\delta_1-\delta_2|\approx 1-e^{-|s^1-s^2|}\approx 1\wedge|s_1-s_2|\,.$$

Proof. As $h(z)>e^{|z^1|\vee|z^2|}$ and $|\langle\delta,z\rangle|\le|\delta|(|z^1|\vee|z^2|)$, where $z^j=\langle\lambda_j,z\rangle$ are the coordinates of z in the basis $\{\alpha_1,\alpha_2\}$, we have

$$\Phi(z)>(1-|\delta|)\,(|z^1|\vee|z^2|)-\log 6\,.$$

Hence $\Phi(z)\to+\infty$ as z tends to infinity in $\mathfrak{a}$. Moreover, as h is W_0-invariant and

$$\langle\delta,z\rangle\ge\langle\delta,w.z\rangle\qquad\forall\,z\in\mathrm{cl}(\mathfrak{a}^+),\ \forall\,w\in W_0\,,$$

Φ reaches its minimum $\phi(\delta)$ in $\mathrm{cl}(\mathfrak{a}^+)$. Notice that $\phi(\delta)>-\log 6$ and $\phi(\delta)\le\Phi(0)=0$. Let us next compute the first two derivatives of $\log h$. As $h=\sum_{\lambda\in\Lambda}e^{\lambda}$ is a sum of exponentials, the gradient of $\log h$ is given by

$$d(\log h)=\tfrac{1}{h}\sum\nolimits_{\lambda\in\Lambda}e^{\lambda}\lambda \tag{3.10}$$

and its Hessian by

$$\begin{aligned} d^2(\log h)&=\tfrac{1}{h}\sum\nolimits_{\lambda\in\Lambda}e^{\lambda}\,\lambda\otimes\lambda-\tfrac{1}{h^2}\sum\nolimits_{\lambda,\lambda'\in\Lambda}e^{\lambda+\lambda'}\lambda\otimes\lambda'\\ &=\tfrac{1}{2h^2}\sum\nolimits_{\lambda,\lambda'\in\Lambda}e^{\lambda+\lambda'}(\lambda-\lambda')\otimes(\lambda-\lambda')\,. \end{aligned} \tag{3.11}$$

Since the vectors $\lambda-\lambda'$ span $\mathfrak{a}$, we conclude that the Hessian $d^2\Phi=d^2(\log h)$ is positive definite, that Φ is strictly convex and that Φ has a single minimum s, which satisfies the stationary equation (3.8). Moreover $s=s(\delta)$ depends analytically on δ, according to the local inversion theorem, and the derivative of

$$\phi(\delta)=\Phi(s(\delta))=\log\tfrac{h(s(\delta))}{6}-\langle\delta,s(\delta)\rangle$$

is given by

$$d\phi(\delta)=d\Phi(s(\delta))\circ ds(\delta)-s(\delta)=-\,s(\delta)\,.$$

This proves (a) and the first part of (b). Apart from (c), which is obvious, all other claims rely on (3.8), which is equivalent to the system

$$\begin{cases} h(s)\,\delta_1=2\sinh s^1+2\sinh(s^1-s^2)=4\sinh\frac{s_1}{2}\cosh\frac{s^2}{2}\,,\\ h(s)\,\delta_2=2\sinh s^2+2\sinh(s^2-s^1)=4\sinh\frac{s_2}{2}\cosh\frac{s^1}{2}\,. \end{cases} \tag{3.12}$$

Firstly, observe that

$$s_1\begin{Bmatrix}>\\=\\<\end{Bmatrix}s_2\quad\Longleftrightarrow\quad s^1\begin{Bmatrix}>\\=\\<\end{Bmatrix}s^2\quad\Longleftrightarrow\quad\delta_1\begin{Bmatrix}>\\=\\<\end{Bmatrix}\delta_2$$

and

$$\delta_j=0\iff s_j=0\,.$$

Secondly, by adding up the equations in (3.12), we get

$$h(s)\,|\delta| = 2\,(\sinh s^1 + \sinh s^2)\,.$$

On the one hand, we deduce that

$$1-|\delta| \,=\, \tfrac{2\,e^{-s^1}+2\,e^{-s^2}+\cosh(s^1-s^2)}{h(s)} \approx \tfrac{e^{|s^1-s^2|}}{e^{s^1\vee s^2}} \approx e^{-s^1\wedge s^2}\,,$$

hence

$$|\delta|\to 1 \quad\Longleftrightarrow\quad |s|\to+\infty\,.$$

On the other hand, given $s\in\mathrm{cl}(\mathfrak{a}^+)$, (3.12) defines $\delta\in\mathrm{cl}(\mathfrak{a}^+)$ with

$$|\delta| = \tfrac{\sinh s^1+\sinh s^2}{\cosh s^1+\cosh s^2+\cosh(s^1-s^2)} < 1\,.$$

Thirdly, by substracting the equations in (3.12), we get

$$\begin{aligned} h(s)\,|\delta_1-\delta_2| &= 2\,|\sinh s^1-\sinh s^2| + 4\,\sinh|s^1-s^2| \\ &= \big(4\cosh\tfrac{s^1+s^2}{2} + 8\cosh\tfrac{s^1-s^2}{2}\big)\sinh\tfrac{|s^1-s^2|}{2}\,, \end{aligned}$$

hence

$$|\delta_1-\delta_2| \approx \frac{e^{\frac{s^1+s^2}{2}}\sinh\frac{|s^1-s^2|}{2}}{e^{s^1\vee s^2}} \approx 1-e^{-|s^1-s^2|} \approx 1\wedge|s^1-s^2| \approx 1\wedge|s_1-s_2|$$

and

$$\delta_1-\delta_2\to 0 \quad\Longleftrightarrow\quad s_1-s_2\to 0\,.$$

Fourthly, by rewriting (3.12) as follows:

$$\begin{cases} \frac{h(s)}{2}\,(1-\delta_1) = \cosh s^2 + e^{-s^1} + e^{s^2-s^1}, \\ \frac{h(s)}{2}\,(1-\delta_2) = \cosh s^1 + e^{-s^2} + e^{s^1-s^2}, \end{cases}$$

we get on the one hand

$$1-\delta_1\vee\delta_2 \approx e^{-|s^1-s^2|}$$

and on the other hand

$$\tfrac{1-\delta_1\vee\delta_2}{\delta_1\vee\delta_2} = e^{-|s^1-s^2|} + \mathcal{O}\big(e^{-s^1\wedge s^2}\big)\,.$$

This concludes the proof of Lemma 3.7 □

By symmetry, we may assume from now that $x_1\ge x_2$, which amounts to either condition $\delta_1\ge\delta_2$, $s^1\ge s^2$ or $s_1\ge s_2$, according to Lemma 3.7.(g). Beside the walls $\{\alpha_1=0\}=\mathbb{R}\lambda_1$ and $\{\alpha_2=0\}=\mathbb{R}\lambda_2$ of the Weyl chamber $\mathfrak{a}^+$, consider the *extra wall*

$$\{\alpha_1=\alpha_2\} = \{\lambda_1=\lambda_2\} = \mathbb{R}\,\rho\,. \tag{3.13}$$

Corollary 3.8. *Assume that δ or equivalently s stays away from 0.*

(a) *The following estimate holds, provided that x^+ stays far enough from the extra wall (3.13):*

$$\big|\,h(s+i\theta)+2\,\tfrac{n+1}{n+3}\big| \gtrsim \tfrac{h(s)}{n}\,.$$

(b) *The improved estimate*

$$\big|\,h(s+i\theta)+2\,\tfrac{n+1}{n+3}\big| \approx h(s)$$

holds in the following two cases:

- *θ is close enough to the extra wall (3.13) and n is large enough,*
- *δ or equivalently s stays away from the extra wall (3.13) and n is large enough.*

Proof. The upper estimate

$$\left|\, h(s+i\theta)+2\,\tfrac{n+1}{n+3}\,\right| \le h(s)+2 \le \tfrac{4}{3}\,h(s)$$

is elementary and holds in full generality. Let us turn to the lower estimates. We deduce from (3.6) that

$$|\,h(s+i\theta)+2\,| \ge \left\{e^{s^1}-1\right\}\left\{1-e^{-s^2}\right\}\left|\,e^{\,i\,(\theta^1-\theta^2)}+e^{-(s^1-s^2)}\right|$$

and we use Lemma 3.7 to estimate the three factors on the right hand side. On the one hand,

$$e^{s^1}-1 \,\approx\, e^{s^1} \approx\, h(s) \quad\text{and}\quad 1-e^{-s^2} \approx\, 1\,.$$

On the other hand,

$$\left|e^{\,i\,(\theta^1-\theta^2)}+e^{-(s^1-s^2)}\right| \ge 1-e^{-(s^1-s^2)} \approx\, \delta_1-\delta_2 = \tfrac{x_1-x_2}{n+2}$$

in general and

$$\left|e^{\,i\,(\theta^1-\theta^2)}+e^{-(s^1-s^2)}\right| \approx 1$$

if $\theta^1-\theta^2$ is small. Consequently,

$$\left|\,h(s+i\theta)+2\,\tfrac{n+1}{n+3}\right| \ge |\,h(s+i\theta)+2\,| - \tfrac{4}{n+3}$$

is bounded from below by $\frac{h(s)}{n}$, if x_1-x_2 is large enough, and by $h(s)$, if n is large enough and if $\delta_1-\delta_2$ stays away from 0 or if $\theta^1-\theta^2$ is sufficiently small. □

3.4. **Complex phase Ψ.** In this subsection, we study the function

$$\Psi(\theta) = \log\tfrac{h(s+i\theta)}{h(s)} - i\,\langle\delta,\theta\rangle\,,$$

which does depend on δ, or equivalently on s, and which is well defined in a neighborhood of the origin in $\mathfrak{a}$, but independently of δ and s. We have $\Psi(0)=0$ by definition and $d\Psi(0)=0$ according to our choice of s. Next lemma describes more precisely the behavior of Ψ near the origin.

