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RANDOM WALKS WITH OCCASIONALLY MODIFIED TRANSITION PROBABILITIES

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ABSTRACT. We study recurrence properties and the validity of the (weak) law of large numbers for (discrete time) processes which, in the simplest case, are obtained from simple symmetric random walk on $\mathbb{Z}$ by modifying the distribution of a step from a fresh point. If the process is denoted as $\{S_n\}_{n\geq 0}$, then the conditional distribution of $S_{n+1}-S_n$ given the past through time n is the distribution of a simple random walk step, provided S_n is at a point which has been visited already at least once during $[0,n-1]$. Thus, in this case, $P\{S_{n+1}-S_n=\pm 1|S_\ell,\ell\leq n\}=1/2$. We denote this distribution by P_1. However, if S_n is at a point which has not been visited before time n, then we take for the conditional distribution of $S_{n+1}-S_n$, given the past, some other distribution P_2. We want to decide in specific cases whether S_n returns infinitely often to the origin and whether $(1/n)S_n\to 0$ in probability. Generalizations or variants of the P_i and the rules for switching between the P_i are also considered.

1. Introduction

There have been a number of investigations of recurrence/transience of "slightly perturbed" random walks. Roughly speaking, we are thinking of processes (in discrete time) whose transition probabilities are "usually" equal to a given transition probability, but "occasionally" make a step according to a different transition probability. Arguably the most challenging of these problems is the question of recurrence vs. transience of "once reinforced" simple random walk on $\mathbb{Z}^d$. In the vertex version of this process, the walk moves at the ℓth step from a vertex x to a neighbor $x+y$ with a probability proportional to a weight $w(\ell,x+y)$. All these weights start out with the

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value 1, but then $w(\ell, x+y)$ is increased to $1+C$ for a given constant $C>0$ at the first time ℓ at which the walk visits $x+y$. After this change the weight of $x+y$ does not change, that is, $w(m, x+y)=1+C$ for all m greater than the time of the first visit to $x+y$ by the process. In general, little is known so far about recurrence or transience of such processes (except on $\mathbb{Z}$; see [5] for some recent results). Other examples include excited or cookie random walks on $\mathbb{Z}^d$, introduced by Benjamini and Wilson [1], which at first visits to a site have a bias in some fixed direction and at further visits make a simple random walk step. These processes have now been well studied in dimension 1 (see [4] and [11] for recent results and references therein), but much less is known in higher dimension.

Benjamini proposed the study of random walks which are perturbed in a somewhat different manner. We describe a slightly generalized version of his setup. Let $P_1, P_2, \dots, P_k$, be $k \le \infty$ probability distributions on $\mathbb{R}$ or on $\mathbb{Z}$, with zero-mean if they have finite first moment, or symmetric. Intuitively speaking, we now consider a process $S_n = S_0 + \sum_{\ell=1}^n X_\ell, n \ge 0$, for which the X_ℓ are chosen in two steps. First, we choose an index $i(\ell) \in \{1, 2, \dots, k\}$ and then, given the past through time $\ell - 1$ and $i(\ell)$, the conditional distribution of X_ℓ is taken to be $P_{i(\ell)}(\cdot)$. More formally, if we set $\mathcal{H}_n = \sigma((S_\ell, i(\ell))_{\ell \le n}) \vee \sigma(S_0)$, then we have for all $n \ge 0$:

$$\text{the conditional law of } X_{n+1} \text{ given } \mathcal{H}_n \vee \sigma\big(i(n+1)\big) \text{ is } P_{i(n+1)}. \tag{1.1}$$

Condition (1.1) is not enough to describe the law of $(S_0, (S_n, i(n))_{n\ge 1})$. This law will be completely described once the way the sequence $i(n)$ is chosen will be given, or equivalently once the conditional law of $i(n+1)$ given $\mathcal{H}_n$ will be given. One way is to choose $i(n+1)$ such that it is $\mathcal{H}_n$-measurable, in which case there exists $f : \mathbb{R} \times \bigcup_{n\ge 0} (\mathbb{R} \times \{1, \dots, k\})^n \to \{1, \dots, k\}$ such that $i(n+1) = f(S_0, (S_\ell, i(\ell))_{\ell \le n})$. In general, there can be added an extra randomness in the choice of $i(n+1)$, in which case, the conditional law of $i(n+1)$ given $\mathcal{H}_n$ is a law ν_n which is a function of $(S_0, (S_\ell, i(\ell))_{\ell \le n})$. Such laws can be described by mean of a random variable A_{n+1} uniformly distributed on $[0,1]$, independent of $\mathcal{H}_n$, and a function $F : [0,1] \times \mathbb{R} \times \bigcup_{n \ge 0} (\mathbb{R} \times \{1, \dots, k\})^n \to \{1, \dots, k\}$, such that ν_n is the conditional law of $F(A_{n+1}, S_0, (S_\ell, i(\ell))_{\ell \le n})$ given $\mathcal{H}_n$. We use here the convention $(\mathbb{R} \times \{1, \dots, k\})^0 = \emptyset$. Note also that there exists a measurable function $G : [0,1] \times \{1, \dots, k\} \to \mathbb{R}$ such that if B is a random variable uniformly distributed on $[0,1]$, then P_i is the law of $G(B, i)$. This function G will be fixed later on.

A convenient way to construct processes satisfying (1.1) will be to start from independent sequences of independent random variables uniformly distributed on $[0,1]$, $(A_n)_{n\ge 1}$ and $(B_n)_{n \ge 1}$, an independent random variable S_0, and a measurable function $F : [0,1] \times \mathbb{R} \times \bigcup_{n \ge 0} (\mathbb{R} \times \{1, \dots, k\})^n \to \{1, \dots, k\}$ (which describes how we choose the law to be used for the next jump). We then define $(S_n, i(n))$ recursively: for $n \geq 0$,

$$\begin{cases} i(n+1) = F\big(A_{n+1}, S_0, \big(S_\ell, i(\ell)\big)_{\ell \leq n}\big), \\ S_{n+1} - S_n = G\big(B_{n+1}, i(n+1)\big). \end{cases} \tag{1.2}$$

Note that all processes $(S_0, (S_n, i(n))_{n\geq 1})$ satisfying (1.1) are equal in law to a process $(S_0, (S_n, i(n))_{n\geq 1})$ defined by (1.2) for a particular choice of function F. The law of this process is thus given by the law μ_0 of S_0, the function F and the sequence $\{P_1, \dots, P_k\}$. We denote the law of this process $\mathbb{P}_{F,\mu_0}$ and simply by $\mathbb{P}_{F,S_0}$ when S_0 is not random.

Another way to construct $(S_n, i(n))$ is as follows. This construction will be used in the last section of this paper. Fix S_0 in some way and let $\{Y(i,n), 1 \leq i \leq k, n \geq 1\}$ be a family of independent random variables such that each $Y(i,n)$ has distribution P_i. These $Y(i,n)$ can be chosen before any $i(\ell)$ is determined. Now define inductively

$$j(i,\ell) = 1 + \text{number of times } P_i \text{ has been used during } [1,\ell],$$

and take for $n \geq 0$,

$$X_{n+1} = S_{n+1} - S_n = Y\big(i(n+1), j\big(i(n+1), n\big)\big).$$

We chose this terminology because we think of the sequence $Y(i,1), Y(i,2), \dots$ as a supply of variables with distribution P_i, and every time $i(\ell) = i$ we "use" one of these variables. When we come to pick the Y variable at time $n+1$ according to $P_{i(n+1)}$, we use the first $Y(i(n+1), \cdot)$ which has not been used yet. This is automatically independent of all Y's used by time n. We define

$$\mathcal{F}_0 = \sigma\text{-field generated by } S_0,$$

and

$$\begin{aligned} \mathcal{F}_{n+1} = \sigma\text{-field generated by } S_0, \mathcal{F}_n, i(n+1), \\ \text{and } Y\big(i(n+1), j\big(i(n+1), n\big)\big). \end{aligned}$$

Once we have observed the variables which generate $\mathcal{F}_n$, we first determine $i(n+1)$ by some rule. This rule may be randomized, but will actually be deterministic in our examples. Then $j(i(n+1), n)$ is also determined by $i(n+1)$ and $\mathcal{F}_n$-measurable functions. Finally, we determine $Y(i(n+1), n)$ and that completes the generators of $\mathcal{F}_{n+1}$. The only conditions on the rule for choosing $i(n+1)$ are that, conditionally on $\mathcal{F}_n$, all the random variables $i(n+1)$ and $\{Y(i,j), j > j(i,n), 1 \leq i \leq k\}$, are independent, with each $Y(i,j)$ with $j > j(i,n)$ having conditional distribution P_i.

Note that if all P_i, $i \leq k$, have finite first moment (and zero-mean), then S_n automatically satisfies the strong law of large numbers, that is, $S_n/n \to 0$ almost surely, as soon as the tails of the P_i are dominated by some fixed distribution with finite first moment (see Lemma 1 in [9]). However, the question of recurrence or transience of S_n is much more delicate, even when

$k=2$. In particular, in [3], Durett, Kesten and Lawler exhibit examples where S_n is transient (see also [10] for some necessary conditions for transience).

Benjamini's questions concerned the case when $k=2$, P_1 puts mass $1/2$ on each of the points $+1$ and -1, while P_2 is a symmetric distribution on $\mathbb{Z}$ in the domain of normal attraction of a symmetric Cauchy law (in particular P_2 does not have finite first moment). As for the $i(n)$, Benjamini made the following choices: $i(n+1)=2$ if S_n is at a "fresh" point, that is, if at time n the process is at a point which it has not visited before. If S_n is at a position which it has visited before, take $i(n+1)=1$. Thus, his process is a perturbation of simple random walk; it takes a special kind of step from each fresh point but is simple random walk otherwise. His principal questions were whether the process $\{S_n\}$ is recurrent and whether it satisfies the weak law of large numbers, that is, whether $(1/n)S_n \to 0$ in probability.

In Section 2.1, we present a general method to attack this kind of problems, which allows us to answer Benjamini's first question affirmatively (see Example 2.7). Our principal tool is a coupling between the S_n of Benjamini's process and a Cauchy random walk. The latter is a random walk with i.i.d. steps, all of which have a symmetric distribution P on $\mathbb{Z}$ which is in the domain of normal attraction of the symmetric Cauchy law. Unfortunately, so far our method works only for very specific P, including the distribution of the first return position to the horizontal axis of symmetric simple random walk on $\mathbb{Z}^2$. It seems that even asymptotically small changes in P cannot be handled by this method. In Section 2.2, we present an analogous method in a continuous setting, that is, when P is a distribution on $\mathbb{R}$, and prove in particular that if P is the usual Cauchy law (with density $1/\pi(1+y^2)$), then $\{S_n\}$ is recurrent (see Proposition 2.15).

Our coupling technique permits also to give sufficient criteria for the process $\{S_n\}$ to be transient.

In the last section, we prove a weak law of large numbers for Benjamini's process.

2. Coupling method

In this section, one wants to construct the process $\{S_n\}$ coupled to another process. If such a coupling exists, then $\{S_n\}$ automatically is recurrent (transience properties will also be considered). The problem now is whether the required coupling exists. The next subsections describe the desired coupling.

