Deviations for the Capacity of the Range of a Random Walk

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Abstract

We obtain estimates for downward deviations for the centered capacity of the range of a random walk on $\mathbb{Z}^d$, in dimension $d \geq 5$. Our regime of deviations runs from large to moderate. We describe path properties of the random walk under the measure conditioned on downward deviations. The proof is based on a martingale decomposition of the capacity, and a delicate analysis of the corrector term. We also obtain a Large Deviation Principle for upward deviations.

Keywords and phrases. Random Walk, Capacity, Range, Large deviations, Moderate deviations.
MSC 2010 *subject classifications.* Primary 60F05, 60G50.

1 Introduction

We consider a simple random walk $\{S_n,\ n \in \mathbb{N}\}$ on $\mathbb{Z}^d$ starting from the origin. The range of the walk between two times $k \leq n$, is denoted as $\mathcal{R}[k,n] := \{S_k, \dots, S_n\}$ with the shortcut $\mathcal{R}_n = \mathcal{R}[0,n]$. Its Newtonian capacity, denoted $\mathrm{Cap}\,(\mathcal{R}_n)$, can be seen as the hitting probability of $\mathcal{R}_n$ by an independent random walk starting *far away* and properly normalized by Green's function, denoted G. Equivalently, using reversibility, it can be expressed as the sum of escape probabilities from $\mathcal{R}_n$ by an independent random walk starting along the range. In other words, $\mathrm{Cap}\,(\mathcal{R}_n)$ is random and has the following representations:

$$\mathrm{Cap}\,(\mathcal{R}_n) \;=\; \lim_{z\to\infty} \frac{\mathbb{P}_{0,z}(\widetilde{H}_{\mathcal{R}_n} < \infty \mid S)}{G(z)} = \sum_{x\in\mathcal{R}_n} \mathbb{P}_{0,x}\big(\widetilde{H}^+_{\mathcal{R}_n} = \infty \mid S\big), \tag{1.1}$$

where $\mathbb{P}_{0,z}$ is the law of two independent walks S and $\widetilde{S}$ starting at 0 and z respectively, and $\widetilde{H}_\Lambda$ (resp. $\widetilde{H}^+_\Lambda$) stands for the hitting (resp. return) time of Λ by the walk $\widetilde{S}$.

In view of (1.1), the study of the capacity of the range is intimately related to the question of estimating probabilities of intersection of random walks. This chapter has grown quite large, with several motivations from statistical mechanics keeping the interest alive (see Lawler's celebrated monograph [Law91]). Let us mention that Norbert Wiener introduced Newtonian capacity in analysis, and later Ito and McKean showed how Wiener's test could be used to decide whether a set is visited infinitely often by a random walk or not. The last decade has witnessed revival interests both after a link between uniform spanning trees and loop erased random walks was discovered (see [LawSW18], [Hut18] for recent results) and after the introduction of random interlacements

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by Sznitman in [S10] which mimic a random walk confined in a region of volume comparable to its time span.

The study of the capacity of the range of a random walk has a long history. Jain and Orey [JO69] show that in any dimension $d \geq 3$, there exists a constant $\gamma_d \in [0, \infty)$, such that almost surely

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Cap}(\mathcal{R}_n) = \gamma_d, \quad \text{and} \quad \gamma_d > 0, \quad \text{if and only if} \quad d \geq 5. \tag{1.2}$$

The first order asymptotics is obtained in dimension 3 in [C17], where $\mathrm{Cap}(\mathcal{R}_n)$ scales like $\sqrt{n}$. Dimension 4 is *the critical dimension*, and a central limit theorem with a non-gaussian limit is established in [ASS19b]. In dimension $d \geq 6$, a standard central limit theorem is proved in [ASS18].

Here, we study the downward deviations for the capacity of the range in dimension $d \geq 5$, in the moderate and large deviations regimes. We also establish a large deviations principle in the upward direction. Our analysis is related to the celebrated large deviation analysis of the volume of the Wiener sausage by van den Berg, Bolthausen and den Hollander [BBH01]. The folding of the Wiener sausage, under squeezing its volume, became a paradigm of *folding*, with localization in a domain with holes of order one (the picture of a Swiss Cheese popularized in [BBH01]). The variational formula for the rate function was shown to have minimizers of different nature in $d = 3$ and in $d \geq 5$ suggesting dimension-dependent optimal scenarii to achieve the deviation. For the discrete analogue of the Wiener sausage, we established in [AS17a] some path properties confirming some observations of [BBH01]. Our present paper is a companion to [AS17a], and the localization obtained by forcing a small Newtonian capacity of the range is of a different nature than the Swiss Cheese picture.

Our first result concerns large and moderate deviations in dimension 5.

Theorem 1.1. *Assume $d = 5$. There exist positive constants ε, $\underline{\kappa}$ and $\overline{\kappa}$, such that for any $n^{16/17}\log n \leq \zeta \leq \varepsilon n$, and n large enough,*

$$\exp\left(-\underline{\kappa}\cdot(\frac{\zeta^2}{n})^{1/3}\right) \;\leq\; \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \leq -\zeta\right) \;\leq\; \exp\left(-\overline{\kappa}\cdot(\frac{\zeta^2}{n})^{1/3}\right).$$

Remark 1.2. In $d = 5$, estimates of the variance and a central limit theorem are still missing for the capacity of the range. We conjecture, based on the analogy between the capacity of the range in $d = 5$ and its volume in $d = 3$, that the variance of $\mathrm{Cap}(\mathcal{R}_n)$ should be of order $n \log n$. Thus, the moderate deviations should pass from a gaussian regime with a speed of order $\zeta^2/(n\log n)$, to a large deviation regime with a speed of order $(\zeta^2/n)^{1/3}$, at a value of ζ where both speeds are equal. That is is for ζ of order $\sqrt{n}(\log n)^{3/4}$. Our techniques do not permit to reach this point. Indeed, our proof relies on a delicate partition of sites of the range according to the occupation times of their neighborhood. This introduces a space-scale r defining the size of the neighborhood each site probes. Even though this space-scale does not appear in our final statements, it is responsible for imposing the limits of the deviations we can study.

Our estimate in dimension 6 and larger requires a notation. Let

$$\chi_d := \frac{d^2 - 3d + 4}{d^2 - 2d}. \tag{1.3}$$

Theorem 1.3. *Assume $d \geq 6$. There exist positive constants ε, $\underline{\kappa}$ and $\overline{\kappa}$ (only depending on the dimension), such that for any $n^{\chi_d}(\log n)^2 \leq \zeta \leq \varepsilon n$, and for n large enough, one has for $d \geq 7$,*

$$\exp\left(-\underline{\kappa}\cdot\zeta^{1-\frac{2}{d-2}}\right) \;\leq\; \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \leq -\zeta\right) \;\leq\; \exp\left(-\overline{\kappa}\cdot\zeta^{1-\frac{2}{d-2}}\right),$$

and for $d = 6$,

$$\exp\left(-\underline{\kappa}\cdot\zeta^{1/2}\right) \;\le\; \mathbb{P}\left(\mathrm{Cap}\left(\mathcal{R}_n\right) - \mathbb{E}[\mathrm{Cap}\left(\mathcal{R}_n\right)] \le -\zeta\right) \;\le\; \exp\left(-\frac{\overline{\kappa}}{\log(n/\zeta)}\cdot\zeta^{1/2}\right).$$

Remark 1.4. In dimension 6 and higher it has been proved [ASS18] that the variance of $\mathrm{Cap}(\mathcal{R}_n)$ is of order n, so the cost of deviations should jump from a gaussian regime with speed ζ^2/n, to a regime with speed of order $\zeta^{1-2/(d-2)}$, with a transition occurring for ζ of order $n^{(d-2)/d}$. As in dimension 5 we do not reach this bound, as one has $\chi_d \ge \frac{d-1}{d}$ in any dimension $d \ge 6$.

Our next results provide path properties of the trajectory under the constraint of moderate deviations. Let $\mathbb{Q}_n$ be the law of the walk conditionally on the event $\{\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \le -\zeta_n\}$, with $(\zeta_n)_{n\ge 1}$ some given sequence. For a subset $\Lambda \subseteq \mathbb{Z}^d$, we denote by $\ell_n(\Lambda)$ the time spent in Λ up to time n. Recall also that for any finite $\Lambda \subseteq \mathbb{Z}^d$, it is known that $\mathrm{Cap}(\Lambda)$ is at least of order $|\Lambda|^{1-2/d}$, with equality when Λ is a ball. Thus a set Λ whose capacity is of order $|\Lambda|^{1-2/d}$ can be considered as being close (in this sense) to a ball.

Theorem 1.5. *Assume $d = 5$. There are positive constants α, c and C, such that for any sequence $(\zeta_n)_{n\ge1}$, satisfying $n^{16/17}(\log n)^2 \le \zeta_n \le n$, one has*

$$\lim_{n\to\infty} \mathbb{Q}_n\left(\exists\Lambda \subseteq \mathbb{Z}^5:\ \ell_n(\Lambda) \ge \alpha n,\ c\,(\frac{n^2}{\zeta_n})^{5/3} \le |\Lambda| \le C\,(\frac{n^2}{\zeta_n})^{5/3},\ \mathrm{Cap}\left(\Lambda\right) \le C\,|\Lambda|^{1-\frac{2}{5}}\right) = 1.$$

In dimension $d \ge 7$ the result reads as follows .

Theorem 1.6. *Assume $d \ge 7$. There are positive constants α, c and C, such that for any sequence $(\zeta_n)_{n\ge1}$, satisfying $n^{\chi_d}(\log n)^4 \le \zeta_n \le n$,*

$$\lim_{n\to\infty} \mathbb{Q}_n\left(\exists\Lambda \subseteq \mathbb{Z}^d:\ \ell_n(\Lambda) \ge \alpha\zeta_n,\ c\,\zeta_n^{\frac{d}{d-2}} \le |\Lambda| \le C\,\zeta_n^{\frac{d}{d-2}},\ \mathrm{Cap}\left(\Lambda\right) \le C|\Lambda|^{1-\frac{2}{d}}\right) = 1.$$

Remark 1.7. We have no path result in dimension six. One of our estimate, Lemma 5.16, needs a very small α, of order $1/\log n$ to make $\xi_{R,+}$ small. Note that even at the level of heuristics, we do not know which of the two scenarii wins: localization of the whole path as in $d = 5$, or localization of a part of it as in $d \ge 7$?

Let us now come to the upward deviations. Our decomposition (1.4) allows us to adapt a beautiful argument of Hamana and Kesten, [HK], written for the volume of the range of a random walk.

Theorem 1.8. *Assume $d \ge 5$. The following limit exists for all $x > 0$:*

$$\psi_d(x) := -\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}\big(\mathrm{Cap}\left(\mathcal{R}_n\right) \ge n\cdot x\big).$$

Furthermore, there exists a constant $\gamma_d^* > \gamma_d$, such that the function ψ_d is continuous and convex on $[0,\gamma_d^*]$, increasing on $[\gamma_d,\gamma_d^*]$, and satisfies

$$\psi_d(x)\begin{cases} = 0 & \text{if } x \le \gamma_d \\ \in (0,\infty) & \text{if } x \in (\gamma_d,\gamma_d^*] \\ = \infty & \text{if } x > \gamma_d^*. \end{cases}$$

Remark 1.9. We also obtain rough upper bounds, in the regime of moderate deviations, see Propositions 3.2 and 3.3.

In order to present some heuristic explanation of the scenarii adopted by the constrained walk, let us present the main steps of our approach.

Our approach to downward deviations. The cornerstone of our approach is a decomposition formula obtained in [ASS19a]:

$$\forall A, B \quad \text{finite sets of } \mathbb{Z}^d, \qquad \mathrm{Cap}\,(A \cup B) = \mathrm{Cap}\,(A) + \mathrm{Cap}\,(B) - \chi_C(A, B), \tag{1.4}$$

where $\chi_C(A, B)$ called *the cross-term* has a nice expression. In this work, the decomposition (1.4) allows us to follow a simple approach devised in [AS17a] to study downward deviations for the volume of the boundary of the range for a random walk in dimensions $d \geq 3$. We partition the time-period of length n into intervals of length $T \leq n$, and write our functional of the range, $\mathrm{Cap}\,(\mathcal{R}_n)$, as a martingale part and a corrector on *scale* T:

$$\mathrm{Cap}\,(\mathcal{R}_n) = \mathrm{Martingale} - \mathrm{Corrector}.$$

The corrector, is obtained by a Doob's like decomposition, and is thus an averaged of cross-terms of the form $\chi_{\mathcal{C}}(\mathcal{R}_{iT}, \widetilde{\mathcal{R}}_T)$, with two independent copies $\mathcal{R}$ and $\widetilde{\mathcal{R}}$, that we integrate over $\widetilde{\mathcal{R}}$. When we impose some downward deviations on an *increasing functional*, (such as the volume or the capacity), we expect some type of *folding* of the walk. We then look for the appropriate time-scale T for which the corrector produces entirely the deviation. For our discussion to be more concrete, let us describe the corrector in more details. It is a sum along the walk's positions, say S_k, of a function probing the occupation density about S_k at *time-scale* T. Indeed, the latter scale enters into the functional $\varphi_T = \frac{1}{T} G_T \star G$, which is the convolution of Green's function with $x \mapsto \frac{1}{T} G_T(x)$, the proportion of the number of visits of site x *up to time* T. The corrector then reads,

$$\xi_n(T) = \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(\widetilde{H}^+_{\mathcal{R}_k} = \infty\big). \tag{1.5}$$

Our first step transfers the deviation of the centered capacity of the range into a deviation for the corrector. In other words, we find $T = T(\zeta, n)$ such that on the event $\{\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \leq -\zeta\}$, one has with high probability that $\xi_n(T)$ must be of order ζ as well, at least for ζ and n large enough. The study of the deviations of $\xi_n(T)$ is more intricate than in our previous study [AS17a], but our general strategy provides a right entry to the problem.

We present now some heuristics to understand the scenarii the constrained walk adopts in different dimensions.

Heuristics. We use the sign $\approx$ to express that two quantities are *of the same order*. As already mentioned, the first step in this work is a simple decomposition for the capacity of a union of sets in term of a cross-term

$$\chi_C(A, B) \;\approx\; 2 \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x\big(H_A^+ = \infty\big) \cdot G(x - y) \cdot \mathbb{P}_y\big(H_B^+ = \infty\big). \tag{1.6}$$

See (2.9) and (2.12) for a precise expression. The key phenomenon responsible for producing a small capacity for the range of a random walk is *an increase of the cross-term on an appropriate scale*. In other words, the walk *folds* into a ball-like domain in order to increase some *self-interaction* captured by the cross-term. Also, in general this cross-term does realize a positive fraction of the deviation, as it turns out to be the case here. Now to be more concrete, let us divide the range $\mathcal{R}[0, 2n]$ into two subsets $\mathcal{R}[0, n]$ and $\mathcal{R}[n, 2n]$. Let us call, for simplicity $\mathcal{R}_n^1 = \mathcal{R}[0, n] - S_n$, and $\mathcal{R}_n^2 = \mathcal{R}[n, 2n] - S_n$ the two subranges translated by S_n so that they become independent. By translation invariance of the capacity, we obtain

$$\mathrm{Cap}\,(\mathcal{R}[0, 2n]) = \mathrm{Cap}\,\big(\mathcal{R}_n^1\big) + \mathrm{Cap}\,\big(\mathcal{R}_n^2\big) - \chi_C(\mathcal{R}_n^1, \mathcal{R}_n^2).$$

Now, assume that both walks stay inside a ball of radius R a time of order $\tau \leq n$, and are unconstrained afterward. Thus, under the strategy we mentioned,

$$\begin{aligned} \chi_C(\mathcal{R}_n^1, \mathcal{R}_n^2) &\approx G(R) \times \mathrm{Cap}\left(\mathcal{R}_\tau^1\right) \times \mathrm{Cap}\left(\mathcal{R}_\tau^2\right) + \mathcal{O}\big(G(\sqrt{n})n^2\big) \\ &\approx G(R)\big(\min(\tau, R^{d-2})\big)^2 + \mathcal{O}\big(n^{\frac{6-d}{2}}\big). \end{aligned} \tag{1.7}$$

The term $\mathcal{O}\big(G(\sqrt{n})n^2\big)$ appears if τ is smaller than n, and accounts for the unconstrained contribution to the cross-term. In obtaining (1.7), we have used that if $\mathcal{R}_\tau^1, \mathcal{R}_\tau^2$ are inside a ball of radius R, their capacity is bounded by the capacity of the ball, which is of order R^{d-2}, as well as by their volume bounded by τ. Thus, it is useless to consider τ larger than R^{d-2}, since then τ no more affects the cross-term and increasing τ (or decreasing R below τ) only makes the cost of the strategy larger. Now, to reach a deviation of order ζ, we have

$$\frac{1}{R^{d-2}}\tau^2 \approx \zeta. \tag{1.8}$$

Recall that the cost of being localized a time τ in a ball of radius R is of order $\exp(-\tau/R^2)$ (up to a constant in the exponential). So we need to find a choice of (τ, R) which minimizes this cost under the constraint (1.8). In other words one needs to maximize $\sqrt{\zeta} \cdot R^{(d-6)/2}$. This leads to two regimes.

