The Heckman-Opdam Markov processes

Bruno Schapira*

30th April 2006

Université d'Orléans,
Fédération Denis Poisson, Laboratoire MAPMO
B.P. 6759, 45067 Orléans cedex 2, France .

Université Pierre et Marie Curie,
Laboratoire de Probabilités et Modèles Aléatoires,
4 place Jussieu,
F-75252 Paris cedex 05, France.

Abstract

We introduce and study the natural counterpart of the Dunkl Markov processes in a negatively curved setting. We give a semimartingale decomposition of the radial part, and some properties of the jumps. We prove also a law of large numbers, a central limit theorem, and the convergence of the normalized process to the Dunkl process. Eventually we describe the asymptotic behavior of the infinite loop as it was done by Anker, Bougerol and Jeulin in the symmetric spaces setting in [1].

Key Words: Markov processes, Jump processes, root systems, Dirichlet forms, Dunkl processes, limit theorems.

A.M.S. Classification: 58J65, 60B15, 60F05, 60F17, 60J35, 60J60, 60J65, 60J75.

e-mail: bruno.schapira@univ-orleans.fr

*partially supported by the European Commission (IHP Network HARP 2002 − 2006).

1 Introduction

In the last few years, some processes living in cones have played an important role in probability. The cones that we consider here are associated to a root system. Roughly speaking a root system is a set of vectors, satisfying a few conditions, in a Euclidean space. The set of hyperplanes orthogonal to the vectors of the root system delimit cones, which are called the Weyl chambers. We usually choose arbitrarily one of them which we call the positive Weyl chamber. One of the first example of process with value in a Weyl chamber is the intrinsic Brownian motion introduced by Biane in [3]. It may be defined as the radial part (in the Lie algebras terminology) of the Brownian motion on a complex Riemannian flat symmetric space. In the particular case where the root system is of type A_n, it is the process of eigenvalues of the Brownian motion on Hermitian matrices with trace null. It was also proved recently by Biane, Bougerol and O'Connell [4], that it is a natural generalization of the Bessel-3 process in dimension n, in the sense that it may be obtained by a transform of the Brownian motion in $\mathbb{R}^n$, which coincides in dimension 1 with the Pitman transform $2S - B$. Another example of processes associated to a root system is the radial part of the Brownian motion on a Riemannian symmetric space of noncompact type [1], [2]. This is the analogue of the intrinsic Brownian motion in a negatively curved setting. In [1] Anker, Bougerol and Jeulin study in fact other processes naturally attached to a symmetric space, and they show some surprising link between them and the intrinsic Brownian motion. More recently, Rösler [26] and Rösler, Voit [27] have introduced a new type of processes related to Weyl chambers of root systems, the Dunkl processes. These processes are Markov processes as well as martingales, but they no longer have continuous paths. They may jump from a chamber to another. Nevertheless the projection on the positive Weyl chamber of these processes, which is called the radial part, has continuous paths. Moreover for a particular choice of the parameter, this radial part is in fact the intrinsic Brownian motion. These processes were studied recently more deeply by Gallardo and Yor [15] [16] [17], and by Chybiryakov [11] [12], who have obtained many interesting properties, such as the time inversion property, a Wiener chaos decomposition, or a skew product decomposition.

In this paper, we introduce and study the natural counterpart of the Dunkl processes in the negatively curved setting, which we call the Heckman-Opdam processes. These processes are also discontinuous, and have a continuous radial part, which coincides with the radial part (in the Lie groups terminology) of the Brownian motion on some symmetric spaces for particular choices of the parameter. We will show that many known results in probability theory (see [1] [2] [5]) in the symmetric spaces setup, can be generalized to these new processes. We also prove that the Dunkl processes are limits of normalized Heckman-Opdam processes.

2 Preliminaries

Let $\mathfrak{a}$ be a Euclidean vector space of dimension n, equipped with an inner product $(\cdot,\cdot)$. Let $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}$. For $\alpha \in \mathfrak{a}$ let $\alpha^\vee = \frac{2}{|\alpha|^2}\alpha$, and let

$$r_\alpha(x) = x - (\alpha^\vee, x)\alpha,$$

be the corresponding orthogonal reflection. Let $\mathcal{R} \subset \mathfrak{a}$ be an integral (or crystallographic) root system, which by definition (cf [7]) satisfies the following hypothesis

1. $\mathcal{R}$ is finite, does not contain 0 and generates $\mathfrak{a}$.
2. $\forall \alpha \in \mathcal{R}$, $r_\alpha(\mathcal{R}) = \mathcal{R}$.
3. $\forall \alpha \in \mathcal{R}$, $\alpha^\vee(\mathcal{R}) \subset \mathbb{Z}$.

We choose a set of positive roots $\mathcal{R}^+$ (it can be taken as the subset of roots $\alpha \in \mathcal{R}$ such that $(\alpha, u) > 0$, for some arbitrarily chosen vector $u \in \mathfrak{a}$ satisfying $(\alpha, u) \neq 0$, for all $\alpha \in \mathcal{R}$). We denote by W the Weyl group associated to $\mathcal{R}$, i.e. the group generated by the r_α's, with $\alpha \in \mathcal{R}$. If C is a subset of $\mathfrak{a}$, we call *symmetric* of C any image of C under the action of W. Let $k : \mathcal{R} \to \mathbb{R}^+$ be a multiplicity function, i.e. a W-invariant function on $\mathcal{R}$. We will assume in this paper that $k(\alpha)$ (also denoted by k_α in the sequel) is strictly positive for all $\alpha \in \mathcal{R}^+$.
Let

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} \mid \forall \alpha \in \mathcal{R}^+,\ (\alpha, x) > 0\},$$

be the positive Weyl chamber. We denote by $\overline{\mathfrak{a}_+}$ its closure, and by $\partial\mathfrak{a}_+$ its boundary. Let also $\mathfrak{a}_{\mathrm{reg}}$ be the subset of regular elements in $\mathfrak{a}$, i.e. those elements which belong to no hyperplane $\{x \in \mathfrak{a} \mid (\alpha, x) = 0\}$.
For $\xi \in \mathfrak{a}$, let T_ξ be the Dunkl-Cherednik operator. It is defined, for $f \in C^1(\mathfrak{a})$ and $x \in \mathfrak{a}_{\mathrm{reg}}$, by

$$T_\xi f(x) = \partial_\xi f(x) - (\rho, \xi) f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \frac{(\alpha, \xi)}{1 - e^{-(\alpha, x)}} \{f(x) - f(r_\alpha x)\},$$

where

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.$$

The Dunkl-Cherednik operators form a commutative family of differential-difference operators (see [10]). The Laplacian $\mathcal{L}$ is defined by

$$\mathcal{L} = \sum_{i=1}^n T_{\xi_i}^2,$$

where $\{\xi_1, \ldots, \xi_n\}$ is any orthonormal basis of $\mathfrak{a}$ ($\mathcal{L}$ is independent of the chosen basis). Here is an explicit expression of $\mathcal{L}$ (see [28]), which holds for $f \in C^2(\mathfrak{a})$

and $x \in \mathfrak{a}_{\text{reg}}$:

$$\begin{aligned}\mathcal{L}f(x) &= \Delta f(x) + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \coth\frac{(\alpha,x)}{2}\partial_\alpha f(x) + |\rho|^2 f(x) \qquad (1)\\ &+ \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{4\sinh^2\frac{(\alpha,x)}{2}}\{f(r_\alpha x) - f(x)\},\end{aligned}$$

where Δ denotes the Euclidean Laplacian. Let μ be the measure on $\mathfrak{a}$ given by

$$d\mu(x) = \delta(x)dx,$$

where

$$\delta(x) = \prod_{\alpha\in\mathcal{R}^+} |\sinh\frac{(\alpha,x)}{2}|^{2k_\alpha}.$$