2.1. Discrete case.

2.1.1. *Successfull coupling.* The following properties, which a Markov chain with transition matrix Q on $\mathbb{Z}^2$ may or may not have, will be useful. If (U,V) is a Markov chain on $\mathbb{Z}^2$ starting from $(0,0)$, with transition matrix Q, let

$$T=T(Q):=\inf\{n>0 : V(n)=0\}.$$

If

$$T \text{ is a.s. finite,} \tag{2.1}$$

then the law P of U_T is well defined. This P equals $\varphi(Q)$ for some function φ. The following definition may differ slightly from the definition the reader knows. A process on $\mathbb{Z}$ is said to be *recurrent* if for any $u \in \mathbb{Z}$ the process visits u infinitely often. We say that a law P on $\mathbb{Z}$ is recurrent, if the random walk whose steps have distribution P is recurrent.

Another property is invariance under horizontal translations, that is,

$$Q\big[(u,v),\big(u',v'\big)\big] = Q\big[(0,v),\big(u'-u,v'\big)\big] \quad \text{for all } \big(u,u',v,v'\big). \tag{2.2}$$

The next property is that Q can be coupled with a certain given transition matrix Q_0 on $\mathbb{Z}^2$ in such a way that "paths chosen according to Q lie below paths chosen according to Q_0." The precise meaning of this is that (2.5) below holds. Assume that Q and Q_0 are translation invariant in the sense of (2.2). We say that Q can be *successfully coupled with* Q_0 if there exists a transition matrix $\widehat{Q}$ on $\mathbb{Z}^3$ such that

$$\begin{aligned} &\sum_{w'} \widehat{Q}\big((u,v,w),\big(u',v',w'\big)\big) \\ &\quad = Q_0\big((u,v),\big(u',v'\big)\big) \quad \text{for all } \big(u,v,w,u',v'\big), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\sum_{v'} \widehat{Q}\big((u,v,w),\big(u',v',w'\big)\big) \\ &\quad = Q\big((u,w),\big(u',w'\big)\big) \quad \text{for all } \big(u,v,w,u',w'\big), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \widehat{Q}\big((u,v,w),\big(u',v',w'\big)\big) = 0 \quad &\text{for all } \big(u,v,w,u',v',w'\big) \\ &\text{such that } |w| \le |v|+1 \text{ and } |w'| > |v'|+1. \end{aligned} \tag{2.5}$$

In this case, we say that Q is *successfully coupled with* Q_0 *by* $\widehat{Q}$. Condition (2.5) implies that if (U,V,W) is a Markov chain with transition matrix $\widehat{Q}$ such that $|W_0| \le |V_0|+1$, then a.s. for all $n \ge 0$, $|W_n| \le |V_n|+1$. Condition (2.3) (resp. (2.4)) implies that (U,V) (resp. (U,W)) is a Markov chain with transition matrix Q_0 (resp. Q). Note finally, even though this will not be needed, that (2.3) with (2.4) implies that U is a Markov chain.

Let (U,W) be a Markov chain on $\mathbb{Z}^2$ with transition matrix Q. In order to prove recurrence properties, we shall need a kind of irreducibility condition. Set

$$\begin{aligned} \mathcal{B} := \big\{&(U,W) \text{ visits the horizonal axis at some time} \ge 1 \\ &\text{and does so first at the origin}\big\}, \end{aligned} \tag{2.6}$$

and for $p > 0$, write $C(p) = C(p,Q)$ for the property

$$Q^*_{\pm}\{\mathcal{B}\} \ge p, \tag{2.7}$$

where Q_+^* (resp. Q_-^*) denotes the law of the Markov chain (U,W) when it starts at $(U_0,W_0)=(0,1)$ (resp. when it starts at $(U_0,W_0)=(0,-1)$). This property will be used to prove certain stopping times (the τ_i below) are finite. We remind the reader that recurrence is defined in the lines right after (2.1).

LEMMA 2.1. *Let Q and Q_0 be two translation invariant transition matrices on $\mathbb{Z}^2$, in the sense of (2.2), such that Q is successfully coupled with Q_0. Assume that (2.1) holds for Q_0. If $\varphi(Q_0)$ is recurrent, and (2.7) holds for Q for some $p>0$, then (2.1) holds for Q and $\varphi(Q)$ is recurrent.*

Proof. Assume that Q is successfully coupled with Q_0 by some $\widehat{Q}$. Let (U,V,W) be a Markov chain on $\mathbb{Z}^3$ with transition matrix $\widehat{Q}$ starting at $(0,0,0)$ and denote by $\mathbb{P}$ the law of this Markov chain. Let u be arbitrary in $\mathbb{Z}$. Since $\varphi(Q_0)$ is recurrent, $(U_n,V_n)=(u,0)$ infinitely often $\mathbb{P}$-a.s. But since Q is successfully coupled with Q_0, it must hold at every time n at which $(U_n,V_n)=(u,0)$, that $|W_n|\le 1$. This implies that the event $\mathcal{E}_n:=\{U_n=u,W_n\in\{-1,0,+1\}\}$ occurs infinitely often $\mathbb{P}$-a.s. Let $\sigma(1)<\sigma(2)\dots$ be the sequence of the successive times at which $\mathcal{E}_t$ occurs and define the σ-fields

$$\mathcal{K}_t=\sigma\big((U_n,W_n);n\le t\big),\qquad \mathcal{L}_t=\mathcal{K}_{\sigma(t)}.$$

Further, define the events

$$\mathcal{B}_n=\{W_{\sigma(n)}=0\}=\big\{(U_{\sigma(n)},W_{\sigma(n)})=(u,0)\big\},$$

and

$$\begin{aligned}\widetilde{\mathcal{B}}_n&=\mathcal{B}_n\cup\mathcal{B}_{n+1}\\ &=\mathcal{B}_n\cup\{W_{\sigma(n)}=+1,W_{\sigma(n+1)}=0\}\\ &\qquad\cup\{W_{\sigma(n)}=-1,W_{\sigma(n+1)}=0\}.\end{aligned}\tag{2.8}$$

We shall complete the proof by showing that

$$\widetilde{\mathcal{B}}_n \text{ occurs infinitely often } \mathbb{P}\text{-a.s.}\tag{2.9}$$

Clearly this suffices for recurrence, since

$$\bigcup_{n\ge1}\widetilde{\mathcal{B}}_n=\bigcup_{n\ge1}\mathcal{B}_n.$$

Now $\widetilde{\mathcal{B}}_n\in\mathcal{L}_{n+1}$. Moreover, it holds

$$\begin{aligned}\mathbb{P}\{\widetilde{\mathcal{B}}_n\mid\mathcal{L}_n\}&=1_{\{W_{\sigma(n)=0}\}}+\mathbb{P}\{W_{\sigma(n+1)}=0\mid\mathcal{L}_n\}1_{\{W_{\sigma(n)=1}\}}\\ &\qquad+\mathbb{P}\{W_{\sigma(n+1)}=0\mid\mathcal{L}_n\}1_{\{W_{\sigma(n)=-1}\}}\\ &\ge 1_{\{W_{\sigma(n)=0}\}}+Q_+^*\{\mathcal{B}\}1_{\{W_{\sigma(n)=1}\}}+Q_-^*\{\mathcal{B}\}1_{\{W_{\sigma(n)=-1}\}}\\ &\qquad\text{(by Markov property and translation invariance)}\\ &\ge p\quad\text{(by (2.7))}.\end{aligned}$$

Consequently,

$$\sum_{n\geq 1}\mathbb{P}\{\widetilde{\mathcal{B}}_n \mid \mathcal{L}_n\} \geq \sum_{n\geq 1} p = \infty.$$

The conditional Borel–Cantelli lemma (Theorem 12.15 in [14]) now implies that (2.9) holds. □

This lemma proves recurrence of (the trace on the horizontal axis of) a Markov chain which uses only one transition matrix Q. Benjamini's process is built up by concatenating excursions from Markov chains with more than one transition matrix. We shall use arguments very similar to the preceding lemma, but involving different transition matrices, in Theorem 2.5.

2.1.2. *Coupling of a modified walk with a Markov process in* $\mathbb{Z}^2$. Throughout this subsection, we let $(Q_i, 0 \leq i \leq k)$, with $k \leq \infty$, be a sequence of transition matrices on $\mathbb{Z}^2$, translation invariant in the sense of (2.2), and such that for all $i \in [1,k]$, Q_i is successfully coupled with Q_0 by some $\widehat{Q}_i$. We assume that for some $p > 0$ independent of i, (2.7) holds for all Q_i, $i \leq k$. Note that by Lemma 2.1, if Q_0 satisfies (2.1) and if $\varphi(Q_0)$ is recurrent, then the Q_i for $1 \leq i \leq k$ automatically satisfy (2.1) as well. Set $P_i = \varphi(Q_i)$ for all $i \leq k$, and let F and G be the functions as defined in the Introduction. We denote by $(S_n, i(n))$ the process defined by (1.2).

Let us now define the coupling between the (generalized version of) the Benjamini process $\{S_n\}$ and the Markov process with transition matrix Q_0 on $\mathbb{Z}^2$. In order to carry this out, we note that for all i, there exists $G_i : \mathbb{Z}^3 \times [0,1] \to \mathbb{Z}^3$, such that if R is a uniformly distributed random variable on $[0,1]$, then

$$\widehat{Q}_i\big((u,v,w),(u',v',w')\big) = \mathbb{P}\big[G_i\big((u,v,w),R\big) = (u',v',w')\big].$$

Here and in the sequel we write $\mathbb{P}$ for the measure governing the choice of one or several uniform random variables on $[0,1]$. It will be clear from the context to which random variables this applies. Let $\mathbb{N}_k := \{1,\dots,k\}$ and define $\widehat{F} : \mathbb{Z} \times \mathbb{N}_k \times \mathbb{Z} \times \bigcup_{n\geq 0}(\mathbb{Z}\times\mathbb{N}_k)^n \times [0,1] \to \mathbb{N}_k$, by

$$\widehat{F}\big((w,i),u_0,(u_\ell,i_\ell)_{\ell\leq n},a\big) = i \quad \text{if } |w| \geq 1 \tag{2.10}$$

and by

$$\widehat{F}\big((0,i),u_0,(u_\ell,i_\ell)_{\ell\leq n},a\big) = F\big(a,u_0,(u_\ell,i_\ell)_{\ell\leq n}\big). \tag{2.11}$$

This function $\widehat{F}$ determines the index i in Q_i which will govern the steps in our modified random walk over a certain random time interval, as we make more precise now. Let $(A_\ell)_{\ell\geq 1}$ and $(B_\ell)_{\ell\geq 1}$ be two independent sequences of i.i.d., uniformly distributed random variables on $[0,1]$. Let U_0 be a random variable distributed like S_0, independent of $(A_\ell)_{\ell\geq 1}$ and $(B_\ell)_{\ell\geq 1}$. Let

$$\mathcal{F}_n = \sigma\big((A_\ell,B_\ell);\ell\leq n\big) \vee \sigma(U_0).$$

Define $\widehat{U}_\ell = (U_\ell, V_\ell, W_\ell)$ and I_ℓ for $\ell \geq 1$ by the following: set $\tau_0 = 0$, $V_0 = W_0 = 0$ and for $n \geq 1$

$$\tau_{n+1} = \inf\{\ell > \tau_n : W_\ell = 0\}. \tag{2.12}$$

In Lemma 2.3, we shall show that if $\varphi(Q_0)$ is recurrent, and the Q_i satisfy (2.7), then these stopping times are $\mathbb{P}$-a.s. finite. For $m \geq 0$, set (one can take $I_0 = 1$, or any other value, since $W_0 = 0$, I_1 will not depend on I_0)

$$I_{m+1} = \widehat{F}\big((W_m, I_m), U_0, (U_{\tau_\ell}, I_{\tau_\ell})_{\{\tau_\ell \leq m\}}, A_{m+1}\big), \tag{2.13}$$

with $\widehat{F}$ as defined in (2.10), and

$$\widehat{U}_{m+1} = G_{I_{m+1}}(\widehat{U}_m, B_{m+1}). \tag{2.14}$$

Note that $(\widehat{U}_m, (\tau_l)_{\{\tau_l \leq m\}}, I_m)$ is $\mathcal{F}_m$-measurable. Note also that (2.10) implies that $I_{m+1} = I_m$ when $|W_m| \geq 1$. This ensures that for all $m \in [\tau_\ell + 1, \tau_{\ell+1}]$, $I_m = I_{\tau_\ell+1}$.

