A localisation phase transition for the catalytic branching random walk

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Abstract

We show the existence of a phase transition between a localisation and a non-localisation regime for a branching random walk with a catalyst at the origin. More precisely, we consider a continuous-time branching random walk that jumps at rate one, with simple random walk jumps on $\mathbb{Z}^d$, and that branches (with binary branching) at rate $\lambda > 0$ everywhere, except at the origin, where it branches at rate $\lambda_0 > \lambda$. We show that, if λ_0 is large enough, then the occupation measure of the branching random walk localises (i.e. converges almost surely without spatial renormalisation), whereas, if λ_0 is close enough to λ, then localisation cannot occur, at least not in a strong sense. The case $\lambda = 0$ (when branching only occurs at the origin) has been extensively studied in the literature and a transition between localisation and non-localisation was also exhibited in this case. Strikingly, the transition that we observe, conjecture, and partially prove in this paper occurs at the same threshold as in the case $\lambda = 0$.

1 Introduction and main results

In this paper, we consider the following continuous-time branching random walk on $\mathbb{Z}^d$ ($d \geq 1$): at time zero, there is one particle alive in the system, and its position is the origin. Each particle carries two independent Poisson clocks: the "jump"-clock that rings at rate 1, and the "branch"-clock that rings at rate λ_0 when the particle is at the origin and at rate λ everywhere else. When the jump-clock of a particle rings, then this particle moves to a neighbouring site chosen uniformly at random among the $2d$ neighbouring sites. When the branch-clock of a particle rings, it gives birth to a new particle at the same location. This model is a variant of the so-called "catalytic branching random walk", which corresponds to taking $\lambda = 0$, and which has been extensively studied in the literature (we give a literature review on this model in Section 1.4).

We are interested in the occupation measure of the process, i.e. the process $(\Pi_t)_{t\geq 0}$ where, for all $t \geq 0$,

$$\Pi_t = \sum_{i=1}^{N(t)} \delta_{X_i(t)},$$

with $N(t)$ the number of particles alive at time t, and $(X_i(t))_{1\leq i\leq N(t)}$ their respective positions. We assume that $\lambda_0 \geq \lambda > 0$.

1.1 Results

Our main result is as follows:

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Theorem 1.1. *If $\lambda_0 > 2d-1+2d\lambda$, then, there exists a deterministic measure ν on $\mathbb{Z}^d$ such that, almost surely as $t \uparrow \infty$,*

$$\widehat{\Pi}_t := \frac{\Pi_t}{N(t)} \to \nu,$$

for the topology of weak convergence on the space of probability measures on $\mathbb{Z}^d$.

The strength of this result is that it gives almost sure convergence of the occupation measure of the branching random walk. In comparison, existing results in the case $\lambda = 0$, such as in [DR13], only give convergence of all moments (see Section 1.4 for a more detailed discussion and more references on the case $\lambda = 0$). Another notable difference is that we use a random normalisation, namely we divide the empirical distribution of particles by $N(t)$, and our limit is a deterministic measure. This contrasts for instance with limiting results in the case $\lambda = 0$, such as in [DR13] or [Wat67] in a continuous setting, where the normalisation is deterministic and the limiting measure is multiplied by a random factor. We believe that our results (Theorem 1.1 and Lemma 2.1 below) may imply convergence of the occupation measure normalised by a deterministic factor. Proving this would require to estimate the speed of convergence in Theorem 1.1, which we have not tried to do so far.

The following results indicate that the condition $\lambda_0 > 2d-1+2d\lambda$ may not be optimal: i.e. we believe that localisation should occur for smaller values of λ_0. First we show that any candidate for the limiting measure ν satisfies some balance equations:

Proposition 1.2. *Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$. If $\widehat{\Pi}_t \to \nu$ almost surely for the topology of weak convergence, then ν satisfies*

$$(1 + (\lambda_0 - \lambda)\nu_0)\nu_x = \frac{1}{2d}\sum_{i=1}^{d}(\nu_{x+e_i} + \nu_{x-e_i}), \quad (\forall x \neq 0) \tag{1.1}$$

and

$$(1 - (\lambda_0 - \lambda)(1 - \nu_0))\nu_0 = \frac{1}{2d}\sum_{i=1}^{d}(\nu_{e_i} + \nu_{-e_i}). \tag{1.2}$$

In fact, (1.2) is redundant. Indeed, by summing over $x \neq 0$ the left and right hand sides of the equalities in (1.1), it is readily seen that any probability measure satisfying (1.1) also satisfies (1.2). Finally, our last result identifies a phase transition for the existence of solutions to the balance equations (1.1) and (1.2), which we conjecture to coincide with the transition between a localisation and a delocalisation regimes, see Conjecture 1.4 below.

Theorem 1.3. *Let γ_d be the probability that a random walk in $\mathbb{Z}^d$ started at zero never returns to zero.*

(i) If $0 \le \lambda_0 - \lambda \le \gamma_d$, then there exist no probability measure satisfying (1.1).

(ii) If $\lambda_0 - \lambda > \gamma_d$, then there exists a unique probability measure satisfying (1.1).

In other words, we prove that, (i) if $\lambda_0 > 2d-1+2d\lambda$, then the occupation measure of the branching random walk "localises" (see Theorem 1.1), and (ii) if $0 \le \lambda_0 - \lambda \le \gamma_d$, then localisation cannot occur in the strong sense of (i) (see Theorem 1.3(i)). We stress that the value of the threshold for the transition observed in Theorem 1.3, namely the constant γ_d, is exactly the same as in the case $\lambda = 0$, see below. However, while the phase transition is elementary to identify in the latter case, using a direct comparison between the total number of particles ever visiting the origin and a standard Galton-Watson process, it seems less immediate to use such a comparison when λ is positive. Actually, it may even seem counter-intuitive that the threshold for the phase transition between localisation and delocalisation holds at the same critical value (for the parameter $\lambda_0 - \lambda$), no matter the value of λ. Indeed, one could expect that increasing λ would favour the natural diffusive behaviour of the process, while our results tend to show that, on the contrary, it does not play any role in this respect.

1.2 Discussion of the proofs (and plan of the paper)

The proof of localisation (see Section 4), i.e. Theorem 1.1, relies on proving that the occupation measure taken at the times when either a branching or a reproduction event occurs is a measure-valued Pólya process (MVPP). MVPPs were introduced in [MM17] and [BT16] (also see [BT17] were the model was first defined on a particular example) as a generalisation of Pólya urns to infinitely-many colours. Mailler and Marckert [MM17] proved convergence in probability of a large class of MVPPs under an assumption of balance (in terms of urns, the balance assumption requires that the total number of balls added at any time-step in the urn is constant and deterministic). They also required that the replacement rule is deterministic, but this assumption was later removed by Janson [Jan19]. To prove Theorem 1.1, we use convergence results proved by Mailler and Villemonais [MV20], which apply without the balance assumption, for random replacements, and hold almost surely. Besides showing that our model is an MVPP, our main contribution is to check that our MVPP satisfies the assumptions of [MV20]; to do this, we need to show that a continuous-time sub-Markovian process admits a quasi-stationary distribution, and this is done using results of Champagnat and Villemonais [CV23]. Interestingly, this quasi-stationary distribution is the limit ν in Theorem 1.1, and our Proposition 1.2 provides a characterisation of this measure for λ_0 large enough.

The proof of Proposition 1.2 (see Section 2) relies on (1) finding a good approximation for the total number of particles alive at large times t (see Lemma 2.1 - this is done by finding a martingale and proving that it converges in L^2), which allows to replace the random normalisation of the occupation measure by a deterministic one (see Lemma 2.2), and then (2) writing some forward evolution equation for the occupation measure and studying their equilibrium.

Finally, the proof of Theorem 1.3 (see Section 3) is done by studying the existence (or lack of thereof) solutions to the balance equations of Proposition 1.2. To do this, we use the optional stopping theorem applied to a martingale that naturally arises from the balance equations.

1.3 Conjectures

As briefly mentioned before, when $\lambda_0 = \lambda$, it is well-known that, almost surely as $t \uparrow \infty$

$$\widehat{\Pi}_t\big(\cdot\sqrt{t}\big) \to \mathcal{N}(0,1),$$

for the topology of weak convergence on the space of probability measures on $\mathbb{R}$. This implies in particular that, for all $B \subset \mathbb{Z}$, almost surely as $t \uparrow \infty$,

$$\widehat{\Pi}_t(B) \to 0.$$

We conjecture that the phase transition is sharp, i.e. that this "diffusive" behaviour stays true for λ_0 larger than λ, up to a critical threshold after which the occupation measure localises. In view of Theorem 1.3, we conjecture that this threshold equals $\lambda + \gamma_d$:

Conjecture 1.4. *(i) For all $\lambda \le \lambda_0 \le \lambda + \gamma_d$, for all $B \subset \mathbb{Z}$, almost surely as $t \uparrow \infty$,*

$$\widehat{\Pi}_t(B) \to 0.$$

(ii) For all $\lambda_0 > \lambda + \gamma_d$, almost surely as $t \uparrow \infty$,

$$\widehat{\Pi}_t \to \nu,$$

for the topology of weak convergence on the space of probability measures on $\mathbb{Z}^d$, where ν is the unique probability measure satisfying (1.1).