Lemma 3.9. *The following results hold uniformly with respect to δ and s :*
(a) *The Hessian $d^2\Psi(0)$ is negative definite and $B=-\,d^2\Psi(0)$ satisfies*

$$B(\theta,\theta) \approx e^{-(s^1-s^2)}\,(\theta^1-\theta^2)^2 + e^{-s^2}\,(\theta^1+\theta^2)^2 \quad \forall\,\theta\in\mathfrak{a}\,,$$

with $e^{-(s^1-s^2)} \ge e^{-s^2}$.
(b) *For θ small,*

$$-\operatorname{Re}\Psi(\theta) \approx B(\theta,\theta) \quad\textit{and}\quad \operatorname{Im}\Psi(\theta) = \mathcal{O}\big(|\theta|\,B(\theta,\theta)\big)\,.$$

Proof. Let us compute the Hessian of Ψ, as we did for $\log h$ in (3.11):

$$\begin{aligned} d^2\Psi(\theta) &= \tfrac{-1}{2\,h(s+i\theta)^2}\sum\nolimits_{\lambda,\lambda'\in\Lambda} e^{\langle\lambda+\lambda',\,s+i\theta\rangle}\,(\lambda-\lambda')\otimes(\lambda-\lambda') \\ &= \tfrac{-1}{2\,|h(s+i\theta)|^4}\sum\nolimits_{\lambda,\lambda',\mu,\mu'\in\Lambda} e^{\langle\lambda+\lambda'+\mu+\mu',\,s\rangle}\,e^{\,i\,\langle\lambda+\lambda'-\mu-\mu',\,\theta\rangle}\,(\lambda-\lambda')\otimes(\lambda-\lambda')\,. \end{aligned} \tag{3.14}$$

Observe that, in the nonnegative quadratic form

$$B(\theta,\theta) = \tfrac{1}{2\,h(s)^2}\sum\nolimits_{\lambda,\lambda'\in\Lambda} e^{\,\langle\lambda+\lambda',\,s\rangle}\,\langle\lambda-\lambda',\theta\rangle^2\,,$$

the leading coefficients arise when

$$\{\lambda,\lambda'\} = \begin{cases}\{\lambda_1,\lambda_2\}\,,\\ \{\lambda_1,\lambda_1-\lambda_2\}\,.\end{cases} \tag{3.15}$$

Consequently,

$$B(\theta,\theta) \approx h(s)^{-2}\left\{e^{\langle\rho,s\rangle}\langle\lambda_1-\lambda_2,\theta\rangle^2+e^{\langle\alpha_1,s\rangle}\langle\lambda_2,\theta\rangle^2\right\}, \tag{3.16}$$

with $h(s)\approx e^{\langle\lambda_1,s\rangle}$. Furthermore, as $e^{\langle\rho,s\rangle}\geq e^{\langle\alpha_1,s\rangle}$, one may replace in (3.16) $\langle\lambda_2,\theta\rangle^2$ by $\langle\lambda_1+\lambda_2,\theta\rangle^2$. This proves (a) and the proof of (b) is somewhat similar. According to Taylor's formula and (3.14), we have indeed

$$\Psi(\theta)=\int_0^1\langle d^2\Psi(t\theta)\theta,\theta\rangle\,(1-t)\,dt\,,$$

with

$$\langle d^2\Psi(t\theta)\theta,\theta\rangle=\tfrac{-1}{2\,|h(s+it\theta)|^4}\sum_{\lambda,\lambda',\mu,\mu'\in\Lambda}e^{\langle\lambda+\lambda'+\mu+\mu',s\rangle}\,e^{i\langle\lambda+\lambda'-\mu-\mu',t\theta\rangle}\langle\lambda-\lambda',\theta\rangle^2. \tag{3.17}$$

The leading coefficients in (3.17) are obtained by taking $\{\lambda,\lambda'\}$ as in (3.15) and $\mu=\mu'=\lambda_1$. Hence, if θ is small,

$$\begin{aligned}-\operatorname{Re}\langle d^2\Psi(t\theta)\theta,\theta\rangle &= |h(s+it\theta)|^{-4}\\ &\times\sum_{\lambda,\lambda',\mu,\mu'\in\Lambda}e^{\langle\lambda+\lambda'+\mu+\mu',s\rangle}\underbrace{\cos\langle\lambda+\lambda'-\mu-\mu',t\,\theta\rangle}_{\geq\,\text{const.}>0}\langle\lambda-\lambda',\theta\rangle^2\end{aligned}$$

is comparable to (3.16), while

$$\begin{aligned}-\operatorname{Im}\langle d^2\Psi(t\theta)\theta,\theta\rangle &= |h(s+it\theta)|^{-4}\\ &\times\sum_{\lambda,\lambda',\mu,\mu'\in\Lambda}e^{\langle\lambda+\lambda'+\mu+\mu',s\rangle}\underbrace{\sin\langle\lambda+\lambda'-\mu-\mu',t\,\theta\rangle}_{\mathcal{O}(|\theta|)}\langle\lambda-\lambda',\theta\rangle^2\end{aligned}$$

is $\mathcal{O}\big(|\theta|\,B(\theta,\theta)\big)$. □

Beside the local behavior of Ψ, we shall need the following global estimate.

Lemma 3.10. *For every $\theta\in U$,*

$$-\log\tfrac{|h(s+i\theta)|}{h(s)}\gtrsim B(\theta,\theta).$$

Remark 3.11. *Such a global estimate is hard to obtain for general random walks on affine buildings (see [26]).*

Proof of Lemma 3.10. Let us expand

$$\begin{aligned}h(s)^2-|h(s+i\theta)|^2&=\textstyle\sum_{\lambda,\lambda'\in\Lambda}e^{\langle\lambda+\lambda',s\rangle}\left\{1-\cos\langle\lambda-\lambda',\theta\rangle\right\}\\&=2\textstyle\sum_{\lambda,\lambda'\in\Lambda}\cosh\langle\lambda+\lambda',s\rangle\sin^2\frac{\langle\lambda-\lambda',\theta\rangle}{2}\,.\end{aligned}$$

By taking

$$\{\lambda,\lambda'\}=\begin{cases}\pm\{\lambda_1,\lambda_2\},\\ \pm\{\lambda_1,\lambda_1-\lambda_2\},\end{cases}$$

we get the lower bound

$$h(s)^2-|h(s+i\theta)|^2\geq 8\cosh\langle\rho,s\rangle\sin^2\tfrac{\langle\lambda_1-\lambda_2,\theta\rangle}{2}+8\cosh\langle\alpha_1,s\rangle\sin^2\tfrac{\langle\lambda_2,\theta\rangle}{2}\,.$$

As $\|\lambda_1-\lambda_2\|^2=\frac{2}{3}$ and $\|\lambda_2\|^2=\frac{2}{3}$, we have

$$\left|\tfrac{\langle\lambda_1-\lambda_2,\theta\rangle}{2}\right|\leq\tfrac{2\pi}{3}\quad\text{and}\quad\left|\tfrac{\langle\lambda_2,\theta\rangle}{2}\right|\leq\tfrac{2\pi}{3}$$

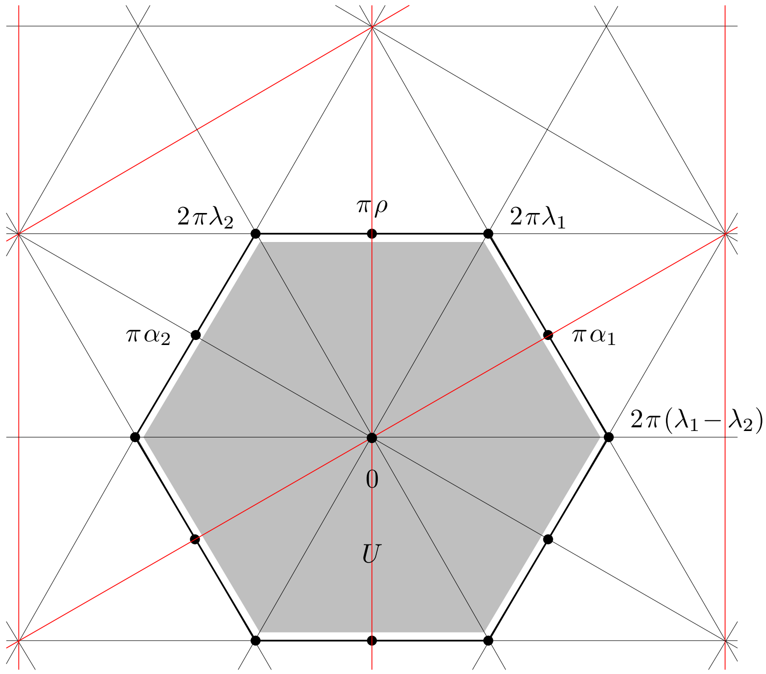

FIGURE 2. Picture for the proof of Lemma 3.10

on U (see Figure 2), hence

$$\sin^2\tfrac{\langle\lambda_1-\lambda_2,\theta\rangle}{2}\approx\langle\lambda_1-\lambda_2,\theta\rangle^2\quad\text{and}\quad\sin^2\tfrac{\langle\lambda_2,\theta\rangle}{2}\approx\langle\lambda_2,\theta\rangle^2.$$

By using $h(s)\approx e^{\langle\lambda_1,s\rangle}$, we deduce that

$$\tfrac{h(s)^2-|h(s+i\theta)|^2}{2\,h(s)^2}\gtrsim e^{-\langle\lambda_1-\lambda_2,s\rangle}\langle\lambda_1-\lambda_2,\theta\rangle^2+e^{-\langle\lambda_2,s\rangle}\langle\lambda_2,\theta\rangle^2,$$

where we may again replace $\langle\lambda_2,\theta\rangle^2$ by $\langle\lambda_1{+}\lambda_2,\theta\rangle^2$. We conclude by using Lemma 3.9.(a) and the elementary estimate

$$-\log\tfrac{|h(s+i\theta)|}{h(s)}=-\tfrac12\log\Big\{1-\tfrac{h(s)^2-|h(s+i\theta)|^2}{h(s)^2}\Big\}\geq\tfrac{h(s)^2-|h(s+i\theta)|^2}{2\,h(s)^2}\,.$$

□

3.5. **Amplitudes.** In this subsection, we study the following amplitudes occurring in (3.4) :

$$a_1(\theta)=|\mathbf{c}(i\theta)|^{-2}\,, \tag{3.18}$$

$$a_2(\theta)=\frac{e^{i\langle x^++\rho,\theta\rangle}}{\boldsymbol{\pi}(x^++\rho)}\,\boldsymbol{\pi}\big(i\tfrac{\partial}{\partial\theta}\big)\,\frac{e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(s+i\theta)}\,, \tag{3.19}$$

$$a_3(\theta)=\frac{h(s+i\theta)\,e^{i\langle x^++\rho,\theta\rangle}}{\boldsymbol{\pi}(x^++\rho)}\,\boldsymbol{\pi}\big(i\tfrac{\partial}{\partial\theta}\big)\,\frac{e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(s+i\theta)\big[h(s+i\theta)+2\,\frac{n+1}{n+3}\big]}\,. \tag{3.20}$$

Lemma 3.12. (a) *The function* (3.18) *has the following behavior:*

$$a_1(\theta)=\boldsymbol{\pi}(\theta)^2\Big\{\big(\tfrac{q}{q-1}\big)^6+\mathcal{O}(|\theta|)\Big\}\,.$$

(b) *The function* (3.19) *is uniformly bounded, as well as its derivatives. Moreover* $|a_2(0)|\approx1$ *provided that* x *is large enough.*