- When $d = 5$, R (and then τ) is as large as possible. So, $\tau = n$ and $R^{d-2} = n^2/\zeta$ by (1.8). The strategy is time homogeneous for any ζ!
- When $d \geq 7$, then τ is as small as possible, that is $\tau = R^{d-2} = \zeta$. The strategy is time-inhomogeneous.

When $d = 6$, the strategy remains unknown, but the cost should be of order $\exp(-\sqrt{\zeta})$.

Application to a polymer melt. The model of random interlacements, introduced by Sznitman [S10], is roughly speaking the union of the ranges of trajectories obtained by a Poisson point process on the space of doubly infinite trajectories, and is such that the probability of avoiding a set K is $\exp(-u \cdot \mathrm{Cap}\,(K))$, where $u > 0$ is a fixed parameter. With this in mind, let us consider the following model of polymer among a polymer melt interacting by exclusion. We distinguish one polymer, a simple random walk, interacting with a cloud of other random walk trajectories modeled by random interlacements which we call for short *the melt*. The interaction is through exclusion: the walk and the melt do not intersect. When integrating over the interlacements law, the measure on the walk with the effective interaction has a density proportional to $\exp(-u \cdot \mathrm{Cap}\,(\mathcal{R}_n))$, with respect to the law of a simple random walk.

As a corollary of our deviation estimates, one can address key issues on this polymer. Since this follows in the same way as the study of the Gibbs measure tilted by the volume of the range was a corollary of [AS17a], we repeat neither the statements, nor the proofs here. The simplest and most notable difference is that the proper scaling of the temperature which provides a phase transition is when it is of order $n^{-2/(d-2)}$ in dimension $d \geq 5$. Moreover, our polymer measure in dimension d behaves similarly as the polymer measure of [AS17a] in dimension $d-2$. Theorem 1.8 of [AS17a] is true here also after the drop in dimension is performed.

Organization. Let us explain how the rest of the paper is organized while sketching the skeleton of our approach. The key decomposition relation (1.4) is given for arbitrary sets in (2.9) and for the range in (3.1). Section 3 makes the link between capacity and corrector. The cross term (3.1) is written as a martingale and a corrector in Proposition 3.1 following a Doob decomposition. Section 5 is the technical core of the paper. Section 5.3 transfers deviations of the capacity of the range into deviations of the corrector. The corrector itself is studied in Section 5.4 ($d = 5$) and in Section 5.5 ($d \geq 6$). Propositions 5.8 and 5.13 imply respectively Theorems 1.1 and 1.3. The path properties are addressed in Section 6. This latter section recall also the steps of the approach of [AS17a] in order to obtain information on the capacity of the region where the walk localizes. Apart from the highly interconnected steps leading to the path properties, the following sections can be read independently. In Section 2, we recall basic facts on Green's function, and on the Newtonian capacity. In Section 4, we prove the lower bounds in Theorems 1.1 and 1.3. Finally, we prove Theorem 1.8 concerning the upward deviations in Section 7.

2 Preliminaries

2.1 Further notation

For $z \in \mathbb{Z}^d$, we denote by $\mathbb{P}_z$ the law of the simple random walk starting from z, and let

$$G(z) := \mathbb{E}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S_n = z\}\right],$$

be the Green's function. It is known that there are positive constants c and C, such that in any dimension $d \geq 3$ (see [Law91]),

$$\frac{c}{\|z\|^{d-2}+1} \;\leq\; G(z) \;\leq\; \frac{C}{\|z\|^{d-2}+1}, \quad \text{for all } z \in \mathbb{Z}^d, \tag{2.1}$$

with $\|\cdot\|$ the Euclidean norm. We also consider for $T > 0$, and $z \in \mathbb{Z}^d$,

$$G_T(z) := \mathbb{E}\left[\sum_{n=0}^{T} \mathbf{1}\{S_n = z\}\right].$$

In particular for any $z \in \mathbb{Z}^d$, and $T \geq 1$,

$$\mathbb{P}(z \in \mathcal{R}_T) \leq G_T(z). \tag{2.2}$$

For $A \subset \mathbb{Z}^d$, we denote by $|A|$ the cardinality of A, and by

$$H_A := \inf\{n \geq 0 \,:\, S_n \in A\}, \quad \text{and} \quad H_A^+ := \inf\{n \geq 1 \,:\, S_n \in A\},$$

respectively the hitting time of A and the first return time to A.

We also need the following well known fact, see [Law91]. There exists a constant $C > 0$, such that for any $R > 0$ and $z \in \mathbb{Z}^d$,

$$\mathbb{P}_z\left(\inf_{k\geq 0} \|S_k\| \leq R\right) \leq C \cdot \left(\frac{R}{\|z\|}\right)^{d-2}. \tag{2.3}$$

2.2 On the capacity

The capacity of a finite subset $A \subset \mathbb{Z}^d$, with $d \geq 3$, is defined by

$$\mathrm{Cap}(A) := \lim_{\|z\| \to \infty} \frac{1}{G(z)} \mathbb{P}_z(H_A < \infty). \tag{2.4}$$

It is well known, see Proposition 2.2.1 of [Law91], that the capacity is monotone for inclusion:

$$\mathrm{Cap}(A) \leq \mathrm{Cap}(B), \quad \text{for any } A \subset B, \tag{2.5}$$

and satisfies the sub-additivity relation

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B) - \mathrm{Cap}(A \cap B), \quad \text{for all } A, B \subset \mathbb{Z}^d. \tag{2.6}$$

Another equivalent definition of the capacity is the following (see (2.12) of [Law91]).

$$\mathrm{Cap}(A) = \sum_{x \in A} \mathbb{P}_x(H_{A^+} = \infty). \tag{2.7}$$

In particular it implies that

$$\mathrm{Cap}(A) \leq |A|, \quad \text{for all } A \subset \mathbb{Z}^d. \tag{2.8}$$

The starting point for our decomposition is the definition (2.4) of the capacity in terms of a hitting time. It implies that for any two finite subsets $A, B \subset \mathbb{Z}^d$,

$$\mathrm{Cap}\,(A \cup B) = \mathrm{Cap}\,(A) + \mathrm{Cap}\,(B) - \chi_{\mathcal{C}}(A, B), \tag{2.9}$$

with

$$\chi_{\mathcal{C}}(A, B) := \lim_{z \to \infty} \frac{1}{G(z)} \mathbb{P}_z\big(\{H_A < \infty\} \cap \{H_B < \infty\}\big). \tag{2.10}$$

Note that by (2.4) and the latter formula, one has

$$0 \leq \chi_C(A, B) \leq \min(\mathrm{Cap}(A), \mathrm{Cap}(B)). \tag{2.11}$$

Now, we have shown in [ASS19b] that $\chi_{\mathcal{C}}(A, B) = \chi(A, B) + \chi(B, A) - \varepsilon(A, B)$, with

$$\chi(A, B) = \sum_{x \in A} \sum_{y \in B} \mathbb{P}_x\big(H^+_{A \cup B} = \infty\big) \cdot G(x - y) \cdot \mathbb{P}_y\big(H^+_B = \infty\big), \tag{2.12}$$

and,

$$0 \leq \varepsilon(A, B) \leq \mathrm{Cap}\,(A \cap B) \leq |A \cap B|,$$

where the last inequality follows from (2.8).

We will need some control on the speed of convergence in (1.2).

Lemma 2.1. *Assume $d \geq 5$. One has*

$$|\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] - \gamma_d n| = \mathcal{O}(\psi_d(n)), \tag{2.13}$$

with

$$\psi_d(n) = \begin{cases} \sqrt{n} & \text{if } d = 5 \\ \log n & \text{if } d = 6 \\ 1 & \text{if } d \geq 7. \end{cases} \tag{2.14}$$

Proof. By Proposition 1.2 in [ASS18], one has the rough lower bound:

$$\mathrm{Cap}(\mathcal{R}_{n+m}) \geq \mathrm{Cap}(\mathcal{R}_n) + \mathrm{Cap}(\mathcal{R}[n, n+m]) - 2\sum_{k=0}^{n}\sum_{\ell=n}^{n+m} G(S_k - S_\ell),$$

for any integers $n, m \geq 1$. Even though a better inequality is used in (2.9) below, this result together with the subadditivity relation (2.6) are enough to conclude the proof, using Hammersley's lemma and Lemma 3.2 in [ASS18], which controls the moments of the error term in the lower bound. For the details, we refer to the proof of (1.13) in [AS17b], which is entirely similar. □

The next result provides some bound on the variance of the capacity of the range.

Lemma 2.2. *There exists a constant $C > 0$, such that for any $n \geq 2$,*

$$\mathrm{var}(\mathrm{Cap}(\mathcal{R}_n)) \leq Cn(\log n)^2.$$

Proof. We use the same argument as Le Gall for the size of the range [LG86, Lemma 6.2]. Let us recall it for the sake of completeness. Write $\|\cdot\|_2$ for the square root of the variance, and for $k \geq 1$, set

$$a_k := \sup\{\|\mathrm{Cap}(\mathcal{R}_n)\|_2 \ : \ 2^k < n \leq 2^{k+1}\}.$$

Let $k \geq 2$, and $n \in (2^k, 2^{k+1}]$ be given. Set $\ell = \lfloor n/2 \rfloor$, and write

$$\mathrm{Cap}(\mathcal{R}_n) = \mathrm{Cap}(\mathcal{R}_\ell) + \mathrm{Cap}(\widetilde{\mathcal{R}}_{n-\ell}) - \chi_C(\mathcal{R}_n, \widetilde{\mathcal{R}}_{n-\ell}),$$

with $\widetilde{R}_{n-\ell} = \mathcal{R}[\ell, \dots, n]$. By using the triangle inequality, we get

$$\|\mathrm{Cap}(\mathcal{R}_n)\|_2 \leq \sqrt{\mathrm{var}(\mathrm{Cap}(\mathcal{R}_\ell)) + \mathrm{var}(\mathrm{Cap}(\widetilde{R}_{n-\ell}))} + \|\chi_C(\mathcal{R}_\ell, \widetilde{\mathcal{R}}_{n-\ell})\|_2,$$

using also independence between $\mathrm{Cap}(\mathcal{R}_\ell)$ and $\mathrm{Cap}(\widetilde{\mathcal{R}}_{n-\ell})$. Using next that $\chi_C(A, B) \leq 2\sum_{x\in A}\sum_{y\in B} G(x, y)$, by (2.12), and Lemma 3.2 in [ASS18], one obtains

$$\|\chi_C(\mathcal{R}_\ell, \widetilde{\mathcal{R}}_{n-\ell})\|_2 \leq C\sqrt{n},$$

for some constant $C > 0$. As a consequence,

$$a_k \leq \sqrt{2}a_{k-1} + C2^{k/2},$$

from which the result follows by induction. □

3 Martingale decomposition and Concentration

We give here a martingale decomposition of the capacity of the range, and deduce rough estimates on upward deviations.

3.1 Martingale Decomposition

The possibility of establishing the heuristic picture described in the introduction stems from writing the capacity of a union of sets as a sum of capacities and a cross-term. The latter though typically small is nonetheless responsible for the fluctuations. Iterating this decomposition leads to an expression of the capacity of the range as a sum of i.i.d. terms plus a martingale part, minus a corrector. The main result of this section, Proposition 3.1 below, gives an explicit expression for this corrector in terms of a sum of convoluted Green's functions taken along the trajectory and weighted by escape probability terms. Thus, the strategy is similar to the one used to treat downward deviations for the range (or its boundary) developed in [AS17a]. However the form of the corrector is quite different and its treatment is the main original part in our present analysis.

A detailed analysis of this corrector is carried out in Sections 5.4 and 5.5. Before we can state precisely the result, some preliminary steps are required.

For $I \subset \mathbb{N}$, we write $\mathcal{R}(I) := \{S_k,\ k \in I\}$, for the set of visited sites during times $k \in I$. Since for any two intervals $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I \cup J) = \mathcal{R}(I) \cup \mathcal{R}(J)$, (2.9) gives

$$\mathrm{Cap}\,(\mathcal{R}(I \cup J)) = \mathrm{Cap}\,(\mathcal{R}(I)) + \mathrm{Cap}\,(\mathcal{R}(J)) - \chi_{\mathcal{C}}\big(\mathcal{R}(I), \mathcal{R}(J)\big). \tag{3.1}$$

Next, given two sets A and B, their symmetric difference is defined as $A\Delta B := (A \cap B^c) \cup (B \cap A^c)$. Note in particular that for any $I, J \subset \mathbb{N}$, one has $\mathcal{R}(I)\Delta\mathcal{R}(J) \subset \mathcal{R}(I\Delta J)$. Moreover, it follows from (2.5), (2.6) and (2.8) that for any $A, B \subset \mathbb{Z}^d$,

$$|\mathrm{Cap}(A) - \mathrm{Cap}(B)| \le \mathrm{Cap}(A\Delta B) \le |A\Delta B|.$$

Applying this inequality to ranges on some intervals I and J, we get

$$|\mathrm{Cap}(\mathcal{R}(I)) - \mathrm{Cap}(\mathcal{R}(J))| \le |I\Delta J|. \tag{3.2}$$

Now assume that some integer T is fixed, which is carefully chosen later. Then for $j \ge 0$ and $k \ge 1$, write

$$I_{j,k} := [j + (k-1)T, j + kT], \quad \text{and} \quad \widetilde{I}_{j,k} := I_{j,1} \cup \cdots \cup I_{j,k}.$$

It follows from (3.2) that almost surely

$$|\mathrm{Cap}(\mathcal{R}_n) - \frac{1}{T}\sum_{j=0}^{T-1} \mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,[n/T]}))| \le T. \tag{3.3}$$

In particular, taking expectation also gives

$$|\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] - \frac{1}{T}\sum_{j=0}^{T-1} \mathbb{E}[\mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,[n/T]}))]| \le T. \tag{3.4}$$

On the other hand, applying (3.1) recursively we obtain for any $j = 0, \dots, T-1$,

$$\mathrm{Cap}(\mathcal{R}(\widetilde{I}_{j,[n/T]})) = \sum_{k=1}^{[n/T]} \mathrm{Cap}(\mathcal{R}(I_{j,k})) - \sum_{k=1}^{[n/T]-1} \chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,k}), \mathcal{R}(I_{j,k+1})\big). \tag{3.5}$$

We use now (3.5) to obtain a relation between centered variables. Set

$$\Sigma_{j,k} := \sum_{\ell=1}^{k} \mathrm{Cap}(\mathcal{R}(I_{j,\ell})) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}(I_{j,\ell}))],$$

and, if $\mathcal{F}_n$ is the $\sigma-$field generated by the increments of the walk up to time n,

$$M_{j,k} := \sum_{\ell=1}^{k} \chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big) - \mathbb{E}\left[\chi_{\mathcal{C}}\big(\mathcal{R}(\widetilde{I}_{j,\ell}), \mathcal{R}(I_{j,\ell+1})\big) \mid \mathcal{F}_{j+\ell T}\right].$$

Note that for any $j = 0, \dots, T-1$, the term $\Sigma_{j,k}$ is a sum of k i.i.d. random variables, while $(M_{j,k})_{k\geq 1}$ is a martingale. One can now state the main result of this Section.