In the case $\lambda \le \lambda_0 \le \lambda + \gamma_d$ (assuming $\gamma_d > 0$), one could even wonder if there exists a renormalisation function $(u_t)_{t\ge 0}$ and a limiting measure μ such that $\widehat{\Pi}_t(\cdot\, u_t) \to \mu$. As discussed above, we know from the literature that, if $\lambda = \lambda_0$, then such a result holds with $u_t = \sqrt{t}$ and μ the standard Gaussian. But the question of how do u_t and μ change when λ_0 increases remains open.

1.4 Discussion of the literature

The catalytic branching random walk. The case when branching only happens at the origin (thus $\lambda_0 > \lambda = 0$) has been long studied in the literature, under the name of catalytic branching random walk. As far as we know, most of the results concerning the localisation/delocalisation of the process are about the asymptotic behaviour of the moments of the number of particles at each site as time goes to infinity.

The first papers on the catalytic branching random walk focused on particles performing simple random walks on $\mathbb{Z}^d$: see, e.g., Albeverio, Bogachev and Yarovaya [ABY98]. Efforts were later made to generalise these results allowing the particles to perform more general Markovian trajectories (between branching events). Several methods were used to analyse the aforementioned moments: Bellman-Harris branching processes in [TV03, VT05, Bul10, Bul11], operator theory in [Yar10], renewal theory in [HTV12], and spine decomposition techniques in Döring and Roberts [DR13].

More recently, some of the focus has shifted to understanding the spread of the catalytic branching random walk at large times see, e.g. [CH14, MY12, Bul18, Bul20]. To the best of our knowledge, this interesting question is open in our setting (i.e. with branching outside the origin).

Most of these papers consider a more general branching mechanism than our binary fission: the offspring distribution can be any distribution with finite second moment. They also consider the case when the offspring distribution is (sub-)critical (i.e. the average number of offspring is less than or equal to 1). In this paper, we focus on a particular super-critical case. However, we allow arbitrary rates λ and λ_0 for the branching events, which, at least phenomenologically, plays a similar role as allowing a general average number of offspring.

To compare the aforementioned results with ours, we now summarise the results of, e.g. [DR13]. For all $y \in \mathbb{Z}^d$, $t \geq 0$, and $k \geq 1$, we let $M^k(t, y)$ be the k-th moment of the number of particles at position y at time t, and $M^k(t)$ be the k-th moment of the total number of particles in the system at time t. Note that Döring and Roberts assume that branching at zero happens at rate 1, but allow the offspring distribution to be more general than binary fission. However, as explained above, one can interpret m, the average number of offspring, as the rate of branching at 0, i.e. $\lambda_0 = m - 1$. Indeed, to create m offspring per unit of time from binary fissions, we need to have $m-1$ binary fissions per unit of time. Roberts and Döring's results can thus be interpreted as:

- if $\lambda_0 < \gamma_d$, then, for all $y \in \mathbb{Z}^d$,
$$\lim_{t\uparrow\infty} M^1(t, x) \in (0, \infty) \quad \text{and} \quad \lim_{t\uparrow\infty} M^1(t, x, y) = 0,$$
which corresponds to a non-localisation phase as most particles in the system have drifted away from the origin.
- If $\lambda_0 > \gamma_d$, then, there exists a positive constant $\rho > 0$ such that, for all $y \in \mathbb{Z}^d$,
$$\lim_{t\uparrow\infty} \mathrm{e}^{-k\rho t} M^1(t, y) \in (0, \infty) \quad \text{and} \quad \lim_{t\uparrow\infty} \mathrm{e}^{-k\rho t} M^1(t) \in (0, \infty),$$
which corresponds to a localisation phase, as the number of particles at any site y is of the same order as the total number of particles in the system.

Note that the critical case, when $\lambda_0 = \gamma_d$, is also studied in [DR13], and the growth rate of the number of particles in this case is sub-exponential (see also [VT11]).

Generalisations of the catalytic branching random walk: Since its introduction, the catalytic branching random walk (with branching only at the origin) has been generalised in several ways. One direction of generalisation is to add more catalysts (sites at which branching happens): instead of having one catalyst at the origin, one can have N catalysts at positions $x_1, \dots, x_N$ (see, e.g. [AB00]). Another direction is to allow the catalysts to move following their own Markovian motion and branching only happens for particles at the same position as one of the catalysts, with a rate that may depend on the number of catalysts there (see, e.g. [GdH06, GH06, KS03, CGM11]). Finally let us mention the model of inhomogeneous branching random walk, where the branching rate depends on the position of the particle, as e.g. in [BH14b, GKW99].

Catalytic branching Brownian motion/superprocesses, mutually catalytic processes: Various models of catalytic branching Brownian motions or superprocesses have also been considered. The case of inhomogeneous branching Brownian motion, where the branching rate depends continuously on the position, has received quite some attention, starting with Watanabe [Wat67], who proves a similar result as our Theorem 1.1, yet with a different (and deterministic) normalisation and a random limit. More recent results concern the position of the right-most particle in dimension one (see, e.g., [LS88, LS89]). The question of localisation as well as other questions are also addressed in a model of catalytic branching Brownian motion where branching only occurs at the origin in [BH14a, BH16]. Finally, catalytic or mutually catalytic superprocesses have been extensively studied, e.g. in [CGDG04, DF94, FLG95].

Parabolic Anderson model: To conclude this discussion, let us mention that the model of catalytic branching random walks bears some similarities with the parabolic Anderson model in a heavy-tail environment. In particular, in this setting a phenomenon of strong localisation also arises [KLMS09].

2 Proof of Proposition 1.2 ("balance" equations)

We start by introducing a martingale which lies at the heart of the proof.

Lemma 2.1. *For all $t \geq 0$, we let $\rho_t = \lambda + (\lambda_0 - \lambda)\widehat{\Pi}_t(0)$ and $M_t = N_t \cdot \mathrm{e}^{-\int_0^t \rho_s \mathrm{d}s}$. Then the process $(M_t)_{t\geq 0}$ is a martingale bounded in L^2.*

Proof. The fact that N_t is square integrable for all $t \geq 0$ follows from the fact that $(N_t)_{t\geq 0}$ is stochastically dominated by the total population of a binary branching process with reproduction rate λ_0, whose square is well-known to be integrable. For all $0 \leq s \leq t$, we let $f_s(t) = \mathbb{E}[M_t|\mathcal{F}_s]$, where $(\mathcal{F}_s)_{s\geq 0}$ is the natural filtration of the process $(\Pi_s)_{s\geq 0}$. It is standard to see, using the definition of the process $(\Pi_t)_{t\geq 0}$ as a Markov jump process, that, for all $s \geq 0$, $t \mapsto f_s(t)$ is continuous on $[s, +\infty)$, and moreover, that for all $t > s$, $\frac{\mathrm{d}}{\mathrm{d}t} f_s(t) = 0$, from which we deduce that $(M_t)_{t\geq 0}$ is indeed a martingale. To see that it is bounded in L^2, note that for any fixed $t \geq 0$,

$$\begin{aligned}\mathbb{E}[N_{t+h}^2 \mathrm{e}^{-2\int_0^t \rho_s\, ds}] &= h\mathbb{E}[\rho_t M_t (N_t+1)^2 \mathrm{e}^{-\int_0^t \rho_s\,\mathrm{d}s}] + \mathbb{E}[(1 - h\rho_t N_t) M_t^2] + \mathcal{O}(h^2)\\ &= \mathbb{E}[M_t^2] + 2h\mathbb{E}[M_t^2 \rho_t] + h\mathbb{E}[M_t \rho_t \mathrm{e}^{-\int_0^t \rho_s\,\mathrm{d}s}] + \mathcal{O}(h^2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}[M_{t+h}^2] &= \mathbb{E}[N_{t+h}^2 \cdot \mathrm{e}^{-2\int_0^t \rho_s\,\mathrm{d}s}] - 2h\mathbb{E}[M_t^2 \rho_t] + \mathcal{O}(h^2)\\ &= \mathbb{E}[M_t^2] + h\mathbb{E}[M_t \rho_t \mathrm{e}^{-\int_0^t \rho_s\,\mathrm{d}s}] + \mathcal{O}(h^2).\end{aligned}$$

Using Cauchy-Schwarz inequality and the fact that, for all $t \geq 0$, $\lambda \leq \rho_t \leq \lambda_0$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[M_t^2] = \mathbb{E}[M_t \rho_t \mathrm{e}^{-\int_0^t \rho_s\,\mathrm{d}s}] \leq \lambda_0 \mathbb{E}[M_t^2]^{1/2} \mathrm{e}^{-\lambda t},$$

which implies

$$\sup_{t\geq 0} \mathbb{E}[M_t^2] \leq \left(1 + \frac{\lambda_0}{\lambda}\right)^2,$$

concluding the proof. ☐

For the next step we need some additional notation: For all $x \in \mathbb{Z}^d$, we let Π_t^x be the empirical measure of the particles in the catalytic branching random walk starting with one particle at site x. Given any measure μ on $\mathbb{Z}^d$, we let $\Pi_t \cdot \mu = \sum_{x \in \mathbb{Z}^d} \mu(x) \Pi_t^x$.