LEMMA 2.2. *The process $(U_n, V_n)_{n\geq 0}$ is a Markov chain with transition matrix Q_0.*

Proof. For $n \geq 0$ and (u', v') in $\mathbb{Z}^2$,

$$\begin{aligned}
&\mathbb{P}\big\{(U_{n+1}, V_{n+1}) = \big(u', v'\big) \mid \mathcal{F}_n\big\}\\
&\quad= \sum_{w',i} \mathbb{P}\big\{\widehat{U}_{n+1} = \big(u', v', w'\big) \text{ and } I_{n+1} = i \mid \mathcal{F}_n\big\}\\
&\quad= \sum_{w',i} \mathbb{P}\big\{G_i(\widehat{U}_n, B_{n+1}) = \big(u', v', w'\big) \text{ and}\\
&\qquad\quad \widehat{F}\big((W_n, I_n), U_0, (U_{\tau_\ell}, I_{\tau_\ell})_{\{\tau_\ell \leq n\}}, A_{n+1}\big) = i \mid \mathcal{F}_n\big\}\\
&\quad= \sum_{w',i} \widehat{Q}_i\big(\widehat{U}_n, \big(u', v', w'\big)\big)\mathbb{P}\{I_{n+1} = i \mid \mathcal{F}_n\}\\
&\quad= Q_0\big((U_n, V_n), \big(u', v'\big)\big),
\end{aligned}$$

from which we deduce the Markov property. □

LEMMA 2.3. *Let (U_m, I_m) be the process defined by (2.13) and (2.14). Assume that all Q_i, $1 \leq i \leq k$, satisfy (2.7) and are successfully coupled with Q_0. Finally, assume that (2.1) holds for Q_0 and that $\varphi(Q_0)$ is recurrent. Then $\mathbb{P}$-a.s. it holds $\tau_n < \infty$ for all $n \geq 0$. In particular all Q_i, $1 \leq i \leq k$, satisfy (2.1), and we set $P_i = \varphi(Q_i)$. Moreover,*

(a) *For all $n \geq 0$, the law of $U_{\tau_{n+1}} - U_{\tau_n}$ given $\mathcal{G}_n := \mathcal{F}_{\tau_n} \vee \sigma(I_{\tau_n+1})$ is $P_{I_{\tau_n+1}}$.*

(b) *For all $n \geq 0$,*

$$I_{\tau_{n+1}} = I_{\tau_n+1} = F\big(A_{\tau_n+1}, U_0, (U_{\tau_\ell}, I_{\tau_\ell})_{\{\tau_\ell \leq n\}}\big). \tag{2.15}$$

(c) *For* $(i,u)\in\{1,\dots,k\}\times\mathbb{Z}$, *write* $\mathbb{P}_{i,u}$ *for the law of the Markov chain on* $\mathbb{Z}^2$ *with transition matrix* Q_i, *starting from* $(u,0)$ *and stopped at the first time the* w*-coordinate returns to* 0. *For all* $n\geq 0$, *given* $\mathcal{G}_n$, *the law of the excursion from the* U*-axis*

$$(U_{\tau_n+\ell},W_{\tau_{n+\ell}})_{0\leq\ell\leq\tau_{n+1}-\tau_n}$$

is $\mathbb{P}_{I_{\tau_n},U_{\tau_n}}$.

Proof. The proof is by induction on n. First, take $n=0$. Then $\tau_1<\infty$ a.s. by virtue of Lemma 2.1. Now part (a) for $n=0$ is contained in part (c) for $n=0$. Part (b) for $n=0$ follows from (2.10), (2.11) and (2.13). In particular, it follows from (2.10) and from the definition of the τ's that I_m can only change when $W_m=0$, so that I_m is constant on the intervals $[\tau_n+1,\tau_{n+1}]$ for $n=0$. Equation (2.15) follows from (2.10) and (2.13). The proof of part (c) for $n=0$ is very similar to the one of Lemma 2.2. We skip the details.

Now assume that $\tau_N<\infty$ and parts (a)–(c) have been proven for $n\leq N$. Then given $\mathcal{G}_n$, on the event $\{I_{\tau_N+1}=i\}$, $\tau_{N+1}-\tau_N$ is equal in law to τ_1 for the Markov chain with transition matrix Q_i started at $(U_{\tau_N},0)$. Lemma 2.1 implies that this τ_1 is finite a.s. Thus, $\tau_{N+1}<\infty$ $\mathbb{P}$-a.s. Now statements (a)–(c) for $n=N+1$ can be proven as in the case $n=0$. Again we skip the details. □

The following lemma is almost immediate from Lemma 2.3 and the strong Markov property. The lemma shows that a sample path of Benjamini's process can be built up from a sequence of excursions, by identifying the initial point of each excursion with the endpoint of the preceding excursion. This leads to our principal recurrence result, Theorem 2.5, which deduces recurrence of a Benjamini process from simple and known recurrence properties of some of the excursions.

LEMMA 2.4. *The processes* $(S_0,(S_n,i(n))_{n\geq 1})$ *defined by* (1.2) *and* $(U_0,(U_{\tau_n},I_{\tau_n})_{n\geq 1})$ *have the same distribution.*

2.1.3. *Recurrence properties and examples.*

THEOREM 2.5. *Let* $(Q_i,0\leq i\leq k)$ *be a sequence of transition matrices on* $\mathbb{Z}^2$ *which are translation invariant in the sense of* (2.2). *Assume that for all* $1\leq i\leq k, Q_i$ *is successfully coupled with* Q_0. *Assume further that* Q_0 *satisfies* (2.1), $P_0=\varphi(Q_0)$ *is recurrent and that all* Q_i, $1\leq i\leq k$, *satisfy* (2.7) *for some* $p>0$, *independent of* i. *Then for any process* $(S_0,(S_n,i(n))_{n\geq 1})$ *that satisfies* (1.1) *with* $P_i=\varphi(Q_i)$, $\{S_n\}_{n\geq 0}$ *is recurrent.*

Proof. Let $(S_0,(S_n,i(n))_{n\geq 1})$ be a process satisfying (1.1). Without loss of generality, we can assume that $S_0=0$. Such process can be defined by (1.2) for some functions F and G. Let $(\widehat{U}_m,I_m)$ be the process defined by (2.13) and (2.14) with $\widehat{F}$ defined by (2.10) and (2.11), and with $\widehat{U}_0=(0,0,0)$. Let $\mathbb{P}$

be the measure governing the choice of the independent uniformly distributed random variables used to define the process $(\widehat{U}_m, I_m)$. We still denote by τ_n, $n \geq 0$, the successive return times to 0 of W, as defined in (2.12). Lemma 2.4 states that $(S_n, i(n))_{n\geq 0}$ is equal in law to $(U_{\tau_n}, I_{\tau_n})_{n\geq 0}$.

We now prove that $\{U_{\tau_n}\}$ is recurrent on $\mathbb{Z}$. To this end, observe that (U, V) is a Markov chain with transition matrix given by Q_0 and that $P_0 = \varphi(Q_0)$ is recurrent. This implies that for any fixed u, $(U_\ell, V_\ell) = (u, 0)$ for infinitely many ℓ with $\mathbb{P}$-probability 1. Moreover, by construction, $|W| \leq |V| + 1$. So

$$(U_\ell, W_\ell) \in \{(u,0), (u,-1), (u,1)\} \quad \text{infinitely often},$$

still with $\mathbb{P}$-probability 1. Denote by σ_n, $n \geq 0$, the successive return times to $\{(u,0),(u,-1),(u,1)\}$ of (U, W).

From here on, we can follow the proof of Lemma 2.1 (which is the case $k = 1$). We redefine

$$\mathcal{K}_t := \sigma\big((U_n, W_n, I_n); n \leq t\big), \qquad \mathcal{L}_t = \mathcal{K}_{\sigma(t)},$$

and we replace the condition (2.7) by (with the event $\mathcal{B}$ as in (2.6))

$$Q_{i,\pm 1}\{\mathcal{B}\} \geq p, \tag{2.16}$$

where $Q_{i,1}$ (resp. $Q_{i,-1}$) denotes the law of the Markov chain with transition matrix Q_i when it starts at $(0,1)$ (resp. at $(0,-1)$). We further redefine the events

$$\mathcal{B}_n = \{W_{\sigma(n)} = 0\} = \{(U_{\sigma(n)}, W_{\sigma(n)}) = (u, 0)\},$$

and $\widetilde{\mathcal{B}}_n = \mathcal{B}_n \cup \mathcal{B}_{n+1}$. The proof will be complete if we show that

$$\widetilde{\mathcal{B}}_n \text{ occurs infinitely often } \mathbb{P}\text{-a.s.}$$

Now $\widetilde{\mathcal{B}}_n \in \mathcal{L}_{n+1}$ and on $\{I_{\sigma(n)} = i\}$, it holds

$$\begin{aligned}\mathbb{P}\{\widetilde{\mathcal{B}}_n \mid \mathcal{L}_n\} = 1_{\{W_{\sigma(n)}=0\}} &+ \mathbb{P}\{W_{\sigma(n+1)} = 0 \mid \mathcal{L}_n\} 1_{\{W_{\sigma(n)}=1\}} \\ &+ \mathbb{P}\{W_{\sigma(n+1)} = 0 \mid \mathcal{L}_n\} 1_{\{W_{\sigma(n)}=-1\}} \\ \geq 1_{\{W_{\sigma(n)}=0\}} &+ Q_{i,1}\{\mathcal{B}\} 1_{\{W_{\sigma(n)}=1\}} + Q_{i,-1}\{\mathcal{B}\} 1_{\{W_{\sigma(n)}=-1\}},\end{aligned}$$

by using that given $\mathcal{L}_n$ and on $\{I_{\sigma(n)} = i\}$, the law of $(U_{\sigma(n)+k}, W_{\sigma(n)+k})_k$ stopped at the first positive time W reaches 0, is the same as the law of the Markov chain with transition matrix Q_i starting at $(u, W_{\sigma(n)})$ and stopped at the first time W reaches 0, and then by using the translation invariance of Q_i. Next, (2.16) implies

$$\mathbb{P}\{\widetilde{\mathcal{B}}_n \mid \mathcal{L}_n\} \geq p.$$

We conclude by using the conditional Borel–Cantelli lemma as in the proof of Lemma 2.1. □

We state now an analogous result which can give examples of transient processes. We say that a process is *transient* if almost surely it comes back a finite number of times to each site. A law is said to be transient if the associated random walk is transient.