Let $\lambda \in \mathfrak{h}$. We denote by G_λ the unique analytic function on $\mathfrak{a}$, which satisfies the differential and difference equations

$$T_\xi G_\lambda = (\lambda,\xi)G_\lambda, \quad \forall\xi\in\mathfrak{a}$$

and which is normalized by $G_\lambda(0) = 1$ (see [24]). Let F_λ be the function defined for $x \in \mathfrak{a}$ by

$$F_\lambda(x) = \frac{1}{|W|}\sum_{w\in W} G_\lambda(wx).$$

These functions were introduced in [18]. Let $C_0(\mathfrak{a})$ be the space of continuous functions on $\mathfrak{a}$ which vanish at infinity, and let $C_0^2(\mathfrak{a})$ be its subset of twice differentiable functions (with analogue definitions for $\overline{\mathfrak{a}_+}$ in place of $\mathfrak{a}$). We denote by $\mathcal{C}(\mathfrak{a})$ the Schwartz space on $\mathfrak{a}$ associated to the measure μ, i.e. the space of infinitely differentiable functions f on $\mathfrak{a}$ such that for any polynomial p, and any $N \in \mathbb{N}$,

$$\sup_{x\in\mathfrak{a}}(1+|x|)^N e^{(\rho,x^+)}|p(\frac{\partial}{\partial x})f(x)| < +\infty,$$

where x^+ is the unique symmetric of x which lies in $\overline{\mathfrak{a}_+}$. We denote by $\mathcal{C}(\mathfrak{a})^W$ the subspace of W-invariant functions, which we identify with their restriction to $\overline{\mathfrak{a}_+}$. We have seen in [28] that

$$\mathcal{D} := \frac{1}{2}(\mathcal{L} - |\rho|^2),$$

densely defined on $\mathcal{C}(\mathfrak{a})$, has a closure on $C_0(\mathfrak{a})$, which generates a Feller semigroup $(P_t, t \geq 0)$. We have also obtained the formula for $f \in C_0(\mathfrak{a})$ and $x \in \mathfrak{a}$:

$$P_t f(x) = \int_{\mathfrak{a}} p_t(x,y)f(y)d\mu(y),$$

where $p_t(\cdot,\cdot)$ is the heat kernel. It is defined for $x, y \in \mathfrak{a}$ and $t > 0$, by

$$p_t(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} G_\lambda(x) G_\lambda(-y) d\nu(\lambda),$$

where ν is the asymmetric Plancherel measure (see [28]). We denote by D the differential part of $\mathcal{D}$, which is equal for $f \in \mathcal{C}(\mathfrak{a})^W$ and $x \in \mathfrak{a}_{\mathrm{reg}}$ to

$$Df(x) = \frac{1}{2}\Delta f(x) + (\nabla \log \delta^{\frac{1}{2}}, \nabla f)(x).$$

It has also a closure on $C_0(\overline{\mathfrak{a}_+})$, which generates a Feller semigroup $(P_t^W, t \geq 0)$. It is associated to a kernel p_t^W, which is defined for $x, y \in \overline{\mathfrak{a}_+}$ and $t > 0$, by

$$p_t^W(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(x) F_\lambda(-y) d\nu'(\lambda),$$

where ν' is the symmetric Plancherel measure (see [28]).

3 Definition and first properties

3.1 The Heckman-Opdam processes

The Heckman-Opdam process (also denoted by HO-process) is defined as the càdlàg Feller process $(X_t, t \geq 0)$ on $\mathfrak{a}$ with semigroup $(P_t, t \geq 0)$. Remember that it is characterized as the unique (in law) solution of the martingale problem associated to $(\mathcal{D}, \mathcal{C}(\mathfrak{a}))$ on $C_0(\mathfrak{a})$, see e.g. Theorem 4.1 and Corollary 4.3 in [13]. Observe that, by elementary calculation, the generator of the HO-process is also the closure of $\mathcal{D}$ on $C_c^\infty(\mathfrak{a})$, the space of infinitely differentiable functions with compact support on $\mathfrak{a}$. The multiplicity k is called the parameter of the HO-process. Moreover a.s., for any $t \geq 0$, $X_t \in \mathfrak{a}$, i.e. the exploding time of $(X_t, t \geq 0)$ is almost surely infinite. This results for example from Proposition 2.4 in [13]. Similarly, we define the radial process (or radial part of the HO-process), as the Feller process on $\overline{\mathfrak{a}_+}$ with semigroup $(P_t^W, t \geq 0)$. It is also characterized as the unique solution of the martingale problem associated to $(D, \mathcal{C}(\mathfrak{a})^W)$ on $C_0(\overline{\mathfrak{a}_+})$. Consider now the process $(X_t^W, t \geq 0)$ on $\overline{\mathfrak{a}_+}$, defined as the projection on $\overline{\mathfrak{a}_+}$ under the Weyl group W (for any t, X_t^W is the unique symmetric of X_t which lies in $\overline{\mathfrak{a}_+}$).

Proposition 3.1 *The process $(X_t^W, t \geq 0)$ is the radial process.*

Proof of the proposition: Remember that if $f \in \mathcal{C}(\mathfrak{a})^W$, then $\mathcal{D}f = Df \in \mathcal{C}(\mathfrak{a})^W$. Thus for $f \in \mathcal{C}(\mathfrak{a})^W$, and $t \geq 0$,

$$f(X_t^W) - f(X_0^W) - \int_0^t Df(X_s^W) ds = f(X_t) - f(X_0) - \int_0^t \mathcal{D}f(X_s) ds.$$

Therefore $(f(X_t^W) - f(X_0^W) - \int_0^t Df(X_s^W) ds, t \geq 0)$ is a local martingale. And we conclude by the uniqueness of the martingale problem associated to D. $\square$

Let us note eventually that when $2k$ equals the multiplicity associated with a Riemannian symmetric space of noncompact type G/K, then the radial HO-process coincide with the radial part of the Brownian motion on this symmetric space (see [18], or [28]). For instance if $G = SL_n$, then $k = \frac{1}{2}$ in the real case, $k = 1$ in the complex case, and $k = 2$ in the quaternionic case. So most of our results about radial processes generalize known results of the probabilistic theory on symmetric spaces.

3.2 The Dunkl processes

We recall now the definition of the Dunkl process and of its radial part. Let $\mathcal{R}'$ be a reduced root system (i.e. such that $\forall\alpha \in \mathcal{R}'$, $2\alpha \notin \mathcal{R}'$), but non necessarily integral (i.e. we do not assume condition 3 in the definition), and let k' be a multiplicity function on $\mathcal{R}'$. The Dunkl Laplacian $\mathcal{L}'$ is defined for $f \in C^2(\mathfrak{a})$, and $x \in \mathfrak{a}_{\mathrm{reg}}$ by

$$\begin{aligned} \mathcal{L}'f(x) &= \frac{1}{2}\Delta f(x) + \sum_{\alpha\in\mathcal{R}'^+} k'_\alpha \frac{1}{(\alpha,x)}\partial_\alpha f(x) \\ &+ \sum_{\alpha\in\mathcal{R}'^+} k'_\alpha \frac{1}{(\alpha,x)^2}\{f(r_\alpha x) - f(x)\}. \end{aligned}$$

It was proved by Rösler in [26] that $\mathcal{L}'$ defined on $\mathcal{S}(\mathfrak{a})$, the classical Schwartz space on $\mathfrak{a}$, is a closable operator, which generates a Feller semigroup on $C_0(\mathfrak{a})$. Naturally the Dunkl process is the Feller process defined by this semigroup. Now our proof (in [28]) that the operator D defined on $\mathcal{C}(\mathfrak{a})^W$ is closable, also holds in the Dunkl setting. Thus we may define the radial Dunkl process as the Feller process on $\overline{\mathfrak{a}_+}$ with generator the closure of $(L', \mathcal{S}(\mathfrak{a})^W)$, where L' is the differential part of $\mathcal{L}'$, and $\mathcal{S}(\mathfrak{a})^W$ is the subspace of $\mathcal{S}(\mathfrak{a})$ of W-invariant function (identified with their restriction to $\overline{\mathfrak{a}_+}$). In this way we get a new characterization of the radial Dunkl process as solution of a martingale problem. Naturally Proposition 3.1 holds as well in the Dunkl setting, thus our definition of the radial Dunkl process agrees with the usual one. Eventually the intrinsic Brownian motion is by definition the radial Dunkl process of parameter $k' = 1$.