(c) *The function* (3.20) *is* $\mathcal{O}(n^4)$, *provided that* x_1-x_2 *is large enough. It is actually bounded, as well as its derivatives, in the following two cases*:
- $|\theta_1-\theta_2|$ *is small enough and* n *is large enough,*
- $\delta_1-\delta_2$ *or equivalently* s_1-s_2 *stays away from* 0 *and* n *is large enough.*

Moreover $|a_3(0)|\approx 1$ *provided that* x *and* s *are large enough.*

Proof. (a) is elementary, as well as the first claim in (b). The first claims in (c) are deduced similarly from Corollary 3.8. Let us turn to the lower estimate

$$|a_2(0)|\gtrsim 1 \tag{3.21}$$

in (b) and let us expand for this purpose the expression (3.19) at $\theta=0$. The main term $\mathbf{b}(s)^{-1}\approx 1$ is obtained by applying $\boldsymbol{\pi}\big(i\frac{\partial}{\partial\theta}\big)\big|_{\theta=0}$ to $e^{-i\langle x^+ +\rho,\theta\rangle}$. All other terms are $\mathcal{O}\big(\frac{1}{x_1+1}\big)$, except for

$$\frac{1}{x_2+1}\,\frac{\partial_{\alpha_2}\mathbf{b}(s)}{\mathbf{b}(s)^2}\,.$$

This term, which is obtained by differentiating $\mathbf{b}(s+i\theta)^{-1}$ in the direction of α_2 and $e^{-i\langle x^+ +\rho,\theta\rangle}$ in the directions of α_1 and ρ, happens to be positive, as

$$\begin{aligned}\frac{\partial_{\alpha_2}\mathbf{b}(s)}{\mathbf{b}(s)}&=-\,\frac{q^{-1}e^{-s_1}}{1-q^{-1}e^{-s_1}}+\frac{2\,q^{-1}e^{-s_2}}{1-q^{-1}e^{-s_2}}+\frac{q^{-1}e^{-s_1-s_2}}{1-q^{-1}e^{-s_1-s_2}}\\&>\frac{q^{-1}e^{-s_2}}{1-q^{-1}e^{-s_2}}-\frac{q^{-1}e^{-s_1}}{1-q^{-1}e^{-s_1}}\geq 0\,.\end{aligned}$$

Hence

$$|a_2(0)|\geq\frac{1}{\mathbf{b}(s)}+\frac{1}{x_2+1}\,\frac{\partial_{\alpha_2}\mathbf{b}(s)}{\mathbf{b}(s)^2}-|\text{remainder}|\geq\frac{1}{\mathbf{b}(s)}-\mathcal{O}\big(\tfrac{1}{x_1+1}\big)$$

is $\geq\frac{1}{2\,\mathbf{b}(s)}\approx 1$, provided that x is large enough. This concludes the proof of (3.21). The lower estimate

$$|a_3(0)|\gtrsim 1 \tag{3.22}$$

in (c) is proved similarly. In the expansion of the expression (3.20) at $\theta=0$, the main term is now

$$T_1=\frac{h(s)}{\mathbf{b}(s)\,\big[h(s)+2\,\frac{n+1}{n+3}\big]}\approx 1$$

and all other terms are $\mathcal{O}\big(\frac{1}{x_1+1}\big)$, except for

$$\frac{1}{x_2+1}\,\frac{h(s)}{\mathbf{b}(s)\,\big[h(s)+2\,\frac{n+1}{n+3}\big]}\,\bigg\{\frac{\partial_{\alpha_2}\mathbf{b}(s)}{\mathbf{b}(s)}+\frac{\partial_{\alpha_2}h(s)}{h(s)+2\,\frac{n+1}{n+3}}\bigg\}\,. \tag{3.23}$$

As

$$\frac{\partial_{\alpha_2}[h(s)+2]}{h(s)+2}=-\frac{e^{-s^2}}{e^{-s^2}+1}+\frac{e^{s^2-s^1}}{e^{s^2-s^1}+1}\,,$$

(3.23) is the difference of the positive expressions

$$T_2=\frac{1}{x_2+1}\,\frac{h(s)}{\mathbf{b}(s)\,\big[h(s)+2\,\frac{n+1}{n+3}\big]}\,\bigg\{\frac{\partial_{\alpha_2}\mathbf{b}(s)}{\mathbf{b}(s)}+\frac{h(s)+2}{h(s)+2\,\frac{n+1}{n+3}}\,\frac{e^{s^2-s^1}}{e^{s^2-s^1}+1}\bigg\}$$

and

$$T_3=\frac{1}{x_2+1}\,\frac{h(s)\,[h(s)+2]}{\mathbf{b}(s)\,\big[h(s)+2\,\frac{n+1}{n+3}\big]^2}\,\frac{e^{-s^2}}{e^{-s^2}+1}=\mathcal{O}\big(e^{-s^2}\big)\,.$$

Hence

$$|a_3(0)|\geq T_1+T_2-T_3-|\text{remainder}|\geq T_1-\mathcal{O}\big(e^{-s^2}\big)-\mathcal{O}\big(\tfrac{1}{x_1+1}\big)$$

is $\geq\frac{T_1}{2}\approx 1$, provided that x and s are large enough. This concludes the proof of (3.22). $\square$

3.6. **Proof of Theorem 3.1 when n remains bounded.** Then (3.1) reduces to

$$p_n(x) \approx 1 \qquad \forall\ |x| \leq n\,. \tag{3.24}$$

While the upper bound in (3.24) is trivial, the lower bound amounts to the nonvanishing of $p_n(x)$. This follows in turn from the fact, already used in the proof of Corollary 3.5, that the random walk is aperiodic and hence that the random walk may join in n steps any two points at distance $\leq n$, as soon as $n \geq 2$.

3.7. **Proof of Theorem 3.1 when x remains bounded while n is large.** Then (3.1) amounts to

$$p_n(x) \approx\, n^{-4}\, \boldsymbol{\sigma}^n \tag{3.25}$$

as, in this case, the Gaussian type factor

$$e^{\,n\,\phi(\delta)} \approx e^{\,(n+2)\,\phi(\delta)}$$

is bounded both from above and from below. The latter claim follows indeed from the mean value theorem, applied to the function ϕ, from the vanishing $\phi(0)=0$ and from the boundedness of $d\phi$, according to (3.9).
Although (3.25) is a consequence of the general local limit theorem in [19], we include a short proof, which will be refined in the next two subsections. First of all, according to Corollary 3.5, we can reduce to $x=0$. Next, by setting $x=0$ in (3.3), we see that (3.25) amounts to the estimate $J(n) \approx n^{-4}$ for the nonnegative expression

$$J(n) \,=\, \int_U \big[\tfrac{h(i\theta)}{6}\big]^n\, a_1(\theta)\, d\theta\,. \tag{3.26}$$

Let us collect some information about (3.26). On the one hand, for θ small, the phase function $-\Psi(\theta) = -\log \frac{h(i\theta)}{6}$ and the amplitude $a_1(\theta)$ are nonnegative and behave as follows, according to Lemma 3.9.(b) and Lemma 3.12.(a) :

$$-\,\Psi(\theta) \approx |\theta|^2 \quad \text{and} \quad a_1(\theta) \approx \boldsymbol{\pi}(\theta)^2\,. \tag{3.27}$$

On the other hand, for $\theta \in U$, the following global estimates hold, according to Lemma 3.10 and Lemma 3.12.(a) :

$$\tfrac{|h(i\theta)|}{6} \lesssim e^{-\,\mathrm{const.}\,|\theta|^2} \qquad \text{and} \qquad a_1(\theta) \lesssim \boldsymbol{\pi}(\theta)^2\,. \tag{3.28}$$

The upper bound of (3.26) is easily deduced from (3.28) :

$$J(n) \lesssim \int_U e^{-\,\mathrm{const.}\,n\,|\theta|^2}\, \boldsymbol{\pi}(\theta)^2\, d\theta \,\lesssim\, n^{-4}\,.$$

In order to prove the lower bound, let us split up

$$\int_U \,=\, \int_{\varepsilon U} + \int_{U \smallsetminus \varepsilon U}$$

and

$$J(n) = J_1(n) + J_2(n)$$

accordingly, where $\varepsilon \in (0,1)$ is chosen small enough, so that (3.27) holds for $\theta \in \varepsilon U$. Then it follows from (3.27) and (3.28) that

$$J_1(n) \approx n^{-4} \qquad \text{while} \qquad J_2(n) \lesssim e^{-\,\mathrm{const.}\,n}\,.$$

In conclusion, $J(n) \approx n^{-4}$ provided that n is large enough.

Assume from now on that x and hence n are large.

3.8. **Proof of Theorem 3.1 when $\frac{|x|}{n}$ stays away from 1.** Assume that $|\delta|\le 1-\eta$, with $\eta\in(0,1)$ small. Then the stationary point s considered Subsection 3.3 remains bounded, according to Lemma 3.7.e.

To begin with, let us modify the integral expression (3.3). Firstly, by using (2.1) and (2.2), together with the W_0-invariance of h and U, we get

$$p_n(x)=\tfrac{1}{4\pi^2}\,\sigma^n\,q^{-\langle\rho,x^+\rangle}\int_U h(i\theta)^n\,\tfrac{\Delta(i\theta)\,e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(i\theta)}\,d\theta\,.$$

Secondly, by deforming the contour of integration in $\mathfrak{a}_{\mathbb{C}}$ and by using the $2\pi Q$-periodicity in θ, we get

$$p_n(x)=\tfrac{1}{4\pi^2}\,\sigma^n\,q^{-\langle\rho,x^+\rangle}\,e^{-\langle x^++\rho,s\rangle}\int_U h(s+i\theta)^n\,\tfrac{\Delta(s+i\theta)\,e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(s+i\theta)}\,d\theta\,.$$

Thirdly, after performing an integration by parts based on (3.7), we get

$$p_n(x)=C(n,x^+)\,J(n,x^+)\,,\tag{3.29}$$

where

$$C(n,x^+)=\tfrac{1}{4\pi^2}\,\tfrac{\boldsymbol{\pi}(x^++\rho)}{(n+3)^2\,(n+2)}\,\sigma^n\,q^{-\langle\rho,x^+\rangle}\,h(s)^{n+2}\,e^{-\langle x^++\rho,s\rangle}\tag{3.30}$$

and

$$J(n,x^+)\;=\int_U\big[\,\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\langle\delta,\theta\rangle}\big]^{n+2}\,a_3(\theta)\,d\theta\,.\tag{3.31}$$