Proposition 3.1. *For any $1 \leq T \leq n$, one has*

$$\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \ \geq \ \frac{1}{T}\sum_{j=0}^{T-1}(\Sigma_{j,[n/T]} - M_{j,[n/T]}) - 2\xi_n(T) + \mathcal{O}(T),$$

with

$$\xi_n(T) := \sum_{k=0}^{n}\sum_{x\in\mathcal{R}_k} \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big)\cdot \frac{G\star G_T(x-S_k)}{T}. \tag{3.6}$$

Proof. By (2.12), for any sets A and B,

$$\chi(A,B) \leq \widetilde{\chi}(A,B) := \sum_{x\in A}\sum_{y\in B} \mathbb{P}_x\big(H_A^+ = \infty\big)\cdot G(x-y)\cdot \mathbb{P}_y\big(H_B^+ = \infty\big).$$

Note that $\widetilde{\chi}$ is symmetric in the sense that $\widetilde{\chi}(A,B) = \widetilde{\chi}(B,A)$, for any A, B. Bounding the last probability term by one, we get

$$\chi(A,B) + \chi(B,A) \leq 2\overline{\chi}(A,B), \quad \text{with} \quad \overline{\chi}(A,B) := \sum_{x\in A}\sum_{y\in B}\mathbb{P}_x\big(H_A^+ = \infty\big)\cdot G(x-y).$$

Now for any j,k, the Markov property and translation invariance of the simple random walk give

$$\begin{aligned}\mathbb{E}\left[\overline{\chi}\big(\mathcal{R}(\widetilde{I}_{j,k}), \mathcal{R}(I_{j,k+1})\big) \mid \mathcal{F}_{j+kT}\right] &= \sum_{x\in\mathcal{R}(\widetilde{I}_{j,k})} \mathbb{P}_x(H^+_{\mathcal{R}(\widetilde{I}_{j,k})} = \infty)\sum_{y\in\mathbb{Z}^d} G(x-y)\cdot\mathbb{P}\big(y\in\mathcal{R}(I_{j,k+1}) \mid \mathcal{F}_{j+kT}\big) \\ &\overset{(2.2)}{\leq} \sum_{x\in\mathcal{R}(\widetilde{I}_{j,k})} \mathbb{P}_x(H^+_{\mathcal{R}(\widetilde{I}_{j,k})} = \infty)\cdot G\star G_T(x - S_{j+kT}).\end{aligned}$$

The proposition follows then from (3.3), (3.4), and (3.5). $\square$

3.2 Concentration Inequalities

We use here the results established in the previous section to deduce a rough estimate for the moderate deviations of the capacity of the range in the upward direction. We start with the case of dimension 5. For a random subset $\mathcal{R} \subseteq \mathbb{Z}^d$, we use the short-hand notation $\overline{\mathrm{Cap}}\,(\mathcal{R}) = \mathrm{Cap}\,(\mathcal{R}) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R})]$.

Proposition 3.2. *Assume $d = 5$. There exist positive constants K and c, such that for any $n \geq 1$, and any $b \geq K\cdot n^{2/3}$,*

$$\mathbb{P}\big(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) > b\big) \leq \exp(-c\cdot\frac{b^3}{n^2}).$$

Proof. The subadditivity relation (2.6), together with (3.2) and Lemma 2.1 lead to

$$\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \;\leq\; \sum_{k=1}^{[n/T]} \overline{\mathrm{Cap}}\,(\mathcal{R}[kT,(k+1)T]) + \mathcal{O}\left(T + \frac{n}{\sqrt{T}}\right).$$

We apply this inequality with $T = C(n/b)^2$, and $C > 0$ a constant such that the $\mathcal{O}(T + \frac{n}{\sqrt{T}})$ is smaller than $b/2$ (note that this is possible if one chooses the constant K large enough, in the hypothesis of the proposition). Then observe that the other part in the upper bound is a sum of $\lfloor n/T \rfloor$ terms, which are i.i.d. centered random variables bounded by T by (2.8), and whose variance are $\mathcal{O}(T\log^2 T)$, by Lemma 2.2. Therefore McDiarmid's inequality (see for instance Theorem 3.4 in [CL]) gives,

$$\mathbb{P}\big(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) > b\big) \leq \exp\left(-\frac{b^2}{8(C_0 n \log^2 T + Tb/6)}\right), \tag{3.7}$$

for some constant $C_0 > 0$. By taking larger $C > 0$ if necessary, one can ensure that in the denominator of the fraction above, the second term dominates the first one, and then the proposition follows. □

The cases of dimensions 6 and higher are treated similarly.

Proposition 3.3. *Assume $d \geq 6$. There exist constants K and c (depending on the dimension), such that for any $n \geq 1$, and any*

$$b \geq K \cdot \begin{cases} \sqrt{n \log n}, & \text{if } d = 6 \\ \sqrt{n} & \text{if } d \geq 7, \end{cases}$$

one has

$$\mathbb{P}\big(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) > b\big) \leq \exp\left(-c \cdot \frac{b^2}{n \log^2 n}\right).$$

Proof. One can argue exactly as in the case of the dimension 5, taking this time $T = C(n \log n)/b$ in dimension 6, and $T = Cn/b$ in higher dimension. □

4 Lower Bounds

4.1 Case of dimension 5

We prove here the lower bound in Theorem 1.1. In fact we show a result which holds under more general hypotheses on ζ.

Proposition 4.1. *Assume $d = 5$. There exist positive constants ε_0, C_0, and $\underline{\kappa}$, such that for any $C_0\, n^{5/7} \leq \zeta \leq \varepsilon_0\, n$, one has*

$$\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) \geq \exp(-\underline{\kappa} \cdot \zeta^{2/3}\, n^{-1/3}).$$

Proof. The proof of (2.12) in [ASS19b] reveals that for any finite $A, B \subset \mathbb{Z}^d$, one has also

$$\mathrm{Cap}(A \cup B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi_0(A, B), \tag{4.1}$$

with

$$\chi_0(A,B) := \sum_{x\in A\setminus B}\sum_{y\in B} \mathbb{P}_x(H^+_{A\cup B}=\infty) G(y-x)\mathbb{P}_y(H^+_B=\infty).$$

Now given $n\ge 1$, set $\ell = \lfloor \frac{n}{10}\rfloor$, and $m = n-\ell$. We apply (4.1) with $A=\mathcal{R}_m$ and $B=\mathcal{R}[m,n]$. Fix $\varepsilon_0>0$ (later chosen small enough), and define

$$E := \left\{\overline{\mathrm{Cap}}\,(\mathcal{R}_n)\ge -\varepsilon_0 n\right\}.$$

Using (4.1), Lemma 2.1, and Corollary 3.2, we deduce that for some constant $c>0$,

$$\begin{aligned}\mathbb{P}\left(-\varepsilon_0 n\le \overline{\mathrm{Cap}}\,(\mathcal{R}_n)\le -\zeta\right) \ \ge\ & \mathbb{P}(E,\, \chi_0(\mathcal{R}_m,\mathcal{R}[m,n])\ge 3\zeta) - \mathbb{P}(\overline{\mathrm{Cap}}\,(\mathcal{R}_m)\ge\zeta)\\ & - \mathbb{P}(\overline{\mathrm{Cap}}\,(\mathcal{R}[m,n])\ge\zeta)\\ \ge\ & \mathbb{P}(E,\, \chi_0(\mathcal{R}_m,\mathcal{R}[m,n])\ge 3\zeta) - 2\exp(-c\zeta^3/n^2).\end{aligned} \tag{4.2}$$

Note that $\zeta\ge C_0\, n^{5/7}$, is equivalent to $\zeta^3/n^2\ge C_0^{7/3}\,\zeta^{2/3}/n^{1/3}$, therefore by taking C_0 large enough, one ensures that the last term is negligible. Now, let $\rho>0$ be some small constant (to be fixed later) and consider the event

$$F:=\{\|S_k\|\le \rho\cdot n^{2/3}\,\zeta^{-1/3},\quad \text{for all } k\le n\}.$$

Note that by (2.1) and (2.7), on the event F,

$$\chi_0(\mathcal{R}_m,\mathcal{R}[m,n])\ \ge\ c_\rho\cdot\frac{\zeta}{n^2}\cdot \mathrm{Cap}(\mathcal{R}[m,n])\cdot\left(\mathrm{Cap}(\mathcal{R}_n)-\mathrm{Cap}(\mathcal{R}[m,n])\right), \tag{4.3}$$

for some constant $c_\rho>0$, going to infinity as ρ goes to zero. Furthermore, by (2.6), one has

$$\mathrm{Cap}(\mathcal{R}_n)\le \mathrm{Cap}(\mathcal{R}_m)+\mathrm{Cap}(\mathcal{R}[m,n]),$$

and thus by Lemma 2.1 and Corollary 3.2, by taking ε_0 small enough, we get for n large enough,

$$\begin{aligned}\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}[m,n])\le\gamma_5\frac{\ell}{2},\, E\right)\ &\le\ \mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_m)\ge\gamma_5(m+\ell/3)\right)\\ &\le\ \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_m)\ge\gamma_5\frac{\ell}{10}\right)\le\ e^{-c'n},\end{aligned}$$

for some constant $c'>0$, and with γ_5 as in (1.2). Similarly one has for some possibly smaller constant $c'>0$,

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n)-\mathrm{Cap}(\mathcal{R}[m,n])\le\gamma_5\frac{n}{4},\, E\right)\ \le\ \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}[m,n])\ge\gamma_5\ell\right)\le\ e^{-c'n}.$$

Then (4.3) gives

$$\begin{aligned}\mathbb{P}(\chi_0(\mathcal{R}_m,\mathcal{R}[m,n])\ge\frac{c_\rho\gamma_5^2}{100}\cdot\zeta,\, E)\ &\ge\ \mathbb{P}(\chi_0(\mathcal{R}_m,\mathcal{R}[m,n])\ge\frac{c_\rho\gamma_5^2}{100}\cdot\zeta,\, E\cap F)\\ &\ge\ \mathbb{P}(E\cap F)-2\exp(-c'n)\\ &\ge\ \mathbb{P}(F)-\mathbb{P}(E^c)-2\exp(-c'n).\end{aligned}$$

Coming back to (4.2), and choosing ρ, such that $c_\rho\ge 300/\gamma_5^2$, we deduce that

$$\begin{aligned}\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n)\le-\zeta\right)\ &=\ \mathbb{P}\left(-\varepsilon_0 n\le\overline{\mathrm{Cap}}\,(\mathcal{R}_n)\le-\zeta\right)+\mathbb{P}(E^c)\\ &\ge\ \mathbb{P}(F)-2\exp(-c'n)-2\exp(-c\,C_0^{7/3}\cdot\zeta^{2/3}n^{-1/3}).\end{aligned}$$

Moreover, it is well known that for any $\rho>0$, there exists $\kappa>0$, such that

$$\mathbb{P}(F)\ \ge\ \exp(-\kappa\cdot\zeta^{2/3}n^{-1/3}).$$

By combining the last two displays and taking C_0 large enough, we get the desired result. $\square$

4.2 Case of dimension $d \geq 6$

In this case we prove the following result:

Proposition 4.2. *Assume $d \geq 6$. There exist positive constants ε_0, C_0, and $\underline{\kappa}$ (only depending on the dimension), such that for any $C_0\,(n\log^2 n)^{\frac{d-2}{d}} \leq \zeta \leq \varepsilon_0\, n$, one has*

$$\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) \;\geq\; \exp\left(-\underline{\kappa}\cdot\zeta^{1-\frac{2}{d-2}}\right).$$

Proof. Fix $C > 0$ (later chosen large enough), and let $\ell = \lfloor C\zeta \rfloor$. Using (2.6), Lemma 2.1, and Corollary 3.2, we obtain, at least for n large enough, and c as in Corollary 3.3,

$$\begin{aligned}\mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_n) \leq -\zeta\right) &\geq \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_\ell) \leq -2\zeta\right) - \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}[\ell,n]) \geq \zeta\right)\\ &\geq \mathbb{P}\left(\overline{\mathrm{Cap}}\,(\mathcal{R}_\ell) \leq -2\zeta\right) - \exp(-c\cdot\frac{\zeta^2}{n\log^2 n}).\end{aligned}$$

Now using exactly the same argument as in the proof of Proposition 4.1 (and taking large enough C), one can see that the first term on the right-hand side is of the right order (which is of the order of the event F where the walk stays confined in a ball of radius $\zeta^{1/(d-2)}$ during the whole time ℓ), and by choosing C_0 large enough, we see that the second term is negligible, since by hypothesis, $\zeta^2/(n\log^2 n) \geq C_0^{d/(d-2)}\cdot\zeta^{1-2/(d-2)}$. This concludes the proof of the proposition. □

5 Upper Bounds

We prove here the upper bounds in Theorems 1.1 and 1.3. This is the main technical part of the paper. The proof involves a delicate analysis of the corrector term $\xi_n(T)$, which we have to cut into several pieces, corresponding to different scales and different densities.

Our main technical tools are elementary though. The first one, Lemma 5.1, bounds the sum of the function $\|x\|^{4-d}$ (resp. $\|x\|^{2-d}$) on the points of a finite set of $\mathbb{Z}^d$, weighted by the escape probabilities of the set, in terms of its diameter squared (resp. the logarithm of its diameter). The second one is an estimate on the function $\varphi_T(x)$ entering the definition of $\xi_n(T)$, which depends on whether $\|x\|$ is smaller than $\sqrt{T}$ or not. The last one bounds the sum of the function $x \mapsto \|x-z\|^{2-d}$, for an arbitrary fixed z, on any set whose density inside cubes of a certain partition of space is controlled. The proof is a simple consequence of a rearrangement inequality.

This section is organized as follows. In Section 5.1, we present and prove the technical tools that we just mentioned. In Section 5.2, we state one key technical ingredient of the proof, Lemma 5.4 taken from our previous paper [AS17a]. It provides an estimate on the size of the walk positions where the occupation times (of some neighborhood) exceeds a given threshold. Then, in Section 5.3, we show that downward deviations of the capacity of the range imply with high probability upward deviations of the corrector $\xi_n(T)$, for a well chosen T. In Section 5.4, we conclude the proof in the case $d = 5$, by cutting $\xi_n(T)$ into several pieces and analyzing each of them carefully. The main difficulty here concerns the regions, which we call high density regions, where the typical density of points in the range exceeds the typical one, called here $\overline{\rho}$, see (5.9). In Section 5.5 we perform a similar analysis in dimension 6 and higher. The analysis here is more intricate than in $d = 5$, and the main difficulty is now with regions of low density.

5.1 Basic estimates

For $r > 0$, and $x \in \mathbb{R}^d$, the discrete cube with center x and side length r, is denoted

$$Q(x,r) := [x - r/2, x + r/2)^d \cap \mathbb{Z}^d,$$

and for simplicity we write $Q(r) := Q(0,r)$.

We start with two basic inequalities, which play an important role in our analysis.