4 The radial HO-process as a Dirichlet process

The goal of this section is to obtain an explicit semimartingale decomposition of the radial HO-process. In the case of root systems of type A, it was obtained by Cépa and Lépingle (see [8] and [9] Theorem 2.2). We present here another approach, which is based on the theory of Dirichlet processes. Our reference for this theory will be [14].
We consider $(D, \mathcal{C}(\mathfrak{a})^W)$ as a symmetric operator on $L^2(\overline{\mathfrak{a}_+}, \mu)$ (simply denoted by L^2 in the sequel). We have seen in [28] that this operator is closable. We

denote by $(D, \mathcal{D}_2(D))$ its closure. Its associated semigroup is just the extension of $(P_t^W, t \geq 0)$ on L^2. It is defined for $f, g \in L^2$ and $t \geq 0$ by

$$P_t^W f(x) = \int_{\overline{\mathfrak{a}_+}} p_t^W(x,y) f(y) d\mu(y).$$

We denote by $\mathcal{E}$ the associated Dirichlet form, and by $\mathcal{F}$ its domain ($\mathcal{D}_2(D) \subset \mathcal{F}$). It is determined for $f, g \in \mathcal{D}_2(D)$ by

$$\mathcal{E}(f,g) := -\int_{\overline{\mathfrak{a}_+}} f(x) Dg(x) d\mu(x) = -\int_{\overline{\mathfrak{a}_+}} Df(x) g(x) d\mu(x).$$

The fact that $\mathcal{E}$ is a regular Dirichlet form with special standard core the algebra $\mathcal{C}(\mathfrak{a})^W$, results from the density of this space in $C_0(\overline{\mathfrak{a}_+})$ (see Lemma 5.1 in [28]) and Theorem 3.1.2 in [14]. We have seen in [28] that when $f \in L^1(\mathfrak{a}, \mu)$, then $Gf : x \mapsto \int_0^\infty P_t^W f(x) dt$ is a.e. finite. In the terminology of [14], this means that the Dirichlet form $\mathcal{E}$ (or the semigroup P_t^W) is transient. This implies in particular that the process tends to infinity when $t \to \infty$. In the sequel we will prove a law of large numbers which makes this fact precise. It implies also that we may consider $\mathcal{F}$, equipped with its inner product $\mathcal{E}$, as a Hilbert space (see [14] chapter 2). For $i = 1, \ldots, n$, we denote by $\varphi_i : x \mapsto x_i$ the coordinate functions on $\overline{\mathfrak{a}_+}$. For $A > 0$, let $\varphi_i^A \in C^\infty(\overline{\mathfrak{a}_+})$ be a function which coincides with φ_i on $\{|x| \leq A\}$, and which is null on $\{|x| \geq A+1\}$.

Lemma 4.1 *For all $A > 0$, $\varphi_i^A \in \mathcal{F}$, and for all $v \in \mathcal{C}(\mathfrak{a})^W$,*

$$\mathcal{E}(\varphi_i^A, v) = -\int_{\overline{\mathfrak{a}_+}} D\varphi_i^A v \, d\mu.$$

Proof of the lemma: Let $(u_n)_n \in \mathcal{C}(\mathfrak{a})^W$ which converges uniformly to φ_i^A as in Lemma 5.1 in [28]. We will assume that $|u_n - \varphi_i^A|_\infty \leq \frac{1}{n}$ for all n. We have to prove that it is an $\mathcal{E}$-Cauchy sequence. Let $n < m$ be two integers. We have

$$\begin{aligned}
\mathcal{E}(u_n - u_m, u_n - u_m) &= \int_{d(x,\partial\mathfrak{a}_+) \leq \frac{1}{m}} D(u_n - u_m)(u_n - u_m) d\mu \\
&+ \int_{\frac{1}{m} \leq d(x,\partial\mathfrak{a}_+) \leq \frac{1}{n}} D(u_n - u_m)(u_n - u_m) d\mu \\
&+ \int_{d(x,\partial\mathfrak{a}_+) \geq \frac{1}{n}} D(u_n - u_m)(u_n - u_m) d\mu.
\end{aligned}$$

By Lemma 5.1 in [28], the integrand in the first integral is bounded, up to a constant, by $\frac{m}{n}$. But $\mu(\{d(x, \partial\mathfrak{a}_+) \leq \frac{1}{m}, |x| \leq A+1\})$ is bounded, up to a constant, by $\frac{1}{m}$. Thus the first integral tends to 0 when $n \to \infty$. The same argument applies for the second integral. The third integral is naturally bounded, up to a constant, by $\frac{1}{n}$. Now always by Lemma 5.1 in [28], the sequence $(Du_n)_n$ is dominated, up to a constant, by $x \mapsto \frac{1}{d(x,\partial\mathfrak{a}_+)} + \sum_\alpha \coth \frac{\alpha}{2}$ on $|x| \leq A+1$, which is μ-integrable since k is strictly positive. Thus the last statement of the lemma is a consequence of the dominated convergence theorem. □

The lemma implies (in the terminology of [14]), that the functions φ_i are in $\mathcal{F}_{b,\mathrm{loc}}$. Thanks to Theorem 5.5.1 p.228 of [14], there exist martingale additive functionals locally of finite energy M^i, and additive functionals locally of zero energy N^i such that, for every i,

$$\varphi_i(X^W) = M^i + N^i. \tag{2}$$

For $A > 0$, we denote by ν_i^A the measure defined by $d\nu_i^A = -D\varphi_i^A d\mu$. Observe that $1_{(|x|\leq A)}d\nu_i^A(x) = \sum_\alpha k_\alpha \coth(\frac{\alpha}{2}, x)\varphi_i(\alpha)1_{(|x|\leq A)}d\mu(x)$. We denote by ν_i the Radon measure defined by $d\nu_i(x) = \sum_\alpha k_\alpha \coth(\frac{\alpha}{2}, x)\varphi_i(\alpha)d\mu(x)$. Thanks to Theorem 5.5.4 p.229 of [14], and the preceding lemma, we see that N^i is an additive functional of bounded variation, and that it is the unique continuous additive functional associated to the measure ν^i (ν^i is called the Revuz measure of N^i). Moreover from Theorem 5.1.3 (iii), we get for all i and all $t \geq 0$,

$$N_t^i = \sum_{\alpha\in\mathcal{R}^+} k_\alpha \phi_i(\alpha)\int_0^t \coth\frac{(\alpha, X_s^W)}{2}ds.$$