Let us make two observations about the latter expressions. On the one hand, as

$$\begin{cases}\boldsymbol{\pi}(x^++\rho)=(x_1+1)(x_2+1)(|x|+1)\,,\\ \boldsymbol{\sigma}^n=6^n\,\sigma^n\,,\\ e^{n\,\phi(\delta)}\approx e^{(n+2)\,\phi(\delta)}=e^{(n+2)\,\Phi(s)}=6^{-n-2}\,h(s)^{n+2}\,e^{-\langle x^++\rho,s\rangle}\,,\end{cases}$$

we have

$$C(n,x^+)\approx\tfrac{(1+|x|)\,(1+x_1)\,(1+x_2)}{n^3}\,\boldsymbol{\sigma}^n\,q^{-\langle\rho,x^+\rangle}\,e^{n\,\phi(\delta)}\,.\tag{3.32}$$

Hence (3.1) amounts to

$$J(n,x^+)\,\approx\,\tfrac{1}{n}\tag{3.33}$$

under the current assumptions. On the other hand, (3.31) is meaningful as long as the denominator $h(s+i\theta)+2\,\frac{n+1}{n+3}$ in (3.20) doesn't vanish, which may happen when x^+ gets close to the extra wall (3.13), according to Corollary 3.8.(a). We get around this problem by considering

$$\widetilde{p}_n(x)=\widetilde{C}(n,x^+)\,\widetilde{J}(n,x^+)$$

instead of (3.29), where

$$\widetilde{p}_n(x)=p_n(x)+2\,\tfrac{n}{n+2}\,\sigma\,p_{n-1}(x)\,,\tag{3.34}$$

$$\widetilde{C}(n,x^+)=\tfrac{(n+3)^2}{(n+2)\,(n+1)}\,C(n,x^+)\approx C(n,x^+)\,,$$

$$\widetilde{J}(n,x^+)=\int_U\big[\,\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\langle\delta,\theta\rangle}\big]^{n+2}\,a_2(\theta)\,d\theta\,.\tag{3.35}$$

Notice indeed that

$$p_n(x)\le\widetilde{p}_n(x)\lesssim p_{n+2}(x).\tag{3.36}$$

Here the first inequality is elementary while the second one follows from the local Harnack inequality (see Remark 3.6).

Let us prove the estimate (3.33) for the integral (3.35) and, to this end, let us resume the analysis carried out for (3.26) in Subsection 3.7. The upper bound of

(3.35) follows again easily from Lemma 3.10 and Lemma 3.12.(b). More precisely, the Gaussian estimate

$$\tfrac{|h(s+i\theta)|}{h(s)} \lesssim e^{-\,\mathrm{const.}\,B(\theta,\theta)} \le e^{-\,\mathrm{const.}\,|\theta|^2} \qquad \forall\,\theta\in U \tag{3.37}$$

and the uniform boundedness of a_2 yield

$$\widetilde{J}(n,x^+) \lesssim \int_U e^{-\,\mathrm{const.}\,n\,|\theta|^2}\,d\theta \lesssim \tfrac{1}{n}\,.$$

In order to prove the lower bound, let us split up this time

$$\widetilde{J}(n,x^+) = \sum\nolimits_{k=1}^{4} \widetilde{J}_k(n,x^+)\,, \tag{3.38}$$

where

$$\begin{aligned} \widetilde{J}_1(n,x^+) &= a_2(0)\int_{\varepsilon U} e^{(n+2)\,\mathrm{Re}\,\Psi(\theta)}\,d\theta\,,\\ \widetilde{J}_2(n,x^+) &= a_2(0)\int_{\varepsilon U}\big\{e^{(n+2)\,\Psi(\theta)} - e^{(n+2)\,\mathrm{Re}\,\Psi(\theta)}\big\}\,d\theta\,,\\ \widetilde{J}_3(n,x^+) &= \int_{\varepsilon U} e^{(n+2)\,\Psi(\theta)}\,\big\{a_2(\theta)-a_2(0)\big\}\,d\theta\,,\\ \widetilde{J}_4(n,x^+) &= \int_{U\smallsetminus\varepsilon U}\big[\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\langle\delta,\theta\rangle}\big]^{n+2}\,a_2(\theta)\,d\theta\,. \end{aligned}$$

Here $\varepsilon\in(0,1)$ is chosen small enough, so that Lemma 3.9.(b) holds for $\theta\in\varepsilon U$. The first term in (3.38) which yields the main contribution, is estimated as follows. On the one hand, according to Lemma 3.12.(b), we have $|a_2(0)|\approx 1$, provided that x is large enough. On the other hand, we deduce from Lemma 3.9 that

$$\int_{\varepsilon U} e^{(n+2)\,\mathrm{Re}\,\Psi(\theta)}\,d\theta \approx \int_{\varepsilon U} e^{-\,\mathrm{const.}\,(n+2)\,B(\theta,\theta)}\,d\theta \approx \int_{\varepsilon U} e^{-\,\mathrm{const.}\,n\,|\theta|^2}\,d\theta \approx \tfrac{1}{n}\,.$$

Hence

$$|\,\widetilde{J}_1(n,x^+)| \approx \tfrac{1}{n}\,.$$

As

$$e^{\,i\,(n+2)\,\mathrm{Im}\,\Psi(\theta)} - 1 = \mathcal{O}\big(n|\theta|^3\big) \qquad\text{and}\qquad a_2(\theta)-a_2(0) = \mathcal{O}(|\theta|)\,,$$

the next two terms in (3.38) are estimated similarly from above:

$$|\,\widetilde{J}_2(n,x^+)| \lesssim n^{-\frac{3}{2}} \qquad\text{and}\qquad |\,\widetilde{J}_3(n,x^+)| \lesssim n^{-\frac{3}{2}}\,.$$

For the last term in (3.38), we obtain

$$|\,\widetilde{J}_4(n,x^+)\,| \lesssim e^{-\,\mathrm{const.}\,n}$$

by using again the Gaussian estimate (3.37) and the uniform boundedness of a_2. Thus,

$$\widetilde{J}(n,x^+) \ge |\,\widetilde{J}_1(n,x^+)| - \sum\nolimits_{k=2}^{4} |\,\widetilde{J}_k(n,x^+)| \gtrsim \tfrac{1}{n}\,,$$

provided that n and x are large enough.
In summary, we have obtained the following estimates:

$$\begin{cases} p_n(x) \le \widetilde{p}_n(x) \lesssim \frac{1}{n}\,C(n,x^+)\,,\\ p_n(x) \gtrsim \widetilde{p}_{n-2}(x) \gtrsim \frac{1}{n-2}\,C(n-2,x^+) \end{cases} \tag{3.39}$$

In order to conclude, observe that the right hand sides in (3.39) are comparable and more precisely the Gaussian type factors entering these expressions. Setting $\widetilde{\delta}=\frac{x^++\rho}{n}$ and writing

$$\phi(\delta)-\phi(\widetilde{\delta})\,=\,-\int_{\frac{1}{n+2}}^{\frac{1}{n}}\langle d\phi\big(t(x^+\!+\rho)\big),\,x^+\!+\rho\rangle\,dt\,,$$

we deduce indeed from (3.9) that $0\le\phi(\delta)-\phi(\widetilde{\delta})\lesssim\frac{1}{n}$, hence

$$e^{(n+2)\,\phi(\delta)}\,\approx\,e^{n\,\phi(\widetilde{\delta})}\,. \tag{3.40}$$

3.9. **Proof of Theorem 3.1 when $\frac{|x|}{n}$ gets close to 1, while $n-|x|$ and x_1-x_2 remain large.** This is the most delicate range to handle. Assume that $1-\eta<|\delta|<1$, with $\eta\in(0,1)$ small. Then (3.1) amounts to the estimate

$$J(n,x^+)\approx(n-|x|)^{-\frac{1}{2}}\,(n-x_1)^{-\frac{1}{2}} \tag{3.41}$$

for the integral (3.31).

The upper bound in (3.41) is proved as in Subsection 3.8. Firstly, we deduce from (3.36) that $J(n,x^+)\lesssim\widetilde{J}(n,x^+)$. Secondly, the following global estimate is obtained by combining Lemma 3.10, Lemma 3.9.(a), Lemma 3.7.(e) and Lemma 3.7.(f) :

$$\Big|\tfrac{h(s+i\theta)}{h(s)}\Big|^{n+2}\le e^{-\,\mathrm{const.}\,(n+2)\,B(\theta,\theta)}\qquad\forall\,\theta\in U\,,$$

with

$$(n+2)\,B(\theta,\theta)\,\approx\,(n-x_1)(\theta^1-\theta^2)^2+(n-|x|)(\theta^1+\theta^2)^2\,. \tag{3.42}$$

Thirdly, recall from Lemma 3.12.(b) that the amplitude (3.19) is uniformly bounded. Hence the upper bound

$$\begin{aligned}J(n,x^+)&\lesssim\widetilde{J}(n,x^+)\\&\lesssim\int_U e^{-\,\mathrm{const.}\,\left\{(n-|x|)(\theta^1+\theta^2)^2+(n-x_1)(\theta^1-\theta^2)^2\right\}}\,d\theta\\&\lesssim(n-|x|)^{-\frac{1}{2}}\,(n-x_1)^{-\frac{1}{2}}\,.\end{aligned}$$

Let us turn to the lower bound in (3.41), which is harder to prove. Our task consists in analyzing the integral

$$J(n,x^+)\,=\int_U\Big[\,\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\,\langle\delta,\theta\rangle}\Big]^{n+2}a_3(\theta)\,d\theta\,, \tag{3.31}$$

which cannot be replaced anymore by the simpler expression $\widetilde{J}(n-2,x^+)$, as the crucial estimate $e^{(n+2)\,\phi(\delta)}\lesssim e^{n\,\phi(\widetilde{\delta})}$ fails to hold in (3.40), when s becomes unbounded. For this purpose, let us split up

$$J(n,x^+)\,=\,\sum\nolimits_{k=1}^{5}J_k(n,x^+)\,, \tag{3.43}$$

where (see Figure 3)

$$\begin{aligned} J_1(n,x^+) &= a_3(0)\int_{\varepsilon U} e^{(n+2)\operatorname{Re}\Psi(\theta)}\,d\theta\,,\\ J_2(n,x^+) &= a_3(0)\int_{\varepsilon U}\bigl\{e^{(n+2)\Psi(\theta)}-e^{(n+2)\operatorname{Re}\Psi(\theta)}\bigr\}\,d\theta\,,\\ J_3(n,x^+) &= \int_{\varepsilon U} e^{(n+2)\Psi(\theta)}\,\{a_3(\theta)-a_3(0)\}\,d\theta\,,\\ J_4(n,x^+) &= \int_{U\cap\varepsilon(S\smallsetminus U)}\bigl[\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\langle\delta,\theta\rangle}\bigr]^{n+2}\,a_3(\theta)\,d\theta\,,\\ J_5(n,x^+) &= \int_{U\smallsetminus\varepsilon(S\cup U)}\bigl[\tfrac{h(s+i\theta)}{h(s)}\,e^{-i\langle\delta,\theta\rangle}\bigr]^{n+2}\,a_3(\theta)\,d\theta\,. \end{aligned}$$