Lemma 5.1. *Assume that $d \geq 5$. There exists a constant $C_1 > 0$, such that for any $r \geq 1$, and any $\Lambda \subset Q(r)$,*

$$\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \leq C_1 r^2. \tag{5.1}$$

Furthermore, for any $1 \leq r \leq R$ and any $\Lambda \subset Q(R) \setminus Q(r)$,

$$\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-2} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \leq C_1 \log(R/r). \tag{5.2}$$

Proof. We only prove (5.1), as (5.2) is entirely similar. Without loss of generality, one can assume $r \geq 2$. For $i \geq 0$, write

$$\Lambda_i := \Lambda \cap \left(Q(r2^{-i}) \setminus Q(r2^{-i-1})\right),$$

and define $L := \lfloor \log r \rfloor$. Then, for some positive constants C_0 and C_1,

$$\begin{aligned}
\sum_{x \in \Lambda} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) &\leq \sum_{i=0}^{L} \sum_{x \in \Lambda_i} \frac{1}{\|x\|^{d-4} + 1} \cdot \mathbb{P}_x(H_\Lambda^+ = \infty) \\
&\leq \sum_{i=0}^{L} \left(\frac{2^{i+1}}{r}\right)^{d-4} \mathrm{Cap}\,(\Lambda_i) \leq \sum_{i=0}^{L} \left(\frac{2^{i+1}}{r}\right)^{d-4} \mathrm{Cap}\left(Q\left(\frac{r}{2^i}\right)\right) \\
&\leq C_0 \sum_{i=0}^{L} \left(\frac{2^{i+1}}{r}\right)^{d-4} \cdot \left(\frac{r}{2^i}\right)^{d-2} \leq C_1 r^2.
\end{aligned}$$

□

The second basic inequality we need is the following.

Lemma 5.2. *Assume $d \geq 5$. There exists a constant $C_2 > 0$, such that for any $x \in \mathbb{Z}^d$, and any $T \geq 1$,*

$$\varphi_T(x) := \frac{G \star G_T(x)}{T} \leq C_2 \cdot \min\left(\frac{1}{1 + \|x\|^{d-2}}, \frac{1}{T(1 + \|x\|^{d-4})}\right).$$

Proof. First $G_T \leq G$, so that $G \star G_T \leq G \star G$, and an elementary computation gives that $G \star G(x) \leq C_2/(1 + \|x\|^{d-4})$, for all $x \in \mathbb{Z}^d$, and some $C_2 > 0$. This already proves one of the two desired bounds.

For the other one write, by definition of G_T,

$$G \star G_T(x) = \sum_{y \in \mathbb{Z}^d} G(x-y) G_T(y) = \sum_{k=1}^{T} \mathbb{E}[G(x - S_k)]. \tag{5.3}$$

Let τ be the hitting time of the cube $Q(x,2)$ for the walk starting at 0, and note that one can assume $\|x\| \geq 4$. Since G is harmonic on $\mathbb{Z}^d \backslash \{0\}$, we have for any $k \geq 0$, $\mathbb{E}[G(x - S_{k\wedge\tau})] = G(x)$. This entails

$$G(x) = \mathbb{E}[\mathbf{1}_{\{\tau\geq k\}}G(x-S_k)] + \mathbb{E}[\mathbf{1}_{\{\tau<k\}}G(x-S_\tau)] \geq \mathbb{E}[G(x-S_k)] - \mathbb{E}[\mathbf{1}_{\{\tau<k\}}\ G(x-S_k)].$$

Now, we use that $G(x)$ is bounded by $G(0)$, so that the previous inequality gives

$$\mathbb{E}[G(x-S_k)] \leq G(x) + G(0)\mathbb{P}(\tau<\infty) \overset{(2.3)}{\leq} (1+CG(0))\cdot G(x),$$

for some constant $C>0$. Injecting this in (5.3) and using (2.1), proves the second inequality. $\square$

Our last estimates are gathered in the following lemma. For a (deterministic) function $S : \mathbb{N} \to \mathbb{Z}^d$ (not necessarily to the nearest neighbor), and for any $\mathcal{K} \subset \mathbb{N}$, we define for any $\Lambda \subset \mathbb{Z}^d$,

$$\ell_{\mathcal{K}}(\Lambda) := \sum_{k\in\mathcal{K}} \mathbf{1}(S(k)\in\Lambda).$$

Lemma 5.3. *Assume $d \geq 3$. Let $S : \mathbb{N} \to \mathbb{Z}^d$, and $\mathcal{K} \subset \mathbb{N}$, be such that for some $\rho \in (0,1)$ and $r \geq 1$,*

$$\ell_{\mathcal{K}}(Q(x,r)) \leq \rho|Q(r)|, \qquad \textit{for all } x \in r\mathbb{Z}^d.$$

There exists a constant $C_3 > 0$ (independent of ρ, r, S, and $\mathcal{K}$), such that for any $R \geq 2r$, and any $z \in \mathbb{Z}^d$,

$$\sum_{k\in\mathcal{K}} \frac{\mathbf{1}(2r \leq \|S(k)-z\| \leq R)}{\|S(k)-z\|^{d-2}} \ \leq\ C_3\,\rho\,R^2, \tag{5.4}$$

and

$$\sum_{k\in\mathcal{K}} \frac{\mathbf{1}(\|S(k)-z\| \geq 2r)}{\|S(k)-z\|^{d-2}} \ \leq\ C_3\,\rho^{1-\frac{2}{d}}\,|\mathcal{K}|^{2/d}. \tag{5.5}$$

Proof. We start with the proof of (5.4). Consider a covering of the cube $Q(z,R)$ by a partition made of smaller cubes homeomorphic to $Q(r)$, with centers in the set $z+r\mathbb{Z}^d$. For each $x \in z+r\mathbb{Z}^d$, with $x \neq z$, the contribution of the points $S(k)$ lying in $Q(x,r)$ to the sum we need to bound, is upper bounded (up to some constant) by $\rho r^d \cdot \|x-z\|^{2-d}$. The result follows as we observe that $\sum_{x\in z+r\mathbb{Z}^d} \|z-x\|^{2-d}\mathbf{1}(r \leq \|z-x\| \leq R) \leq CR^2/r^d$, for some constant $C>0$.

The second inequality (5.5) follows as well, once we observe that by rearranging the points $(S(k))_{k\in\mathcal{K}}$, one can increase the second sum (at least up to a multiplicative constant) by assuming they are all in $Q(z, 2(\frac{|\mathcal{K}|}{\rho})^{1/d})$, and still satisfy the hypothesis of the lemma. $\square$

5.2 The sets $\mathcal{K}_n$ and $\mathcal{A}_n$

We recall now our main technical estimate for downward deviations, which we proved in [AS17a] (it is the lemma 2.4 from this reference), and then derive some useful corollary.

For $n \geq 0$, and $\Lambda \subset \mathbb{Z}^d$, define the time spent in Λ by the walk up to time n as

$$\ell_n(\Lambda) := \sum_{k=0}^{n} \mathbf{1}(S_k \in \Lambda).$$

Then given $\rho > 0$, $r \geq 1$, and $n \geq 1$, we set

$$\mathcal{K}_n(r,\rho) := \{k \leq n :\ \ell_n(Q(S_k,r)) \geq \rho|Q(r)|\}. \tag{5.6}$$

Lemma 5.4 ([AS17a]). *Assume that $d \geq 3$. There exist positive constants C and κ, such that for any $n \geq 2$, $\rho > 0$, $r \geq 1$, and $L \geq 1$, satisfying*

$$\left(1 + \frac{L}{\rho r^d}\right)^{2/d} \cdot \log n \;\leq\; \kappa\, \rho\, r^{d-2}, \tag{5.7}$$

one has

$$\mathbb{P}\big(|\mathcal{K}_n(r,\rho)| \geq L\big) \;\leq\; C \exp\left(-\kappa\, \rho^{\frac{2}{d}}\, L^{1-\frac{2}{d}}\right).$$

Note that the parameter ρ represents an occupation density. Under the event of moderate deviations considered here (when the capacity of the range up to time n is reduced by an amount ζ from its mean value), the walk folds its trajectory during a typical time τ in a region of space of typical diameter R with

$$R^{d-2} := \frac{\tau^2}{\zeta}, \quad \text{and} \quad \tau := \begin{cases} n & \text{if } d = 5 \\ \zeta & \text{if } d \geq 6. \end{cases} \tag{5.8}$$

In the *folding region*, its occupation density is thus typically of order $\overline{\rho} := \tau / R^d$, also given in terms of ζ and n by

$$\overline{\rho} = \begin{cases} \zeta^{5/3}\, n^{-7/3} & \text{if } d = 5 \\ \zeta^{-\frac{2}{d-2}} & \text{if } d \geq 6. \end{cases} \tag{5.9}$$

Set $J := \log_2(\frac{n}{\overline{\rho}})$. Then for $r \geq 1$, $\alpha > 0$, $i \in \mathbb{Z}$ with $i \leq J$, and $1 \leq \zeta \leq n$, define

$$\mathcal{A}_n(r,i,\alpha) := \bigcap_{j=i}^{J} \left\{ |\mathcal{K}_n(r, 2^j \overline{\rho})| \leq \frac{\alpha\, \tau}{2^{\frac{2j}{d-2}}} \right\}.$$

As a consequence of Lemma 5.4 we get the following.

Corollary 5.5. *Assume $d \geq 5$. There exists positive constants C and κ (depending only on the dimension), such that the following holds. For any $\alpha > 0$, there exists $K = K(\alpha)$, such that for any $i \geq 0$, any $n \geq 2$, and any $\zeta \leq n$ and $r \geq 1$ satisfying respectively*

$$\zeta \geq \begin{cases} \sqrt{n} & \textit{if } d = 5 \\ (\log n)^3 & \textit{if } d \geq 6, \end{cases} \tag{5.10}$$

and

$$r \geq K\, 2^{\frac{i}{d-2}} \cdot (\log n) \cdot \begin{cases} (n^{11} \zeta^{-7})^{\frac{1}{15}} & \textit{if } d = 5 \\ \zeta^{\frac{4}{d(d-2)}} & \textit{if } d \geq 6, \end{cases} \tag{5.11}$$

one has

$$\mathbb{P}\big(\mathcal{A}_n(r,-i,\alpha)^c\big) \leq C\,(J+i) \cdot \begin{cases} \exp\left(-\kappa\, \alpha^{3/5} \cdot (\frac{\zeta^2}{n})^{1/3}\right) & \textit{if } d = 5, \\ \exp\left(-\kappa\, \alpha^{1-\frac{2}{d}} \cdot \zeta^{1-\frac{2}{d-2}}\right) & \textit{if } d \geq 6. \end{cases}$$

Proof. First note that one can always assume that n is large enough, since small values of n can be ruled out by taking C large enough. Now one has to verify that the hypothesis of Lemma 5.4 is satisfied for each of the sets entering the definition of $\mathcal{A}_n$. The worst case is actually when $j = -i$, so one just needs to check that

$$\left(1 + \frac{\alpha \tau 2^{i\frac{d}{d-2}}}{\overline{\rho} r^d}\right)^{2/d} \log n \leq \kappa 2^{-i} \overline{\rho} r^{d-2}. \tag{5.12}$$

We consider two cases: either $\tau 2^{i\frac{d}{d-2}} \leq \overline{\rho} r^d$ holds, in which case (5.12) is implied by

$$r^{d-2} \geq K(\alpha)^{d-2} \cdot \frac{2^i}{\overline{\rho}} \cdot \log n, \tag{5.13}$$

or it holds instead $\tau 2^{i\frac{d}{d-2}} \geq \overline{\rho} r^d$, and then (5.12) is weaker than

$$r^d \geq K(\alpha)^d \cdot \frac{\tau^{\frac{2}{d}} 2^{i\frac{d}{d-2}}}{\overline{\rho}^{1+\frac{2}{d}}} \cdot \log n, \tag{5.14}$$

for some well chosen constant $K(\alpha)$. Assume first that $d = 5$, in which case we recall that $\tau = n$, and we observe that (5.14) is exactly (5.11) (up to the power of the logarithm). Furthermore, when $\zeta \geq \sqrt{n}$, one can check that (5.14) is stronger than (5.13) (again up to the power of the logarithm), so one can forget the latter condition.

Assume now that $d \geq 6$. If $r \geq (2^i\zeta)^{1/(d-2)}$ (which is equivalent to $\tau 2^{i\frac{d}{d-2}} \leq \overline{\rho} r^d$), and $\zeta \geq (\log n)^3$, then (5.13) is automatically satisfied (at least provided n is large enough). If on the other hand $r \leq (2^i\zeta)^{1/(d-2)}$, then one can simply observe that (5.14) is exactly (5.11) up to the power of the logarithm, by definition of $\overline{\rho}$, see (5.9).

Finally one can observe that the upper bound given by Lemma 5.4 for the probability of the events entering the definition of $\mathcal{A}_n$ is independent of r and j, and thus the corollary follows by a union bound. □

5.3 Transferring the deviations to the corrector

We now set the value of T in the rest of the whole section as a function of R, τ in (5.8) and of some small constants $\{\beta_d,\ d \geq 5\}$ fixed just after Proposition 5.6 below.

$$T := 2 + \lfloor T_0 \rfloor, \quad \text{with} \quad T_0 := \beta_d R^2 \cdot \frac{\zeta^2}{\tau n} = \begin{cases} \beta_5 \cdot \zeta^{4/3} n^{-2/3} & \text{if } d = 5 \\ \beta_d \cdot \frac{\zeta^{\frac{d}{d-2}}}{n} & \text{if } d \geq 6. \end{cases} \tag{5.15}$$

This definition of T is motivated by the fact that it is the largest possible value that makes the contribution of the martingale part negligible, as shown in Proposition 5.6.

Our concern is to show that downward moderate deviations of the capacity imply with high probability upward deviations of the corrector $\xi_n(T)$, which has been defined in (3.6), by roughly the same amount. A precise statement is given in Proposition 5.6 below, which is actually slightly stronger than what we really need, but we state the result in this form, since it might be interesting in itself and for further analysis of downward deviations.

We will then later interpret the upward deviations of $\xi_n(T)$ in terms of the sets $\mathcal{A}_n$ which were introduced in the previous subsection, see Propositions 5.8 and 5.13 below.

Proposition 5.6. *Assume $d \geq 5$. For any $\delta \in (0,1)$, there exists $\beta_d > 0$, such that*

$$\lim_{n\to\infty} \mathbb{P}\big(\xi_n(T) > (1-\delta)\zeta \mid \mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \leq -\zeta\big) = 1,$$

with T as in (5.15), and where the convergence holds uniformly in ζ satisfying the hypotheses of Propositions 4.1 and 4.2 (respectively for $d = 5$ and $d \geq 6$).

Proof. Using (3.1), we get that for T as in (5.15),

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n)] \le -\zeta\right) \le \mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1}(\Sigma_{j,T} - M_{j,T}) \le -\delta\zeta\right) + \mathbb{P}\left(\xi_n(T) \ge (1-\delta)\zeta\right),$$

and it just remains to see that the first term on the right-hand side is negligible with respect to the probability on the left-hand side.

For the sum of the terms $\Sigma_{j,T}$, we can apply McDiarmid's inequality (see Theorem 6.1 in [CL]), since each $\Sigma_{j,T}$ is a sum of i.i.d. terms, which are almost surely bounded by T and whose variance are $\mathcal{O}(T(\log T)^2)$, by Lemma 2.2. It shows that, for n large enough, and for some constant $c > 0$ (only depending on δ),

$$\mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1}\Sigma_{j,T} \le -(\delta/2)\zeta\right) \le T\exp\left(-c\cdot\frac{\zeta^2}{n(\log T)^2 + \zeta T}\right) \le \begin{cases} \exp(-2\underline{\kappa}\cdot\zeta^{2/3}n^{-1/3}) & \text{if } d=5\\ \exp(-2\underline{\kappa}\cdot\zeta^{1-\frac{2}{d-2}}) & \text{if } d\ge 6,\end{cases}$$

with $\underline{\kappa}$ as in Propositions 4.1 and 4.2 respectively for $d = 5$ and $d \ge 6$, and taking β_d in (5.15) small enough.