In the same way, it is immediate from Theorem 5.5.2 p.229, and the identity 3.2.14 p.110, that the Revuz measure of $< M^i >$ is μ for each i. Therefore $M := \sum_i M^i e_i$ is necessarily a Brownian motion on $\mathfrak{a}$ (we have denoted by e_i the i^{th} vector of the canonical basis). Now, since coth is positive on $(0, +\infty)$, and X^W does not explode, we see with (2) that necessarily, for all $t > 0$, $\sum_{i=1}^n N_t^i e_i \in \overline{\mathfrak{a}_+}$ (in particular it does not explode). Thus for any α, $(\int_0^t \coth\frac{(\alpha,X_s^W)}{2}ds, t\geq 0)$ is in fact a positive additive functional and its expectation is therefore finite for each time $t \geq 0$ and for q.e. starting point x (q.e. stands for quasi everywhere, as explained in [14]). But it results from [9] Theorem 2.2 that it is in fact true for all $x \in \overline{\mathfrak{a}_+}$. Indeed there it is proved that $\mathbb{E}_x[\int_0^t |\nabla \log \delta^{\frac{1}{2}}(X_s^W)|ds] < +\infty$. But $\nabla \log \delta^{\frac{1}{2}}$ is equal to $k_\alpha \alpha \coth\frac{\alpha}{2} + z$, where z lies in the cone, let say C^*, generated by the convex hull of $\mathcal{R}^+$. Thus (since $-\alpha \notin C^*$) there exists a constant $c > 0$ (independent of x) such that, $c\coth\frac{(\alpha,x)}{2} \leq d(0, k_\alpha\alpha\coth\frac{(\alpha,x)}{2} + C^*) \leq |\nabla \log \delta^{\frac{1}{2}}(x)|$, for $x \in \overline{\mathfrak{a}_+}$. Finally we have proved the following result

Proposition 4.1 *The radial Heckman-Opdam process $(X_t^W, t \geq 0)$ starting at $x \in \overline{\mathfrak{a}_+}$, is a continuous semimartingale, and is the unique solution of the following SDE*

$$X_t^W = x + \beta_t + \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha}{2}\int_0^t \coth\frac{(\alpha, X_s^W)}{2}ds,\ t \geq 0, \tag{3}$$

where $(\beta_t, t \geq 0)$ is a Brownian motion on $\mathfrak{a}$. Moreover for any $t \geq 0$, any $x \in \overline{\mathfrak{a}_+}$ and any $\alpha \in \mathcal{R}^+$,

$$\mathbb{E}_x\left[\int_0^t \coth\frac{(\alpha, X_s^W)}{2}ds\right] < +\infty.$$

The uniqueness in law of the SDE (3) is just a consequence of the uniqueness of solutions to the martingale problem associated to $(D, \mathcal{C}(\mathfrak{a})^W)$. In fact there is also strong uniqueness. This results simply from the fact that coth is decreasing. Indeed if (X, B) and (X', B) are two solutions of (3), then for all $t \geq 0$,

$$\frac{d}{dt}(|X_t - X'_t|^2) = 2(X_t - X'_t, \nabla \log \delta^{\frac{1}{2}}(X_t) - \nabla \log \delta^{\frac{1}{2}}(X'_t)) \leq 0,$$

which proves that X and X' are indistinguishable. With Theorem (1.7) p. 368 in [25], this implies also that each solution is strong.

Remark 4.1

1. The finiteness of the expectation in the proposition will be used in the next section for the study of the jumps.

2. We could ask whether the processes considered by Cépa and Lepingle in [9] coincide with ours. In fact they prove existence of a solution for the same EDS but with an additional local time term. The question is therefore to know if this local time must be 0. Cépa and Lépingle have proved this for root systems of type A. But we can prove it now for the other root systems. Indeed by the Itô formula, their process is a solution of the martingale problem associated to D, since for any W-invariant function f, $(\nabla f(x), n) = 0$ for all $x \in \partial \mathfrak{a}_+$ and n a normal vector. Thus they coincide with the radial HO-process whose local time on $\partial \mathfrak{a}_+$ is 0.

3. In fact Proposition 4.1 is also valid in the Dunkl setting (with the same proof), where it was proved in the same time, but with a completely different method, by Chybiryakov (see [12]).

A first consequence of this proposition is an absolute continuity relation between the laws of the radial HO-process and the corresponding radial Dunkl process. More precisely, let $\mathbb{P}^W$ be the law of $(X_t^W, t \geq 0)$ with parameter k on $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$. Let $\mathcal{R}' := \{\frac{\sqrt{2}\alpha}{|\alpha|} \mid \alpha \in \mathcal{R}\}$, and if $\beta = \frac{\sqrt{2}\alpha}{|\alpha|} \in \mathcal{R}'$, let $k'_\beta := k_\alpha + k_{2\alpha}$. Let $\mathbb{Q}^W$ be the law of the radial Dunkl process $(Z_t^W, t \geq 0)$ associated to the root system $\mathcal{R}'$ and with parameter k'. Let $(L_t, t \geq 0)$ be the process defined by

$$L_t := \int_0^t \nabla \log \frac{\delta^{\frac{1}{2}}}{\pi}(X_s) d\beta_s, \ t \geq 0,$$

where $(\beta_t, t \geq 0)$ is a Brownian motion under $\mathbb{Q}^W$, and

$$\pi(x) = \prod_{\beta \in \mathcal{R}'} (\beta, x)^{k'_\beta}.$$

As the function $x \mapsto \frac{1}{x} - \coth(x)$ is bounded on $\mathbb{R}$ we get that, for all $t \geq 0$, $\mathbb{Q}^W[\exp(\frac{1}{2} < L >_t)] < \infty$. Thus $M := \exp(L - \frac{1}{2} < L >)$, the stochastic exponential of L, is a $\mathbb{Q}^W$-martingale. Moreover as mentioned in Remark 4.1 we have also an explicit decomposition of the radial Dunkl process as "Brownian motion + term with bounded variation". Therefore, by using the Girsanov theorem [25], we get that for any $t \geq 0$, if $(\mathcal{F}_t, t \geq 0)$ is the canonical filtration on $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$, then

$$\mathbb{P}^W_{|\mathcal{F}_t} = M_t \cdot \mathbb{Q}^W_{|\mathcal{F}_t}. \tag{4}$$

As a consequence we obtain for instance that when $k_\alpha + k_{2\alpha} \geq 1/2$, then the HO-process starting at any $x \in \mathfrak{a}_+$ a.s. does not touch the walls (i.e. the subspaces of the type $\{\alpha = 0\}$). This follows from the similar result for the Dunkl processes proved in [11]. Now if it starts at some $x \in \partial\mathfrak{a}_+$ then a.s., by the Markov property, it will never touch the walls in strictly positive times (observe that $\mu(\partial\mathfrak{a}_+) = 0$, thus at any $t > 0$, a.s. $X_t \in \mathfrak{a}_+$).

We will now prove a law of large numbers and a central limit theorem for the radial Heckman-Opdam processes. These results are well known in the setting of symmetric spaces of noncompact type (see for instance Babillot [2]).

Proposition 4.2 *The radial process satisfies the law of large numbers*

$$\lim_{t\to\infty} \frac{X^W_t}{t} \to \rho \ a.s.,$$

and there is the convergence in $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$

$$\left(\frac{X^W_{tT} - \rho t T}{\sqrt{T}}, t \geq 0\right) \xrightarrow[T\to\infty]{} (\beta_t, t \geq 0).$$

Proof of the proposition: The first step is to prove that $(\alpha, X^W_t) \to +\infty$, when $t \to +\infty$, for all $\alpha \in \mathcal{R}^+$, or it is enough, for all the simple roots. We denote by $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots. From Proposition 4.1 we see that the radial process (starting at x) satisfies for any $t \geq 0$,

$$X^W_t = x + \beta_t + \rho t + \frac{1}{2}\sum_{\alpha\in\mathcal{R}^+} k_\alpha \alpha \int_0^t [\coth\frac{(\alpha, X^W_s)}{2} - 1]ds. \tag{5}$$