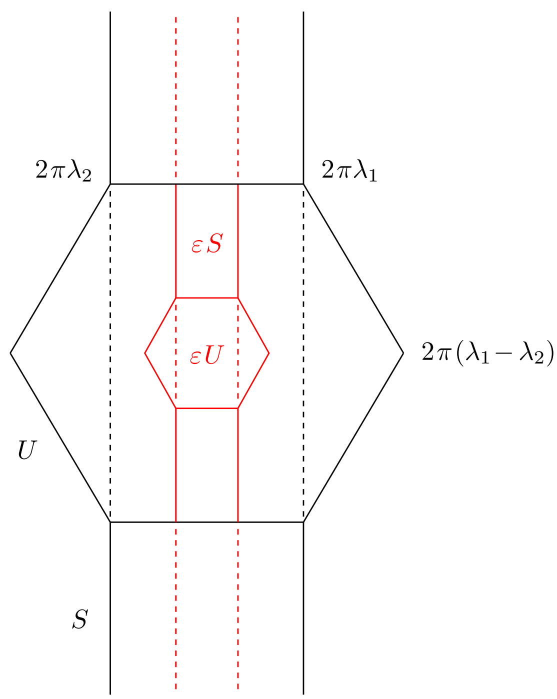

FIGURE 3. Picture for the decomposition (3.43)

Here, S denotes the vertical strip

$$\{\,\theta\in\mathfrak{a}\mid|\langle\lambda_1-\lambda_2,\theta\rangle|\le\tfrac{2}{3}\pi\,\}$$

and $\varepsilon\in(0,1)$ is chosen small enough, so that

- Lemma 3.9.(b) holds for $\theta\in\varepsilon U$,
- $a_3(\theta)$ and $da_3(\theta)$ are uniformly bounded for $\theta\in\varepsilon(S\cup U)$ and n large, according to Lemma 3.12.(c).

We use again (3.42) to estimate the five integrals $J_k(n,x^+)$ occurring in (3.43). Let us elaborate. By arguing as for the first three terms in (3.38), we obtain

$$|J_1(n,x^+)| \approx (n-x_1)^{-\frac12}(n-|x|)^{-\frac12},$$
$$|J_2(n,x^+)| \lesssim (n-x_1)^{-\frac12}(n-|x|)^{-1},$$
$$|J_3(n,x^+)| \lesssim (n-x_1)^{-\frac12}(n-|x|)^{-1},$$

provided that η is small enough and n is large enough, which we assume from now on. We obtain similarly

$$|J_4(n,x^+)| \lesssim (n-x_1)^{-\frac12}\, e^{-\,\text{const.}\,(n-|x|)}$$

by using the uniform boundedness of $a_3(\theta)$ for $\theta\in\varepsilon S$ and the estimate

$$\Big[\tfrac{|h(s+i\theta)|}{h(s)}\Big]^{n+2} \le e^{-\,\text{const.}\,(n-x_1)(\theta^1-\theta^2)^2}\, e^{-\,\text{const.}\,(n-|x|)} \qquad \forall\,\theta\in U\cap\varepsilon(S\smallsetminus U)\,,$$

which follows from Lemma 3.10 and (3.42). For the last integral, which is most troublesome, we use the estimate

$$\Big[\tfrac{|h(s+i\theta)|}{h(s)}\Big]^{n+2} \le e^{-\,\text{const.}\,(n-x_1)} \qquad \forall\,\theta\in U\smallsetminus\varepsilon S\,,$$

which follows again from Lemma 3.10 and (3.42), together with the estimates of $a_3(\theta)$ contained in Lemma 3.12.(c). On the one hand, if $\delta_1-\delta_2$ stays away from 0, say $\delta_1-\delta_2\ge\frac14$, then $a_3(\theta)$ is uniformly bounded, hence

$$|J_5(n,x^+)| \lesssim e^{-\,\text{const.}\,(n-x_1)}.$$

On the other hand, if $\delta_1-\delta_2<\frac14$, then

$$x_1 = \tfrac{x_1+x_2}{2}+\tfrac{x_1-x_2}{2} < \tfrac{n-1}{2}+\tfrac{n+2}{4} = \tfrac34\,n\,,$$

hence $n-x_1\approx n$. As $a_3(\theta)=\mathcal{O}(n^4)$, under the additionnal assumption that x_1-x_2 is large enough, we obtain again

$$|J_5(n,x^+)| \lesssim e^{-\,\text{const.}\,(n-x_1)}.$$

In conclusion, we obtain the expected bound

$$J(n,x^+) \ge |J_1(n,x^+)| - \sum\nolimits_{k=2}^{5}|J_k(n,x^+)| \gtrsim (n-x_1)^{-\frac12}(n-|x|)^{-\frac12},$$

provided that η is small enough and that n, $n-|x|$, x_1-x_2 are all large enough.

3.10. **Completion of the proof of Theorem 3.1 when $\frac{|x|}{n}$ gets close to 1, while $n-|x|$ remains large.** In this subsection, we extend up to the extra wall (3.13) the lower bound proved in Subsection 3.9. Specifically, assume that the lower bound in (3.1) holds in the range

$$\begin{cases} x_1-x_2\ge m\,,\\ n-|x|\ge m\,,\end{cases}$$

for some fixed $m\in\mathbb{N}^*$, and let us deduce it under the following conditions:

$$\begin{cases} 0\le x_1-x_2<m\,,\\ n-|x|\ge m\,,\\ 1-\eta<\frac{|x|}{n}<1,\ \text{ with }\ 0<\eta<1\ \text{ small enough},\\ n\ \text{ or equivalently }\ |x|\ \text{ is large enough}.\end{cases}$$

In this case, x_1 and x_2 are close to $\frac{|x|}{2}$. Thus the lower estimate in (3.1), which we aim for, amounts to

$$p_n(x) \gtrsim n^{-\frac12}(n-|x|)^{-\frac12}\,\boldsymbol{\sigma}^n\, q^{-\langle\rho,x^+\rangle}\, e^{n\,\phi(\delta)}. \tag{3.44}$$

Consider $\widetilde{n}=n-m$ and $\widetilde{x}=x^+-m\lambda_2$. Then

$$p_n(x) \gtrsim p_{\widetilde{n}}(\widetilde{x}),$$

according to the local Harnack inequality (3.5). Besides $\widetilde{x}\in\mathfrak{a}^+$ provided that $|x|$ is large enough. Moreover

$$\begin{cases} \widetilde{x}_1-\widetilde{x}_2 = x_1-x_2+m \geq m\,,\\ \widetilde{n}-|\widetilde{x}| = n-|x| \geq m\,. \end{cases}$$

Thus (3.44) holds for $p_{\widetilde{n}}(\widetilde{x})$ by assumption, hence

$$p_{\widetilde{n}}(\widetilde{x}) \gtrsim n^{-\frac12}(n-|x|)^{-\frac12}\,\boldsymbol{\sigma}^n\,q^{-\langle\rho,x^+\rangle}\,e^{\widetilde{n}\,\phi(\widetilde{\delta})},$$

where $\widetilde{\delta}=\frac{\widetilde{x}+\rho}{\widetilde{n}+2}$. In order to conclude the proof of (3.44), it remains for us to compare

$$e^{\widetilde{n}\,\phi(\widetilde{\delta})} \gtrsim e^{n\,\phi(\delta)}\,.$$

Firstly, $\widetilde{n}\,\phi(\widetilde{\delta})\geq n\,\phi(\widetilde{\delta})$, as $\phi\leq 0$. Secondly, we claim that $\phi(\widetilde{\delta})\geq\phi(\delta)$, if η is small enough and n large enough. For this purpose, let us write

$$\phi(\widetilde{\delta})-\phi(\delta) = \int_0^1\langle(d\phi\circ\delta)(t),\widetilde{\delta}-\delta\,\rangle\,dt\,, \tag{3.45}$$

where $\delta(t)=(1-t)\delta+t\widetilde{\delta}$. In this expression, $(d\phi\circ\delta)(t)=-\,s(t)$, according to (3.9), and

$$\widetilde{\delta}-\delta = \tfrac{m}{(n+2)\,(n+2-m)}\,\bigl\{(x_1+1)\,\lambda_1+(x_2-n-1)\,\lambda_2\bigr\}\,.$$

Hence,

$$\langle(d\phi\circ\delta)(t),\widetilde{\delta}-\delta\,\rangle = \tfrac{m}{(n+2)\,(n+2-m)}\,\bigl\{(n-|x|)\,s^2(t)-(x_1+1)\,[s^1(t)-s^2(t)]\bigr\}\,. \tag{3.46}$$

On the one hand,

$$\delta_1(t)-\delta_2(t) = (1-t)\,\tfrac{x_1-x_2}{n+2}+t\,\tfrac{x_1-x_2+m}{n+2-m} = \mathcal{O}\bigl(\tfrac1n\bigr)\,.$$

Thus, if n is large enough,

$$(x_1+1)\,[s^1(t)-s^2(t)] = \mathcal{O}(1)\,, \tag{3.47}$$

according to Lemma 3.7.(g). On the other hand,

$$1-|\delta(t)| = (1-t)\,\tfrac{n-|x|}{n+2}+t\,\tfrac{n-|x|}{n+2-m} \leq \tfrac{n-|x|}{n+2-m} = \bigl(1-\tfrac{m-2}{n}\bigr)^{-1}\bigl(1-\tfrac{|x|}{n}\bigr)$$

is smaller than 2η, if n is large enough, and

$$e^{s^2(t)} \approx \tfrac{1}{1-|\delta(t)|}\,,$$

according to Lemma 3.7.(e). Thus,

$$(n-|x|)\,s^2(t) \geq m\,\bigl[-\log(2\eta)-\text{const.}\bigr]$$

is positive and even larger than (3.47), if η is small enough. In conclusion, (3.46) is positive and (3.45) too, provided that η is small enough and n large enough.