For the martingale part on the other hand, we do not have at our disposal a better bound than T^2 for the conditional variance of its increments. More precisely we only know that the increments are bounded by T almost surely, as it follows from (2.11) and (2.8). Then Azuma's inequality, shows that for n large enough,

$$\mathbb{P}\left(\frac{1}{T}\sum_{j=0}^{T-1}M_{j,T} \ge (\delta/2)\zeta\right) \le T\exp(-c\cdot\frac{\zeta^2}{nT}) \le \begin{cases} \exp(-2\underline{\kappa}\cdot\zeta^{2/3}n^{-1/3}) & \text{if } d=5\\ \exp(-2\underline{\kappa}\cdot\zeta^{1-\frac{2}{d-2}}) & \text{if } d\ge 6,\end{cases}$$

taking again β_d small enough for the second inequality. We conclude the proof using the lower bounds given in Propositions 4.1 and 4.2. □

Remark 5.7. The fact that we do not have better estimates for the conditional variance of the increments of the martingales $M_{j,T}$ is responsible for our choice of T. Better estimates, such as a $\mathcal{O}(T(\log T)^2)$, would allow to take a larger T, which in turn would lead to an extension of our result to lower values of ζ (in fact one would reach, at least in high dimension, the optimal bound $\zeta \ge n^{1-2/d}$, up to a logarithmic factor).

We now fix in the rest of this section the values of $(\beta_d)_{d\ge 5}$ as those given by the previous proposition, with say $\delta = 1/2$. Without loss of generality one will also assume that $\beta_d \le 1$.

We then have to relate the upward deviations of $\xi_n(T)$ in terms of the sets $\mathcal{A}_n$, in order to apply Corollary 5.5. Since the arguments are quite different in dimension 5 on one hand and in dimension 7 and higher on the other hand, we treat these two cases in two separate subsections. The case of dimension 6 is in a sense critical, and could be treated in both subsections, but we chose to include it in the latter.

5.4 The case of dimension five

We assume here that $d = 5$ and prove the upper bound in Theorem 1.1. We start with the following result. Recall the definition (5.15) of T.

Proposition 5.8. *There exist $i_0 \geq 0$, and $\alpha_0 > 0$, such that for any $1 \leq \zeta \leq n$, one has*

$$\mathcal{A}_n(\alpha_0\sqrt{T}, -i_0, \alpha_0) \;\subseteq\; \{\xi_n(T) \leq \zeta\}.$$

Proof of the upper bound in Theorem 1.1. Let i_0 and α_0 be as in Proposition 5.8, and let $K = K(\alpha_0)$ be the constant appearing in Corollary 5.5. A simple computation shows that the condition (5.11), with $r = \alpha_0\sqrt{T}$ and $i = i_0$, is equivalent to $\zeta \geq (K2^{i_0/3}(\log n)/\alpha_0)^{15/17} n^{16/17}$, which is satisfied, at least for n large enough, under the hypotheses of the theorem. Thus the desired upper bound follows from Corollary 5.5, and Propositions 5.6 and 5.8. $\square$

The rest of this section is devoted to the proof of Proposition 5.8. Given $r \geq 1$, we write $\xi_n(T)$ as the sum of two terms: $\xi_n(T) := \xi_{r,-} + \xi_{r,+}$, where

$$\xi_{r,-} := \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \mathbf{1}(\|x - S_k\| \leq r)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big),$$

and

$$\xi_{r,+} := \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \mathbf{1}(\|x - S_k\| > r)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big).$$

Proposition 5.8 is then a consequence of the two following lemmas.

Lemma 5.9. *There exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, any $1 \leq \zeta \leq n$, and any $1 \leq r \leq \alpha_1\sqrt{T}$, one has*

$$\mathcal{A}_n(r, 0, \alpha) \;\subseteq\; \{\xi_{r,-} \leq \frac{1}{2}\zeta\}.$$

Lemma 5.10. *There exist $i_0 \geq 0$, and $\alpha_2 > 0$, such that for any $\alpha \in (0, \alpha_2)$, any $1 \leq \zeta \leq n$, and any $r \geq 1$, one has*

$$\mathcal{A}_n(r, -i_0, \alpha) \;\subseteq\; \{\xi_{r,+} \leq \frac{1}{2}\zeta\}.$$

Remark 5.11. It is interesting to notice that the proof of Lemma 5.10 is independent of the choice of T. It is only in Lemma 5.9 that this choice is important, and determines the maximal value of the parameter r that one can afford (namely $\sqrt{T}$, up to constant). In turn this is responsible for the limitation of our result regarding the range of moderate deviations that we cover.

Proof of Proposition 5.8. It suffices to apply the two previous lemmas with any $\alpha < \min(\alpha_1, \alpha_2)$, and $r = \alpha\sqrt{T}$, since $\mathcal{A}_n(r, -i_0, \alpha) \subseteq \mathcal{A}_n(r, 0, \alpha)$, for any $r \geq 1$ and $\alpha > 0$, by definition. $\square$

It remains to prove Lemmas 5.9 and 5.10.

Proof of Lemma 5.9. We further decompose $\xi_{r,-}$ as a sum of three terms. Let $s := \min(r, c_s\sqrt{\zeta T/n})$, with c_s a small positive constant to be fixed later, and assume first that $s \geq 1$. Then, write $\xi_{r,-} := \xi_{s,-} + \xi^{\text{high}}_{s,r} + \xi^{\text{low}}_{s,r}$, where

$$\xi_{s,-} := \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \mathbf{1}(\|x - S_k\| \leq s)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big),$$

and for some $j \geq 0$ to be fixed later,

$$\xi^{\text{high}}_{s,r} = \sum_{k \in \mathcal{K}_n(r, 2^j\overline{\rho})} \sum_{x \in \mathcal{R}_k} \mathbf{1}(s < \|x - S_k\| \leq r)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big),$$

$$\xi_{s,r}^{\mathrm{low}} = \sum_{k \notin \mathcal{K}_n(r, 2^j \overline{\rho})} \sum_{x \in \mathcal{R}_k} \mathbf{1}(s < \|x - S_k\| \le r)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k} = \infty\big).$$

Using (5.1) and Lemma 5.2, we get

$$\xi_{s,-} \le \frac{C_2}{T} \cdot \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k \cap Q(S_k, s)} \frac{1}{1 + \|x - S_k\|} \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k \cap Q(S_k,s)} = \infty\big) \le C_1 C_2\,(n+1)\,\frac{s^2}{T} \le \frac{\zeta}{6}, \quad (5.16)$$

by taking the constant c_s in the definition of s small enough for the last inequality (for instance such that $c_s^2 = 1/(12 C_1 C_2)$).

We next estimate the term $\xi_{s,r}^{\mathrm{high}}$. Note that for any $\alpha > 0$ and $j \ge 0$, one has

$$\mathcal{A}_n(r, 0, \alpha) \ \subseteq\ \{|\mathcal{K}_n(r, 2^j \overline{\rho})| \le 2^{-2j/3} \alpha n\},$$

(observe in particular that for $j > J$, the set $\mathcal{K}_n(r, 2^j \overline{\rho})$ is empty). Therefore using (5.1) and Lemma 5.2 we have on the event $\mathcal{A}_n(r, 0, \alpha)$, for any $j \ge 0$,

$$\xi_{s,r}^{\mathrm{high}} \ \le\ \frac{C_2}{T} \sum_{k \in \mathcal{K}_n(r, 2^j \overline{\rho})} \sum_{x \in \mathcal{R}_k \cap Q(S_k, r)} \frac{1}{1 + \|x - S_k\|} \cdot \mathbb{P}_x\big(H^+_{\mathcal{R}_k \cap Q(S_k, r)} = \infty\big) \ \le\ C_1 C_2 \cdot \frac{\alpha r^2 n}{2^{2j/3} T}.$$

We now choose for j the smallest positive integer such that $2^{2j/3} \ge n/\zeta$, and we obtain that on $\mathcal{A}_n(r, 0, \alpha)$, with α small enough, and $r \le \sqrt{T}$,

$$\xi_{s,r}^{\mathrm{high}} \le \frac{1}{6}\zeta. \quad (5.17)$$

Finally, we estimate $\xi_{s,r}^{\mathrm{low}}$. We assume that $s < r$, as otherwise this sum is zero by definition. Bounding the probability terms by 1 and the cardinality of the set $\mathcal{K}_n(r, 2^j \overline{\rho})^c$ by $n+1$, we obtain using Lemma 5.2, that for some constant $C > 0$,

$$\begin{aligned} \xi_{s,r}^{\mathrm{low}} \ &\le\ \frac{C_2}{T} \sum_{k \notin \mathcal{K}_n(r, 2^j \overline{\rho})} \sum_{x \in \mathcal{R}_k \cap Q(S_k, r)} \frac{\mathbf{1}(\|x - S_k\| \ge s)}{1 + \|x - S_k\|} \\ &\le\ \frac{C_2}{T} \sum_{k \notin \mathcal{K}_n(r, 2^j \overline{\rho})} \frac{|\mathcal{R}_k \cap Q(S_k, r)|}{1 + s} \ \le\ C\, \frac{n 2^j \overline{\rho} \cdot r^5}{T s} \ \le\ C\, \frac{n^{2/3}}{\zeta^{1/3}} \cdot \frac{r^5}{T^{3/2}}. \end{aligned}$$

Taking next $r \le \alpha\sqrt{T}$, with α small enough, gives

$$\xi_{s,r}^{\mathrm{low}} \le C\alpha^5 \cdot \frac{n^{2/3}}{\zeta^{1/3}} \cdot T \le \frac{1}{6}\zeta.$$

Combining this with (5.16), and (5.17) concludes the proof of the lemma, in the case $s \ge 1$.

When $s < 1$, one can just write $\xi_{r,-}$ as the sum of two terms $\xi_{r,-}^{\mathrm{low}}$ and $\xi_{r,-}^{\mathrm{high}}$ defined respectively as $\xi_{s,r}^{\mathrm{low}}$ and $\xi_{s,r}^{\mathrm{high}}$, except that we remove the condition $\{\|x - S_k\| > s\}$ in the indicator functions, and the rest of the proof applies mutatis mutandis. □

Proof of Lemma 5.10. It will be convenient here to write the expression of $\xi_{r,+}$ in a more symmetric way. One has using Lemma 5.2,

$$\xi_{r,+} \le C_2 \cdot \sum_{k=0}^{n} \sum_{k'=0}^{n} \frac{\mathbf{1}(\|S_k - S_{k'}\| \ge r)}{\|S_k - S_{k'}\|^3}.$$

Now given some $i_0 \geq 0$ (which will be fixed later), we get using symmetry that $\xi_{r,+} \leq C_2(2\xi_{r,+}^{\mathrm{low}} + \xi_{r,+}^{\mathrm{high}})$, where

$$\xi_{r,+}^{\mathrm{low}} := \sum_{k=0}^{n} \sum_{k' \notin \mathcal{K}_n(r, 2^{10-i_0}\overline{\rho})} \frac{\mathbf{1}(\|S_k - S_{k'}\| \geq r)}{\|S_k - S_{k'}\|^3},$$

and

$$\xi_{r,+}^{\mathrm{high}} := \sum_{k,k' \in \mathcal{K}_n(r, 2^{10-i_0}\overline{\rho})} \frac{\mathbf{1}(\|S_k - S_{k'}\| \geq r)}{\|S_k - S_{k'}\|^3}.$$

We first treat the term $\xi_{r,+}^{\mathrm{low}}$. Fix $k \leq n$, and set $r' := r/2$. Consider a subdivision of space into cubes with centers in $r'\mathbb{Z}^5$, and of side length r'. Note that if $S_{k'} \in Q(x, r')$, for some $x \in r'\mathbb{Z}^5$ and $k' \notin \mathcal{K}_n(r, 2^{10-i_0}\overline{\rho})$, then $Q(x, r') \subseteq Q(S_{k'}, r)$ and therefore

$$\ell_n(Q(x, r')) \leq \ell_n(Q(S_{k'}, r)) \leq 2^{10-i_0}\overline{\rho}|Q(r)|.$$

Thus one can apply (5.5), with $\mathcal{K} = \mathcal{K}_n(r, 2^{10-i_0}\overline{\rho})^c$, and this shows that for any $k \leq n$,

$$\sum_{k' \notin \mathcal{K}_n(r, 2^{10-i_0}\overline{\rho})} \frac{\mathbf{1}(\|S_k - S_{k'}\| \geq r)}{\|S_k - S_{k'}\|^3} \;\leq\; C_3 \cdot \frac{2^6}{2^{3i_0/5}}\, \overline{\rho}^{3/5} n^{2/5}.$$

Then recalling the definition (5.9) of $\overline{\rho}$, this shows that by taking i_0 large enough, one has

$$\xi_{r,+}^{\mathrm{low}} \leq \frac{\zeta}{4}. \tag{5.18}$$

We deal now with the contribution of *high densities*. We start with dividing space into regions of distinct densities. We fix the value of i_0 as above, and we define for $j \geq -i_0 + 10$,

$$\mathcal{C}_j := \{x \in r\mathbb{Z}^5 : \; 1 \leq \frac{\ell_n(Q(x, 2r))}{2^j \overline{\rho}|Q(r)|} < 2\}.$$

Note that $\mathcal{C}_j$ is empty for $j > J$, by definition of J. We divide each box $Q(x, 2r)$, with $x \in \mathcal{C}_j$, into 2^{10} disjoint sub-boxes of side-length $r' = r/2$. The pigeonhole principle tells us that one of them, say Q_x, is visited at least $2^{j-10}\overline{\rho}|Q(r)|$ times. Furthermore, since Q_x is a cube of side-length r', one has $Q_x \subseteq Q(S_k, r)$, for any $k \leq n$, such that $S_k \in Q_x$, so that for any such k, it holds $k \in \mathcal{K}_n(r, 2^{j-10}\overline{\rho})$. Using also that Q_x belongs to exactly 2^5 boxes $Q(x', 2r)$, with $x' \in r\mathbb{Z}^5$, it follows that

$$|\mathcal{C}_j| \cdot 2^{j-15}\overline{\rho} \cdot |Q(r)| \leq |\mathcal{K}_n(r, 2^{j-10}\overline{\rho})|. \tag{5.19}$$

Therefore, on the event $\mathcal{A}_n(r, -i_0, \alpha)$, one has for any $-i_0 + 10 \leq j \leq J$,

$$|\mathcal{C}_j| \leq \frac{2^{25}\alpha n}{2^{5j/3}\overline{\rho}|Q(r)|}. \tag{5.20}$$

On the other hand, for any $j \geq -i_0 + 10$, applying again the pigeonhole principle gives that for any $k \in \mathcal{K}_n(r, 2^j\overline{\rho})$, one has $S_k \in Q(x, 2r)$, for some $x \in \mathcal{C}_{j'}$, with $j' \geq j$. Therefore, for some constant $C > 0$,

$$\begin{aligned}
\xi_{r,+}^{\mathrm{high}} &\leq C \cdot \sum_{\substack{-i_0+10 \leq i \leq J \\ -i_0+10 \leq j \leq i}} \sum_{\substack{x \in \mathcal{C}_i \\ y \in \mathcal{C}_j}} \ell_n(Q(x, 2r))\ell_n(Q(y, 2r)) \frac{\mathbf{1}(\|x - y\| \geq r)}{\|x - y\|^3} \\
&\leq C \cdot \sum_{\substack{-i_0+10 \leq i \leq J \\ -i_0+10 \leq j \leq i}} 2^{i+j+2}(\overline{\rho}|Q(r)|)^2 \sum_{\substack{x \in \mathcal{C}_i \\ y \in \mathcal{C}_j \setminus \{x\}}} \frac{1}{\|x - y\|^3}.
\end{aligned} \tag{5.21}$$

We now use spherical rearrangement to obtain that there is a constant C', such that for any $j \le i$, and any $x \in \mathcal{C}_i$,

$$\sum_{y \in \mathcal{C}_j \setminus \{x\}} \frac{1}{\|x-y\|^3} \ \le\ C' \cdot \frac{|\mathcal{C}_j|^{2/5}}{|Q(r)|^{3/5}}.$$

Thus, for any $-i_0+10 \le j \le i$, on the event $\mathcal{A}_n(r, -i_0, \alpha)$, using (5.20),

$$\sum_{x \in \mathcal{C}_i} \sum_{y \in \mathcal{C}_j \setminus \{x\}} \frac{1}{\|x-y\|^3} \ \le\ \frac{C'}{|Q(r)|^2} 2^{-\frac{5}{3}i - \frac{2}{3}j} \left(\frac{\alpha n}{\overline{\rho}}\right)^{1+\frac{2}{5}}, \tag{5.22}$$

for some possibly different constant $C' > 0$. Thus, using (5.21) and (5.22), we obtain on $\mathcal{A}_n(r, -i_0, \alpha)$,

$$\xi_{r,+}^{\text{high}} \le\ C''\,(\alpha n)^{7/5} \cdot \overline{\rho}^{3/5} \sum_{\substack{-i_0+10 \le i \le J \\ -i_0+10 \le j \le i}} 2^{-\frac{2}{3}i + \frac{1}{3}j} \le\ C''\,(\alpha n)^{7/5} \cdot \overline{\rho}^{3/5} 2^{i_0/3},$$

for some constant $C'' > 0$. By taking α small enough in the last inequality, namely such that $C''\,2^{i_0/3}\alpha^{7/5} \le 1/4$, this shows that $\xi_{r,+}^{\text{high}} \le \zeta/4$. Together with (5.18), this completes the proof. □

Remark 5.12. Note that the last series written while keeping the dimension as a parameter would read after summing over j,

$$\sum_{-i_0+10 \le i \le J} 2^{i(d-6)/(d-2)}.$$

In dimension 6, this sum is of order $\log(n/\zeta)$, which explains the logarithmic correction that we have in this case in Theorem 1.3. The same problem appears with the proof in the next section confirming that $d = 6$ is indeed critical.