Let $u \in \overline{\mathfrak{a}_+}$. From (5) we get that for all $t \geq 0$, $(u, X^W_t) - (u, x) - (u, \beta_t) \geq (u, \rho)t$, because $\coth(x) \geq 1$ for $x \geq 0$. Thus $(u, X^W_t) \to +\infty$, when $t \to +\infty$. In particular $(\rho, X^W_t) \to +\infty$. It implies that $\max_{i=1,\dots,n}(\alpha_i, X^W_t) \to +\infty$. For $t > 0$, let $i_1, \dots, i_n$ be such that $(\alpha_{i_1}, X^W_t) \geq \cdots \geq (\alpha_{i_n}, X^W_t)$ (we forget the dependance in t in the notation). We prove now that $(\alpha_{i_2}, X^W_t) \to +\infty$. Let $\epsilon > 0$ and let T_0 be such that $\coth(\alpha_{i_1}, X^W_t) - 1 \leq \epsilon$ and $|x + \beta_t| \leq \epsilon t$ for $t \geq T_0$. Let $\mathcal{R}_2$ be the root system generated by $\{\alpha_{i_2}, \dots, \alpha_{i_n}\}$ and let $\rho_2 = \sum_{\alpha\in\mathcal{R}_2^+} k_\alpha \alpha$. Observe in particular that if $\alpha \in \mathcal{R}_2^+$, then $(\alpha, \rho_2) \geq 0$, whereas if $\alpha \notin \mathcal{R}_2^+$, then $\alpha - \alpha_{i_1} \in \mathcal{R}^+$ and thus $\coth\frac{(\alpha, X^W_t)}{2} \leq 1 + \epsilon$. Now from (5) we get for $t \geq T_0$,

$$(\rho_2, X^W_t) \geq ((\rho, \rho_2) - \epsilon)t + f(t),$$

where $f(t) = \sum_{\alpha\in\mathcal{R}^+} k_\alpha(\alpha,\rho_2)\int_0^t [\coth\frac{(\alpha,X_s^W)}{2} - 1]ds$. Hence by our choice of ρ_2, we have for $t \geq T_0$, $f'(t) \geq -C\epsilon$ for some constant $C > 0$. Then we get another constant $C' > 0$ such that $f(t) \geq -C' - C\epsilon t$ for $t \geq T_0$. Thus we conclude that $(\rho_2, X_t^W) \to +\infty$ and $(\alpha_{i_2}, X_t^W) \to +\infty$. In the same way we deduce that $(\alpha_i, X_t^W) \to +\infty$ for all $1 \leq i \leq n$. Eventually we get immediately the law of large numbers from (5).
For the second claim of the proposition, we will show that a.s.

$$|\frac{1}{\sqrt{T}}(X_{tT}^W - x - \beta_{tT} - tT\rho)| \xrightarrow[T\to\infty]{} 0,$$

uniformly in $t \in \mathbb{R}^+$. Let $\epsilon > 0$. By the first claim, we know that there is some N such that for every $s \geq N$, $|\coth\frac{(\alpha,X_s^W)}{2} - 1| \leq e^{-cs}$ for some strictly positive constant c. Then we have

$$\begin{aligned}\frac{1}{\sqrt{T}}\int_0^{tT}[\coth\frac{(\alpha,X_s^W)}{2} - 1]ds &= \frac{1}{\sqrt{T}}\int_0^{N\wedge tT}[\coth\frac{(\alpha,X_s^W)}{2} - 1]ds \\ &+ \frac{1}{\sqrt{T}}1_{(tT\geq N)}\int_N^{tT}[\coth\frac{(\alpha,X_s^W)}{2} - 1]ds.\end{aligned}$$

But the both integrals can be made smaller than ϵ by choosing T sufficiently large. The second claim follows using the scaling property of the Brownian motion. □

5 Jumps of the process

We will now study the behavior of the jumps of the Heckman-Opdam process. We use essentially the same tool as in [17] for the Dunkl processes, i.e. we use the predictable compensators of some discontinuous functionals. However in our setting we obtain a more precise result when $k_\alpha + k_{2\alpha} \geq \frac{1}{2}$ for all α. In fact in this case, there is almost surely a finite random time, after which the process does not jump anymore. This allows to prove for such multiplicity k a law of large numbers and a central limit theorem for the HO-process.
Let us first recall the definition of the Lévy kernel $N(x,dy)$ of a homogeneous Markov process with a transition semigroup $(P_t)_{t\geq 0}$ and generator $\mathcal{D}$ (see Meyer [23]). It is determined, for any $x \in \mathbb{R}^d$ by:

$$\mathcal{D}f(x) = \lim_{t\to 0}\frac{P_tf(x)}{t} = \int_{\mathfrak{a}} N(x,dy)f(y),$$

for f a function in the domain of the infinitesimal generator which vanishes in a neighborhood of x. The following lemma describes the Lévy kernel of the HO-process. It is an immediate consequence of the explicit expression (1) of $\mathcal{L}$. First we introduce some notation. If I is a subset of $\mathcal{R}^+$, we denote by

$$\mathfrak{a}^I = \{x \in \mathfrak{a} \mid \forall\alpha \in I,\ (\alpha,x) = 0\},$$

the face associated to I. We denote also by $\mathcal{R}_I$ the set of positive roots which vanish on $\mathfrak{a}^I$.

Lemma 5.1 *The Lévy kernel of the HO-process has the following form*

$$N(x,dy) = \begin{cases} \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{8} \frac{\epsilon_{r_\alpha x}(dy)}{\sinh^2\frac{(\alpha,x)}{2}} & if \quad x \in \mathfrak{a}_{reg} \\ \sum_{\alpha\in\mathcal{R}^+\setminus\mathcal{R}_I} k_\alpha \frac{|\alpha|^2}{8} \frac{\epsilon_{r_\alpha x}(dy)}{\sinh^2\frac{(\alpha,x)}{2}} & if \quad x \in \mathfrak{a}^I, \end{cases}$$

where I is a subset of $\mathcal{R}^+$, and for $x \in \mathfrak{a}$, ϵ_x is the Dirac measure in x.

Remark 5.1 The lemma implies that when there is a jump at a random time s, i.e. $X_s \neq X_{s-}$, then almost surely there exists $\alpha \in \mathcal{R}^+$ such that $X_s = r_\alpha X_{s-}$ (see [17]). In this case we have

$$\Delta X_s := X_s - X_{s-} = -(\alpha^\vee, X_{s-})\alpha.$$

Using the finiteness of the expectation of the time integrals appearing in (3), we can show that the sum over any time interval of the amplitudes of the jumps is finite.

Proposition 5.1 *Let $(X_t, t \geq 0)$ be a Heckman-Opdam process. For every $t > 0$,*

$$\mathbb{E}\left[\sum_{s\leq t} |\Delta X_s|\right] < +\infty.$$

Proof of the proposition: From the above remark we get

$$\sum_{s\leq t} |\Delta X_s| = \sum_{\alpha\in\mathcal{R}^+} \sum_{s\leq t} f_\alpha(X_{s-}, X_s),$$

where

$$f_\alpha(x,y) = \frac{2}{|\alpha|} |(\alpha,x)| 1_{(y=r_\alpha x\neq x)}.$$

Now, the positive discontinuous functional $\sum_{s\leq t} f_\alpha(X_{s-}, X_s)$ is compensated by the process $\int_0^t ds \int_{\mathfrak{a}} N(X_{s-}, dy) f_\alpha(X_{s-}, y)$. As a consequence, the proposition will be proved if we know that the expectation of the compensator is finite at all time $t \geq 0$. Thus we have to show that for every $\alpha \in \mathcal{R}^+$,

$$\mathbb{E}\left[\int_0^t \left|\frac{(\alpha, X_s)}{\sinh^2\frac{(\alpha,X_s)}{2}}\right| ds\right] < +\infty.$$

But for every $x > 0$, $\frac{x}{\sinh^2 x} \leq 2\coth x$. Therefore the above condition follows from Proposition 4.1. □

For $\alpha \in \mathcal{R}^+$, we denote by $(M_t^\alpha, t \geq 0)$ the process defined for $t \geq 0$ by:

$$M_t^\alpha = \sum_{s\leq t} -(\alpha^\vee, X_{s-}) 1_{(r_\alpha X_{s-} = X_s)} + \frac{k_\alpha}{4} \int_0^t \frac{(\alpha, X_s)}{\sinh^2\frac{(\alpha,X_s)}{2}} ds. \tag{6}$$

By the martingale characterization of $(X_t, t \geq 0)$, we know that $(f(X_t), t \geq 0)$ is a local semimartingale for all $f \in C_c^\infty(\mathfrak{a})$. Thus $(X_t, t \geq 0)$ itself is a local semimartingale. In the next proposition we give its explicit decomposition. It is the analogue of a result of Gallardo and Yor on the decomposition of the Dunkl processes. The proof is very similar (and uses Proposition 5.1), so we refer to [17] for more details.