3.11. **Proof of Theorem 3.1 close to the boundary $|x|=n$.** In this subsection, we give a combinatorial proof of Theorem 3.1 in the range $n-m\leq|x|\leq n$, where m is any fixed positive integer and n is large. Still assuming that $x_1\geq x_2$, let us first show that (3.1) amounts then to

$$p_n(x) \approx \sigma^n\,q^{-n}\,n^{n+d}\,x_1^{-x_1}\,(x_2+1)^{-x_2-d-\frac12}\,, \tag{3.48}$$

where $d=n-|x|$. Firstly,

$$\frac{(1+|x|)\,(1+x_1)\,(1+x_2)}{n^3\sqrt{n-|x|}\,\sqrt{n-x_1}} \approx \frac{\sqrt{x_2+1}}{n}$$

as $x_1\approx|x|\approx n$ and $n-x_1=x_2+d\approx x_2+1$. Secondly,

$$\widetilde{\sigma}^n = 6^n\sigma^n \qquad\text{and}\qquad q^{-\langle\rho,x^+\rangle} = q^{-|x|}\approx q^{-n}\,.$$

Thirdly,

$$e^{n\,\phi(\delta)} \approx e^{(n+2)\,\phi(\delta)} \approx 6^{-n}\,h(s)^{n+2}\,e^{-\langle x^+ +\rho,\,s\rangle}\,,$$

with $h(s)=e^{s^1}\big\{1+e^{-(s^1-s^2)}+\mathcal{O}\big(e^{-s^2}\big)\big\}$. According to Lemma 3.7,

$$h(s)=e^{s^1}\big\{\tfrac{1}{\delta_1}+\mathcal{O}(1-|\delta|)\big\}=\tfrac{e^{s^1}}{\delta_1}\big\{1+\mathcal{O}(\tfrac{1}{n})\big\}\,,$$

hence

$$h(s)^{n+2}\,e^{-\langle x^+ +\rho,\,s\rangle} \approx n^{n+1}\,x_1^{-n-1}\,e^{(x_2+d+1)(s^1-s^2)}\,e^{d\,s^2}\,,$$

with $e^{ds^2}\approx(1-|\delta|)^{-d}\approx n^d$. On the one hand, if x_2 is large enough,

$$\begin{aligned} e^{(x_2+d+1)(s^1-s^2)} &\approx \big\{\tfrac{1-\delta_1}{\delta_1}+\mathcal{O}(1-|\delta|)\big\}^{-x_2-d-1}\\ &\approx \big(\tfrac{1-\delta_1}{\delta_1}\big)^{-x_2-d-1}\big\{1+\mathcal{O}\big(\tfrac{1}{x_2+d+1}\big)\big\}^{-x_2-d-1}\\ &\approx \big(\tfrac{x_1+1}{x_2+d+1}\big)^{x_2+d+1} \approx \big(\tfrac{x_1}{x_2+1}\big)^{x_2+d+1}\,. \end{aligned}$$

On the other hand, as long as x_2 is bounded,

$$e^{(x_2+d+1)(s^1-s^2)} \approx \big(\tfrac{1-\delta_1}{\delta_1}\big)^{-x_2-d-1}=\big(\tfrac{x_1+1}{x_2+d+1}\big)^{x_2+d+1}\approx\big(\tfrac{x_1}{x_2+1}\big)^{x_2+d+1}\,.$$

Thus

$$h(s)^{n+2}\,e^{-\langle x^+ +\rho,\,s\rangle}\approx n^{n+d+1}\,x_1^{-x_1}\,(x_2+1)^{-x_2-d-1}$$

in all cases and the right hand side of (3.1) is comparable to (3.48), as claimed.

Let us next turn to the proof of (3.48). Instead of the simple random walk in $\mathcal{X}$, it is more convenient to work with the corresponding radial random walk in P^+, whose transition probability is given by the following table where, let us recall, $\sigma=\frac{1}{2\,(q+1+q^{-1})}$.

$\lambda\in P^{++}$	$p^+(\lambda,\lambda+\lambda_1)=p^+(\lambda,\lambda+\lambda_2)=\sigma\,q$ $p^+(\lambda,\lambda+\lambda_1-\lambda_2)=p^+(\lambda,\lambda-\lambda_1+\lambda_2)=\sigma$ $p^+(\lambda,\lambda-\lambda_1)=p^+(\lambda,\lambda-\lambda_2)=\sigma\,q^{-1}$
$\lambda\in\mathbb{N}^*\lambda_1$	$p^+(\lambda,\lambda+\lambda_2)=\sigma\,(q+1)$ $p^+(\lambda,\lambda+\lambda_1)=\sigma\,q$ $p^+(\lambda,\lambda-\lambda_1+\lambda_2)=\sigma\,(1+q^{-1})$ $p^+(\lambda,\lambda-\lambda_1)=\sigma\,q^{-1}$
$\lambda\in\mathbb{N}^*\lambda_2$	$p^+(\lambda,\lambda+\lambda_1)=\sigma\,(q+1)$ $p^+(\lambda,\lambda+\lambda_2)=\sigma\,q$ $p^+(\lambda,\lambda+\lambda_1-\lambda_2)=\sigma\,(1+q^{-1})$ $p^+(\lambda,\lambda-\lambda_2)=\sigma\,q^{-1}$
$\lambda=0$	$p^+(\lambda,\lambda+\lambda_1)=p^+(\lambda,\lambda+\lambda_2)=\frac{1}{2}$

We claim that, in the range $n-m\le|x|\le n$, (3.48) amounts to showing that

$$M\approx n^{n+d}\,x_1^{-x_1}\,(x_2+1)^{-x_2-d-\frac{1}{2}}\,,\tag{3.49}$$

where M denotes the number of paths in P^+ between 0 and $\lambda=x^+$. This claim is obtained by combining the following two facts. On the one hand, $p_n^+(0,\lambda)=N_\lambda\,p_n(\lambda)$, where $N_\lambda\approx q_{t_\lambda}=q^{2\langle\rho,\lambda\rangle}=q^{2|\lambda|}\approx q^{2n}$. On the other hand, according to the table above, the transition probability of the radial random walk equals $\sigma\,q$ or $\sigma\,(q+1)$ at each step, except for finitely many.

Let us turn to the proof of (3.49) and, for this purpose, consider the sequence of increments of the radial random walk up to time n:

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \quad \text{with} \quad \varepsilon_j \in \{\pm\lambda_1, \pm\lambda_2, \pm\lambda_1 \mp \lambda_2\}\,.$$

On the one hand, choose $x_1 - d$ times λ_1, $x_2 + d$ times λ_2, and d times $\lambda_1 - \lambda_2$, the latter occurring after at least d increments λ_2, in order to remain within P^+. The number of such choices equals

$$\tfrac{n!}{(x_1-d)!\,(x_2+2d)!}\,\tfrac{(x_2+d)!}{x_2!\,d!}\,,$$

which is comparable to

$$n^{n+\frac{1}{2}}\,x_1^{-x_1+d-\frac{1}{2}}\,(x_2+1)^{-x_2-d-\frac{1}{2}}\,,$$

according to Stirling's formula, hence to $n^{n+d}x_1^{-x_1}(x_2+1)^{-x_2-d-\frac{1}{2}}$. This proves the lower bound. On the other hand, in order to reach $\lambda = x_1\lambda_1 + x_2\lambda_2$ in n steps, one needs k increments λ_1, ℓ increments λ_2 and $n-k-\ell$ increments in $\{-\lambda_1, -\lambda_2, \pm\lambda_1 \mp \lambda_2\}$. Notice that $|x| \le k+\ell \le n$ and $|k - x_1| \le d$, $|\ell - x_2| \le d$. Thus

$$M \;\le\; \sum\nolimits_{\substack{k,\ell\in\mathbb{N}\\ |k-x_1|\le d,\,|\ell-x_2|\le d}} \tfrac{n!}{k!\,\ell!}\,. \tag{3.50}$$

According to Stirling's formula, each term $\frac{n!}{k!\,\ell!}$ on the right hand side of (3.50) is comparable to $n^{n+\frac{1}{2}}\,k^{-k-\frac{1}{2}}\,(\ell+1)^{-\ell-\frac{1}{2}}$ hence to $n^n\,x_1^{-k}\,(x_2+1)^{-\ell-\frac{1}{2}}$. The upper bound

$$M \lesssim n^{n+d}\,x_1^{-x_1}\,(x_2+1)^{-x_2-d-\frac{1}{2}}$$

follows from the fact that the sum in (3.50) is finite and

$$x_1^{x_1-k}\,(x_2+1)^{x_2+d-\ell} \lesssim n^{n-k-\ell} \le n^d\,.$$

This concludes the proof of Theorem 3.1. □

Remark 3.13. *Most of the analysis carried out in this section applies actually to any isotropic nearest neighbor random walk*

$$A = c_1\,A_{\lambda_1} + c_2\,A_{\lambda_2}\,,$$

where $c_1 > 0$, $c_2 > 0$ and $c_1 + c_2 = 1$. More precisely, such a random walk has transition density

$$p_n(x) = \tfrac{1}{4\pi^2}\,\sigma^n\,q^{-\langle\rho,x^+\rangle}\int_U h(i\theta)^n\,\frac{\Delta(i\theta)\,e^{-i\langle x^+ + \rho,\,\theta\rangle}}{\mathbf{b}(i\theta)}\,d\theta$$

with

$$h = 2c_1\sum\nolimits_{\lambda\in W_0.\lambda_1} e^{\lambda} + 2c_2\sum\nolimits_{\lambda\in W_0.\lambda_2} e^{\lambda}\,.$$

As for the simple random walk, its spectral radius is $\boldsymbol{\sigma} = 6\,\sigma = \frac{3}{q+1+q^{-1}}$ and h enjoys remarkable product and differentiation formulae (see Appendix A). Thus the same analysis yields again the upper and lower bound

$$n^{-4}\,\boldsymbol{\sigma}^n\,F_0(x)\,e^{n\,\phi(\delta)}$$

in the range $|x| \le (1-\eta)n$. In the range $(1-\eta)n < |x| < n$, there is a problem with the lower bound, but we get the upper bound

$$\frac{e^{C(n-|x|)}}{n^3\sqrt{n-|x|}\,\sqrt{n-x_1\vee x_2}}\;\boldsymbol{\sigma}^n\,F_0(x)\,e^{n\,\phi(\delta)}$$

by arguing as in Subsection 4.2. Here C is a positive constant.

4. Heat kernel estimates in higher rank

Let us generalize the notation of Section 3. Given an integer $n\geq 2$ and $x\in\mathcal{X}$ with $|x|<n$, set $\delta=\frac{x^++\rho}{n+r}$ and denote by $\phi(\delta)\in(-\log N,0]$ the minimum of the function

$$\Phi(z)=\log\tfrac{h(z)}{N}-\langle\delta,z\rangle\qquad\forall\,z\in\mathfrak{a},\tag{4.1}$$

where

$$N=h(0)=\sum\nolimits_{j=1}^{r}|W_0.\lambda_j|=2\,(2^r\!-1)\,.$$

Here is a partial generalization of Theorem 3.1.