Lemma 2.2. *If there exists a probability measure ν such that $\lim_{t\to\infty} \widehat{\Pi}_t \to \nu$, almost surely for the topology of weak convergence, then $\mathbb{E}[\Pi_t \cdot \nu] \cdot \mathrm{e}^{-\rho t} = \nu$, for all $t \geq 0$, with $\rho = \lambda + (\lambda_0 - \lambda)\nu_0$.*

Proof. Fix $t \geq 0$. On one hand, we know by Lemma 2.1 that $(M_t)_{t\geq 0}$ converges almost surely and in L^2 towards some random variable M_∞. It follows that, for any $x \in \mathbb{Z}^d$, almost surely,

$$\Pi_{t+s}(x)\mathrm{e}^{-\int_0^{t+s} \rho_u \,\mathrm{d}u} = M_{t+s}\widehat{\Pi}_{t+s}(x) \underset{s\to\infty}{\longrightarrow} M_\infty \nu_x.$$

Moreover, since, by definition, $|\widehat{\Pi}_{t+s}(x)| \leq 1$, this convergence also holds in L^1, which implies

$$\lim_{s\to\infty} \mathbb{E}[\Pi_{t+s}(x)\mathrm{e}^{-\int_0^{t+s} \rho_u \,\mathrm{d}u}] = \nu_x,$$

because $\mathbb{E}[M_\infty] = 1$. Furthermore, because by assumption $\rho_t \to \rho$ almost surely as $t \uparrow \infty$, we get $\int_s^{t+s} \rho_u \,\mathrm{d}u = \rho t + o(1)$ as $s \uparrow \infty$. Hence,

$$\lim_{s\to\infty} \mathbb{E}[\Pi_{t+s}(x)\mathrm{e}^{-\rho t - \int_0^s \rho_u \,\mathrm{d}u}] = \nu_x. \tag{2.1}$$

Now note that, for all $s, t \geq 0$, if we let $(\widetilde{\Pi}_t)_{t\geq 0}$ be an independent copy of the process $(\Pi_t)_{t\geq 0}$, then $\Pi_{t+s} = \widetilde{\Pi}_t \circ \Pi_s$ in distribution, where, with a slight abuse of notation, we consider that $\widetilde{\Pi}_t$ acts as well on a particle configuration and on its empirical measure. By linearity of the expectation, this implies

$$\mathbb{E}[\Pi_{t+s}(x)\mathrm{e}^{-\int_0^s \rho_u \,\mathrm{d}u}] = \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\Pi_t^y(x)]\mathbb{E}[\Pi_s(y)\mathrm{e}^{-\int_0^s \rho_u \,\mathrm{d}u}].$$

Applying the convergence result (2.1) with $t = 0$, we get that, for all $y \in \mathbb{Z}^d$,

$$\lim_{s\to\infty} \mathbb{E}[\Pi_s(y)\mathrm{e}^{-\int_0^s \rho_u \,du}] = \nu_y.$$

It just remains now to invert the summation over $y \in \mathbb{Z}^d$ and the limit $s \uparrow \infty$, which we do using the dominated convergence theorem: First note that

$$\sup_{y\in\mathbb{Z}^d} \sup_{s\geq 0} \mathbb{E}[\Pi_s(y)\mathrm{e}^{-\int_0^s \rho_u \,\mathrm{d}u}] \leq \sup_{s\geq 0} \mathbb{E}[M_s] < \infty.$$

Also, by Cauchy-Schwarz's inequality, for all $y \in \mathbb{Z}^d$,

$$\mathbb{E}[\Pi_t^y(x)] \leq \mathbb{E}[(N_t^y)^2]^{1/2}\mathbb{P}(\Pi_t^y(x) \neq 0)^{1/2}.$$

Moreover, when $\|y\| > 2\|x\|$, we can bound the probability on the right-hand side by the probability that a simple random starting from y enters the ball $B(0, \|x\|)$. Thus there exist two positive constants c and C such that, for all $y \in \mathbb{Z}^d$ satisfying $\|y\| > 2\|x\|$,

$$\mathbb{P}(\Pi_t^y(x) \neq 0) \leq C \exp(-c\|y\|^2/t).$$

Finally, because the catalytic branching random walk is stochastically dominated by a branching random walk reproducing at constant rate λ_0 everywhere, we get that, for any fixed $t \geq 0$,

$$\sup_{y\in\mathbb{Z}^d} \mathbb{E}[(N_t^y)^2] < \infty.$$

Thus, the dominated convergence theorem applies and gives that

$$\lim_{s\to\infty} \mathbb{E}[\Pi_{t+s}(x) \cdot \mathrm{e}^{-\int_0^s \rho_u \,\mathrm{d}u}] = \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\Pi_t^y(x)] \cdot \nu_y.$$

Together with (2.1), this concludes the proof. ∎

We are now in a position to conclude the proof of our main result in this section:

Proof of Proposition 1.2. By Lemma 2.2, for all $x \in \mathbb{Z}^d$ and $t \geq 0$,

$$\nu_x = \sum_{y \in \mathbb{Z}^d} \nu_y \cdot \mathbb{E}[\Pi_t^y(x)] \cdot \mathrm{e}^{-\rho t}.$$

Taking the differential in t at $t = 0$, we get that, for all $x \neq 0$,

$$0 = \nu_x(\lambda - 1 - \rho) + \frac{1}{2d} \sum_{y \sim x} \nu_y,$$

and when $x = 0$,

$$0 = \nu_0(\lambda_0 - 1 - \rho) + \frac{1}{2d} \sum_{y \sim 0} \nu_y,$$

which after simplifying give respectively (1.1) and (1.2), as desired. □

3 Proof of Theorem 1.3: existence of a "stationary" measure

3.1 Proof of Theorem 1.3(i): non-localisation

First note that any probability measure ν satisfying (1.1) must satisfy $\nu_0 > 0$. Indeed, if $\nu_0 = 0$, then (1.1) and (1.2) (which follows from (1.1) and the fact that $\sum_{x \in \mathbb{Z}^d} \nu_x = 1$) show that ν is harmonic. However, it is well-known that any bounded harmonic function on $\mathbb{Z}^d$ is constant, hence $\nu_x = 0$ for all $x \in \mathbb{Z}^d$, which is a contradiction.

Now, let ν be a probability measure on $\mathbb{Z}^d$ satisfying (1.1). To simplify notation, we set $\varepsilon = \lambda_0 - \lambda$. Consider a simple random walk $(S_n)_{n \geq 0}$ on $\mathbb{Z}^d$, and let τ_0 be the first return time to 0, i.e. $\tau_0 = \inf\{n \geq 1 : S_n = 0\}$. Equation (1.1) shows that for any $x \neq 0$, on the event that $S_0 = x$, the process $(M_n)_{n \geq 0}$, defined for $n \geq 0$ by

$$M_n = \frac{\nu_{S_{n \wedge \tau_0}}}{(1 + \varepsilon \nu_0)^{n \wedge \tau_0}},$$

is a bounded martingale. Hence, the optional stopping theorem gives

$$\nu_x = \mathbb{E}_x \Big[\frac{\nu_0 \cdot \mathbf{1}\{\tau_0 < \infty\}}{(1 + \varepsilon \nu_0)^{\tau_0}} \Big], \tag{3.1}$$

where $\mathbb{E}_x$ denotes expectation with respect to the law of the random walk starting from x. Using now the symmetry of the walk, and letting τ_x denote the hitting time of x, we get

$$\nu_x = \nu_0 \cdot \mathbb{E} \Big[\frac{\mathbf{1}\{\tau_x < \infty\}}{(1 + \varepsilon \nu_0)^{\tau_x}} \Big] = \nu_0 \sum_{k=1}^{\infty} \frac{1}{(1 + \varepsilon \nu_0)^k} \cdot \mathbb{P}(\tau_x = k).$$

Summing over $x \in \mathbb{Z}^d \setminus \{0\}$, and taking into account that $(\nu_x)_{x \in \mathbb{Z}^d}$ is a probability measure, we get

$$1 = \nu_0 + \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \nu_x = \nu_0 + \nu_0 \sum_{k=1}^{\infty} \frac{1}{(1 + \varepsilon \nu_0)^k} \cdot \mathbb{P}(S_k \notin \{S_0, \dots, S_{k-1}\}).$$

Reversing time again, we deduce that

$$\begin{aligned} 1 &= \nu_0 + \nu_0 \sum_{k=1}^{\infty} \frac{1}{(1 + \varepsilon \nu_0)^k} \cdot \mathbb{P}(0 \notin \{S_1, \dots, S_k\}) \\ &= \nu_0 + \nu_0 \sum_{k=1}^{\infty} \frac{1}{(1 + \varepsilon \nu_0)^k} \cdot \mathbb{P}(\tau_0 \geq k + 1). \end{aligned} \tag{3.2}$$

Note now that for any $k \geq 1$, one has $\mathbb{P}(\tau_0 \geq k+1) \geq \mathbb{P}(\tau_0 = \infty) = \gamma_d$. Thus (3.2) gives

$$1 \geq \nu_0 + \nu_0 \gamma_d \sum_{k=1}^{\infty} \frac{1}{(1+\varepsilon\nu_0)^k} = \nu_0 + \frac{\gamma_d}{\varepsilon},$$

which yields a contradiction if $\varepsilon \leq \gamma_d$, since we recall that $\nu_0 > 0$.