THEOREM 2.6. *Let $(Q_i, 0 \le i \le k)$ be a sequence of transition matrices on $\mathbb{Z}^2$ which are translation invariant in the sense of (2.2) and satisfy (2.1). Assume that for all $1 \le i \le k$, Q_0 is successfully coupled with Q_i. Assume further that $P_0 = \varphi(Q_0)$ is transient. Then for any process $(S_0, (S_n, i(n))_{n\ge 1})$, which satisfies (1.1) with $P_i = \varphi(Q_i)$, $\{S_n\}_{n\ge 0}$ is transient.*

The proof of this result is analogous to the proof of Theorem 2.5 and left to the reader. Note the asymmetry. The hypothesis is that Q_0 is successfully coupled with Q_i, instead of Q_i with Q_0.

Theorem 2.5 solves in particular the recurrence part in Benjamini's original question. This is explained in the following example. Here and in the remainder of this paper "Cauchy law" will always be short for "symmetric Cauchy law."

EXAMPLE 2.7. Let Q_0 be the transition matrix of a simple random walk on $\mathbb{Z}^2$: $Q_0((u,v),(u',v')) = 1/4$ if $(u',v') \in \{(u, v\pm 1), (u\pm 1, v)\}$. We call $P_0 := \varphi(Q_0)$ the "discrete Cauchy law." Observe that P_0 is recurrent. Benjamini's process uses in an arbitrary order jumps of law P_0 and jumps of law P_1, with $P_1(1) = P_1(-1) = 1/2$. Proving Benjamini's process is recurrent using Theorem 2.5 would require finding Q_1 such that $P_1 = \varphi(Q_1)$, and then to prove that Q_0 and Q_1 are both successfully coupled with Q_0. Such Q_1 does not exist. So instead we will define $\widetilde{Q}_1$, such that $\widetilde{P}_1 = \varphi(\widetilde{Q}_1)$ satisfies

$$\widetilde{P}_1\{\pm 1\} = 1/4 \quad \text{and} \quad \widetilde{P}_1\{0\} = 1/2. \tag{2.17}$$

As far as recurrence is concerned, there is no difference between using P_1 or $\widetilde{P}_1$, as we show in Lemma 2.8 below.

So let us now define $\widetilde{Q}_1$ and the two different couplings. Assume $(u, v, w) \in \mathbb{Z}^3$ are given. Let (U, V) be a simple random walk on $\mathbb{Z}^2$ starting from (u, v) and define the process W by $W_0 = w$ and $W_n = 0$ for all $n > 0$. W is deterministic and hence independent of (U, V). Then $\widehat{U} = (U, V, W)$ and (U, W) are Markov chains and $\widetilde{Q}_1$, the transition matrix of (U, W), has entries

$$\widetilde{Q}_1\big[(u,w),(u\pm 1, 0)\big] = 1/4 \quad \text{and} \quad \widetilde{Q}_1\big[(u,w),(u,0)\big] = 1/2 \quad \text{for all } u, w.$$

Moreover, it is straightforward that $\widetilde{Q}_1$ is successfully coupled with Q_0 and satisfies (2.7) for $p = 1/2$. Observe also that (2.17) holds, as claimed.

Next we define the coupling of Q_0 with itself. We still let (U, V) be a simple random walk on $\mathbb{Z}^2$ starting from (u, v). But this time W is defined by $W_0 = w$ and for $n \geq 0$, by

$$W_{n+1} - W_n = \begin{cases} V_{n+1} - V_n & \text{if } W_n V_n > 0 \text{ or if } V_n = 0 \text{ and } W_n > 0, \\ -(V_{n+1} - V_n) & \text{otherwise.} \end{cases} \tag{2.18}$$

Then $\widehat{U} = (U, V, W)$ and (U, W) are Markov chains, and the transition matrix of (U, W) is Q_0. Moreover, it is straightforward that this gives a successful coupling of Q_0 with itself, and that Q_0 satisfies (2.7) for $p = 1/4$. Thus, the hypotheses of Theorem 2.5 are satisfied by $(\widetilde{P}_1, P_2)$, where $P_2 = P_0 = \varphi(Q_0)$. Therefore for all processes $(S_0, (S_n, i(n))_{n\geq 1})$ satisfying (1.1) (or equivalently, for all processes $(S_0, (S_n, i(n))_{n\geq 1})$ defined by (1.2)), the resulting processes S will be recurrent. The fact that Benjamini's process is recurrent is now a consequence of Lemma 2.8.

REMARK. In Section 4, we shall use some consequences of this example in the special case when $k = 2$ and the corresponding distributions $\widetilde{Q}_1$ and $Q_2 = Q_0$ are as defined a few lines before (2.18). Let now I_n and (U_n, V_n, W_n) be the processes defined by (2.13) and (2.14). Recall (U_n, V_n) is a simple random walk on $\mathbb{Z}^2$. First, it needs to be pointed out that in this special case, the function G_2 can be defined such that (2.18) is valid for $n \in [\tau_\ell, \tau_{\ell+1})$ for some ℓ with $I_{n+1} = I_{\tau_\ell + 1} = 2$, and the function G_1 is defined such that when $I_{n+1} = 1$, then $W_{n+1} = W_n = 0$. We claim that

$$\begin{aligned} U_n = V_n = 0, \quad & V_{n+1} = -1 \text{ and } W_n \in \{-1, 0, 1\} \\ & \text{together imply } U_n = W_n = 0 \text{ or } U_{n+1} = W_{n+1} = 0. \end{aligned} \tag{2.19}$$

To see this, assume that $U_n = V_n = 0$ and $V_{n+1} = -1$. Then $V_{n+1} - V_n = -1$. If $W_n = 0$, then $U_n = W_n = 0$ by assumption and there is nothing to prove. Assume then that $W_n = +1$. This excludes $I_{n+1} = 1$, because when $I_{n+1} = 1$, then $W_n = W_{n+1} = 0$. So $I_{n+1} = 2$ and (2.18) applies. Thus,

$$W_{n+1} - W_n = V_{n+1} - V_n = -1, \quad \text{whence } W_{n+1} = W_n - 1 = 0.$$

Moreover, the jump from (U_n, V_n) to (U_{n+1}, V_{n+1}) can only be of size 1 (because (U_n, V_n) is a simple random walk on $\mathbb{Z}^2$). But there already is a change of size 1 in the V-direction. Thus, we can only have $U_{n+1} - U_n = 0$. This proves our claim in case $W_n = 1$. The case $W_n = -1$ is entirely similar, since now $W_{n+1} - W_n = -(V_{n+1} - V_n)$. Thus, (2.19) holds in general.

LEMMA 2.8. *Let $(\widetilde{P}_i, 1 \leq i \leq k)$ be a sequence of probability distributions on $\mathbb{Z}$. Assume that for all processes $(S_0, (S_n, i(n))_{n\geq 1})$ satisfying (1.1) with $\widetilde{P}_i$ instead of P_i, the process S is recurrent. Let now $\mathcal{I} \subset \mathbb{N}_k$ be given and let $(P_i, 1 \leq i \leq k)$ be defined by $P_i = \widetilde{P}_i$ if $i \notin \mathcal{I}$, and if $i \in \mathcal{I}$, $P_i\{u\} = \widetilde{P}_i\{U = u \mid U \neq 0\}$, for $u \neq 0$, with U a random variable of law $\widetilde{P}_i$.*

Then for all processes $(S_0, (S_n, i(n))_{n\geq 1})$ satisfying (1.1), S is recurrent as well.

Proof. First note that the hypothesis on the $\widetilde{P}_i$'s means that for any choice of $\widetilde{F} : [0,1] \times \bigcup_{n\geq 0}(\mathbb{Z}\times\mathbb{N}_k)^n \to \mathbb{N}_k$, the process defined by (1.2) (with $\widetilde{F}$ and $\widetilde{G}$ in place of F and G respectively, and $\widetilde{G}$ associated to the $\widetilde{P}_i$'s) is recurrent.

The intuition for this lemma is clear. A walker using $\widetilde{P}_i$ as distribution for his displacement stands still with probability $\widetilde{P}_i(0)$. In fact when he arrives at a new site, he stands still a geometric number of times and then makes a displacement with distribution P_i. The standing still has no influence on the collection of sites visited by the walker and hence does not influence recurrence. Recurrence will be the same whether $\widetilde{P}_i$ or P_i is used. A complication arises because we have to deal not with sequences (S_n) but with sequences $(S_n, i(n))$, and even the latter sequences are not Markovian.

Let now $(S_0, (S_n, i(n))_{n\geq 1})$ be a process satisfying (1.1). To simplify, we take $S_0 = 0$. As is explained in the Introduction, such a process can be constructed with functions $F : [0,1]\times\mathbb{Z}\times\bigcup_{n\geq 0}(\mathbb{Z}\times\mathbb{N}_k)^n \to \mathbb{N}_k$, $G : [0,1]\times\mathbb{N}_k \to \mathbb{Z}$ and independent sequences $(A_\ell)_{\ell\geq 0}$ and $(B_\ell)_{\ell\geq 0}$ of i.i.d. uniformly distributed random variables on $[0,1]$: for $n \geq 0$,

$$i(n+1) = F\bigl(A_{n+1}, S_0, \bigl(S_\ell, i(\ell)\bigr)_{\ell\leq n}\bigr)$$

and

$$S_{n+1} - S_n = G\bigl(B_{n+1}, i(n+1)\bigr).$$

Here G is such that the law of $G(B_1, i)$ is P_i.

Let now $\widetilde{F} : [0,1]\times\mathbb{Z}\times\bigcup_{n\geq 0}(\mathbb{Z}\times\mathbb{N}_k)^n \to \mathbb{N}_k$ be defined by

$$\widetilde{F}\bigl(a, s(0), \bigl(s(\ell), j(\ell)\bigr)_{\ell\leq n}\bigr) = j(n), \tag{2.20}$$

if $j(n) \in \mathcal{I}$ and $s(n) = s(n-1)$, and otherwise by

$$\widetilde{F}\bigl(a, s(0), \bigl(s(\ell), j(\ell)\bigr)_{\ell\leq n}\bigr) = F\bigl(a, s(0), \bigl(s(t_\ell), j(t_\ell)\bigr)_{\ell\leq m}\bigr), \tag{2.21}$$

where $t_0 = 0$,

$$t_\ell = \inf\bigl\{r \in (t_{\ell-1}, n] : s(r) \neq s(r-1) \text{ or } j(r) \notin \mathcal{I}\bigr\} \quad \text{for } \ell \geq 1, \tag{2.22}$$

and

$$m = \sup\{\ell : t_\ell < \infty\}. \tag{2.23}$$

Note that (2.20)–(2.23) are merely the *definitions* of the non-random functions t_ℓ, m and $\widetilde{F}$ at a generic point $(a, s(0), (s(\ell), j(\ell))_{\ell\leq n})$ of their domains. Note also that, by convention, $t_\ell = \infty$ if the set in the right hand side of (2.22) is empty. In particular, this is the case for $\ell > n$.