5.5 Dimension six and larger

We prove here the upper bound in Theorem 1.3. As in dimension 5, we will see that it follows from Corollary 5.5, Proposition 5.6, and the following counterpart of Proposition 5.8 when $d \ge 6$. Before we state it, recall (see the heuristic part of the introduction), that now the walk typically folds its trajectory a time ζ in a region of typical diameter R, whose value is

$$R = R(\zeta) := \zeta^{\frac{1}{d-2}}. \tag{5.23}$$

Recall also that $T = \beta_d(\zeta/n)R^2$. In particular, since we assumed $\beta_d \le 1$, one has $T \le R^2$, when $\zeta \le n$.

Proposition 5.13. *Assume $d \ge 6$. There exist positive constants α_0, c_1, and c_2 (only depending on the dimension), such that for any $1 \le \zeta \le n$, one has when $d \ge 7$,*

$$\mathcal{A}_n(\alpha_0 \overline{r}, -i_1, \alpha_0) \cap \mathcal{A}_n(R, -i_2, \alpha_0) \ \subseteq\ \{\xi_n(T) \le \zeta\},$$

with T and R as defined in (5.15) *and* (5.23) *respectively,*

$$\overline{r} := (\zeta/n)^{\frac{d-4}{2d}} \sqrt{T}, \tag{5.24}$$

and i_1 and i_2 the smallest integers satisfying respectively

$$2^{i_1} \ge c_1 \cdot \frac{n}{\zeta}, \quad \textit{and} \quad 2^{i_2} \ge c_2 \cdot (\frac{n}{\zeta})^{\frac{d+2}{d-2}}.$$

When $d = 6$, it holds (with the same notation),

$$\mathcal{A}_n(\alpha_0 \overline{r}, -i_1, \alpha_0) \cap \mathcal{A}_n(R, -i_2, \frac{\alpha_0}{\log(2n/\zeta)}) \ \subseteq\ \{\xi_n(T) \le \zeta\}.$$

Proof of the upper bound in Theorem 1.3. As in dimension 5, one just need to verify that the hypotheses of Corollary 5.5 are satisfied for the parameters given in Proposition 5.13. In other words one needs, with the notation of Proposition 5.13,

$$\overline{r} \geq K \cdot 2^{\frac{i_1}{d-2}} \log n \cdot \zeta^{\frac{4}{d(d-2)}}, \tag{5.25}$$

and

$$R \geq K \cdot 2^{\frac{i_2}{d-2}} \log n \cdot \zeta^{\frac{4}{d(d-2)}}, \tag{5.26}$$

with $K = K(\alpha_0)$, the constant from Corollary 5.5. Condition (5.25) is satisfied when

$$\zeta \geq K n^{\chi_d} \log n, \quad \text{with} \quad \chi_d = \frac{d^2 - 3d + 4}{d^2 - 2d},$$

and a possibly larger constant K. Note that $\frac{d-1}{d} < \chi_d < 1$, for any $d \geq 1$.

On the other hand, Condition (5.26) is satisfied when

$$\zeta^{\frac{1}{d-2} + \frac{d+2}{(d-2)^2} - \frac{4}{d(d-2)}} \geq K n^{\frac{d+2}{(d-2)^2}} \log n,$$

(with a possibly larger K), which is itself weaker than requiring (at least for n large enough),

$$\zeta \geq n^{\chi_d'} (\log n)^d, \quad \text{with} \quad \chi_d' := \frac{d(d+2)}{2(d^2 - 2d + 4)}.$$

We note that $\chi_d' \leq \frac{d-2}{d}$, for $d \geq 9$, and that when $d = 6, 7, 8$, one also has $\chi_d' < \chi_d$, so that (5.26) is weaker (at least for n large enough) than (5.25) in any dimension $d \geq 6$. This concludes the proof of Theorem 1.3. $\square$

The rest of this subsection is devoted to the proof of Proposition 5.13.

Given any $1 \leq r \leq R$, we write now $\xi_n(T)$ as the sum of three terms $\xi_n(T) := \xi_{r,-} + \xi_{r,R} + \xi_{R,+}$, with $\xi_{r,-}$ as defined in dimension 5,

$$\xi_{r,R} := \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \mathbf{1}(r < \|x - S_k\| \leq R)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H_{\mathcal{R}_k}^+ = \infty\big),$$

and

$$\xi_{R,+} := \sum_{k=0}^{n} \sum_{x \in \mathcal{R}_k} \mathbf{1}(\|x - S_k\| > R)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H_{\mathcal{R}_k}^+ = \infty\big).$$

Then Proposition 5.13 is a consequence of the following lemmas.

Lemma 5.14. *There exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, any $1 \leq \zeta \leq n$, and any $1 \leq r \leq \alpha_1 \overline{r}$,*

$$\mathcal{A}_n(r, 0, \alpha) \ \subseteq\ \{\xi_{r,-} \leq \frac{1}{3}\zeta\}.$$

Lemma 5.15. *There exists $\alpha_2 > 0$, and $c_1 > 0$, such that for any $\alpha \in (0, \alpha_2)$, any $1 \leq \zeta \leq n$, and any $1 \leq r \leq R$, one has*

$$\mathcal{A}_n(r, -i_1, \alpha) \ \subseteq\ \{\xi_{r,R} \leq \frac{1}{3}\zeta\},$$

with $i_1 = \lfloor c_1 + \log_2(n/\zeta) \rfloor$.

Lemma 5.16. *There exists $\alpha_3 > 0$, and $c_2 > 0$, such that for any $\alpha \in (0, \alpha_3)$, and any $1 \le \zeta \le n$, one has*

$$\{\xi_{R,+} \le \frac{1}{3}\zeta\} \;\supseteq\; \begin{cases} \mathcal{A}_n(R, -i_2, \frac{\alpha}{\log(2n/\zeta)}) & \text{if } d = 6 \\ \mathcal{A}_n(R, -i_2, \alpha) & \text{if } d \ge 7, \end{cases}$$

with $i_2 := \lfloor c_2 + \frac{d+2}{d-2} \cdot \log_2(n/\zeta) \rfloor$.

Remark 5.17. Similarly as in dimension 5, the proofs of Lemmas 5.15 and 5.16 are independent of the value of T, which only matters for the proof of Lemma 5.14 and the determination of $\overline{r}$.

Proof of Proposition 5.13. One just has to notice that $\overline{r} \le R$, for any $\zeta \le n$, so that one can apply Lemmas 5.14 and 5.15 with some $\alpha_0 < \min(\alpha_1, \alpha_2, \alpha_3)$, and $r = \alpha_0 \overline{r}$. $\square$

We prove now the three lemmas in the next subsections.

5.5.1 Proof of Lemma 5.14

This part is entirely similar to the case of dimension 5, so we only briefly indicate the necessary changes. Define $s = \min(r, c_s\sqrt{\zeta T/n})$, with c_s a small constant to be defined in a moment. Assume first that $s \ge 1$, and then split $\xi_{r,-}$ into three parts: $\xi_{r,-} = \xi_{s,-} + \xi_{s,r}^{\mathrm{high}} + \xi_{s,r}^{\mathrm{low}}$, with the same notation as in the proof of Lemma 5.9, and with the integer j entering their definitions that we take here to be 0.

Observe that (5.1) and Lemma 5.2 give

$$\xi_{s,-} \;\le\; \frac{C_1 C_2}{T}(n+1)s^2 \;\le\; \frac{\zeta}{9}, \tag{5.27}$$

choosing the constant c_s small enough for the last inequality.

Next, using the same argument as in dimension 5, we get that for some constant $C > 0$, for any $r \le \alpha \overline{r}$

$$\xi_{s,r}^{\mathrm{low}} \;\le\; C\frac{n\overline{\rho} r^d}{T s^{d-4}} \;\le\; C\,\zeta \cdot (\frac{r}{\overline{r}})^d \;\le\; \frac{\zeta}{9}, \tag{5.28}$$

taking α small enough for the last inequality.

Finally, notice that $\overline{r} \le \sqrt{T}$, so that on the event $\mathcal{A}_n(r, 0, \alpha)$, we have for some constant $C' > 0$, and any $r \le \overline{r}$,

$$\xi_{s,r}^{\mathrm{high}} \;\le\; C' \cdot \frac{\alpha \zeta r^2}{T} \;\le\; \frac{\zeta}{9}, \tag{5.29}$$

taking again α small enough at the end. Then the lemma follows from (5.27), (5.28) and (5.29).

In the case when s is smaller than one, we just write $\xi_{r,-}$ as the sum of two terms $\xi_{r,-}^{\mathrm{low}}$ and $\xi_{r,-}^{\mathrm{high}}$, defined respectively as $\xi_{s,r}^{\mathrm{low}}$ and $\xi_{s,r}^{\mathrm{high}}$, except that we remove the indicator functions of the events $\{\|S_k - x\| > s\}$ in their definitions. The rest of the proof remains unchanged.

5.5.2 Proof of Lemma 5.15

We start by writing $\xi_{r,R} = \xi_{r,R}^{\text{low},*} + \xi_{r,R}^{\text{high}}$, with

$$\xi_{r,R}^{\text{low},*} := \sum_{k\notin\mathcal{K}_n(r,\overline{\rho})} \sum_{x\in\mathcal{R}_k} \mathbf{1}(r \le \|S_k - x\| \le R)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H_{\mathcal{R}_k}^+ = \infty\big),$$

and

$$\xi_{r,R}^{\text{high}} := \sum_{k\in\mathcal{K}_n(r,\overline{\rho})} \sum_{x\in\mathcal{R}_k} \mathbf{1}(r \le \|S_k - x\| \le R)\, \varphi_T(x - S_k) \cdot \mathbb{P}_x\big(H_{\mathcal{R}_k}^+ = \infty\big).$$

Consider first the term $\xi_{r,R}^{\text{low},*}$. Bounding the sum over x by a sum over indices $k' \in \{0,\dots,n\}$, and cutting it again into two pieces, one obtains $\xi_{r,R}^{\text{low},*} \le C_2(\xi_{r,R}^{\text{low}} + \xi_{r,R}^{\text{mix}})$, with

$$\xi_{r,R}^{\text{low}} := \sum_{k,k'\notin\mathcal{K}_n(r,\overline{\rho})} \frac{\mathbf{1}(r \le \|S_k - S_{k'}\| \le R)}{\|S_k - S_{k'}\|^{d-2}},$$

and

$$\xi_{r,R}^{\text{mix}} := \sum_{k'\in\mathcal{K}_n(r,\overline{\rho})} \sum_{k\notin\mathcal{K}_n(r,\overline{\rho})} \frac{\mathbf{1}(r \le \|S_k - S_{k'}\| \le R)}{\|S_k - S_{k'}\|^{d-2}}.$$

Now, exactly as for the proof of (5.18) one has using the first inequality in Lemma 5.3, that on the event $\mathcal{A}_n(r,0,\alpha)$, for some constant $C > 0$,

$$\xi_{r,R}^{\text{mix}} \;\le\; C|\mathcal{K}_n(r,\overline{\rho})|\,\overline{\rho}R^2 \;\le\; \frac{\zeta}{9C_2}, \tag{5.30}$$

taking α small enough for the last inequality. We next consider the term $\xi_{r,R}^{\text{low}}$. Define i_1 as the smallest integer, such that $2^{i_1} \ge c_1 n/\zeta$, with c_1 a large constant to be fixed later. First, a similar argument as above gives that

$$2\sum_{k\notin\mathcal{K}_n(r,\overline{\rho})} \sum_{k'\notin\mathcal{K}_n(r,2^{-i_1+2d}\overline{\rho})} \frac{\mathbf{1}(r \le \|S_k - S_{k'}\| \le R)}{\|S_k - S_{k'}\|^{d-2}} \le Cn\frac{\overline{\rho}}{2^{i_1}}R^2 \le \frac{\zeta}{18C_2}, \tag{5.31}$$

choosing c_1 large enough in the last inequality.