3.2 Proof of Theorem 1.3(ii): existence of a stationary measure

As proved in Section 3.1, any probability measures ν satisfying (1.1) satisfies $\nu_0 > 0$. For any positive real number ν_0, we use (3.1) to define a positive measure ν on $\mathbb{Z}^d$ that satisfies (1.1), by the Markov property. Now note that a simple random walk on $\mathbb{Z}^d$ started at some x cannot reach 0 before time $\|x\|_\infty$. Thus, $\nu_x \leq \nu_0(1+\varepsilon\nu_0)^{-\|x\|_\infty}$, which is summable. Hence it only remains to show that, for all $\varepsilon > \gamma_d$, there exists a unique value of ν_0 making ν a probability measure. This is equivalent to showing that there exists a unique value of ν_0 such that (3.2) is satisfied. Letting $u = \varepsilon\nu_0$, we see that this is equivalent to the existence and uniqueness of a solution to the equation $f(u) = \varepsilon$, where $\varepsilon > \gamma_d$ is fixed and for $u > 0$,

$$f(u) = u + u \sum_{k=1}^{\infty} \frac{1}{(1+u)^k} \cdot \mathbb{P}(\tau_0 \geq k+1). \tag{3.3}$$

Noting that $\mathbb{P}(\tau_0 \geq k+1) = \gamma_d + \sum_{i \geq k+1} \mathbb{P}(\tau_0 = i)$, and inverting the order of summation, we get

$$f(u) = u + \gamma_d + \sum_{i=2}^{\infty} \mathbb{P}(\tau_0 = i) \cdot \big(1 - \frac{1}{(1+u)^{i-1}}\big).$$

Noting also that $\tau_0 \geq 2$ almost surely, this gives

$$f(u) = 1 + u - \sum_{i=2}^{\infty} \frac{\mathbb{P}(\tau_0 = i)}{(1+u)^{i-1}} = 1 + u - \mathbb{E}\Big[\frac{1}{(1+u)^{\tau_0 - 1}}\Big].$$

(Note that this last expression of f could also have been derived directly from (1.2) and (3.1).) In particular f is strictly increasing on $(0,\infty)$, converges to $+\infty$, as u goes to infinity, and by dominated convergence, we can see that it converges to γ_d as u goes to zero, which proves well that for any $\varepsilon > \gamma_d$, the equation $f(u) = \varepsilon$ has a unique solution.

Remark. Note that the last expression of f above rewrites as

$$f(u) = (1+u) \cdot \big(1 - G(\frac{1}{1+u})\big),$$

where $G(z) = \mathbb{E}[z^{\tau_0}]$, is the generating function of τ_0. Now for $n \geq 1$, denote by τ_n the n-th return time to the origin of the walk, and let $G_n(z) = \mathbb{E}[z^{\tau_n}]$. By independence between the successive return times to the origin, one has $G_n(z) = G(z)^n$, for any $n \geq 1$. In particular one has for any $0 < z < 1$,

$$\sum_{n \geq 1} G_n(z) = \frac{G(z)}{1 - G(z)}. \tag{3.4}$$

Now in [DR13], branching occurs only at the origin and at rate 1, which corresponds to choosing $\lambda = 0$ and $\lambda_0 = 1$ in our notation. In this case, when furthermore each particle splits into two particles at each branching event (as in our case), they show that the exponential growth rate ρ of the catalytic branching random walk (the so-called Malthusian parameter) is solution of the equation

$$\int_0^\infty e^{-\rho t}\, p_t(0,0)\, dt = 1, \tag{3.5}$$

where $p_t(0,0)$ is the probability that a continuous time simple random walk is at the origin at time t. Since by definition the time between any consecutive jumps is a mean one exponential random variable, one also has

$$\int_0^\infty e^{-\rho t}\, p_t(0,0)\, dt = \frac{1}{1+\rho} + \frac{1}{1+\rho}\sum_{n\geq 1} G_n\Big(\frac{1}{1+\rho}\Big).$$

Hence, using (3.4), we see that (3.5) is equivalent to

$$G\Big(\frac{1}{1+\rho}\Big) = \frac{\rho}{1+\rho}.$$

In particular we recover well the same equation for ρ as our equation defining ν_0, namely $f(\nu_0) = 1$, which agrees with the fact that in our setting, when $\lambda_0 = 1$, and in the limit $\lambda \to 0$, the exponential growth rate is given by ν_0.

4 Proof of localisation (proof of Theorem 1.1)

In this section, we assume that $\lambda_0 > 2d - 1 + 2d\lambda$. For the proof of Theorem 1.1, we use the fact that $(\Pi_t)_{t\geq 0}$, taken at the times when it changes values is a "measure-valued Pólya process" (or, in other words, an infinitely-many-colour Pólya urn); we thus start the section with some useful background and existing results on these processes.

4.1 Measure-valued Pólya processes

Definition 4.1. Let E be a Polish space, π a finite measure on E, $R^{(1)} = (R^{(1)}_x)_{x\in E}$ be a random kernel on E (i.e., for all $x \in E$, $R^{(1)}_x$ is a random measure on E, almost surely finite), and $P = (P_x)_{x\in E}$ is a kernel on E (i.e., for all $x \in E$, P_x is a finite measure on E). The measure-valued Pólya process (MVPP) of initial composition π, replacement kernel $R^{(1)}$ and weight kernel P is the sequence of random measures $(m_n)_{n\geq 0}$ defined recursively as follows: $m_0 = \pi$ and, for all $n \geq 0$,

$$m_{n+1} = m_n + R^{(n+1)}_{\xi(n+1)},$$

where, given m_n, $\xi(n+1)$ is a random variable of distribution $m_nP/m_nP(E)$, and where, given $\xi(n+1)$, $R^{(n+1)}_{\xi(n+1)}$ is an independent copy of $R^{(1)}_{\xi(n+1)}$.

Lemma 4.2. *For all $n \geq 0$, let τ_n be the time of the n-th event (jump of a particle or birth of a particle). Also let $m_n = \frac{1}{\kappa}\Pi_{\tau_n}$ for all $n \geq 0$, where*

$$\kappa = \lambda_0 - \frac{\lambda_0 - \lambda}{1 + \lambda_0}. \tag{4.1}$$

Then, $(m_n)_{n\geq 0}$ is an MVPP with the following parameters:

- *initial composition $m_0 = \frac{1}{\kappa}\delta_{\mathbf{0}}$;*
- *replacement kernel $(R^{(1)}_x)_{x\in\mathbb{Z}^d}$ where, for all $x \in \mathbb{Z}^d$,*

$$R^{(1)}_x = \frac{1}{\kappa}\big(B_x\delta_x + (1 - B_x)(\delta_{x+\Delta} - \delta_x)\big) = \frac{1}{\kappa}\big((2B_x - 1)\delta_x + (1 - B_x)\delta_{x+\Delta}\big),$$

  where B_x is a Bernoulli random variable of parameter $\frac{\lambda_x}{1+\lambda_x}$, with $\lambda_x = \lambda_0$ if $x = 0$ and $\lambda_x = \lambda$ otherwise, and where Δ is a simple symmetric random walk increment independent of B_x;
- *weight kernel $((1 + \lambda_x)\delta_x)_{x\in\mathbb{Z}^d}$.*

Remark. The reason why we divide Π_{τ_n} by κ in the definition of m_n is technical, and will be discussed later (we need that $\sup_x \mathbb{E}[R^{(1)}_x(\mathbb{Z}^d)] \leq 1$). Note, however, that $\kappa > 0$.