Let $\widetilde{G} : [0,1]\times\mathbb{N}_k \to \mathbb{Z}$ be such that the law of $\widetilde{G}(B_1, i)$ is $\widetilde{P}_i$. Define the random quantities $\widetilde{S}_0$ and $(\widetilde{S}_n, \widetilde{i}(n))$ by $\widetilde{S}_0 = 0$ and for $n \geq 0$,

$$\widetilde{i}(n+1) = \widetilde{F}\bigl(A_{n+1}, \widetilde{S}_0, \bigl(\widetilde{S}_\ell, \widetilde{i}(\ell)\bigr)_{\ell\leq n}\bigr), \tag{2.24}$$

and

$$\widetilde{S}_{n+1} - \widetilde{S}_n = \widetilde{G}\big(B_{n+1}, \widetilde{i}(n+1)\big). \tag{2.25}$$

Equation (2.20) implies that

$$\widetilde{i}(n+1) = \widetilde{i}(n) \quad \text{if } \widetilde{i}(n) \in \mathcal{I} \text{ and } \widetilde{S}_n = \widetilde{S}_{n-1}. \tag{2.26}$$

Let $\rho_0 = 0$ and $\rho_\ell = \inf\{r > \rho_{\ell-1} : \widetilde{S}_r \neq \widetilde{S}_{r-1} \text{ or } \widetilde{i}(r) \notin \mathcal{I}\}$ for $\ell \geq 1$. Note that ρ_n is essentially the value of t_n at the random place $(\widetilde{S}_\ell, \widetilde{i}(\ell))_{\ell \leq n}$. By definition

$$\widetilde{S}_r = \widetilde{S}_{\rho_\ell} \quad \text{for all } r \in [\rho_\ell, \rho_{\ell+1}), \tag{2.27}$$

and

$$\widetilde{S}_{\rho_{\ell+1}} \neq \widetilde{S}_{\rho_\ell} \quad \text{if } \widetilde{i}(\rho_\ell + 1) \in \mathcal{I}. \tag{2.28}$$

Moreover, (2.26) implies (by induction on r) that $\widetilde{i}(r) = \widetilde{i}(\rho_\ell + 1)$ for all $r \in (\rho_\ell, \rho_{\ell+1}]$, and

$$\begin{aligned} \widetilde{i}(\rho_{\ell+1}) = \widetilde{i}(\rho_\ell + 1) &= \widetilde{F}\big(A_{\rho_\ell+1}, \widetilde{S}_0, \big(\widetilde{S}_r, \widetilde{i}(r)\big)_{r \leq \rho_\ell}\big) \\ &= F\big(A_{\rho_\ell+1}, \widetilde{S}_0, \big(\widetilde{S}_{\rho_r}, \widetilde{i}(\rho_r)\big)_{r \leq \ell}\big), \end{aligned} \tag{2.29}$$

where the last equality follows from (2.21). Now, for any $i \in \mathbb{N}_k$ and $u \in \mathbb{Z}$, if $\mathcal{F}_n = \sigma((A_\ell, B_\ell)_{\ell \leq n})$,

$$\mathbb{P}\big\{\widetilde{S}_{\rho_{\ell+1}} - \widetilde{S}_{\rho_\ell} = u \,|\, \widetilde{i}(\rho_\ell + 1) = i, \mathcal{F}_{\rho_\ell}\big\} = P_i(u). \tag{2.30}$$

Indeed when $i \in \mathcal{I}$ and $u \neq 0$, the left-hand side is equal to

$$\sum_{K \geq 1} \mathbb{P}\big\{\big\{\widetilde{G}(B_{\rho_\ell+K}, i) = u\big\} \cap \{\rho_{\ell+1} - \rho_\ell = K\} \,|\, \widetilde{i}(\rho_\ell + 1) = i, \mathcal{F}_{\rho_\ell}\big\},$$

which is equal to

$$\sum_{K \geq 1} \widetilde{P}_i(0)^{K-1} \widetilde{P}_i(u) = P_i(u).$$

If $i \in \mathcal{I}$ and $u = 0$, both sides of (2.30) equal 0 by (2.28) and the definition of $\mathcal{I}$. When $i(\rho_\ell + 1) = i$ and $i \notin \mathcal{I}$, then $\rho_{\ell+1} = \rho_\ell + 1$ and (2.30) follows from the fact that $\widetilde{P}_i = P_i$.

Finally, we claim that (2.29) and (2.30) show that $(\widetilde{S}_{\rho_\ell}, \widetilde{i}(\rho_\ell))$ has the same law as $(S_\ell, i(\ell))$: indeed we shall show by induction on ℓ that for any sequence $j(1), \ldots, j(\ell) \in \mathbb{N}_k$, $u_1, \ldots, u_\ell \in \mathbb{Z}$,

$$\mathbb{P}\big\{\widetilde{S}_{\rho_{\ell+1}} - \widetilde{S}_{\rho_\ell} = u_{\ell+1}, \widetilde{i}(\rho_{\ell+1}) = j(\ell+1), \ldots, \widetilde{S}_{\rho_1} = u_1, \widetilde{i}(\rho_1) = j(1)\big\}, \tag{2.31}$$

is equal to

$$\mathbb{P}\big\{S_{\ell+1} - S_\ell = u_{\ell+1}, i(\ell+1) = j(\ell+1), \ldots, S_1 = u_1, i(1) = j(1)\big\}. \tag{2.32}$$

But (2.31) is equal to

$$P_{j(\ell+1)}(u_{\ell+1}) \mathbb{P}\big\{\widetilde{i}(\rho_{\ell+1}) = j(\ell+1), \ldots, \widetilde{S}_{\rho_1} = u_1, \widetilde{i}(\rho_1) = j(1)\big\},$$

by (2.30). By using (2.29), we see that the second factor in this last expression is equal to

$$\mathbb{P}\{F(A_{\rho_\ell+1}, s_0, (s_r, j(r))_{r\le\ell}) = j(\ell+1)\} \\ \times \mathbb{P}\{\widetilde{S}_{\rho_\ell} - \widetilde{S}_{\rho_{\ell-1}} = u_\ell, \widetilde{i}(\rho_\ell) = j(\ell), \dots, \widetilde{S}_{\rho_1} = u_1, \widetilde{i}(\rho_1) = j(1)\},$$

where for all r, $s_r := u_1 + \cdots + u_r$. Then an induction procedure shows that (2.31) is equal to (2.32), as claimed.

Moreover, by assumption $\widetilde{S}$ is recurrent and (2.27) implies that $(\widetilde{S}_{\rho_\ell}, \ell \ge 0)$ is also recurrent. This proves the lemma. □

We finish with this last class of examples.

EXAMPLE 2.9. Take for Q_0 the transition matrix of a Markov chain (U, V) such that U and V are both Markovian and independent of each other, and such that $P_0 = \varphi(Q_0)$ is recurrent. Assume that for all $i \in [1, k]$, Q_i is the transition matrix of a Markov chain (U, W_i) such that U and W_i are both Markovian and independent of each other (the chain U being the same for Q_0 and for Q_i). Assume that all Q_i's are translation invariant (note that this hypothesis only concerns the Markov chain U). Suppose also that for all i it is possible to couple the chains V and W_i such that (V, W_i) is Markovian and such that if $|W_i(0)| \le |V(0)| + 1$ then for all $n \ge 0$, $|W_i(n)| \le |V(n)| + 1$. Let now U be a chain independent of this Markov process (V, W_i). Then (U, V) and (U, W_i) are both Markovian respectively with transition matrices Q_0 and Q_i. This coupling of (U, V) and (U, W_i) shows that Q_i is successfully coupled with Q_0. Assume also that the Q_i's, $i \le k$, satisfy (2.7) for some positive p, uniformly in i. Then the hypotheses of Theorem 2.5 are satisfied.

This can be applied to the following: let (A_n) and (B_n) be two independent sequences of i.i.d. random variables uniformly distributed on $[0, 1]$. Let $p \in [0, 1/2)$ and let $U(n) = \sum_{i=1}^n (1_{\{A_i \ge p\}} - 1_{\{A_i < 1-p\}})$. Let V be the simple random walk on $\mathbb{Z}$ defined by $V(0) = v_0$ and

$$V(n) - V(n-1) = 1_{\{B_n < 1/2\}} - 1_{\{B_n \ge 1/2\}}.$$

Let $(p_i(w) : i \ge 1 \text{ and } w \ge 0)$ be such that $p_i(w) \in [0, 1/2]$ for all $w \in \mathbb{Z}$. Define W_i by $W_i(0) = 0$ and on the event $\{W_i(n-1) = w\}$,

$$\begin{aligned} W_i(n) - w = {} & [1_{\{B_n < p_i(w)\}} - 1_{\{B_n \ge p_i(w)\}}] 1_{\{w \ge 1\}} \\ & + [1_{\{B_n \ge p_i(w)\}} - 1_{\{B_n < p_i(w)\}}] 1_{\{w \le -1\}} \\ & + [1_{\{B_n < 1/2 - p_i(0)\}} - 1_{\{B_n \ge 1/2 + p_i(0)\}}] 1_{\{w = 0\}}. \end{aligned}$$

Then one immediately checks that $|W_i(n)| \le |V(n)| + 1$ for all $n \ge 0$, and the resulting transition matrices Q_i are successfully coupled with Q_0. Moreover, Condition (2.16) is satisfied for all Q_i's with $(1 - 2p)/2$ instead of p. Thus, for any such choice of $(p_i(w))$, we can apply Theorem 2.5 and find in this way many examples of recurrent processes. However, given the $(p_i(w))$'s, it is usually not easy to describe explicitly the associated laws P_i.

2.2. Continuous case. We present now an analogous coupling method (in the spirit of Example 2.9) when the laws P_i are defined on $\mathbb{R}$, because in this case, by using stochastic calculus, we can give more explicit examples of P_i's, which can be used to construct recurrent processes $\{S_n\}$ (see Proposition 2.15 below).

Let $B^{(1)}$ and $B^{(2)}$ be two independent Brownian motions started at 0. Let (U_0, V_0, W_0) be a random variable in $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, independent of $B = (B^{(1)}, B^{(2)})$. For all $t > 0$, set $U_t = U_0 + B_t^{(1)}$. Let $(\sigma_0, b_0) : [0, +\infty) \to \mathbb{R}^2$ be some Lipschitz functions and $v_0 \geq 0$ some constant. Then (see Exercice 2.14 p. 385 in [13]) the stochastic differential equation

$$V_t = V_0 + \int_0^t \sigma_0(V_s)\, dB_s^{(2)} + \int_0^t b_0(V_s)\, ds + L_t, \tag{2.33}$$

with L the local time in 0 of V, admits a unique solution which is measurable with respect to the filtration generated by $B^{(2)}$.

Consider next $(\sigma, b) : (0, +\infty) \to \mathbb{R}^2$ some locally Lipschitz functions and the stochastic differential equation

$$W_t = W_0 + \int_0^t \sigma(W_s)\, dB_s^{(2)} + \int_0^t b(W_s)\, ds, \quad t < T \wedge e, \tag{2.34}$$

where

$$e = \inf\{t \geq 0 : W_t = +\infty\} \quad \text{and} \quad T = \inf\{t \geq 0 : W_t = 0\}.$$

It is known (see for instance Exercise 2.10 p. 383 in [13]) that if σ and b are locally Lipschitz, then equation (2.34) admits a unique solution W which is measurable with respect to the filtration generated by $B^{(2)}$. When $(U_0, W_0) = (0, 1)$ and when (σ, b) is such that

$$T < e \quad \text{almost surely}, \tag{2.35}$$

we denote by P the law of U_T. Then, like in the discrete case, we have $P = \varphi(\sigma, b)$ for some function φ. In the following, all (σ, b) will be assumed to be locally Lipschitz and such that (2.35) is satisfied. Moreover, for $w > 0$ we will denote by $\mathbb{P}_w^{(\sigma,b)}$ the law of $(W_t)_{t \leq T}$ when $W_0 = w$.