Theorem 4.1. *Let $0<\eta<1$ (small). Then*

$$p_n(x)\approx n^{-\frac{r}{2}-|R^+|}\,\boldsymbol{\sigma}^n\,F_0(x)\,e^{n\,\phi(\delta)}\tag{4.2}$$

uniformly in the range $|x|\leq(1-\eta)n$. Moreover,

$$p_n(x)\leq n^{-\frac{r}{2}-|R^+|}\,\boldsymbol{\sigma}^n\,F_0(x)\,e^{n\,\phi(\delta)}\,\frac{e^{\,n\,(1-|\delta|)}}{\prod_{\alpha\in R^+}\sqrt{1-\langle\alpha,\delta\rangle}}\tag{4.3}$$

in the range $|x|<n$.

Remark 4.2. *Notice that the power $\frac{r}{2}+|R^+|$ is half of the pseudo-dimension, as in the symmetric space case.*

The proof of Theorem 4.1 is similar to the proof of Theorem 3.1 in Section 3. Therefore, we just outline the proof, elaborating on higher rank features. Let us begin with analogs of (3.6), (3.7) and (A.3).

Lemma 4.3. *The following product and differentiation formulae hold:*

$$h+2=\prod_{\lambda\in W_0.\lambda_1}(e^{\lambda}+1)=\prod_{1\leq j\leq r+1}^{r}(e^{\lambda_j-\lambda_{j-1}}+1)\,,\tag{4.4}$$

where we set $\lambda_0=\lambda_{r+1}=0$, and

$$\boldsymbol{\pi}(\partial)\,h^{n+|R^+|}=\underbrace{\tfrac{(n+|R^+|)\,!}{n\,!}}_{\approx\,n^{|R^+|}}\,r_n(h)\,h^n\,\Delta\,,\tag{4.5}$$

where

$$r_n(h)=(h+2)^{|R^+|-r}+\sum_{r\leq k<|R^+|}\overbrace{c_k\,\tfrac{n\,!}{(n+|R^+|-k)\,!}}^{\mathcal{O}\left(n^{-(|R^+|-k)}\right)}(h+2)^{k-r}\,h^{|R^+|-k}$$

is a polynomial in h with coefficients $c_k\in\mathbb{Z}$. Moreover,

$$\boldsymbol{\pi}(\partial)\,(h+2)^{n+r}=d_n\,(h+2)^n\,\Delta\,,\tag{4.6}$$

where d_n is a constant, which is positive and $\approx n^{-|R^+|}$ for n large enough.

Proof. Let us first prove (4.4). Recall that the fundamental weights satisfy

$$\langle\lambda_j,z\rangle=z_1+\ldots+z_j\quad\text{ hence }\quad\langle\lambda_j-\lambda_{j-1},z\rangle=z_j$$

for every $1\leq j\leq r+1$ and $z\in\mathfrak{a}$. We deduce on the one hand that

$$\sum\nolimits_{\lambda\in W_0.\lambda_j}e^{\langle\lambda,\,z\rangle}=\sum\nolimits_{1\leq k_1<\ldots<k_j\leq r+1}e^{z_{k_1}+\ldots+z_{k_j}}\tag{4.7}$$

and on the other hand that

$$\prod\nolimits_{\lambda\in W_0.\lambda_1}\bigl(e^{\langle\lambda,\,z\rangle}\!+1\bigr)=\prod\nolimits_{1\leq j\leq r+1}\bigl(e^{z_j}\!+1\bigr)=\prod\nolimits_{1\leq j\leq r+1}\bigl(e^{\langle\lambda_j-\lambda_{j-1},\,z\rangle}\!+1\bigr).\tag{4.8}$$

By adding up (4.7) over $1 \leq j \leq r$, we obtain that $h(z)+2$ is equal to the sum of products $\prod_{j\in J} e^{z_j}$, where J runs through all subsets of $\{1,\dots,r+1\}$, which is equal in turn to

$$\prod\nolimits_{1\leq j\leq r+1}\bigl(e^{z_j}+1\bigr).$$

Together with (4.8), this concludes the proof of (4.4). Let us turn to the proof of (4.5). Setting

$$\boldsymbol{\pi}_I(\lambda)=\prod\nolimits_{\alpha\in I}\langle\alpha,\lambda\rangle \qquad \forall\, I\subset R^+,$$

we have

$$\boldsymbol{\pi}(\partial)\,h^{n+|R^+|}=\sum\nolimits_{k=1}^{|R^+|}(n+|R^+|)\cdots(n+|R^+|-k+1)\,h^{n+|R^+|-k}\,f_k\,,$$

where

$$f_k=\sum_{\substack{I_1\sqcup I_2\sqcup\ldots\sqcup I_k=R^+\\ I_1\neq\emptyset,\ldots,I_k\neq\emptyset}}\bigl[\pi_{I_1}(\partial)(h+2)\bigr]\cdots\bigl[\pi_{I_k}(\partial)(h+2)\bigr] \tag{4.9}$$

is an exponential polynomial, which belongs to the $\mathbb{Z}$-span of $\{e^\lambda\,|\,\lambda\in P\}$. As f_k is skew, it is divisible by Δ and the quotient g_k is a symmetric exponential polynomial of degree $\leq k-r$. Here, the *degree* of an exponential polynomial

$$g=\sum\nolimits_{\lambda\in P} g(\lambda)\,e^\lambda$$

is defined by

$$\deg g=\begin{cases}\max\{|\lambda|\,|\,g(\lambda)\neq 0\}\in\mathbb{N} & \text{if } g\neq 0,\\ -\infty & \text{if } g=0.\end{cases}$$

We deduce in particular that g_k vanishes when $k<r$ and it remains for us to show that g_k is proportional to $(h+2)^{k-r}$ when $r\leq k\leq |R^+|$. For this purpose, notice that, for every positive root $\alpha=e_j-e_k$ $(1\leq j<k\leq r+1)$ and for every weight $\langle\lambda,z\rangle=z_\ell$ $(1\leq \ell\leq r+1)$ in $W_0.\lambda_1$, we have

$$\langle\alpha,\lambda\rangle=\begin{cases}1 & \text{when } \ell=j,\\ -1 & \text{when } \ell=k,\\ 0 & \text{otherwise.}\end{cases} \tag{4.10}$$

Consequently,

(a) for every $\alpha\in R^+$, there are exactly two weights $\lambda\in W_0.\lambda_1$ such that $\langle\alpha,\lambda\rangle\neq 0$,
(b) for every $\lambda\in W_0.\lambda_1$, there are exactly r roots $\alpha\in R^+$ such that $\langle\alpha,\lambda\rangle\neq 0$.

On the one hand, we deduce from (4.4) and (4.10) that

$$\tfrac{\partial_\alpha[h(z)+2]}{h(z)+2}=\tfrac{e^{z_j}}{e^{z_j}+1}-\tfrac{e^{z_k}}{e^{z_k}+1}=e^{\frac{z_j+z_k}{2}}\tfrac{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}{(e^{z_j}+1)(e^{z_k}+1)} \qquad (\alpha=e_j-e_k)$$

and obtain this way the leading term

$$f_{|R^+|}=\prod\nolimits_{\alpha\in R^+}\partial_\alpha(h+2)=(h+2)^{|R^+|-r}\,\Delta\,.$$

On the other hand, we deduce from (b) above that each term on the right hand side of (4.9) is divisible by $(e^\lambda+1)^{k-r}$, for every $\lambda\in W_0.\lambda_1$, and hence by $(h+2)^{k-r}$. In summary, f_k is divisible by and hence proportional to $(h+2)^{k-r}\Delta$. This concludes the proof of (4.5) and (4.6) is proved similarly. □

Let us next generalize partially Lemma 3.7

Lemma 4.4. (a) *The function* Φ *is strictly convex, tends to* $+\infty$ *at infinity and reaches its minimum* $\phi(\delta)\in(-\log N,0]$ *at a single point* $s\in\mathrm{cl}(\mathfrak{a}^+)$*, which is the unique solution to the equation* $\frac{dh(s)}{h(s)}=\delta$*. Moreover* $s\longleftrightarrow\delta$ *is an analytic bijection between* $\mathrm{cl}(\mathfrak{a}^+)$ *and* $\{\delta\in\mathrm{cl}(\mathfrak{a}^+)\,|\,|\delta|<1\}$.

(b) *The following properties hold*:

- $s=0 \iff \delta=0$,
- $h(s)\approx\prod_{1\le j\le r+1}\big(1\vee e^{s_j}\big)=e^{\sum_{1\le j\le r+1}0\vee s_j}$,
- $\frac{2}{h(s)}\le 1-|\delta|$.

Proof. (a) and the first claim in (b) are proved as in Section 3. The second claim in (b) follows from (4.4) and more precisely from

$$h(s)\approx h(s)+2=\prod\nolimits_{1\le j\le r+1}\big(1+e^{s_j}\big). \tag{4.11}$$

Let us turn to the third claim in (b). We use (4.4) and actually the right hand side of (4.11) to compute the logarithmic derivative

$$\tfrac{dh(s)}{h(s)+2}=\sum\nolimits_{1\le j\le r+1}\tfrac{e^{s_j}}{e^{s_j}+1}\,e_j-\tfrac{1}{r+1}\Big(\sum\nolimits_{1\le k\le r+1}\tfrac{e^{s_k}}{e^{s_k}+1}\Big)\Big(\sum\nolimits_{1\le k\le r+1}e_k\Big).$$

As $\delta=\frac{dh(s)}{h(s)}$, we deduce that

$$|\delta|=\sum\nolimits_{1\le j\le r}\langle\alpha_j,\delta\rangle=\delta_1-\delta_{r+1}=\tfrac{h(s)+2}{h(s)}\,\tfrac{e^{s_1}-e^{s_{r+1}}}{(e^{s_1}+1)(e^{s_{r+1}}+1)}\,,$$

hence $h(s)(1-|\delta|)=\varpi(s)-2$, where

$$\begin{aligned}\varpi(s)&=\big[h(s)+2\big]\Big[1-\tfrac{e^{s_1}-e^{s_{r+1}}}{(e^{s_1}+1)(e^{s_{r+1}}+1)}\Big]\\&=\big(e^{s_1+s_{r+1}}+2e^{s_{r+1}}+1\big)\prod\nolimits_{1<j<r+1}\big(e^{s_j}+1\big).\end{aligned}\tag{4.12}$$

By expanding the product (4.12), we obtain a sum of positive terms including

$$\begin{aligned}e^{s_1+s_{r+1}}\times\prod\nolimits_{1<j<r+1}e^{s_j}&=1,\\1\times\prod\nolimits_{1<j<r+1}1&=1,\\e^{s_1+s_{r+1}}\times\prod\nolimits_{1<j<r+1}1&=e^{s_1+s_{r+1}},\\1\times\prod\nolimits_{1<j<r+1}e^{s_j}&=e^{-s_1-s_{r+1}}.\end{aligned}$$