Proposition 5.2 *We have the following semimartingale decomposition:*

$$X_t = \beta_t + \sum_{\alpha \in \mathcal{R}^+} M_t^\alpha \alpha + A_t,$$

for $t \geq 0$*, where*

$$A_t = \sum_{\alpha \in \mathcal{R}^+} \frac{k_\alpha}{2} \alpha \int_0^t \left[\coth \frac{(\alpha, X_s)}{2} - \frac{(\alpha, X_s)}{2 \sinh^2 \frac{(\alpha, X_s)}{2}} \right] ds,$$

and the M^α*'s are purely discontinuous martingales given by (6) which satisfy*

$$[M^\alpha, M^\beta]_t = 0, \ \text{if } \alpha \neq \beta,$$

and

$$< M^\alpha >_t = \frac{k_\alpha}{4|\alpha|^2} \int_0^t \frac{(\alpha, X_s)^2}{\sinh^2 \frac{(\alpha, X_s)}{2}} ds.$$

Another interesting property of the jumps is that, when $k_\alpha + k_{2\alpha} \geq 1/2$ for all α, and when the starting point lies in $\mathfrak{a}_{\text{reg}}$, in which case the HO-process does not touch the walls, the number of jumps N_t up to a fixed time t is a.s. finite. Indeed otherwise the paths of the trajectories would not be càdlàg. Therefore the sequence of stopping times $T_n = \inf\{t > 0, N_t \geq n\}$ converges a.s. to $+\infty$ when n tends to infinity. Thus $(N_t, t \geq 0)$ is a locally integrable (because locally finite) increasing process. We will deduce from this observation and a general result of [19] a more precise result. For $t \geq 0$, we denote by w_t the element of W such that $X_t = w_t X_t^W$.

Proposition 5.3 *Assume that* $k_\alpha + k_{2\alpha} \geq 1/2$ *for all* α*.*

1. *If the starting point lies in* $\mathfrak{a}_{reg}$*, then a.s.*

$$\sup_{t \geq 0} N_t < +\infty.$$

2. *For any starting point in* $\mathfrak{a}$*,* w_t *converges a.s. to* $w_\infty \in W$ *when* $t \to \infty$*. If the process starts from zero, then the law of* w_∞ *is the uniform probability on* W*.*

3. *When* $T \to \infty$*, the sequences* $(\frac{1}{T} X_{tT}, t \geq 0)$ *and* $(\frac{1}{\sqrt{T}}(X_{tT} - w_{tT}\rho tT), t \geq 0)$ *converge in law in* $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ *respectively to* $(w_\infty \rho t, t \geq 0)$*, and to a Brownian motion* $(\beta_t, t \geq 0)$*.*

Proof of the proposition: Let us begin with the first claim. As in the preceding proposition we use the following result of Meyer about the Lévy kernel: the positive discontinuous functional $(N_t, t \geq 0)$ can be compensated by the predictable process $(\sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{|\alpha|^2}{8} \int_0^t \frac{1}{\sinh^2 \frac{(\alpha,X_s)}{2}} ds, t \geq 0)$. Now from the law of large numbers (Proposition 4.2) we deduce that this compensator converges a.s. to a finite value when $t \to \infty$. Thus the corollary (5.20) p.168 of [19] gives the result. Then the second point is simply a consequence of the first point and of the Markov property. The assertion that the limit law is uniform when the process starts from zero results from the fact that for any $w \in W$, $\mathcal{D}$ remains unchanged if we replace $\mathcal{R}^+$ by $w\mathcal{R}^+$. The first convergence result of the last point is straightforward with the second point and Proposition 4.2. For the second convergence result, we can use Proposition 5.2. Indeed it says that for all $t > 0$ and $T > 0$,

$$\begin{aligned}\frac{X_{tT} - w_{tT}\rho tT}{\sqrt{T}} &= \frac{\beta_{tT}}{\sqrt{T}} + \frac{1}{\sqrt{T}} \sum_{s\leq tT} \Delta X_s \\ &+ \sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha}{2\sqrt{T}} \int_0^{tT} \left[\coth \frac{(\alpha, X_s)}{2} - \epsilon^\alpha_{tT}\right] ds,\end{aligned}$$

where the $\epsilon^\alpha_{tT} \in \{\pm 1\}$ are defined by $w_{tT}\rho = \sum_{\alpha\in\mathcal{R}^+} k_\alpha \epsilon^\alpha_{tT}\alpha$, or equivalently by $\epsilon^\alpha_{tT}\alpha \in w_{tT}\mathcal{R}^+$. But by the second point of the proposition, we know that a.s. there exists a random time after which the process stays in the same chamber, which is $w_\infty\mathfrak{a}_+$. Moreover for all $s > 0$ and all $\alpha \in \mathcal{R}^+$, $\epsilon^\alpha_s(\alpha, X_s) \geq 0$. Thus by Proposition 4.2 $\coth \frac{(\alpha,X_s)}{2} - \epsilon^\alpha_{tT}$ tends to 0 exponentially fast when $s \to \infty$ (and $s \leq tT$). Then a.s. $\sum_{\alpha\in\mathcal{R}^+} k_\alpha \frac{\alpha}{2\sqrt{T}} \int_0^{tT} \left[\coth \frac{(\alpha,X_s)}{2} - \epsilon^\alpha_{tT}\right] ds$ tends to 0 when $T \to \infty$, uniformly in $t \in \mathbb{R}^+$. In the same way, by Proposition 5.1, a.s. for any $A > 0$, $\sum_{s\leq A} |\Delta X_s| < +\infty$. By the second point we know that a.s. after some time there is no more jumps, thus a.s. $\frac{1}{\sqrt{T}} \sum_{s\leq tT} |\Delta X_s|$ tends to 0 when $T \to \infty$, uniformly in $t \in \mathbb{R}^+$. This proves the desired result by the scaling property of the Brownian motion. □

6 Convergence to the Dunkl processes

In this section we will show that when it is well normalized, the HO-process of parameter $k > 0$ converges to a certain Dunkl process $(Z_t, t \geq 0)$. The proof uses a general criteria for a sequence of Feller processes with jumps, which can be found in [13] for instance. Roughly speaking it states that it just suffices to prove the convergence of the generator of these processes on a core of the limit. Let us notice that the convergence of the normalized radial HO-process to the corresponding radial Dunkl process is more elementary. It could be proved essentially by using that the laws of the radial HO-process and the radial Dunkl process are absolutely continuous, and that the normalized Radon-Nikodym derivative tends to 1. Let us also observe that the convergence of the normalized

radial process has a natural geometric interpretation in the setting of symmetric spaces of noncompact type. Indeed in this setting the radial Dunkl process is just the radial part of the Brownian motion on the tangent space (or the Cartan motion group, see the more precise description by De Jeu [22], and in [1] in the complex case). From the analytic point of view, it also illustrates the more conceptual principle, that the Dunkl (also called rational) theory is the limit of the Heckman-Opdam (or trigonometric) theory, when "the curvature goes to zero".
We denote by $(X_t^T, t \geq 0)$ the normalized HO-process, which is defined for $t \geq 0$ and $T > 0$ by:

$$X_t^T = \sqrt{T} X_{\frac{t}{T}}.$$

We recall that $\mathcal{R}' = \{\frac{\sqrt{2}\alpha}{|\alpha|},\ \alpha \in \mathcal{R}\}$, and that for $\beta = \frac{\sqrt{2}\alpha}{|\alpha|} \in \mathcal{R}'$, $k'_\beta = k_\alpha + k_{2\alpha}$.