We say that $(\sigma, b) : (0, \infty) \to \mathbb{R}^2$ is *successfully coupled* with (σ_0, b_0) if for any solutions V and W, respectively of (2.33) and (2.34), with $W_0 \leq V_0 + 1$, we have $W_t \leq V_t + 1$ for all $t < T$. Note that, by using a comparison theorem (see [7] Theorem 1.1 p. 437), if for all $v > 0$, $\sigma(v+1) = \sigma_0(v)$ and $b(v+1) \leq b_0(v)$, then (σ, b) is successfully coupled with (σ_0, b_0).

Let $((\sigma_i, b_i), 0 \leq i \leq k)$, $k \leq \infty$, be a sequence of locally Lipschitz functions on $(0, \infty)^2$ such that for all $i \in \mathbb{N}_k$, (σ_i, b_i) is successfully coupled with (σ_0, b_0). For $i \in \mathbb{N}_k$, set $P_i = \varphi(\sigma_i, b_i)$.

Let $F : [0, 1] \times \mathbb{R} \times \bigcup_{n \geq 0} (\mathbb{R} \times \mathbb{N}_k)^n \to \mathbb{N}_k$ be given. This function F determines the index i in (σ_i, b_i) which will govern the steps in our modified random walk over a certain random time interval, as we make more precise now.

Let $(A_n)_{n\geq 1}$ be a sequence of independent random variables uniformly distributed on $[0,1]$. Assume that this sequence is independent of B. Let V be the solution of (2.33), with $V_0=0$. Let (U_0,I_0) be a random variable in $\mathbb{R}\times\mathbb{N}_k$, independent of A and B. Define $(\tau_n)_{n\geq 0}$ an increasing sequence of random times, and the processes $(W_t^*)_{t<\tau_\infty}$ and $(I_t)_{t<\tau_\infty}$, with

$$\tau_\infty := \lim_{n\to\infty} \tau_n, \tag{2.36}$$

by the following: first $\tau_0=0$. Assume then that $(\tau_1,\dots,\tau_n)$ and $(I_t, W_t^*)_{0\leq t\leq \tau_n}$ are defined and measurable with respect to the σ-field

$$\sigma\big((U_s,B_s)_{s\leq\tau_n}\big)\vee\sigma(A_1,\dots,A_n).$$

Assume moreover that $I_t=I_{\tau_{\ell+1}}$ for $t\in(\tau_\ell,\tau_{\ell+1}]$ and $\ell\leq n-1$, and that $W^*_{\tau_\ell}=0$ for all $1\leq\ell\leq n$. For $\ell\leq n$, set $i(\ell)=I_{\tau_\ell}$ and $S_\ell=U_{\tau_\ell}$. Then we define $i(n+1)$ by

$$i(n+1)=F\big(A_{n+1},S_0,\big(S_\ell,i(\ell)\big)_{\ell\leq n}\big),$$

and W^n as the solution of

$$W_t^n = 1+\int_0^t \sigma_{i(n+1)}\big(W_s^n\big)\,dB_s^{\tau_n}+\int_0^t b_{i(n+1)}\big(W_s^n\big)\,ds,\quad t\leq T^{(n)},$$

where $T^{(n)}$ is the first time when W^n reaches 0, and where $B_s^{\tau_n}=B^{(2)}_{\tau_n+s}-B^{(2)}_{\tau_n}$. The process W^n is well defined since B^{τ_n} is independent of $i(n+1)$. Let

$$\tau_{n+1}:=\tau_n+T^{(n)}.$$

Then set

$$W_t^*=W^n_{t-\tau_n}\quad\text{and}\quad I_t=i(n+1)\quad\text{for } t\in(\tau_n,\tau_{n+1}].$$

This defines the sequence τ_n for all n and (W_t^*,I_t) for $t<\tau_\infty$.

Let now $\mathcal{F}_t=\sigma((U_s,B_s,I_s)_{s\leq t\wedge\tau_\infty})$. Then, $(\tau_n)_{n\geq 0}$ is a sequence of $\mathcal{F}_t$-stopping times and like in the discrete setting, we have

LEMMA 2.10. *For all $n\geq 0$, the conditional law of $U_{\tau_{n+1}}-U_{\tau_n}$ given $\mathcal{G}_n:=\mathcal{F}_{\tau_n}\vee\sigma(i(n+1))$ is $P_{i(n+1)}$.*

Proof. Given $\mathcal{G}_n$, the law of $(W_t^n=W^*_{\tau_n+t})_{0\leq t\leq\tau_{n+1}-\tau_n}$ is $\mathbb{P}_1^{(\sigma_{i(n+1)},b_{i(n+1)})}$ and $(U_t^n=U_{\tau_n+t}-U_{\tau_n})_{t\geq 0}$ is a Brownian motion independent of $(W_t^n)_{0\leq t\leq\tau_{n+1}-\tau_n}$. The lemma follows, since by definition $\varphi(\sigma_{i(n+1)},b_{i(n+1)})=P_{i(n+1)}$. □

This lemma implies that the sequence $(S_n,i(n))_{n\geq 0}$ has the same law as the process defined in the Introduction by (1.2) (with $(S_0,i(0))=(U_0,I_0)$). Moreover, we have the following result.

PROPOSITION 2.11. *Assume that there exists positive constants* $0 < \alpha < 1 < \beta$, σ_+ *and* b_+ *such that*

$$0 \le \sigma_i(x) \le \sigma_+ \quad \text{and} \quad |b_i(x)| \le b_+ \quad \text{for all } x \in (\alpha,\beta) \text{ and all } 1 \le i \le k. \tag{2.37}$$

Then τ_∞, *as defined in* (2.36), *is a.s. infinite for any choice of* $(i(n), n \ge 0)$.

Proof. We start with a lemma. For $z \in \mathbb{R}$, let $T_z := \inf\{t : W_t^* = z\}$. In particular, $T = T_0$.

LEMMA 2.12. *Let* (σ, b) *be locally Lipschitz and* $0 < \alpha < 1 < \beta$ *some constants. Then for all* $r \ge 1$, *there exists a constant* $C > 0$ *depending only on* r, α, β *and* $\sigma_{\max} := \sup_{x\in[\alpha,\beta]} |\sigma(x)|$, *such that*

$$\mathbb{P}_1^{(\sigma,b)}\{T < \epsilon\} \le C\epsilon^r \quad \text{for all } \epsilon < \big((1-\alpha)\wedge(\beta-1)\big)/(2b_{\max}),$$

where $b_{\max} := \sup_{x\in[\alpha,\beta]} |b(x)|$.

Proof. First we have

$$\mathbb{P}_1^{(\sigma,b)}\{T < \epsilon\} \le \mathbb{P}_1^{(\sigma,b)}\{T_\alpha < \epsilon\} \le \mathbb{P}_1^{(\sigma,b)}\{T_\alpha \wedge T_\beta < \epsilon\}.$$

Next set, for all $t < T$,

$$H(t) := \int_0^t \sigma(W_s)\,dB_s^{(2)} + \int_0^t b(W_s)\,ds.$$

We have

$$\mathbb{P}_1^{(\sigma,b)}\{T_\alpha \wedge T_\beta < \epsilon\} \le \mathbb{P}\Big\{\sup_{t\le\epsilon}\big|H(t\wedge T_\alpha\wedge T_\beta)\big| \ge (1-\alpha)\wedge(\beta-1)\Big\}.$$

If $\epsilon < ((1-\alpha)\wedge(\beta-1))/(2b_{\max})$, this last term is bounded by

$$\mathbb{P}\left\{\sup_{t\le\epsilon}\left|\int_0^{t\wedge T_\alpha\wedge T_\beta} \sigma(W_s)\,dB_s^{(2)}\right| \ge \frac{(1-\alpha)\wedge(\beta-1)}{2}\right\},$$

which by Doob's inequality (Theorem (1.7) p.54 in [13]) is bounded by $C\epsilon^r$ for some constant $C > 0$, which depends only on r, $\sigma_{\max}$, α and β. This concludes the proof of the lemma. □

Taking $r = 2$ in this lemma, we have that for $n > ((1-\alpha)\wedge(\beta-1))^{-1} 2b_+$

$$\mathbb{P}\big\{\tau_{n+1} - \tau_n < n^{-1} \mid \mathcal{G}_n\big\} \le Cn^{-2}.$$

Proposition 2.11 follows now from the conditional Borel–Cantelli lemma (Theorem 12.15 in [14]) by a standard argument. □

Let us give now $p = (p_\epsilon)_{\epsilon>0}$ such that $p_\epsilon \in (0,1)$ for all $\epsilon > 0$. Let (σ, b) be locally Lipschitz and let W be the solution of (2.34), with $W_0 = w_0$. Remember that $T = \inf\{t > 0 : W_t = 0\}$. We write $C'(p) = C'(p,\sigma,b))$ for the property

$$\mathbb{P}\big\{U_T \in [-\epsilon,\epsilon]\big\} > p_\epsilon \quad \text{for all } \epsilon > 0 \text{ and all } w_0 \in (0,1], \tag{2.38}$$

where U is a Brownian motion starting from 0 independent of W.

We say that a process on $\mathbb{R}$ is recurrent, if for all $\epsilon > 0$ and all $x \in \mathbb{R}$, it returns a.s. infinitely often to $[x-\epsilon, x+\epsilon]$. Similarly a law P is recurrent if the associated random walk is recurrent. The analogue of Theorem 2.5 is then the following theorem.

THEOREM 2.13. *Let $p = (p_\epsilon)_{\epsilon>0}$ be given. Let (σ_0, b_0) be a Lipschitz function and $((\sigma_i, b_i), 1 \le i \le k)$ be a sequence of locally Lipschitz functions. Assume (2.35) holds for $i \in \{0\} \cup \mathbb{N}_k$. Set $P_i = \varphi(\sigma_i, b_i)$. Assume that there exists $0 < \alpha < 1 < \beta$ and positive constants σ_+ and b_+ such that (2.37) holds. Assume moreover that $P_0 = \varphi(\sigma_0, b_0)$ is a recurrent law on $\mathbb{R}$ and that for each $i \in \mathbb{N}_k$, (σ_i, b_i) is successfully coupled with (σ_0, b_0) and satisfies (2.38). Then for any $(S_0, (S_n, i(n))_{n\ge1})$ which satisfies (1.1), the process $\{S_n\}_{n\ge0}$ is recurrent.*

We state now an analogue of Theorem 2.6 which can give examples of transient processes. We say that a process on $\mathbb{R}$ is transient, if for all $a < b$, it returns a.s. a finite number of times in $[a, b]$. Similarly, a law P is transient if the associated random walk is transient.