Thus $\varpi(s)\ge 2+2\cosh(s_1+s_{r+1})\ge 4$ and we conclude that

$$h(s)(1-|\delta|)=\varpi(s)-2\ge 2.$$

□

4.1. Proof of (4.2) for n large.

The inversion formula yields first

$$p_n(x)=C_0\,\sigma^n\,q^{-\langle\rho,x^+\rangle}\int_U h(i\theta)^n\,\tfrac{\Delta(i\theta)\,e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(i\theta)}\,d\theta \tag{4.13}$$

and next

$$p_n(x)=C_0\,\sigma^n\,q^{-\langle\rho,x^+\rangle}e^{-\langle x^++\rho,s\rangle}\int_U h(s+i\theta)^n\,\tfrac{\Delta(s+i\theta)\,e^{-i\langle x^++\rho,\theta\rangle}}{\mathbf{b}(s+i\theta)}\,d\theta \tag{4.14}$$

after a shift of contour. Notice that s remains bounded, under the present assumption $|x| \le (1-\eta)n$, and that the function

$$\Psi(\theta) = \log \tfrac{h(s+i\theta)}{h(s)} - i\langle \delta, \theta \rangle$$

satisfies

$$-\operatorname{Re}\Psi(\theta) \approx |\theta|^2 \quad \text{and} \quad |\operatorname{Im}\Psi(\theta)| \lesssim |\theta|^3 \tag{4.15}$$

in a neighborborhood of the origin, uniformly in s and δ. All these properties are established as in Section 3. With the notation of Lemma 4.3, let us first replace $h^n\Delta$ by

$$r_{n+r-|R^+|}(h)\, h^{n+r-|R^+|}\,\Delta = \tfrac{(n+r-|R^+|)!}{(n+r)!}\,\boldsymbol{\pi}(\partial)\, h^{n+r}$$

in (4.13) or (4.14), and let us denote by $\widetilde{p}_n(x)$ the resulting expression. Then, after an integration by parts, we obtain the desired estimate (4.2) for $|\widetilde{p}_n(x)|$ as we did for (3.35) in Subsection 3.8. For any $r \le k \le |R^+|$, let us next replace h^n by $h^{n+r-k}(h+2)^{k-r}$ in (4.13) or (4.14), and let us denote by $\widetilde{p}_{n,k}(x)$ the resulting expression. By applying the binomial formula to powers of $h+2$, we get

$$p_n(x) = \widetilde{p}_{n,r}(x) \le \ldots \le \widetilde{p}_{n,k}(x) \le \widetilde{p}_{n,k+1}(x) \le \ldots \le \widetilde{p}_{n,|R^+|}(x)\,.$$

With the notation of Lemma 4.3, assume that n is large enough so that

$$\sum\nolimits_{r\le k<|R^+|} |c_k|\, \tfrac{(n+r-|R^+|)!}{(n+r-k)!}\, \widetilde{p}_{n,k}(x) \le \tfrac{1}{2}\, \widetilde{p}_{n,|R^+|}(x)\,.$$

Then

$$\widetilde{p}_n(x) = \widetilde{p}_{n,|R^+|}(x) + \sum\nolimits_{r\le k<|R^+|} c_k\, \tfrac{(n+r-|R^+|)!}{(n+r-k)!}\, \widetilde{p}_{n,k}(x) \ge \tfrac{1}{2}\, \widetilde{p}_{n,|R^+|}(x)$$

is nonnegative with leading term $\widetilde{p}_{n,|R^+|}(x)$. Hence $\widetilde{p}_{n,|R^+|}(x)$ and consequently $\widetilde{p}_n(x)$ satisfy (4.2). □

4.2. **Proof of (4.3) for n large.** Let us replace h^n by $(h+2)^n$ in (4.13) and (4.14) and let us denote by $\widetilde{p}_n(x)$ the resulting kernel. On the one hand, we deduce again from the binomial formula that $\widetilde{p}_n \ge p_n$. On the other hand, by performing an integration by parts based on (4.6) and by resuming our overall strategy, we obtain

$$\widetilde{p}_n(x) \lesssim n^{-\frac{r}{2}-|R^+|}\, \widetilde{\boldsymbol{\sigma}}^{\,n}\, F_0(x)\, \frac{e^{n\widetilde{\phi}(\delta)}}{\prod_{\alpha\in R^+}\sqrt{1-\langle \alpha, \delta\rangle}}\,,$$

where $\widetilde{\boldsymbol{\sigma}} = \sigma(N+2)$, $\widetilde{\Phi}(z) = \log \frac{h(z)+2}{N+2} - \langle \delta, z\rangle$ and $\widetilde{\phi}(\delta) = \min_{z\in\mathfrak{a}} \widetilde{\Phi}(z)$. In order to conclude, let us compare the expressions $e^{n\widetilde{\phi}(\delta)}$ and $e^{n\phi(\delta)}$. It follows from

$$0 \le \big[\widetilde{\Phi}(z) + \log(N+2)\big] - \big[\Phi(z) + \log N\big] = \log\big[1 + \tfrac{2}{h(z)}\big] \le \tfrac{2}{h(z)} \qquad \forall\, z \in \mathfrak{a},$$

that

$$0 \le \big[\widetilde{\phi}(\delta) + \log(N+2)\big] - \big[\phi(\delta) + \log N\big] \le \tfrac{2}{h(s)}\,,$$

which is $\le 1-|\delta|$, according to the last inequality in Lemma 4.4.(b). Thus

$$\widetilde{\boldsymbol{\sigma}}^{\,n}\, e^{n\widetilde{\phi}(\delta)} \le \boldsymbol{\sigma}^n\, e^{n\phi(\delta)}\, e^{n(1-|\delta|)}$$

and this concludes the proof of (4.3). □

This concludes the proof of Theorem 4.1.

Appendix A. Some formulae

This appendix is devoted to the following remarkable formulae in rank 2. Consider

$$h = c_1 \overbrace{(e^{\lambda_1} + e^{-\lambda_2} + e^{\lambda_2-\lambda_1})}^{h_1} + c_2 \overbrace{(e^{\lambda_2} + e^{-\lambda_1} + e^{\lambda_1-\lambda_2})}^{h_2},$$

where c_1 and c_2 are arbitrary real constants.

Lemma A.1. *For every integer* $n\in\mathbb{N}$,

$$\boldsymbol{\pi}(\partial)\, h^n = c_1 c_2\, n^2 (n-1)\, h^{n-2} \Delta + (c_1^3 + c_2^3)\, n(n-1)(n-2)\, h^{n-3} \Delta. \tag{A.1}$$

Proof. We have

$$\begin{aligned} \partial_{\alpha_1}\partial_{\alpha_2}\partial_\rho\, h^n &= n\, h^{n-1} A \\ &\quad + n(n-1)\, h^{n-2} B \\ &\quad + n(n-1)(n-2)\, h^{n-3} C, \end{aligned}$$

where

$$\begin{cases} A = \partial_{\alpha_1}\partial_{\alpha_2}\partial_\rho h\,, \\ B = (\partial_{\alpha_1} h)(\partial_{\alpha_2}\partial_\rho h) + (\partial_{\alpha_2} h)(\partial_{\alpha_1}\partial_\rho h) + (\partial_\rho h)(\partial_{\alpha_1}\partial_{\alpha_2} h)\,, \\ C = (\partial_{\alpha_1} h)(\partial_{\alpha_2} h)(\partial_\rho h)\,. \end{cases}$$

Elementary computations yield first

$$\begin{aligned} \partial_{\alpha_1} h &= c_1\big(e^{\lambda_1} - e^{\lambda_2-\lambda_1}\big) + c_2\big(e^{\lambda_1-\lambda_2} - e^{-\lambda_1}\big) \\ &= \big(e^{\frac{\alpha_1}{2}} - e^{-\frac{\alpha_1}{2}}\big)\big(c_1 e^{\frac{\lambda_2}{2}} + c_2\, e^{-\frac{\lambda_2}{2}}\big)\,, \\ \partial_{\alpha_2} h &= c_1\big(e^{\lambda_2-\lambda_1} - e^{-\lambda_2}\big) + c_2\big(e^{\lambda_2} - e^{\lambda_1-\lambda_2}\big) \\ &= \big(e^{\frac{\alpha_2}{2}} - e^{-\frac{\alpha_2}{2}}\big)\big(c_1 e^{-\frac{\lambda_1}{2}} + c_2\, e^{\frac{\lambda_1}{2}}\big)\,, \\ \partial_\rho h &= c_1\big(e^{\lambda_1} - e^{-\lambda_2}\big) + c_2\big(e^{\lambda_2} - e^{-\lambda_1}\big) \\ &= \big(e^{\frac{\rho}{2}} - e^{-\frac{\rho}{2}}\big)\big(c_1 e^{\frac{\lambda_1-\lambda_2}{2}} + c_2\, e^{\frac{\lambda_2-\lambda_1}{2}}\big)\,, \\ \partial_{\alpha_2}\partial_\rho h &= c_1 e^{-\lambda_2} + c_2\, e^{\lambda_2}\,, \\ \partial_{\alpha_1}\partial_\rho h &= c_1 e^{\lambda_1} + c_2\, e^{-\lambda_1}\,, \\ \partial_{\alpha_1}\partial_{\alpha_2} h &= -\, c_1 e^{\lambda_2-\lambda_1} - c_2\, e^{\lambda_1-\lambda_2}\,, \end{aligned}$$

and next

$$\begin{cases} A = 0\,, \\ B = 2\, c_1 c_2\, \Delta\,, \\ C = \big(c_1 c_2\, h + c_1^3 + c_2^3\big)\Delta\,. \end{cases}$$

This concludes the proof of Lemma A.1. □

Remark A.2. *When* c_1 *or* c_2 *is equal to* 0, *notice that* (A.1) *reduces to*

$$\boldsymbol{\pi}(\partial)\, h_j^n = n(n-1)(n-2)\, h_j^{n-3}\Delta \qquad (j = 1, 2).$$

Lemma A.3. *Assume that* c_1 *and* c_2 *are nonzero. Then the following product and differentiation formulae hold for* $\widetilde{h} = c_1 c_2\, h + c_1^3 + c_2^3$:

$$\widetilde{h} = \big(c_2\, e^{\frac{\lambda_1}{2}} + c_1\, e^{-\frac{\lambda_1}{2}}\big)\big(c_2\, e^{-\frac{\lambda_2}{2}} + c_1\, e^{\frac{\lambda_2}{2}}\big)\big(c_2\, e^{\frac{\lambda_2-\lambda_1}{2}} + c_1\, e^{\frac{\lambda_1-\lambda_2}{2}}\big), \tag{A.2}$$

$$\boldsymbol{\pi}(\partial)\, \widetilde{h}^n = c_1^3 c_2^3\, n^2 (n-1)\, \widetilde{h}^{n-2} \Delta\,. \tag{A.3}$$

Proof. The proof of (A.2) is straightforward and (A.3) is proved as (A.1). □

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