Theorem 6.1 *When $T \to \infty$, the normalized HO-process $(X_t^T, t \geq 0)$ with parameter k starting at 0 converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the Dunkl process $(Z_t, t \geq 0)$ associated with $\mathcal{R}'$ and with parameter k' starting at 0.*

Proof of the theorem: First it is well known that the process $(X_t^T, t \geq 0)$ is also a Feller process with generator $\mathcal{L}^T$ defined for $f \in C^2(\mathfrak{a})$ and $x \in \mathfrak{a}_{\mathrm{reg}}$, by $\mathcal{L}^T f(x) = \frac{1}{T}(\mathcal{L}g)(\frac{x}{\sqrt{T}})$, where $g(x) = f(\sqrt{T}x)$. Thus a core of $\mathcal{L}^T$ is, like for $\mathcal{L}$ and $\mathcal{L}'$, the space $C_c^\infty(\mathfrak{a})$. Moreover it is straightforward that for any $f \in C_c^\infty(\mathfrak{a})$, $\mathcal{L}^T f$ converges uniformly on $\mathfrak{a}$ to $\mathcal{L}' f$. Therefore we can apply Theorem 6.1 p.28 and Theorem 2.5 p. 167 in [13], which give the desired result. $\square$

7 The F_0-process and its asymptotic behavior

We introduce here and study a generalization of the radial part of the Infinite Brownian Loop (abbreviated as I.B.L.) introduced in [1]. Let $\tilde{F}_0(x,t) := F_0(x)e^{\frac{|\rho|^2}{2}t}$, for $(x,t) \in \mathfrak{a} \times [0, +\infty)$. Then $\tilde{F}_0$ is harmonic for the operator $\partial_t + \mathcal{D}$ which is the generator of $(X_t, t)_{t \geq 0}$. We define now the processes $(Y_t, t)_{t \geq 0}$ as the relativized $\tilde{F}_0$-processes in the sense of Doob of $(X_t, t)_{t \geq 0}$. By abuse of language we will call $(Y_t, t \geq 0)$ the F_0-process. We denote by $(Y_t^W, t \geq 0)$ its radial part, that we will call the radial F_0-process. For particular values of k it coincides with the radial part (in the Lie group terminology) of the I.B.L. on a symmetric space.

The goal of this section is to generalize some results of [1] and [5], for any $k > 0$. Essentially we first prove the convergence of the HO-bridge of length T around 0, i.e. the HO-process conditioned to be equal to 0 at time T, to the F_0-process starting at 0, when T tends to infinity. Then we prove the convergence of the normalized F_0-process to a process whose radial part is the intrinsic Brownian motion, but which propagates in a random chamber (independently and uniformly chosen). We begin by the following lemma:

Lemma 7.1 *Let $x, a \in \overline{\mathfrak{a}_+}$. When $T \to \infty$,*

$$\frac{p_{T-t}^W(x,a)}{p_T^W(a,a)} \to \frac{F_0(x)}{F_0(a)} e^{\frac{t}{2}|\rho|^2}.$$

Proof of the lemma: We need the integral formula of the heat kernel:

$$p_t^W(x,y) = \int_{i\mathfrak{a}} e^{-\frac{t}{2}(|\lambda|^2+|\rho|^2)} F_\lambda(x) F_{-\lambda}(y) d\nu'(\lambda), \ x, y \in \mathfrak{a}_+.$$

We make the change of variables $u := \lambda(T-t)$ for p_{T-t}^W, and $v := \lambda T$ for p_T^W. Then we let T tend to $+\infty$ and the result follows. □

Proposition 7.1 *Let $(X_t^{0,T}, t \geq 0)$ be the HO-bridge around 0 of length T. Then when $T \to +\infty$, it converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the F_0-process starting at 0.*

Proof of the proposition: We know that for any $t \geq 0$, and any bounded $\mathcal{F}_t$-measurable function F,

$$\begin{aligned}
\mathbb{E}\left[F(X_s^{0,T}, 0 \leq s \leq t)\right] &= \mathbb{E}\left[F(X_s, 0 \leq s \leq t)\frac{p_{T-t}(X_t,0)}{p_T(0,0)}\right] \\
&= \mathbb{E}\left[F(X_s, 0 \leq s \leq t)\frac{p_{T-t}^W(X_t^W,0)}{p_T^W(0,0)}\right]
\end{aligned}$$

The second equality results from the fact that $p_t(x,0) = \frac{1}{|W|} p_t^W(x,0)$, for all $x \in \mathfrak{a}$ and all $t \geq 0$ (see [28]). Moreover since F_λ is bounded (cf [24]) and the measure ν' is positive, we see from the integral formula of p_t^W, that there exists a constant C such that, $p_t^W(x,y) \leq C p_t^W(0,0)$, for all $t > 0$ and all $x, y \in \overline{\mathfrak{a}_+}$. It follows that $\frac{p_{T-t}^W(x,a)}{p_T^W(a,a)}$ is a bounded function of $(x,T) \in \overline{\mathfrak{a}_+} \times [1,\infty)$. Then we get the result from the preceding lemma and the dominated convergence theorem. □

Remark 7.1 The same proof shows in fact that for all $a \in \overline{\mathfrak{a}_+}$, the radial HO-bridge of length T around a converges in law to the radial F_0-process starting from a.

The next result is an important technical lemma:

Lemma 7.2 *There exist two Bessel processes $(R_t, t \geq 0)$ and $(R'_t, t \geq 0)$ (not necessarily with the same dimension), such that a.s. $|R_0| = |R'_0| = |Y_0|$ and for any $t \geq 0$,*

$$R_t^2 \leq |Y_t|^2 \leq R_t'^2.$$

Proof of the lemma: First the F_0 process and its radial part have the same norm, hence it suffices to prove the result for $(Y_t^W, t \geq 0)$. Next we can follow exactly the same proof as in [1]. We recall it for the convenience of the reader. We know that $(Y_t^W, t \geq 0)$ is solution of the SDE

$$Y_t^W = Y_0^W + \beta_t + \int_0^t \nabla \log(\delta^{\frac{1}{2}} F_0)(Y_s^W) ds,$$

where $(\beta_t, t \geq 0)$ is a Brownian motion. By Itô formula we get

$$|Y_t^W|^2 = |Y_0^W|^2 + 2\int_0^t (Y_s^W, d\beta_s) + tn + 2\int_0^t E[\log(\delta^{\frac{1}{2}} F_0)](Y_s^W) ds,$$

where $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathfrak{a}$. And it was shown in [28] that $E[\log(\delta^{\frac{1}{2}} F_0)]$ is positive and bounded on $\mathfrak{a}$. Thus we can conclude by using comparison theorems. $\square$

Corollary 7.1 *Let $(Y_t, t \geq 0)$ be the F_0-process. Then almost surely,*

$$\lim_{t\to\infty} \frac{|Y_t|}{t} = 0.$$

More precisely (law of the iterated logarithm), a.s.

$$\limsup_{t\to\infty} \frac{|Y_t|}{\sqrt{2t \log\log t}} = 1.$$

Proof of the corollary: It follows from the preceding lemma and known properties of the Bessel processes. $\square$