To prove localisation, we use [MV20, Theorem 1] (see Section 1.4 where the case of $R^{(1)}$ being a signed-measure is discussed). We prove that $(m_n)_{n\geq 0}$ satisfies the assumptions (T), (A1), (A'2), (A3) and (A4) of [MV20], which, for convenience, we copy here: first define $R = \mathbb{E}[R^{(1)}]$, meaning that, for all $x \in E$, for all measurable sets $A \subset E$, $R_x(A) = \mathbb{E}[R^{(1)}_x(A)]$. We also define, for all $x \in E$,

$$Q^{(1)}_x = \sum_{y\in E} R^{(1)}_x(\{y\})P_y \quad \text{and} \quad Q_x = \sum_{y\in E} R_x(\{y\})P_y, \tag{4.2}$$

or, in other words, $Q^{(1)} = R^{(1)}P$ and $Q = RP$.

(T) For all $n \geq 0$, m_n is a positive measure.

(A1) There exists $c > 0$ and $\beta > 1$ such that, for all $x \in E$, almost surely,

$$c \leq \inf_{x\in E} Q_x(E) \leq \sup_{x\in E} Q_x(E) \leq 1 \text{ and } \sup_{x\in E} \mathbb{E}\big|Q^{(1)}_x(E) - Q_x(E)\big|^\beta < +\infty. \tag{4.3}$$

(A'2) there exist a locally bounded function $V : E \to [1, +\infty)$ and some constants $r > 1$, $p > 2$, $q' > q := p/(p-1)$, $\theta \in (0, c)$, $K > 0$, $A \geq 1$, and $B \geq 1$, such that

- (i) for all $N \geq 0$, the set $\{x \in E \colon V(x) \leq N\}$ is relatively compact.
- (ii) for all $x \in E$,
$$Q_x \cdot V \leq \theta V(x) + K \quad \text{and} \quad Q_x \cdot V^{1/q} \leq \theta V^{1/q}(x) + K$$
- (iii) for all continuous functions $f : E \to \mathbb{R}$ bounded by 1 and all $x \in E$,
$$|Q_x \cdot f|^{q'} \vee \mathbb{E}\big[\big|R^{(1)}_x \cdot f - R_x \cdot f\big|^r\big] \vee \mathbb{E}\big[\big|Q^{(1)}_x \cdot f - Q_x \cdot f\big|^p\big] \leq AV(x),$$
- (iv) and
$$|Q_x \cdot V^{1/q}|^q \vee |Q_x \cdot V| \vee \mathbb{E}\left[\left|Q^{(1)}_x \cdot V^{1/q} - Q_x \cdot V^{1/q}\right|^r\right] \leq BV(x).$$

(A'3) the continuous-time pure jump Markov process X with sub-Markovian jump kernel $Q - I$ admits a quasi-stationary distribution $\nu \in \mathcal{P}(E)$ (the set of all probability measures on E). We further assume that the convergence of $\mathbb{P}_\alpha(X_t \in \cdot \,|\, X_t \neq \partial)$ holds uniformly with respect to the total variation norm on a neighbourhood of ν in $\mathcal{P}_C(E) = \{\alpha \in \mathcal{P}(E) \mid \alpha \cdot V^{1/q} \leq C\}$, for each $C > 0$, where $q = p/(p-1)$.

(A4) for all bounded continuous functions $f : E \to \mathbb{R}$, $x \in E \mapsto R_x f$ and $x \in E \mapsto Q_x f$ are continuous.

Remark. By definition, i.e. because m_n is the occupation measure of the catalytic random walk at the n-th jumping or branching event, (T) holds.

Remark. (A1) above is not as in [MV20], but this version of (A1) suffices, as discussed in [MV20, Section 1.5] (see Equation (3) therein).

Remark. In Assumption (A3) in [MV20], it is assumed that the convergence of $\mathbb{P}_\alpha(X_t \in \cdot \,|\, X_t \neq \partial)$ holds uniformly with respect to the total variation norm in $\{\alpha \in \mathcal{P}(\mathbb{Z}^d) \mid \alpha \cdot V^{1/q} \leq C\}$, for each $C > 0$. We are not able to prove that this assumption holds for our model, and hence replace (A3) by the weaker (A'3).

Remark. Because $\mathbb{Z}^d$ is discrete, (A4) holds straightforwardly.

The following theorem is a close adaptation (the only difference being the assumption (A'3)) of Theorem 1 of [MV20]:

Theorem 4.3. *Under Assumptions* (T), (A1), (A'2), (A'3), *and* (A4), *if* $m_0 \cdot V < \infty$ *and* $m_0 P \cdot V < \infty$, *then the sequence of random measures* $(m_n/n)_{n\geq 0}$ *converges almost surely to* νR *with respect to the topology of weak convergence. Furthermore, if* $\nu R(\mathbb{Z}^d) > 0$, *then* $(\tilde{m}_n := m_n/m_n(\mathbb{Z}^d))_{n\in\mathbb{N}}$ *converges almost surely to* $\nu R/\nu R(\mathbb{Z}^d)$ *with respect to the topology of weak convergence.*

Proof. We only need to explain how to adapt the proof of [MV20] to the case when (A'3) holds instead of (A3). To do so, we need to introduce some notation from [MV20]: for all $n \geq 0$, we let

$$\eta_n = \sum_{i=1}^{n} \delta_{\xi(i)}.$$

We also let $\tilde{\eta}_0 = 0$ and, for all $n \geq 1$, $\tilde{\eta}_n = \frac{\eta_n}{n}$. Finally, we set $\tau_n = \gamma_1 + \cdots + \gamma_n$, where $\gamma_n = \frac{1}{n\tilde{\eta}_{n-1}Q(E)}$ for all $n \geq 1$. The proof of [MV20] relies on the fact that $(\tilde{\eta}_n)_{n\geq 0}$ is a stochastic approximation: indeed, for all $n \geq 0$, one can write

$$\tilde{\eta}_{n+1} = \tilde{\eta}_n + \gamma_{n+1}\big(F(\tilde{\eta}_n) + U_{n+1}\big),$$

where $F(\mu) = \mu Q - \mu Q(E)\mu$ and $U_{n+1} = \tilde{\eta}_n Q(E)\delta_{\xi(n+1)} - \tilde{\eta}_n Q$. Remark 12 in [MV20] states that, without Assumption (A3), one can still conclude that $(\tilde{\eta}_t)_{t\geq 0}$ is an asymptotic pseudo-trajectory in $\mathcal{P}_C(E)$ for the semi-flow induced in $\mathcal{P}_C(E)$ by the well-posed dynamical system

$$\frac{\mathrm{d}\mu_t \cdot f}{\mathrm{d}t} = \mu_t Q \cdot f - \mu_t Q(E)\mu_t \cdot f, \tag{4.4}$$

(for all bounded functions $f = E \mapsto \mathbb{R}$). Now, by Theorem 3.7 in [Ben06], the limit set of a pre-compact asymptotic pseudo-trajectory is internally chain transitive and thus (by [Ben06, Proposition 5.3]) it does not contain any proper attractor. Now let $(X_t)_{t\geq 0}$ be the continuous-time pure jump Markov process with sub-Markovian jump kernel $Q - I$; if, for all $t \geq 0$, we let μ_t be the distribution of X_t conditioned on having survived until time t, then $(\mu_t)_{t\geq 0}$ is a solution of (4.4). By Assumption (A3), we get that ν is an attractor for the flow defined by (4.4) on $\mathcal{P}_C(E)$. Also, all trajectories started in $\mathcal{P}_C(E)$ converge to ν. Thus, the limit set of $(\tilde{\eta}_t)_{t\geq 0}$ is $\{\nu\}$. Therefore, $\tilde{\eta}_n \to \nu$ almost surely as $n \uparrow \infty$, for the topology of weak convergence. One can then follow the rest of the arguments in [MV20] to conclude the proof. ☐

4.2 Checking Assumption (A1)

First note that, by definition (see Equation (4.2)), for all $x \in \mathbb{Z}^d$,

$$\kappa Q^{(1)}_x = (2B_x - 1)(1 + \lambda_x)\delta_x + (1 - B_x)(1 + \lambda_{x+\Delta})\delta_{x+\Delta},$$

and thus

$$\kappa Q_x = \Big(\frac{2\lambda_x}{1+\lambda_x} - 1\Big)(1+\lambda_x)\delta_x + \frac{1}{1+\lambda_x}\mathbb{E}[(1+\lambda_{x+\Delta})\delta_{x+\Delta}] = (\lambda_x - 1)\delta_x + \frac{1}{1+\lambda_x}\mathbb{E}[(1+\lambda_{x+\Delta})\delta_{x+\Delta}]. \tag{4.5}$$

If $x = 0$, then $x + \Delta \neq 0$ almost surely and thus $\lambda_{x+\Delta} = \lambda$ almost surely, and hence,

$$\kappa Q_0(\mathbb{Z}^d) = \lambda_0 - 1 + \frac{\gamma + \lambda}{1 + \lambda_0} = \lambda_0 - \frac{\lambda_0 - \lambda}{1 + \lambda_0}.$$