THEOREM 2.14. *Let (σ_0, b_0) be a Lipschitz function and $((\sigma_i, b_i), 1 \le i \le k)$ be a sequence of locally Lipschitz functions. Assume (2.35) holds for $i \in \{0\} \cup \mathbb{N}_k$. Set $P_i = \varphi(\sigma_i, b_i)$. Assume that there exists $0 < \alpha < 1 < \beta$ and positive constants σ_+ and b_+ such that (2.37) holds. Assume moreover that $P_0 = \varphi(\sigma_0, b_0)$ is a transient law on $\mathbb{R}$ and that for each $i \in \mathbb{N}_k$, (σ_0, b_0) is successfully coupled with (σ_i, b_i). Then for any $(S_0, (S_n, i(n))_{n\ge1})$ which satisfies (1.1), the process $\{S_n\}_{n\ge0}$ is transient.*

The proof of these theorems are similar to the discrete case and left to the reader.

As an example of laws which are successfully coupled we give the following result.

PROPOSITION 2.15. *Let P_0 be the Cauchy law on $\mathbb{R}$. Set $(\sigma_0, b_0) = (1, 0)$. Then (σ_0, b_0) satisfies (2.35), is successfully coupled with itself, and $P_0 = \varphi(\sigma_0, b_0)$. Moreover for any $\alpha \in [1, 2]$, there exists $(\sigma_\alpha, b_\alpha)$ locally Lipschitz satisfying (2.35), successfully coupled with (σ_0, b_0) and such that $\varphi(\sigma_\alpha, b_\alpha)$ is in the domain of normal attraction of a symmetric stable law with index α.*

Proof. The fact that (σ_0, b_0) satisfies (2.35) and is successfully coupled with itself is immediate (in the coupling we have $V_t = V_0 + B_t^{(2)} + L_t$ and $W_t = W_0 + B_t^{(2)}$ for $t < T$). So let us concentrate on the second claim. The case $\alpha = 1$ is given for instance by P_0 itself. Now we prove the result for $\alpha = 2$. Take $(\sigma, b) = (0, -1)$ to be constants. Then $W_t = W_0 - t$ for all $t \le T = W_0$. Set $P = \varphi(\sigma, b) = \varphi(0, -1)$, (2.35) being obviously satisfied. Let U be a standard Brownian motion on $\mathbb{R}$. Observe that when $W_0 = 1$, then W reaches 0 at time

$T=1$. Thus P is the law of U at time 1 which is the standard Gaussian and it is immediate that $(\sigma,b)=(0,-1)$ is successfully coupled with $(\sigma_0,b_0)=(1,0)$. This gives the result for $\alpha=2$. It remains to prove the claim for $\alpha\in(1,2)$. For $\nu\in(-1,-1/2)$, let $W^{(\nu)}$ be a Bessel process of index ν starting from 1, that is, $W^{(\nu)}$ is the solution of the SDE:

$$W_t^{(\nu)}=1+B_t^{(2)}+(\nu+1/2)\int_0^t\frac{1}{W_s^{(\nu)}}\,ds\quad\text{for all } t<T,$$

where $B^{(2)}$ is a Brownian motion and T is as always the first time when W reaches 0. It is known (see [13]) that T is a.s. finite when $\nu\in(-1,-1/2)$. Set $\sigma^{(\nu)}=1$ and $b^{(\nu)}(w)=(\nu+1/2)/w$. Then, for $\nu\in(-1,-1/2)$, $(\sigma^{(\nu)},b^{(\nu)})$ satisfies (2.35). Set $P^{(\nu)}=\varphi(\sigma^{(\nu)},b^{(\nu)})$. We claim that if $\nu\in(-1,-1/2)$, then $(\sigma^{(\nu)},b^{(\nu)})$ can be successfully coupled with (σ_0,b_0) and $P^{(\nu)}$ is in the domain of attraction of a stable law with index -2ν. The first part is immediate: since $\nu+1/2\le 0$, it follows from a comparison theorem (see [7] Theorem 1.1 p. 437). For the second part, first observe that

$$\mathbb{E}\{e^{iuU_T}\}=\mathbb{E}\{e^{-\frac{u^2}{2}T}\}\quad\text{for all } u\in\mathbb{R}.$$

So the characteristic function of $P^{(\nu)}$ is related to the Laplace transform of T. For Bessel processes this last function can be expressed in terms of modified Bessel functions: if ϕ_ν is the Laplace transform of T, the hitting time of 0 for a Bessel process of index $\nu<-1/2$ starting from 1, then (see [8] Theorem 3.1):

$$\phi_\nu(s)=\frac{2^{\nu+1}}{\Gamma(-\nu)}\frac{K_\nu(\sqrt{2s})}{(2s)^{\nu/2}}\quad\text{for all } s>0,$$

where Γ is the usual Gamma function and K_ν is a modified Bessel function (to see this from [8], take $a=1$ and let b tend to 0 in Formula (3.7), and use the asymptotic when $x\to 0$ of $K_\nu(x)$ given just above Theorem 3.1). Moreover (see [12] Formula (5.7.1) and (5.7.2)), we have

$$K_\nu(s)=\frac{\pi}{2}\frac{I_{-\nu}(s)-I_\nu(s)}{\sin\nu\pi}\quad\text{for all } s>0,$$

where

$$I_\nu(s)=\sum_{k=0}^{\infty}\frac{(s/2)^{\nu+2k}}{k!\Gamma(k+\nu+1)}\quad\text{for all } s>0.$$

This shows (use also basic identities of the Gamma function given in Formula (1.2.1) and (1.2.2) in [12]) that for u close to 0,

$$\mathbb{E}\{e^{iuU_T}\}=1-cu^{-2\nu}+o(u^{-2\nu})$$

for some constant $c > 0$, which proves our claim. □

3. A weak law of large numbers

The next result answers the second part of Benjamini's original question:

THEOREM 3.1. *Let $(S_n, n \geq 0)$ be the process on $\mathbb{Z}$ starting from 0, which at a first visit to a site makes a discrete symmetric Cauchy jump and at other visits makes a ± 1 Bernoulli jump. Then*

$$\frac{1}{n} \sup_{t \leq n} | S_t | \to 0 \quad \textit{in probability.} \tag{3.1}$$

Proof. We shall first prove (3.1) with S_t replaced by the auxiliary process $\widetilde{S}_t$ which makes a discrete symmetric Cauchy jump at a first visit to a site and at other visits makes a jump with distribution $\widetilde{P}_1$, where

$$\widetilde{P}_1\{\pm 1\} = 1/4, \qquad \widetilde{P}_1\{0\} = 1/2 \quad \text{and} \quad \widetilde{P}_1\{u\} = 0 \quad \text{for } u \notin \{-1, 0, +1\}.$$

Quantities referring to the walk $\{\widetilde{S}_n\}$ will all be decorated with a tilde, but will otherwise be defined in the same way as their analogues without a tilde. We further remind the reader that P_1 is the distribution on $\mathbb{Z}$ which puts mass 1/2 on ± 1 and that P_2 is the discrete Cauchy distribution.

Let $\widetilde{R}_n$ be the range at time n, that is,

$$\widetilde{R}_n = \text{cardinality of } \{\widetilde{S}_0, \widetilde{S}_1, \ldots, \widetilde{S}_{n-1}\}. \tag{3.2}$$

This means that during the time interval $[0, n]$, $\widetilde{S}_\ell$ took exactly $\widetilde{R}_n$ Cauchy jumps and $n - \widetilde{R}_n$ steps with distribution $\widetilde{P}_1$. Let us now use the construction of the $\{\widetilde{S}_\ell\}$ which is the analogue of the one given for $\{S_\ell\}$ in the Introduction. More precisely, let $(\widetilde{Y}(1, \ell))_{\ell \geq 1}$ and $(\widetilde{Y}(2, \ell))_{\ell \geq 1}$ be independent sequences of independent random variables respectively, of law $\widetilde{P}_1$ and P_2, then $(\widetilde{S}_n)_{n \geq 1}$ is such that $\widetilde{S}_0 = 0$ and for $n \geq 1$,

$$\widetilde{S}_n = \sum_{\ell=1}^{n - \widetilde{R}_n} \widetilde{Y}(1, \ell) + \sum_{\ell=1}^{\widetilde{R}_n} Y(2, \ell)$$

with $\widetilde{R}_n$ defined by (3.2).

Consequently, for any $\varepsilon > 0$ it holds that on the event $\{\widetilde{R}_n \leq \varepsilon n\}$,

$$\sup_{t \leq n} |\widetilde{S}_t| \leq \sup_{s \leq n} \left| \sum_{\ell=1}^{s} \widetilde{Y}(1, \ell) \right| + \sup_{r \leq \varepsilon n} \left| \sum_{\ell=1}^{r} Y(2, \ell) \right|.$$

By maximal inequalities (see [2], Theorem 22.5), we therefore have for any $\varepsilon \leq 1, \alpha > 0$,

$$
\begin{aligned}
\text{(3.3)} \quad \mathbb{P}\Big\{\sup_{t\leq n} |\widetilde{S}_t| \geq 8\alpha n\Big\} &\leq \mathbb{P}\{\widetilde{R}_n > \varepsilon n\} + 4\max_{t\leq n} \mathbb{P}\left\{\left|\sum_{\ell=1}^{t} \widetilde{Y}(1,\ell)\right| \geq \alpha n\right\} \\
&\quad + 4\max_{t\leq \varepsilon n} \mathbb{P}\left\{\left|\sum_{\ell=1}^{t} Y(2,\ell)\right| \geq \alpha n\right\}.
\end{aligned}
$$

Now, as is well known (e.g., by Chebyshev's inequality), for each fixed $\alpha > 0$,

$$
\max_{t\leq n} \mathbb{P}\left\{\left|\sum_{\ell=1}^{t} \widetilde{Y}(1,\ell)\right| \geq \alpha n\right\} \to 0 \quad \text{as } n \to \infty. \tag{3.4}
$$

Also, for fixed $\alpha > 0, \varepsilon > 0, t \leq \varepsilon n$,

$$
\mathbb{P}\left\{\left|\sum_{\ell=1}^{t} Y(2,\ell)\right| \geq \alpha n\right\} \leq \mathbb{P}\left\{\left|\sum_{\ell=1}^{t} Y(2,\ell)\right| \geq \frac{\alpha}{\varepsilon} t\right\}, \tag{3.5}
$$

and

$$
\lim_{t\to\infty} \mathbb{P}\left\{\left|\sum_{\ell=1}^{t} Y(2,\ell)\right| \geq \frac{\alpha}{\varepsilon} t\right\} = f\left(\frac{\alpha}{\varepsilon}\right) \tag{3.6}
$$

for some function $f(\cdot)$. Moreover, $f(\alpha/\varepsilon)$ can be made as small as desired by taking α/ε large. In fact,

$$
\frac{1}{m}\sum_{\ell=1}^{m} Y(2,\ell) \text{ converges in distribution to a Cauchy variable,}
$$

as $m \to \infty$ (see [6], Theorem 17.7). It is immediate from (3.3)–(3.6) that

$$
\frac{1}{n}\widetilde{R}_n \to 0 \quad \text{in probability} \tag{3.7}
$$

is a sufficient condition for (3.1) with S_t replaced by $\widetilde{S}_t$.