Then define for $-i_1 + 2d \le j < 0$,

$$\mathcal{C}_j := \{x \in r\mathbb{Z}^d :\ 1 \le \frac{\ell_n(Q(x,2r))}{2^j\overline{\rho}|Q(r)|} < 2\},$$

and for commodity reason call $\mathcal{C}_0$ the set

$$\mathcal{C}_0 := \{x \in r\mathbb{Z}^d :\ \ell_n(Q(x,2r)) \ge \overline{\rho}|Q(r)|\}.$$

For $x \in \mathcal{C}_0$, divide the cube $Q(x,2r)$ into 2^{2d} sub-boxes of side $r' = r/2$, and note that for each of them, the number of times k which are not in the set $\mathcal{K}_n(r,\overline{\rho})$, but such that S_k belongs to this sub-box, cannot exceed $\overline{\rho}|Q(r)|$. Therefore for any $x \in \mathcal{C}_0$,

$$|\{k \notin \mathcal{K}_n(r,\overline{\rho})\ :\ S_k \in Q(x,2r)\}| \;\le\; 2^{2d}\overline{\rho}|Q(r)|.$$

As a consequence,

$$\overline{\xi}_{r,R}^{\text{low}} := \sum_{k,k'\in\mathcal{K}_n(r,2^{-i_1+2d}\overline{\rho})\setminus\mathcal{K}_n(r,\overline{\rho})} \frac{\mathbf{1}(r \le \|S_k - S_{k'}\| \le R)}{\|S_k - S_{k'}\|^{d-2}}$$

$$\leq C \sum_{\substack{-i_1+2d\leq j\leq 0\\ j\leq i\leq 0}} \sum_{\substack{x\in\mathcal{C}_i\\ y\in\mathcal{C}_j}} (\ell_n(Q(x,2r))\wedge 2^{2d}\overline{\rho}|Q(r)|)\cdot(\ell_n(Q(y,2r))\wedge 2^{2d}\overline{\rho}|Q(r)|)\frac{\mathbf{1}(r\leq \|x-y\|\leq R)}{\|x-y\|^{d-2}}$$

$$\leq C\cdot \sum_{\substack{-i_1+2d\leq j\leq 0\\ j\leq i\leq 0}} 2^{i+j}(\overline{\rho}|Q(r)|)^2 \sum_{\substack{x\in\mathcal{C}_i\\ y\in\mathcal{C}_j}} \frac{\mathbf{1}(r\leq \|x-y\|\leq R)}{\|x-y\|^{d-2}}. \tag{5.32}$$

Then, by the same argument as in the proof of Lemma 5.10, we get that on the event $\mathcal{A}_n(r,-i_1,\alpha)$, for $-i_1+2d\leq j\leq 0$, one has

$$|\mathcal{C}_j| \ \leq\ \frac{2^d|\mathcal{K}_n(r,2^{j-2d}\overline{\rho})|}{2^{j-2d}\overline{\rho}\,|Q(r)|} \ \leq\ \frac{2^d\alpha\zeta}{2^{(j-2d)(1+\frac{2}{d-2})}\overline{\rho}|Q(r)|},$$

Furthermore, it holds for some constant $C>0$,

$$\sum_{x\in\mathcal{C}_i}\sum_{y\in\mathcal{C}_j}\frac{\mathbf{1}(r\leq \|x-y\|\leq R)}{\|x-y\|^{d-2}} \leq C|\mathcal{C}_i|\frac{R^2}{|Q(r)|}\leq C\frac{\alpha\zeta R^2}{\overline{\rho}|Q(r)|^2}2^{-i\frac{d}{d-2}}.$$

Injecting this in (5.32) and taking α small enough gives

$$\overline{\xi}^{\mathrm{low}}_{r,R}\leq\ C\,\alpha\zeta\,\overline{\rho}R^2\sum_{-i_1+2d\leq j\leq 0}2^j\sum_{i:j\leq i\leq 0}2^{-i\frac{2}{d-2}}\leq C\,\alpha\zeta\ \leq\ \frac{\zeta}{18C_2}. \tag{5.33}$$

By combining (5.31) and (5.33), we deduce that on the event $\mathcal{A}_n(r,-i_1,\alpha)$, one has

$$\xi^{\mathrm{low}}_{r,R}\leq \frac{\zeta}{9C_2}. \tag{5.34}$$

It remains to bound the term $\xi^{\mathrm{high}}_{r,R}$. For $i\geq 0$, set

$$\mathcal{K}_i := \mathcal{K}_n(r,2^i\overline{\rho})\setminus\mathcal{K}_n(r,2^{i+1}\overline{\rho}).$$

The second part (5.2) in Lemma 5.1 shows that on the event $\mathcal{A}_n(r,0,\alpha)$ (a fortiori on $\mathcal{A}_n(r,-i_1,\alpha)$), one has for some constant $C>0$,

$$\sum_{i\geq 0}\sum_{k\in\mathcal{K}_i}\sum_{x\in\mathcal{R}_k}\frac{\mathbf{1}(\frac{R}{2^i}\leq\|S_k-x\|\leq R)}{\|S_k-x\|^{d-2}}\cdot\mathbb{P}_x\big(H^+_{\mathcal{R}_k}=\infty\big)\ \leq\ C\sum_{i\geq 0}|\mathcal{K}_i|i\ \leq\ \frac{\zeta}{18C_2}, \tag{5.35}$$

at least for α small enough. One the other hand Lemma 5.3 shows that on $\mathcal{A}_n(r,0,\alpha)$,

$$\sum_{i\geq 0}\sum_{k\in\mathcal{K}_i}\sum_{x\in\mathcal{R}_k}\frac{\mathbf{1}(r\leq\|S_k-x\|\leq R/2^i)}{\|S_k-x\|^{d-2}}\ \leq\ \xi^{\mathrm{mix}}_{r,R}+\sum_{i\geq 0}\sum_{k'\in\mathcal{K}_n(r,\overline{\rho})}\sum_{k\in\mathcal{K}_i}\frac{\mathbf{1}(r\leq\|S_k-S_{k'}\|\leq R/2^i)}{\|S_k-S_{k'}\|^{d-2}}$$

$$\leq\ \xi^{\mathrm{mix}}_{r,R}+C\alpha\zeta\sum_{i\geq 0}2^{i+1}\overline{\rho}(R/2^i)^2\ \leq\ \frac{\zeta}{18C_2}, \tag{5.36}$$

for α small enough, using also (5.30). The result follows by combining (5.30), (5.34), (5.35), and (5.36).

5.5.3 Proof of Lemma 5.16

Fix c_2 a large constant to be adjusted below, and define i_2 as the smallest integer, such that

$$2^{i_2} \geq c_2 \cdot \left(\frac{n}{\zeta}\right)^{\frac{d+2}{d-2}}.$$

Then using Lemma 5.2, we see that $\xi_{R,+} \leq \xi_{R,+}^{\text{very low}} + \xi_{R,+}^{\text{low}}$, with

$$\xi_{R,+}^{\text{very low}} := 2C_2 \sum_{k=0}^{n} \sum_{k' \notin \mathcal{K}_n(R, 2^{-i_2+2d}\overline{\rho})} \frac{\mathbf{1}(\|S_k - S_{k'}\| \geq R)}{\|S_k - S_{k'}\|^{d-2}},$$

and

$$\xi_{R,+}^{\text{low}} := C_2 \sum_{k,k' \in \mathcal{K}_n(R, 2^{-i_2+2d}\overline{\rho})} \frac{\mathbf{1}(\|S_k - S_{k'}\| \geq R)}{\|S_k - S_{k'}\|^{d-2}}.$$

We first treat the term $\xi_{R,+}^{\text{very low}}$. By using Lemma 5.3, exactly as it was used in the proof of Lemma 5.10, we get

$$\sum_{k \notin \mathcal{K}_n(R, 2^{-i_2+2d}\overline{\rho})} \frac{\mathbf{1}(\|S_k - z\| \geq R)}{\|z - S_k\|^{d-2}} \leq C_3\, n^{\frac{2}{d}} (2^{-i_2+2d}\overline{\rho})^{1-\frac{2}{d}}.$$

Therefore by taking c_2 large enough in the definition of i_2, we get

$$\xi_{R,+}^{\text{very low}} \;\leq\; 2C_2C_3 \cdot (n+1)^{1+\frac{2}{d}} (2^{-i_2+2d}\overline{\rho})^{1-\frac{2}{d}} \;\leq\; \frac{\zeta}{6}. \tag{5.37}$$

We next treat the term $\xi_{R,+}^{\text{low}}$. Consider j_0, the smallest integer such that $2^{j_0} > 2^{2(d-2)}$. We claim that for any $\alpha \leq 1$, on the event $\mathcal{A}_n(R, 0, \alpha)$, the set $\mathcal{K}_n(R, 2^{j_0}\overline{\rho})$ is empty. Indeed if this was not the case, then there would exist somewhere a cube of side length R visited more than $2^{j_0-d}\overline{\rho}R^d$ times (using the rough bound $|Q(R)| \geq (R/2)^d$). Note that by definition $\overline{\rho}R^d = \zeta$. So by the pigeonhole principle, this would mean that a cube of side length $R/2$ would be visited more than $2^{j_0-2d}\zeta$ times, which in turn would imply $|\mathcal{K}_n(R, 2^{j_0}\overline{\rho})| \geq 2^{j_0-2d}\zeta > \alpha\zeta 2^{-2j_0/(d-2)}$, contradicting the fact that we are on $\mathcal{A}_n(R, 0, \alpha)$.

We now perform a subdivision of the space, similar to the one from the proofs of Lemma 5.10 and 5.15. For $j \geq -i_2 + 2d$, we define

$$\mathcal{C}_j := \{x \in R \cdot \mathbb{Z}^d : \; 1 \leq \frac{\ell_n(Q(x, 2R))}{2^j \overline{\rho} |Q(R)|} < 2\}.$$

By a similar argument as in the proofs of the aforementioned lemmas, we get that on the event $\mathcal{A}_n(R, -i_2, \alpha)$, for $j \leq j_0 + 2d$,

$$|\mathcal{C}_j| \;\leq\; \frac{2^d |\mathcal{K}_n(R, 2^{j-2d}\overline{\rho})|}{2^{j-2d}\overline{\rho}\,|Q(R)|} \;\leq\; \frac{2^d \alpha \zeta}{2^{(j-2d)(1+\frac{2}{d-2})}\overline{\rho}|Q(R)|}, \tag{5.38}$$

and $\mathcal{C}_j$ is empty for $j > j_0 + 2d$. Thus for some constant $C > 0$ (which may change from line to line), on the event $\mathcal{A}_n(R,-i_2,\alpha)$,

$$\begin{aligned}
\xi^{\text{low}}_{R,+} &\le C\cdot \sum_{\substack{-i_2+2d\le j\le j_0+2d\\ j\le i\le j_0+2d}} \sum_{\substack{x\in\mathcal{C}_i\\ y\in\mathcal{C}_j}} \ell_n(Q(x,2R))\ell_n(Q(y,2R))\frac{\mathbf{1}(\|x-y\|\ge R)}{\|x-y\|^{d-2}}\\
&\le C\cdot \sum_{\substack{-i_2+2d\le j\le j_0+2d\\ j\le i\le j_0+2d}} 2^{i+j}(\overline{\rho}|Q(R)|)^2\cdot \sum_{\substack{x\in\mathcal{C}_i\\ y\in\mathcal{C}_j}} \frac{\mathbf{1}(\|x-y\|\ge R)}{\|x-y\|^{d-2}}\\
&\le C\cdot \overline{\rho}^{1-\frac{2}{d}}(\alpha\zeta)^{1+\frac{2}{d}} \sum_{\substack{-i_2+2d\le j\le j_0+2d\\ j\le i\le j_0+2d}} 2^{j\frac{d-4}{d-2}-\frac{2}{d-2}i}\\
&\le C\cdot \alpha^{1+\frac{2}{d}}\zeta\cdot \sum_{\substack{-i_2+2d\le j\le j_0+2d\\ j\le i\le j_0+2d}} 2^{j\frac{d-4}{d-2}-\frac{2}{d-2}i},
\end{aligned}$$

using the same argument (based on spherical rearrangement) as in Lemma 5.10 for the third inequality. When $d>6$, the last series is convergent, and we obtain $\xi^{\text{low}}_{R,+}\le\zeta/6$, by taking α small enough. When $d=6$, on the other hand, by taking α of the form $\alpha=\alpha'\cdot(\log(n/\zeta))^{-\frac{d}{d+2}}$, with α' small enough, we obtain as well

$$\xi^{\text{low}}_{R,+}\ \le\ C\alpha^{1+\frac{2}{d}}\zeta\cdot i_2\ \le\ \frac{\zeta}{6},$$

for some possibly different constant $C>0$, and using that i_2 is of order $\log_2(n/\zeta)$. Together with (5.37), this concludes the proof.

6 Path Properties

In this section we prove Theorems 1.5 and 1.6. We need to show that the walk spends a time of order τ in a region of typical density of order $\overline{\rho}$ under the constraint. First observe that for any $r\ge 1$, $i_0\ge 0$ and $\alpha_0>0$, one has

$$\bigcup_{i=-i_0}^{i_0}\{|\mathcal{K}_n(r,2^i\overline{\rho})|>\frac{\alpha_0\tau}{2^{2i/(d-2)}}\}\ \subseteq\ \{|\mathcal{K}_n(r,2^{-i_0}\overline{\rho})|>\alpha\tau\},\tag{6.1}$$

with $\alpha=2^{-2i_0/(d-2)}\alpha_0$, and this means that a time $\alpha\tau$ is spent in a region of density larger than $2^{-i_0}\overline{\rho}$. Thus, we need to show that under the event $\{\xi_n(T)>\zeta_n\}$, the left-hand side of (6.1) most likely holds, for i_0 and α_0 independent of ζ_n and n. Indeed this would prove that for some $r\ge 1$, $i_0\ge 0$ and $\alpha>0$,

$$\lim_{n\to\infty}\mathbb{P}\Big(|\mathcal{K}_n(r,2^{-i_0}\overline{\rho})|>\alpha\tau\ \Big|\ \mathrm{Cap}\,(\mathcal{R}_n)-\mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)]<-\zeta_n\Big)=1,\tag{6.2}$$

and would settle the first step in establishing the path properties. The two other steps follow very closely the arguments of [AS17a]. Let us review them before establishing the first step. For $r\ge 1$, $\rho>0$, and $m\ge 1$, define

$$\mathcal{G}_n(r,\rho,m)=\{\exists\mathcal{C}\in(\mathbb{Z}^d)^m:\ \|x-y\|\ge 4r,\,\forall x\neq y\in\mathcal{C}\text{ and }\ell_n(Q(x,r))\ge\rho|Q(r)|,\ \forall x\in\mathcal{C}\}.$$

Thus, $\mathcal{G}_n(r,\rho,m)$ is the event that there are at least m (random) disjoint cubes of side-length r, each one visited at least $\rho|Q(r)|$ times. Now Equation (2.24) in Lemma 2.4 of [AS17a], and (6.2) give the existence of some constant $\kappa>0$, such that with $m^* = \lfloor \kappa\,\tau/(\overline{\rho} r^d)\rfloor$,

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{G}_n(r, 2^{-i_0}\overline{\rho}, m^*)\ \big|\ \mathrm{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)] < -\zeta_n\big) = 1.$$

This readily implies that for $\mathcal{V} := \cup_{x\in\mathcal{C}} Q(x,r)$, with $\mathcal{C}$ one set realizing the event $\mathcal{G}_n(r, 2^{-i_0}\overline{\rho}, m^*)$, and some positive constants α, c and C (all independent of n and ζ_n), one has

$$\lim_{n\to\infty} \mathbb{P}\big(\exists \mathcal{V}\subseteq\mathbb{Z}^d:\ \ell_n(\mathcal{V})\ge\alpha\tau,\ c\le\frac{|\mathcal{V}|}{R^d}\le C\,\big|\ \mathrm{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\mathrm{Cap}\,(\mathcal{R}_n)] < -\zeta_n\big) = 1.$$

The last step in proving Theorem 1.5 concerns showing that the capacity of the random set $\mathcal{V}$ is of order $|\mathcal{V}|^{1-2/d}$. One of the key inequality in [AS17a] is its Proposition 1.7 which penalizes the event of visiting a set $\mathcal{V}$ (made of disjoint balls) with capacity much larger than $|\mathcal{V}|^{1-2/d}$. We thus conclude Theorem 1.5 by invoking this latter proposition.

We are now back in establishing the first step mentioned earlier. We first need to strengthen Propositions 5.8 and 5.13 into the following.

Proposition 6.1. *Assume $d=5$. For any $\beta>0$, there exist an integer $i_0\ge 0$, and $\alpha_0>0$, such that for any $1\le\zeta\le n$, and n large enough,*

$$\bigcap_{i=-i_0}^{i_0} \{|\mathcal{K}_n(\widetilde{r}, 2^i\overline{\rho})| \le \alpha_0 \frac{n}{2^{2i/3}}\} \cap \mathcal{A}_n(\widetilde{r}, i_0, \beta)\ \subseteq\ \{\xi_n(T)\le\zeta\}, \tag{6.3}$$

where T is as in (5.15)*, and $\widetilde{r} = \frac{\sqrt{T}}{\log(n+1)}$.*

The important point here is that the parameters i_0, and α_0 are independent of ζ and n.

Our result in dimension $d\ge 7$ is more intricate, since we have to deal with two scales $\widetilde{r}$ and R. However, its pattern is similar. We recall that $\overline{r}$ is defined in (5.24), T and R are defined in (5.15) and (5.23) respectively, and i_1 and i_2 are as in Proposition 5.13. We then define also $\widetilde{r} := \frac{\overline{r}}{\log(n+1)}$.