If $\|x\|_1 > 1$, then $x + \Delta \neq 0$ almost surely, and hence,

$$\kappa Q_x(\mathbb{Z}^d) = \lambda - 1 + 1 = \lambda.$$

Finally, if $\|x\|_1 = 1$, then $x + \Delta = 0$ with probability $1/(2d)$, and hence,

$$\kappa Q_x(\mathbb{Z}^d) = \lambda - 1 + \frac{1}{1+\lambda}\Big(\frac{2d-1}{2d}(1+\lambda) + \frac{1}{2d}(1+\lambda_0)\Big) = \lambda + \frac{1}{2d\cdot(1+\lambda)}(\lambda_0 - \lambda).$$

In summary,

$$\kappa Q_x(\mathbb{Z}^d) = \begin{cases} \lambda_0 - \frac{1}{1+\lambda_0}(\lambda_0 - \lambda) & \text{if } x = 0 \\ \lambda + \frac{1}{2d\cdot(1+\lambda)}(\lambda_0 - \lambda) & \text{if } \|x\|_1 = 1 \\ \lambda & \text{if } \|x\|_1 > 1. \end{cases} \tag{4.6}$$

Note that, under the assumption that $\lambda_0 > \lambda$ and $\lambda_0 > 2d - 1 + 2d\lambda$, we have

$$\lambda_0 - \frac{\lambda_0 - \lambda}{1+\lambda_0} = \lambda + (\lambda_0 - \lambda)\cdot\frac{\lambda_0}{1+\lambda_0} > \lambda - (\lambda_0 - \lambda)\cdot\frac{2d-1+2d\lambda}{2d(1+\lambda)} \geq \lambda - \frac{\lambda_0 - \lambda}{2d(1+\lambda)},$$

because $2d-1+2d\lambda \geq 2d-1 \geq 1$. Thus the maximum of $x \mapsto \kappa Q_x(\mathbb{Z}^d)$ is reached at $x = 0$ and because, by definition, $\kappa = Q_0(E)$ (see (4.1); this is why we chose this value for κ), we get

$$\max_{x\in\mathbb{Z}^d} Q_x(E) = 1. \tag{4.7}$$

This concludes the proof of (4.7). From (4.6), we also get that

$$\min_{x\in\mathbb{Z}^d} Q_x(\mathbb{Z}^d) = \frac{\lambda}{\kappa} < 1. \tag{4.8}$$

Thus, to prove that (A1) holds with $c = \lambda > 0$ and $\beta = 2$, it is enough to show that

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E}\left|Q_x^{(1)}(\mathbb{Z}^d) - Q_x(\mathbb{Z}^d)\right|^2 < +\infty,$$

which holds because, for all $x \in \mathbb{Z}^d$, $Q_x^{(1)}(\mathbb{Z}^d)$ is almost surely bounded (by $-(1+\lambda_0)/\kappa$ from below, and by $2(1+\lambda_0)/\kappa$ from above).

4.3 Checking Assumption (A'2)

We let $V(x) = \|x\|_1$ if $x \neq \mathbf{0}$, and $V(0) = 1$.
(i) For all $N \geq 0$, $\{x \in \mathbb{Z}^d \colon \|x\| \leq N\}$ is finite and thus compact.
(ii) By Equation (4.5), for all $x \in \mathbb{Z}^d$,

$$\kappa Q_x V = (\lambda_x - 1)V(x) + \frac{1}{1+\lambda_x}\mathbb{E}[(1+\lambda_{x+\Delta})V(x+\Delta)].$$

In particular, for all $x \in \mathbb{Z}^d$ such that $\|x\|_1 > 1$,

$$\kappa Q_x V = (\lambda - 1)V(x) + \mathbb{E}[V(x+\Delta)].$$

For all $\|x\|_1 > 1$, we have $x + \Delta \neq \mathbf{0}$ and thus, by the triangular inequality,

$$V(x+\Delta) = \|x+\Delta\|_1 \leq \|x\|_1 + \|\Delta\|_1 = \|x\|_1 + 1.$$

Therefore,

$$\kappa Q_x V \leq (\lambda - 1)V(x) + V(x) + 1 = \lambda V(x) + 1.$$

We thus get that, for all $x \in \mathbb{Z}^d$,

$$\kappa Q_x V \leq \lambda V(x) + 1 + \max_{\|x\|_1 \leq 1} Q_x V. \tag{4.9}$$

Similarly, for all $\alpha \in (1/2, 1)$, for all $x \in \mathbb{Z}^d$,

$$\kappa Q_x V^\alpha = (\lambda - 1)V(x)^\alpha + \mathbb{E}[V(x+\Delta)^\alpha].$$

Almost surely, for all $x \in \mathbb{Z}^d$ such that $\|x\|_1 > 1$,

$$V(x+\Delta)^\alpha = \|x+\Delta\|_1^\alpha \leq (\|x\|_1 + 1)^\alpha.$$

As $\|x\|_1 \uparrow \infty$, $(\|x\|_1+1)^\alpha \sim \|x\|_1^\alpha$. We fix $\varepsilon = \varepsilon(\alpha) > 0$ small enough so that $\lambda + \varepsilon < \kappa$ (this is possible because $\lambda < \kappa$ - see (4.8)). Then, we choose $L(\varepsilon) \geq 1$ large enough so that $\|x\|_1 \geq L(\varepsilon)$ implies $(\|x\|_1+1)^\alpha \leq (1+\varepsilon)\|x\|_1^\alpha$. With these choices, for all $x \in \mathbb{Z}^d$ such that $\|x\|_1 \geq L(\varepsilon)$,

$$V(x+\Delta)^\alpha = \|x+\Delta\|_1^\alpha \leq (1+\varepsilon)V(x)^\alpha,$$

and thus

$$\kappa Q_x V^\alpha \leq (\lambda - 1)V(x)^\alpha + (1+\varepsilon)V(x)^\alpha = (\lambda+\varepsilon)V(x)^\alpha.$$

This implies

$$\kappa Q_x V^\alpha \leq (\lambda+\varepsilon)V(x)^\alpha + \max_{\|x\|_1 \leq L(\varepsilon)} V(x)^\alpha. \tag{4.10}$$

By Equations (4.9) and (4.10), we get that (ii) holds for any $q = 1/\alpha \in (1,2)$: indeed, one can take $\theta = (\lambda + \varepsilon(\alpha))/\kappa < 1$ and $K = \max_{\|x\|_1 \leq L(\varepsilon(\alpha))} V(x)^\alpha$.

(iii) Let $f : \mathbb{Z}^d \to \mathbb{R}$ be a function bounded by 1. First note that, for all $x \in \mathbb{Z}^d$, almost surely, by the triangular inequality and because $B_x \in \{0,1\}$,

$$|\kappa R_x^{(1)} f| = |(2B_x - 1)f(x) + (1-B_x)f(x+\Delta)| \leq 2\|f\|_\infty \leq 2.$$

Similarly,

$$|\kappa Q_x^{(1)} f| = |(2B_x-1)(1+\lambda_x)f(x) + (1-B_x)(1+\lambda_{x+\Delta})f(x+\Delta)| \leq 2(1+\lambda_0)\|f\|_\infty \leq 2(1+\lambda_0).$$

Using Jensen's inequality, these imply that

$$|R_x f| \leq \mathbb{E}[|R_x^{(1)} f|] \leq 2/\kappa \quad \text{and} \quad |Q_x f| \leq \mathbb{E}[|Q_x^{(1)} f|] \leq 2(1+\lambda_0)/\kappa.$$

Thus,

$$|Q_x f|^2 \leq \left(\frac{2(1+\lambda_0)}{\kappa}\right)^2 \leq \frac{4(1+\lambda_0)^2}{\kappa^2} \leq \frac{4(1+\lambda_0)^2}{\kappa^2} \cdot V(x),$$

because $V(x) \geq 1$ for all $x \in \mathbb{Z}^d$. Now, for all $r > 1$, using first the triangular inequality and then the fact that $x \mapsto x^r$ is convex and thus $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ for all $a, b \in \mathbb{R}$, we get

$$\mathbb{E}\big[\big|R_x^{(1)} \cdot f - R_x \cdot f\big|^r\big] \leq \mathbb{E}\big[\big(\big|R_x^{(1)} \cdot f\big| + \big|R_x \cdot f\big|\big)^r\big] \leq 2^{r-1}\big(\mathbb{E}\big[\big|R_x^{(1)} \cdot f\big|^r\big] + \big|R_x \cdot f\big|^r\big) \leq 2^r (2/\kappa)^r \leq (4/\kappa)^r.$$

Thus, for any $r \in (1,2)$,

$$\mathbb{E}\big[\big|R_x^{(1)} \cdot f - R_x \cdot f\big|^2\big] \leq (1 \vee 4/\kappa)^2 \leq (1 \vee 4/\kappa)^2 V(x).$$

Similarly,

$$\mathbb{E}\big[\big|Q_x^{(1)} \cdot f - Q_x \cdot f\big|^4\big] \leq 2^4((1+\lambda_0)/\kappa)^4 \leq 16((1+\lambda_0)/\kappa)^4 V(x).$$

Thus, (iii) holds for $q' = 2$, $p = 4$ and any $r \in (1,2)$.