We now turn to a proof of (3.7). Since $0 \leq \widetilde{R}_n/n \leq 1$, (3.7) is equivalent to

$$
\frac{1}{n}\mathbb{E}[\widetilde{R}_n] = \frac{1}{n}\sum_{t=0}^{n-1} \mathbb{P}\{A_{t,n-t}\} \to 0,
$$

where

$$
\begin{aligned}
A_{t,\ell} &= \big\{\widetilde{S}_t \text{ is not revisited during } [t+1, t+\ell-1]\big\} \\
&= \{\widetilde{S}_t \neq \widetilde{S}_{t+s} \text{ for } 1 \leq s \leq \ell - 1\}.
\end{aligned}
$$

In particular, since $A_{t,\ell}$ is decreasing in ℓ, a sufficient condition for the WLLN for $\widetilde{S}_n$ is that

$$
\lim_{\ell\to\infty} \mathbb{P}\{A_{t,\ell}\} = 0 \quad \text{uniformly in } t. \tag{3.8}
$$

Recurrence essentially is property (3.8), *without the uniformity requirement.* To prove (3.8) with the uniformity, we use the coupling defined in the remark below Example 2.7, as we now explain. Let Q_0 be the transition probability matrix of a simple random walk on $\mathbb{Z}^2$. Denote this walk by $\{(U_n, V_n)\}_{n\geq 0}$ and let its starting point be $(0,0)$. We proved in Example 2.7 that Q_0 is successfully coupled with itself. We shall use a part of that result here. We also need to know that there exists another process $\{(U_n, W_n)\}_{n\geq 0}$ which also starts at $(0,0)$ and takes values in $\mathbb{Z}^2$ and in addition a coupled process $\{(U_n, V_n, W_n)\}_{n\geq 0}$ such that

> the law of the imbedded process of $\{(U_n, W_n)\}$ in the U-axis is the same as the law of Benjamini's process $\{\widetilde{S}_n\}$,

and

$$|W_n| \leq |V_n| + 1. \tag{3.9}$$

We remind the reader that the imbedded process here is $\{U_{\tau_n}\}_{n\geq 0}$, where $\tau_0 = 0$ and for $\ell \geq 1$

$$\tau(\ell) = \inf\{t > \tau(\ell - 1) : W_t = 0\}.$$

In the remainder of this proof, we shall often write $\Gamma(\phi)$ instead of Γ_ϕ for certain Γ and ϕ, in order to avoid double subscripts. Now fix some $t \in \{0, 1, \dots, n-1\}$. For time running from 0 to t we let $U_0^t, U_1^t, \dots, U_t^t$ be a copy of $\{\widetilde{S}_\ell\}_{0\leq \ell\leq t}$. No coupling of this process with another process is needed. However, we shall further need an independent copy of the variables $\{(U_n, V_n, W_n)\}_{n\geq 0}$ with its corresponding sequence of times τ_ℓ at which the walk $\{(U_n, W_n)\}$ visits the U-axis. The successive positions of Benjamini's walk determined by the triple $\{(U_n, V_n, W_n)\}_{n\geq 0}$ itself would be $U(\tau_0), U(\tau_1), \dots$. However, we want to shift those positions to come right after the previous points $\{U_\ell^t\}$. This requires one important change. In the coupling construction by itself, at a time τ at which $W_\tau = 0$, assume that the Benjamini walk arrived in some point, u say, on the U-axis. In order to choose the next step for the walk one must now decide whether the visit to u at τ is the first visit by the walk to u or not. In the construction of Example 2.7, it would be a first visit if and only if $U(\tau_m) \neq u$ for $0 \leq \tau_m < \tau$. Here, we have to modify this. We think of the walk as first traversing $U_0^t, U_1^t, \dots, U_t^t$, and then to start from time t on to use the coupling construction. The visit at time τ to u will therefore be counted as the first visit if and only if

$$U(\tau_m) \neq u - U_t^t \quad \text{for } 0 \leq \tau_m < \tau \quad \text{and} \quad U_s^t \neq u \quad \text{for } 0 \leq s \leq \tau.$$

After this change, the path

$$\begin{array}{llll} U_0^t, & U_1^t, & \dots, & U_t^t = U_t^t + U(\tau_0), \\ U_t^t + U(\tau_1), & U_t^t + U(\tau_2), & \dots & \end{array}$$

is a typical path of a Benjamini walk, but with a modified rule for determining whether a point is fresh or old. To be more precise, let $\Theta = \Theta(t) = \{U_0^t = 0, U_1^t, \dots, U_t^t\}$ be the set of points visited by the Benjamini walk during $[0,t]$. Now first fix $\Theta(t)$. Then $A_{t,\ell}$ occurs if and only if none of the next $\ell - 1$ positions of a Benjamini walk equals U_t^t. However, for this second stage the points of Θ are regarded as old points, even if they have not been visited before. Thus we use a modified Benjamini walk in which the walk takes a simple symmetric walk step when it is at an old point or a point from Θ, and a discrete Cauchy distribution when the walk is at a fresh point outside Θ. We shall call this the Θ-*modified walk*. The original Benjamini walk is the special case of this when $\Theta = \emptyset$. When the dependence on Θ is important we shall indicate this by a superscript Θ. In particular, the law of the walk which we just described (in which we regard the points of Θ as old points) is written as $\mathbb{P}^{\Theta}$. Choosing or modifying Θ merely modifies the rule by which the index i, or equivalently the function F in (2.10) and (2.15) is chosen. However, Lemma 2.3 remains valid for the Θ-modified process. In particular, we can express the conditional law of $\widetilde{S}_{t+s} - \widetilde{S}_t$ given $\mathcal{F}_t$, by means of $\mathbb{P}^{\Theta(t)}$. This gives

$$\begin{aligned}\mathbb{P}\{A_{t,\ell}\} &= \mathbb{E}\big\{\mathbb{P}\{\widetilde{S}_{t+q} \neq \widetilde{S}_t \text{ for all } 1 \le q \le \ell - 1 | \mathcal{F}_t\}\big\}\\ &= \mathbb{E}\big\{\mathbb{P}^{\Theta(t)}\{\widetilde{S}_q \neq 0 \text{ for all } 1 \le q \le \ell - 1\}\big\}\\ &\le \sup_{\Theta} \mathbb{P}^{\Theta}\{\widetilde{S}_q \neq 0 \text{ for all } 1 \le q \le \ell - 1\}.\end{aligned}$$

We now complete the proof of (3.8). We find it useful for this purpose to introduce the events

$$\mathcal{A}_q := \{U_q = V_q = 0\} \cap \{V_{q+1} = -1\}.$$

Since $\{(U_q, V_q)\}_{q \ge 0}$ is a simple random walk it is well known that this walk is recurrent, so that with probability 1, the event $\{U_q = V_q = 0\}$ occurs for infinitely many q. By a straightforward application of a conditional version of the Borel–Cantelli lemma (cf. Theorem 12.15 in [14]), it then follows that, again with probability 1, $\mathcal{A}_q$ occurs infinitely often. For every q with $V_q = 0$, we have $W_q \in \{-1, 0, 1\}$, by virtue of (3.9). The remark following Example 2.7 now shows that if $\mathcal{A}_q$ occurs for some q, then also

$$\mathcal{B}_q := \big\{U(q) = W(q) = 0\big\} \cup \big\{U(q+1) = W(q+1) = 0\big\}$$

occurs for the same q. Note that the event

$$\{\mathcal{B}_q \text{ fails for all } 1 \le q \le \ell - 1\}$$

coincides with the event

$$\bigcap_{q=1}^{\ell} \big\{U(q) \neq 0 \text{ or } W(q) \neq 0\big\}.$$

Note now that (since $\tau_\ell \ge \ell$ and since $W(q) = 0$ implies $q = \tau_r$ for some r) the event $\bigcap_{q=1}^{\ell}\{U(q) \neq 0 \text{ or } W(q) \neq 0\}$ occurs when the event $\{\widetilde{S}_t = U(\tau_t) \neq 0 \text{ for all } 1 \le t \le \ell\}$. Thus, we have

$$\begin{aligned}\mathbb{P}^{\Theta}\{\widetilde{S}_q \neq 0 \text{ for } 1 \le q \le \ell\} &\le \mathbb{P}^{\Theta}\{\mathcal{B}_q \text{ fails for all } 1 \le q \le \ell - 1\}\\ &\le \mathbb{P}\{\mathcal{A}_q \text{ fails for all } 1 \le q \le \ell - 1\}\end{aligned}$$

(use contrapositives for the last inequality). But the right-hand side here is independent of t and Θ, since it involves only the simple random walk (U_n, V_n). In addition this right-hand side tends to 0 as $\ell \to \infty$, since we already proved that with probability 1 infinitely many $\mathcal{A}_q$ occur. This last estimate is uniform in t, Θ, as desired.

This finally proves (3.8) and the WLLN, that is, (3.1) with S_t replaced by $\widetilde{S}_t$. However, this proof is for the $\{\widetilde{X}_n\}$-process which takes a step with distribution $\widetilde{P}_1$ whenever the walk is at an old point. We shall now show that this implies the WLLN for the process $\{S_n\}$, that is, (3.1) itself. Indeed, in the notation of the proof of Lemma 2.8, the processes $\{S_n, i(n)\}_{n\ge1}$ and $\{\widetilde{S}_{\rho_n}, \widetilde{i}(\rho_n)\}_{n\ge1}$ have the same law. In particular,

$$\left(\frac{1}{t}\sup_{\ell\le t} | S_\ell |, i(n)\right) \quad \text{and} \quad \left(\frac{1}{t}\sup_{\ell\le t} | \widetilde{S}_{\rho_\ell} |, \widetilde{i}(\rho_n)\right) \tag{3.10}$$

have the same law. As explained in the proof of Lemma 2.8, we may even assume that all these variables are defined on the same probability space of sequences $\{A_\ell\}_{\ell\ge0}, \{B_\ell\}_{\ell\ge0}$ provided with the measure which makes all these variables i.i.d. uniform on $[0, 1]$. We denote this probability measure by $\mathbb{P}$. It follows from from the definition of the sequence ρ_ℓ that for $q \ge 1$

$$\mathbb{P}\big\{\rho_{\ell+1} - \rho_\ell \ge q \mid \sigma(\rho_1, \rho_2, \dots, \rho_\ell)\big\} \le 2^{-(q-1)}.$$

In turn, this implies that for some constant $C \in (0, \infty)$,

$$\begin{aligned}\limsup_{t\to\infty}\sup_{\ell\le t}\frac{\rho_\ell}{t} &= \limsup_{t\to\infty}\frac{\rho_t}{t}\\ &= \limsup_{t\to\infty}\frac{\sum_{\ell=1}^{t}[\rho_\ell - \rho_{\ell-1}]}{t} \le C \quad \text{with probability 1.}\end{aligned}$$

It follows that

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{\ell\le t} | \widetilde{S}_{\rho_\ell} | > \varepsilon t\Big\}\\ &\quad\le \mathbb{P}\Big\{\max_{\ell\le t}\rho_\ell > (C+1)t\Big\} + \mathbb{P}\Big\{\sup_{\ell\le(C+1)t} | \widetilde{S}_\ell | > \varepsilon t\Big\}.\end{aligned}$$

The last term on the right here tends to 0 as $t \to \infty$ and $\varepsilon > 0$ fixed, by virtue of (3.1) with S_t replaced by $\widetilde{S}_t$. Since also the first term on the right here tends to 0 as $t \to \infty$, we conclude from (3.10) that $(1/t)\sup_{\ell\le t} | S_\ell | \to 0$ in probability, as desired. $\square$

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