Proposition 6.2. *Assume $d\ge 7$. For any $\beta>0$, there exist $i_0\ge 0$, and $\alpha_0>0$, such that for any $1\le\zeta\le n$, and n large enough*

$$\bigcap_{i=-i_0}^{i_0} \{|\mathcal{K}_n(\widetilde{r}, 2^i\overline{\rho})|\vee|\mathcal{K}_n(R, 2^i\overline{\rho})| \le \frac{\alpha_0\zeta}{2^{2i/(d-2)}}\} \cap \mathcal{A}_n(\widetilde{r}, -i_1, \beta) \cap \mathcal{A}_n(R, -i_2, \beta)\ \subseteq\ \{\xi_n(T)\le\zeta\}. \tag{6.4}$$

The proofs of the two latter propositions are close to the proofs from Sections 5.4 and 5.5, and we explain the differences in the case of $d=5$. We omit the details in $d\ge 7$.

Recall that in Section 5.4, we wrote $\xi_n(T)$ as a sum of $\xi_{\widetilde{r},-}$ and $\xi_{\widetilde{r},+}$. On one hand a careful look at the proof of Lemma 5.9 reveals that for any fixed $\beta>0$, and any $i_0\ge 0$, $\mathcal{A}_n(\widetilde{r}, i_0, \beta)\ \subseteq\ \{\xi_{\widetilde{r},-}\le \frac{1}{2}\zeta\}$, provided n is large enough. Thus, we only need to find i_0 and α_0 (independent of ζ and n), such that

$$\bigcap_{i=-i_0}^{i_0} \{|\mathcal{K}_n(\widetilde{r}, 2^i\overline{\rho})| \le \alpha_0 \frac{n}{2^{2i/3}}\} \cap \mathcal{A}_n(\widetilde{r}, i_0, \beta)\ \subseteq\ \{\xi_{\widetilde{r},+}\le \frac{1}{2}\zeta\}.$$

In the proof of Lemma 5.10, $\xi_{\widetilde{r},+}$ was further written as $\xi_{\widetilde{r},+}^{\text{low}} + \xi_{\widetilde{r},+}^{\text{high}}$, and some i_0 was introduced in order to make $\xi_{\widetilde{r},+}^{\text{low}} \leq \zeta/4$ (see (5.18)). This part can be used here as well. Then, we replace the bound on $|\mathcal{C}_j|$ in (5.19) by the following: for $j \geq -i_0$

$$|\mathcal{C}_{j+10}| \cdot 2^{j-5}\overline{\rho} \cdot |Q(\widetilde{r})| \leq |\mathcal{K}_n(\widetilde{r}, 2^j\overline{\rho})| \leq 2^{-\frac{2}{3}j} n\big(\alpha_0 \mathbf{1}(|j| \leq i_0) + \beta \mathbf{1}(j > i_0)\big).$$

When estimating $\xi_{\widetilde{r},+}^{\text{high}}$ as in (5.21), we divide the converging series into a finite number of small indices weighted by α_0, and a converging series starting with index i_0, and weighted by β (large). By choosing a larger i_0 if necessary one can make the sum of the last series small, and then take α_0 small enough so that the finite sum (of about i_0 terms) is in turn small. This makes $\xi_{\widetilde{r},+}^{\text{high}} \leq \zeta/4$. Thus, (6.3) holds.

The interest of these propositions is that by taking β large enough, one can make the probability of the complement of the event $\mathcal{A}_n(\widetilde{r}, i_0, \beta)$ negligible, thanks to Corollary 5.5:

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{A}_n(\widetilde{r}, i_0, \beta)^c \bigm| \text{Cap}\,(\mathcal{R}_n) - \mathbb{E}[\text{Cap}\,(\mathcal{R}_n)] < -\zeta_n\big) = 0. \tag{6.5}$$

It then follows from (6.1), Propositions 5.6 and 6.1, and from (6.5), that for some $i_0 \geq 0$, and $\alpha_0 > 0$, the limit (6.2) holds. In dimension seven and larger, the same arguments are needed. We omit to repeat them.

7 Upward Deviations

We prove here Theorem 1.8. Thanks to our decomposition (2.9), we can adapt the approach of Hamana and Kesten [HK], who proved a similar result for the size of the range.

The approach of Hamana and Kesten is based on first proving an approximate subadditivity relation for the probability of upward deviations, that is the existence of some constants $\chi \in (0,1)$, $c > 0$, and $C > 0$, such that for any $m, n \geq 1$ integers, and y, z positive reals,

$$\mathbb{P}\big(|\mathcal{R}_{m+n}| \geq y + z - Ca(m,n)\big) \;\geq\; c\,\chi^{a(m,n)}\,\mathbb{P}\big(|\mathcal{R}_n| \geq y\big)\mathbb{P}\big(|\mathcal{R}_m| \geq z\big), \tag{7.1}$$

with

$$a(m,n) := (n \cdot m)^{\frac{1}{d+1}}.$$

The second step, which is general and only based on (7.1) and the fact that (when $d \geq 2$) one has $\lim_{m,n\to\infty} \frac{a(m,n)}{n \vee m} = 0$, shows that the following limit exists,

$$\psi(x) := -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(|\mathcal{R}_n| \geq x \cdot n\big), \quad \text{for all } x > 0,$$

and that ψ is continuous and convex on $[0,1]$. Here we prove an analogous result as (7.1), and use their general argument to conclude.

Proof of Theorem 1.8. We first prove an analogous result as (7.1), but with $a(m,n)$ replaced by the function:

$$\widetilde{a}(m,n) = (n \cdot m)^{\frac{1}{d-1}}.$$

In other words we establish the following inequality. There exists $\chi \in (0,1)$, and $C > 0$, such that for any m, n integers and y, z positive reals,

$$\mathbb{P}\left(\text{Cap}\,(\mathcal{R}_{m+n}) \geq y + z - C\,\widetilde{a}(m,n)\right) \geq \frac{1}{2}\chi^{\widetilde{a}(m,n)}\mathbb{P}\left(\text{Cap}\,(\mathcal{R}_n) \geq y\right)\mathbb{P}\left(\text{Cap}\,(\mathcal{R}_m) \geq z\right). \tag{7.2}$$

The first step is to obtain the analogue of the following simple deterministic bound used in [HK]: if $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_m$ are two independent copies of the range, there is a positive constant C, such that for any $r \geq 1$

$$\frac{1}{|Q(r)|} \sum_{z \in Q(r)} |(z + \mathcal{R}_n) \cap \widetilde{\mathcal{R}}_m| \leq C \, \frac{n \cdot m}{r^d}.$$

The corresponding bound in our context reads as follows:

$$\frac{1}{|Q(r)|} \sum_{z \in Q(r)} \sum_{x \in \mathcal{R}_n} \sum_{y \in \widetilde{\mathcal{R}}_m} G(x - y + z) \leq C \, \frac{n \cdot m}{r^{d-2}}, \tag{7.3}$$

and is a direct consequence of (2.1) and the fact that for any $x \in \mathbb{Z}^d$, and for some constant $C > 0$,

$$\sum_{z \in Q(r)} \frac{1}{1 + \|z - x\|^{d-2}} \leq C \, r^2.$$

Now to lighten notation, we simply write $a = \widetilde{a}(m, n) = \lfloor (mn)^{\frac{1}{d-1}} \rfloor$. Using that the capacity is translation-invariant, we deduce

$$\begin{aligned} \mathrm{Cap}\,(\mathcal{R}_{m+n+a}) &\overset{(2.5)}{\geq} \mathrm{Cap}\,(\mathcal{R}_n \cup \mathcal{R}[n+a, n+m+a]) \\ &\overset{(2.9)}{=} \mathrm{Cap}\,(\overline{\mathcal{R}}_n) + \mathrm{Cap}\left(\widetilde{\mathcal{R}}_m\right) - \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + S'_a), \end{aligned} \tag{7.4}$$

with $\overline{\mathcal{R}}_n := \mathcal{R}_n - S_n$, $S'_a := S_{n+a} - S_n$, and $\widetilde{\mathcal{R}}_m := \mathcal{R}[n+a, n+m+a] - S_{n+a}$. The Markov property implies that $\overline{\mathcal{R}}_n$ and $\widetilde{R}_m$ are independent, and distributed as $\mathcal{R}_n$ and $\mathcal{R}_m$ respectively. Furthermore,

$$\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + S'_a) \overset{(2.12)}{\leq} \sum_{x \in \overline{\mathcal{R}}_n} \sum_{y \in \widetilde{\mathcal{R}}_m} G(x - y - S'_a). \tag{7.5}$$

Now, one idea of Hamana and Kesten [HK] is to bound the law of S'_a by a uniform law on the cube $Q(a/d)$. Indeed for any $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$, one has

$$\mathbb{P}(S'_a = x) \geq \frac{1}{(2d)^a}, \tag{7.6}$$

since there is at least one path of length a going from 0 to x. Write $\overline{Q}(a/d)$ for the set of sites $x \in Q(a/d)$, for which $\mathbb{P}(S_a = x) \neq 0$. Then for any $x \in \overline{Q}(a/d)$, and any $\alpha > 0$,

$$\mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - \frac{\alpha}{2}\right) \overset{(7.4)}{\geq} \mathbb{P}(S'_a = x) \cdot \mathbb{P}\left(\mathrm{Cap}(\overline{\mathcal{R}}_n) \geq z, \mathrm{Cap}(\widetilde{\mathcal{R}}_m) \geq y, \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \leq \frac{\alpha}{2}\right).$$

Integrating with respect to the uniform measure on $\overline{Q}(a/d)$, we get

$$\begin{aligned} \mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n+a}) \geq z + y - \frac{\alpha}{2}\big) &\overset{(7.6)}{\geq} \frac{1}{(2d)^a} \\ &\times \frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbb{P}\left(\mathrm{Cap}(\overline{\mathcal{R}}_n) \geq z, \mathrm{Cap}(\widetilde{\mathcal{R}}_m) \geq y, \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \leq \frac{\alpha}{2}\right). \end{aligned} \tag{7.7}$$

We need now to estimate the mean of $\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + \cdot)$ with respect to the uniform measure. According to (7.3), there is a positive constant C, such that

$$\frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \le C \frac{m \cdot n}{a^{d-2}} \le Ca, \tag{7.8}$$

where the last inequality follows from the definition of a. Then by Chebychev's inequality, we obtain

$$\frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbf{1}(\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \le 2Ca) \ge \frac{1}{2}. \tag{7.9}$$

As a consequence,

$$\begin{aligned}
&\mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n}) \ge z + y - a - 4Ca\big) \overset{(2.6),(2.8)}{\ge} \mathbb{P}\big(\mathrm{Cap}(\mathcal{R}_{m+n+a}) \ge z + y - 4Ca\big) \\
&\overset{(7.7)}{\ge} \frac{1}{(2d)^a} \cdot \mathbb{E}\Big[\mathbf{1}(\mathrm{Cap}(\overline{\mathcal{R}}_n) \ge z) \cdot \mathbf{1}(\mathrm{Cap}(\widetilde{\mathcal{R}}_m) \ge y) \times \frac{1}{|\overline{Q}(a/d)|} \sum_{x \in \overline{Q}(a/d)} \mathbf{1}(\chi_C(\overline{\mathcal{R}}_n, \widetilde{\mathcal{R}}_m + x) \le 2Ca)\Big] \\
&\overset{(7.9)}{\ge} \frac{1}{2(2d)^a} \cdot \mathbb{P}\big(\mathrm{Cap}\,(\mathcal{R}_n) \ge z\big)\mathbb{P}\big(\mathrm{Cap}\,(\mathcal{R}_m) \ge y\big),
\end{aligned}$$

proving (7.2), with $\chi = 1/(2d)$.

It then follows from the general arguments of Hamana and Kesten, see Lemma 3 in [HK], that the following limit exists for all $x > 0$:

$$\psi_d(x) \ := \ - \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\mathrm{Cap}(\mathcal{R}_n) \ge nx\right).$$

We now prove that the range for which $\psi_d(x)$ is finite is not empty. Define for $n \ge 0$,

$$c_n := \max_{\gamma : \{0,\dots,n\} \to \mathbb{Z}^d} \mathrm{Cap}(\{\gamma(0), \dots, \gamma(n)\}), \tag{7.10}$$

where the max is taken over all nearest neighbor paths of length $n+1$. By (2.6), it follows that $c_{n+m} \le c_n + c_m$, for all $n, m \ge 0$, so that by Fekete's lemma, the limit $\lim_{n\to\infty} c_n/n$ exists. Call γ_d^* this limit. Note that $\psi_d(x)$ is finite on $[\gamma_d, \gamma_d^*]$, since the probability that the simple random walk follows the path realizing the maximum in (7.10) is larger than or equal to $1/(2d)^{n+1}$, so that $\psi_d(x) \le \log(2d)$, for all $x \le \gamma_d^*$. Conversely, by definition of c_n, one has $\psi_d(x) = \infty$ for all $x > \gamma_d^*$. Furthermore, it follows from Lemma 3 and Proposition 4 in [HK], that ψ_d is continuous, and convex on $(0, \gamma_d^*]$. Now Proposition 3.2 and Lemma 2.1 show that when $d = 5$, $\psi_d(x) \ge c(x - \gamma_5)^3$, for all $x \ge \gamma_d$. Likewise, (3.7) with $b = (x - \gamma_d)n$ and $T = C/(x - \gamma_d)^2$, for some large enough constant $C > 0$, show that $\psi_d(x) \ge c(x - \gamma_d)^3$, for $\gamma_d \le x \le 1$ when $d \ge 6$. Using convexity, this also shows that ψ_d is increasing on $[\gamma_d, \gamma_d^*]$. In addition one has $\psi_d(x) = 0$ for all $x < \gamma_d$, by definition of γ_d as the limit of the (normalized) expected capacity, and using that by (2.8), $\mathrm{Cap}(\mathcal{R}_n) \le n$.

Finally we show that $\gamma_d^* > \gamma_d$.

Consider $\mathcal{D}_n$ the set of *no double backtrack at even times* paths of length $n+1$ that we introduced in [AS17b]. By definition, this is simply the set of paths $\gamma : \{0, \dots, n\} \to \mathbb{Z}^d$, such that for any even $k \le n$, one has $\gamma(k+2) \ne \gamma(k)$. The only important property we need is that from a no-backtrack walk $\widetilde{S}$, and a sum of independent geometric variables $\{\xi_i,\ i \in \mathbb{N}\}$, with parameter $1/(2d)^2$, we can build a simple random walk S such that

$$\mathcal{R}[0, n + 2 \sum_{i \le n/2} \xi_i] = \widetilde{\mathcal{R}}_n.$$

Thus, for any $\alpha > 0$, we have by (2.6) and (2.8),

$$\mathrm{Cap}(\widetilde{\mathcal{R}}_n) \geq \mathrm{Cap}(\mathcal{R}_{(1+\alpha)n}) - \mathbf{1}\left(\sum_{i\leq n/2} \xi_i < \frac{\alpha n}{2}\right) \cdot (1+\alpha)(n+1).$$

By taking the maximum over $\mathcal{D}_n$ on the left hand side, and then the expectation on the right hand side, we obtain

$$c_n \geq \max_{\pi\in\mathcal{D}_n} \mathrm{Cap}(\pi) \geq \mathbb{E}[\mathrm{Cap}(\mathcal{R}_{(1+\alpha)n})] - (1+\alpha)(n+1)\cdot \mathbb{P}\left(\sum_{i\leq n/2} \xi_i < \frac{\alpha n}{2}\right). \tag{7.11}$$

Now take $\alpha < 1/(2d)^2$, and use Chebyshev's inequality, to see that the last term of (7.11) is $\mathcal{O}(1)$. Together with Lemma 2.1 it implies that

$$c_n \geq \gamma_d(1+\alpha)n - \mathcal{O}(\sqrt{n}),$$

which indeed proves that $\gamma_d < \gamma_d^*$. $\square$

Acknowledgements: Perla Sousi participated at an early stage of the project and we thank her for stimulating discussions, and her proof of Lemma 5.1. We also thank Quentin Berger and Julien Poisat for the idea of considering the polymer melt. The authors were partly supported by the French Agence Nationale de la Recherche under grants ANR-17-CE40-0032 and ANR-16-CE93-0003.

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