(iv) For all $q \geq 1$, for all $x \in \mathbb{Z}^d$, by the triangular inequality,

$$\begin{aligned}
|\kappa Q_x^{(1)} V^{1/q}|^q &= |(2B_x-1)(1+\lambda_x)V(x)^{1/q} + (1-B_x)(1+\lambda_{x+\Delta})V(x+\Delta)|^q \\
&\leq (1+\lambda_0)^q (V(x)^\alpha + V(x+\Delta)^{1/q})^q \\
&\leq 2^{q-1}(1+\lambda_0)^q (V(x) + V(x+\Delta)),
\end{aligned}$$

because $q \geq 1$ and thus $x \mapsto x^q$ is convex, implying that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$. We have proved before that, for all $x \in \mathbb{Z}^d$, $V(x+\Delta) \leq V(x) + 1$ almost surely. This implies that, almost surely,

$$|\kappa Q_x^{(1)} V^{1/q}|^q \leq 2^{q-1}(1+\lambda_0)^q(2V(x)+1) \leq 2^{q+1}(1+\lambda_0)^q V(x), \tag{4.11}$$

because $1 \leq V(x) \leq 2V(x)$. By Jensen's inequality, because $|\cdot|^q$ is convex,

$$|Q_x V^{1/q}|^q \leq \mathbb{E}[|Q_x^{(1)} V^{1/q}|^q] \leq 2^{q+1}(1+\lambda_0)^q V(x)/\kappa. \tag{4.12}$$

In particular, taking $q = 1$ gives

$$|Q_x V| \leq 4(1+\lambda_0)V(x)/\kappa. \tag{4.13}$$

For all $q \in (1,2)$, using the triangular inequality and the convexity of $x \mapsto x^q$ we get that, almost surely,

$$\left|Q_x^{(1)} \cdot V^{1/q} - Q_x \cdot V^{1/q}\right|^q \leq \left(\left|Q_x^{(1)} \cdot V^{1/q}\right| + \left|Q_x \cdot V^{1/q}\right|\right)^q \leq 2^{q-1}\left(\left|Q_x^{(1)} \cdot V^{1/q}\right|^q + \left|Q_x \cdot V^{1/q}\right|^q\right).$$

Thus, by (4.11) and (4.12),

$$\left|Q_x^{(1)} \cdot V^{1/q} - Q_x \cdot V^{1/q}\right|^q \leq 2^q(4(1+\lambda_0)/\kappa)^q V(x).$$

Thus, (iv) holds for any $r = q \in (1,2)$, and it thus holds for $r = q = \frac{p}{p-1} = \frac{4}{3}$.

In total, we have showed that (A'2) holds for $q' = 2$, $p = 4$ and $q = r = 4/3$.

4.4 Checking Assumption (A'3)

To check Assumption (A'3), we apply a result of Champagnat and Villemonais (see [CV23, Theorem 5.1]). We first give a statement of this result (this statement is a simplification of the original version in [CV23], which is enough for our purposes): Let $(X(t))_{t\geq 0}$ be a continuous-time Markov process on a space $E \cup \{\partial\}$, absorbed at ∂, with jump rates given by $(q_{x,y})_{x,y\in E\cup\{\partial\}}$ satisfying $\sum_{y\in E\cup\{\partial\}} q_{x,y} < \infty$ for all $x \in E$. The infinitesimal generator of $(X(t))_{t\geq 0}$ acts on non-negative functions $f : E \cup \{\partial\} \to [0,\infty)$ satisfying $\sum_{y\in E\cup\{\partial\}} q_{x,y} f(y) < \infty$ for all $x \in E$, as follows: $\mathcal{L}f(\partial) = 0$ and, for all $x \in E$,

$$\mathcal{L}f(x) = \sum_{y\in E\cup\{\partial\}} q_{x,y}(f(y) - f(x)).$$

Theorem 4.4 ([CV23, Theorem 5.1])**.** *Assume that there exists a finite set $L \subset E$ such that $\mathbb{P}(X(1) = y) > 0$ for all $x, y \in L$, and such that the constant*

$$\eta_2 = \inf\{\eta > 0\colon \liminf_{t\uparrow\infty} \mathrm{e}^{-\eta t}\mathbb{P}_x(X(t) = x)\}$$

is finite and does not depend on $x \in L$. If there exists $C > 0$, $\eta_1 > \eta_2$ and $\varphi\colon E \cup \{\partial\} \to [0,\infty)$ such that $\varphi_{|E} \geq 1$, $\varphi(\partial) = 0$, $\sum_{y\in E\setminus\{x\}} q_{x,y}\varphi(y) < \infty$ for all $x \in E$, $\eta_1 > \sup_{x\in E} q_{x,\partial}$ and

$$\mathcal{L}\varphi(x) \leq -\eta_1\varphi(x) + C\mathbf{1}_{x\in L},$$

then the process $(X(t))_{t\geq 0}$ admits a quasi-stationary distribution ν_{QSD}, which is the unique one satisfying $\nu_{QSD}\varphi < \infty$ and $\mathbb{P}_{\nu_{QSD}}(X(t) \in L) > 0$ for some $t \geq 0$. Moreover, there exist constants $\alpha \in (0,1)$ and $C > 0$ such that, for all probability measures μ on E such that $\int \varphi \,\mathrm{d}\mu < \infty$ and $\mu(L) > 0$,

$$\|\mathbb{P}_\mu(X(t) \in \cdot | t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq C\alpha^t \cdot \frac{\int \varphi \,\mathrm{d}\mu}{\mu(L)}.$$

We first prove that the assumptions of Theorem [CV23] are satisfied: Recall that, for all $x \in \mathbb{Z}^d$,

$$\begin{aligned}\kappa Q_x &= (\lambda_x - 1)\delta_x + \frac{1}{1+\lambda_x}\mathbb{E}[(1+\lambda_{x+\Delta})\delta_{x+\Delta}] \\ &= \begin{cases} (\lambda_0 - 1)\delta_0 + \frac{1+\lambda}{1+\lambda_0}\mathbb{E}[\delta_{x+\Delta}] & \text{if } x = 0 \\ (\lambda - 1)\delta_x + \mathbb{E}[\delta_{x+\Delta}] + \frac{1}{2d}\frac{\lambda_0-\lambda}{1+\lambda}\delta_0 & \text{if } \|x\|_1 = 1 \\ (\lambda - 1)\delta_x + \mathbb{E}[\delta_{x+\Delta}] & \text{if } \|x\|_1 > 1. \end{cases}\end{aligned}$$

Instead of considering the jump process X of generator $(Q_x - \delta_x + (1 - Q_x(\mathbb{Z}^d))\delta_\partial)_{x\in\mathbb{Z}^d}$, we look at the jump process Y of generator $(\kappa Q_x - \kappa\delta_x + \kappa(1 - Q_x(\mathbb{Z}^d))\delta_\partial)_{x\in\mathbb{Z}^d}$. One can couple X and Y so that $(X(\kappa t))_{t\geq 0} = (Y(t))_{t\geq 0}$. In particular, (A3) holds for X if and only if it holds for Y.

Now note that, for all $x \in E$,

$$\begin{aligned}&\kappa Q_x - \kappa\delta_x + \kappa(1 - Q_x(\mathbb{Z}^d))\delta_\partial \\ &= \begin{cases} (\lambda_0 - 1 - \kappa)\delta_0 + \frac{1+\lambda}{1+\lambda_0}\mathbb{E}[\delta_{x+\Delta}] & \text{if } x = 0 \\ (\lambda - 1 - \kappa)\delta_x + \mathbb{E}[\delta_{x+\Delta}] + \frac{1}{2d}\frac{\lambda_0-\lambda}{1+\lambda}\delta_0 + (\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0} - \frac{1}{2d}\frac{1}{1+\lambda}\big)\delta_\partial & \text{if } \|x\|_1 = 1 \\ (\lambda - 1 - \kappa)\delta_x + \mathbb{E}[\delta_{x+\Delta}] + (\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0}\big)\delta_\partial & \text{if } \|x\|_1 > 1. \end{cases} \\ &= \begin{cases} -\frac{1+\lambda}{1+\lambda_0}\delta_0 + \frac{1+\lambda}{1+\lambda_0}\mathbb{E}[\delta_{x+\Delta}] & \text{if } x = 0 \\ -\big(\lambda_0 - \lambda + \frac{1+\lambda}{1+\lambda_0}\big)\delta_x + \mathbb{E}[\delta_{x+\Delta}] + \frac{1}{2d}\frac{\lambda_0-\lambda}{1+\lambda}\delta_0 + (\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0} - \frac{1}{2d}\frac{1}{1+\lambda}\big)\delta_\partial & \text{if } \|x\|_1 = 1 \\ -\big(\lambda_0 - \lambda + \frac{1+\lambda}{1+\lambda_0}\big)\delta_x + \mathbb{E}[\delta_{x+\Delta}] + (\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0}\big)\delta_\partial & \text{if } \|x\|_1 > 1. \end{cases}\end{aligned}$$

In other words, the process Y describes the movement on $\mathbb{Z}^d$ of a particle that behaves as follows:

- When the particle sits at 0, it jumps at rate $\frac{1+\lambda}{1+\lambda_0}$, and when it jumps, it moves to a neighbouring site chosen uniformly at random among the $2d$ possible choices.
- When the particle sits at x and $\|x\|_1 = 1$, it jumps to a uniformly-chosen neighbouring site at rate 1, it jumps to 0 with an additional rate of $\frac{1}{2d}\frac{\lambda_0-\lambda}{1+\lambda}$, and it dies at rate $(\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0} - \frac{1}{2d}\frac{1}{1+\lambda}\big)$.
- When the particle sits at x and $\|x\|_1 > 1$, it jumps to a uniformly-chosen neighbouring site at rate 1 and it dies at rate $(\lambda_0 - \lambda)\big(1 - \frac{1}{1+\lambda_0}\big)$.

Note in particular that the dying rate is the largest when $\|x\|_1 > 1$ and the lowest at $x = 0$ (in fact, the particle does not dies when it sits at the origin). Also note the additional drift towards zero when the particle sits at a neighbouring site of the origin.

To prove (A3), we aim at applying Theorem 4.4. To do this, we need to check that $(Y(t))_{t\geq 0}$ satisfies Assumptions (F0-3). We fix

$$\rho_1 = (\lambda_0 - \lambda)\Big(1 - \frac{1}{1+\lambda_0}\Big) = \frac{\lambda_0(\lambda_0 - \lambda)}{(1+\lambda_0)}, \tag{4.14}$$

and

$$\rho_2 = (\lambda_0 - \lambda)\Big(1 - \frac{1}{1+\lambda_0} - \frac{1}{2d}\cdot\frac{1}{1+\lambda}\Big) + \frac{2d-1}{2d}. \tag{4.15}$$

Note that $\rho_2 < \rho_1$; indeed, this is equivalent to

$$(\lambda_0 - \lambda)\Big(1 - \frac{1}{1+\lambda_0} - \frac{1}{2d}\cdot\frac{1}{1+\lambda}\Big) + \frac{2d-1}{2d} < (\lambda_0 - \lambda)\Big(1 - \frac{1}{1+\lambda_0}\Big)$$

which is equivalent to $2d-1+2d\lambda < \lambda_0$, which is our assumption. We choose $\varepsilon > 0$ small enough so that $\rho_2 < \rho_1 - \varepsilon$. We then choose $K > 1/\varepsilon$ large enough so that, for all $x \geq K$,

$$\Big(1+\frac{1}{x}\Big)^{3/4} \leq 1+\frac{1}{x}. \tag{4.16}$$

We now let $L = \{x \in \mathbb{Z}^d : \|x\|_1 \leq K\}$ and $\varphi(x) = \max(1, \|x\|_1^{3/4})$.

First note that, for any $x, y \in L$, and $t \geq 2$, by Markov's property,

$$\mathbb{P}_x(Y(t)=x) \geq \mathbb{P}_x(Y(1)=y)\mathbb{P}_y(Y(t-2)=y)\mathbb{P}_y(Y(1)=x).$$

Thus, if we let $\mathfrak{m} = \min_{x,y\in L} \mathbb{P}_x(Y(1)=y)$, then

$$\liminf_{t\uparrow\infty} \mathrm{e}^{-\eta t}\mathbb{P}_x(X(t)=x) \geq \mathfrak{m}^2\mathrm{e}^{2\eta} \liminf_{t\uparrow\infty} \mathrm{e}^{-\eta t}\mathbb{P}_y(X(t)=y),$$

which implies that η_0 does not depend on $x \in L$; it only remains to show that it is finite. Now note that, if we let $L_0 = \{\|x\| \leq 1\}$ for all $x \in L$, $t \geq 2$,

$$\begin{aligned}\mathbb{P}_x(Y(t)=x) &\geq \sum_{y,z\in L_0} \mathbb{P}_x(Y(1)=y)\mathbb{P}_y(Y(t-2)=z)\mathbb{P}_z(Y(1)=x)\\ &\geq \mathfrak{m}^2 \sum_{y,z\in L_0} \mathbb{P}_y(Y(t-2)=z) = \mathfrak{m}^2 \sum_{y\in L_0} \mathbb{P}_y(Y(t-2)\in L_0).\end{aligned} \tag{4.17}$$

Note that $\mathbb{P}_y(Y(t-2)\in L_0) \geq \mathbb{P}_y(Y(s) \in L_0$ for all $s \leq t-2)$. Because the jump rate from a site y such that $\|y\| = 1$ outside of L_0 is ρ_2 (see (4.15) for the definition of ρ_2), and because the process has to visit one of these sites before exiting L_0, we have that

$$\mathbb{P}_y(Y(s)\in L_0 \text{ for all } s \leq t-2) \geq \mathrm{e}^{-\rho_2(t-2)}.$$

This implies that, for all $x \in L$,

$$\mathbb{P}_x(Y(t)=x) \geq \mathfrak{m}^2\mathrm{e}^{\rho_2(t-2)},$$

which implies that $\liminf_{t\uparrow\infty} \mathbb{P}_x(Y(t)=x) \geq \mathfrak{m}^2\mathrm{e}^{2\rho_2} > 0$. Thus,

$$\eta_1 \leq \rho_2 < \infty.$$

Now, because the death rate outside of L equals ρ_1 (see (4.14) for the definition of ρ_1), we get that, for all $x \notin L$, for all $t \geq 0$,

$$\mathcal{L}\varphi(x) = \sum_{i=1}^d \frac{1}{2d}\big(\|x+e_i\|^{3/4} + \|x-e_i\|^{3/4} - 2\|x\|^{3/4}\big) - \rho_1\|x\|^{3/4}.$$

By symmetry, we may assume without loss of generality that $x \in \mathbb{N}^d_{\geq 0}$. Under this assumption, $\|x+e_i\| = \|x\|+1$ for all $1 \leq i \leq d$, and $\|x-e_i\| = \|x\|-1$ if $x_i \neq 0$, $\|x-e_i\| = \|x\|+1$ if $x_i = 0$. This gives that, for all $x \in \mathbb{N}^d_{\geq 0}$, for all $1 \leq i \leq d$,

$$\|x+e_i\|^{3/4} + \|x-e_i\|^{3/4} - 2\|x\|^{3/4} \leq 2(\|x\|+1)^{3/4} - 2\|x\|^{3/4},$$

which implies

$$\begin{aligned}\mathcal{L}\varphi(x) &\leq (\|x\|+1)^{3/4} - (1+\rho_1)\|x\|^{3/4} = \Big(\Big(1+\frac{1}{\|x\|}\Big)^{3/4} - 1 - \rho_1\Big)\|x\|^{3/4}\\ &\leq -(\rho_1 - 1/K)\|x\|^{3/4} \leq -(\rho_1-\varepsilon)\varphi(x),\end{aligned}$$

as long as $\|x\| > K$, by (4.16). We thus get

$$\mathcal{L}\varphi(x) \leq -\eta_1\varphi(x) + C\mathbf{1}_{x\in L},$$

where we have set $C = \max_{x\in L}\mathcal{L}\varphi(x) < \infty$ and $\eta_1 = \rho_1 - \varepsilon$. Recall that we have chosen ε such that $\eta_2 = \rho_2 < \rho_1 - \varepsilon = \eta_1$. We have thus shown that the assumptions of Theorem 4.4 hold for $L = \{\|x\| \leq K\}$ for some K large enough and $\varphi_1 = \|\cdot\|^{3/4} = V^{1/q}$ (for the choices of V and q made in Section 4.3). We thus get that X admits a quasi-stationary distribution ν as required in (A'3). Furthermore, for any $\mu \in \mathcal{P}_C(E)$ such that $\|\mu - \nu\|_{TV} \leq \nu(L)/2$, we get that

$$\|\mathbb{P}_\mu(X(t) \in \cdot | t < \tau_\partial) - \nu\|_{TV} \leq C\alpha^t \cdot \frac{\int \varphi \, \mathrm{d}\mu}{\mu(L)} \leq C\alpha^t \cdot \frac{C}{\nu(L)/2},$$

and thus (A'3) holds.

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