In the complex case, i.e. when $2k$ is equal to the multiplicity function on some complex Riemannian symmetric space of noncompact type (or equivalently when $\mathcal{R}$ is reduced and $k = 1$), then it was proved in [1] that the radial F_0-process coincides with the intrinsic Brownian motion. It was also proved in [1] that in the real case, i.e. for other choices of k, a normalization of the radial F_0-process converges to the intrinsic Brownian motion. The next theorem gives a generalization of this result for any multiplicity $k > 0$ and for the (non radial) F_0-process also. We denote by $(Y_t^T, t \geq 0)$ the process defined for $t \geq 0$ and $T > 0$, by

$$Y_t^T := \frac{1}{\sqrt{T}} Y_{tT},$$

and we denote by $(Y_t^{W,T}, t \geq 0)$ its radial part. Let $(I_t, t \geq 0)$ be the intrinsic Brownian motion starting from 0. We denote by $(I_t^*, t \geq 0)$ the continuous process starting from 0, whose radial part is the intrinsic Brownian motion, but which propagates in a random chamber $w\mathfrak{a}_+$, where w is chosen independently and with respect to the uniform probability on W. This is a typical example of a strong Markov process which is not Feller (for instance it does not satisfy the Blumenthal's zero-one law). We have

Theorem 7.1 *Let $k > 0$. The normalized F_0-process $(Y_t^T, t \geq 0)$ starting at any $x \in \mathfrak{a}$ converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to $(I_t^*, t \geq 0)$.*

Proof of the theorem: The first step is to prove that $(Y_t^{W,T}, t \geq 0)$ converges in law in $C(\mathbb{R}^+, \overline{\mathfrak{a}_+})$ to $(I_t, t \geq 0)$. Thanks to Lemma 7.2 we can use the same proof as for Theorem 5.5 in [1]. The result about the convergence of the semigroup needed in the proof was established in [28]. Now let $\mathbb{P}^0$ be the law of the F_0-process, and let $\mathbb{P}$ be the one of the HO-process. By definition we have the absolute continuity relation

$$\mathbb{P}^0_{|\mathcal{F}_t} = \frac{F_0(Y_t)}{F_0(x)} e^{\frac{|\rho|^2}{2} t} \cdot \mathbb{P}_{|\mathcal{F}_t}.$$

Since F_0 is bounded (cf [24]) Proposition 5.1 implies that for any $t > 0$, $\mathbb{E}^0[\sum_{s \leq t} |\Delta Y_s|] < +\infty$. Thus by Girsanov theorem (see [25]) and Proposition 5.2 we get the semimartingale decomposition of the F_0-process:

$$\begin{aligned} Y_t &= x + \beta_t + \underbrace{\sum_{\alpha \in \mathcal{R}^+} M_t^\alpha \alpha}_{:= M_t} + \int_0^t \nabla \log \delta^{\frac{1}{2}} F_0(Y_s) ds \\ &- \sum_{\alpha \in \mathcal{R}^+} \frac{k_\alpha}{2} \alpha \int_0^t \frac{(\alpha, Y_s)}{2 \sinh^2 \frac{(\alpha, Y_s)}{2}} ds, \end{aligned}$$

where $(\beta_t, t \geq 0)$ is a $\mathbb{P}^0$-Brownian motion and the M^α's are defined by (6) (with Y_s in place of X_s). We set $M_t^T := \frac{1}{\sqrt{T}} M_{tT}$. By Proposition 5.2 we know that

$$< M^T >_t := \sum_{i=1}^n < M_i^T >_t = \sum_{\alpha \in \mathcal{R}^+} \frac{k_\alpha}{4} \int_0^t \frac{(\alpha, \sqrt{T} Y_s^T)^2}{\sinh^2 \frac{\sqrt{T}}{2} (\alpha, Y_s^T)} ds,$$

where M_i^T is the i^{th} coordinate of M^T in the canonical basis. Now for all $w \in W$ the preceding sum remains unchanged if $\mathcal{R}^+$ is replaced by $w\mathcal{R}^+$. Therefore we can replace Y_s^T by $Y_s^{W,T}$ in the last equality. Moreover $\sinh x \geq x + \frac{x^3}{6}$ on $\mathbb{R}^+$. Hence

$$< M^T >_t \leq \sum_{\alpha \in \mathcal{R}^+} k_\alpha \int_0^t \frac{1}{1 + \frac{T}{12} (\alpha, Y_s^{W,T})^2} ds.$$

Thus by using the first step, we see that for any fixed $t > 0$, $\mathbb{E}[< M^T >_t] \to 0$ when $T \to +\infty$. It implies by Doob's L^2-inequality (see [25]), that $(M_t^T, t \geq 0)$ converges in law in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to 0. Now the triangular inequality implies that for any $A > 0$, $\epsilon > 0$ and $\alpha > 0$,

$$\begin{aligned} \mathbb{P}\left[\sup_{|s-t| \leq \epsilon,\ s \leq t \leq A} |Y_s^T - Y_t^T| \geq \alpha \right] &\leq \mathbb{P}\left[\sup_{|s-t| \leq \epsilon,\ s \leq t \leq A} |Y_s^{W,T} - Y_t^{W,T}| \geq \frac{\alpha}{2N} \right] \\ &+ \mathbb{P}\left[\sum_{s \leq A} |\Delta Y_s^T| \geq \frac{\alpha}{2} \right], \end{aligned}$$

where N is the number of Weyl chambers. Thus using that $\Delta Y^T = \Delta M^T$, tightness of $(Y_t^{W,T}, t \geq 0)$, and standard results (see Theorem 3.21 p.314 and Proposition 3.26 p.315 in [20] for instance), we see that the sequence $(Y_t^T, t \geq 0)$ is C-tight in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$, i.e. it is tight and any limit law of a subsequence is supported on $C(\mathbb{R}^+, \mathfrak{a})$. By the first step each limit has a radial part equal to the intrinsic Brownian motion. Since we know that the intrinsic Brownian motion does not touch the walls (in strictly positive time), each limit process is necessarily of the type $(wI_t, t \geq 0)$ where w is some random variable on W. Thus in order to identify the limit, we need to prove that the law of w has to be the uniform probability on W, and that w is independent of the radial part. For the law of w first, let us just observe that when the process starts from 0, the result is immediate since by W-invariance of $\mathcal{D}$ and F_0, the law of the F_0-process is W-invariant, and thus the law of any limit also. However when $x \neq 0$ we can not argue like this, and we need to prove for instance that the law of Y_1^T converges to the law of I_1^*, when $T \to \infty$. Let $f : \mathfrak{a} \to \mathbb{R}$ be continuous and bounded. We have

$$\mathbb{E}_{\frac{x}{\sqrt{T}}}\left[f(\frac{Y_T}{\sqrt{T}})\right] = \int_{\mathfrak{a}} p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)\frac{F_0(\sqrt{T}y)}{F_0(\frac{x}{\sqrt{T}})}e^{\frac{|\rho|^2}{2}T} f(y)d\mu(\sqrt{T}y).$$

Then it results from the asymptotic of $p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)$ and of $F_0(\sqrt{T}y)$ proved in [28], that

$$p_T(\frac{x}{\sqrt{T}}, \sqrt{T}y)\frac{F_0(\sqrt{T}y)}{F_0(\frac{x}{\sqrt{T}})}e^{\frac{|\rho|^2}{2}T}d\mu(\sqrt{T}y) \to \text{const} \cdot e^{-\frac{|y|^2}{2}} \prod_{\alpha \in \mathcal{R}^+} (\alpha, y)^2 dy,$$

which gives the density of the law of I_1^* (see [1] for instance for the law of I_1). We conclude by Sheffé's lemma. Now the only missing part is the independence of w and $(I_t, t \geq 0)$. Observe first that since any limit process is adapted, w is $\mathcal{F}_{0^+}$-measurable. On the space $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$, we denote by $(\mathcal{F}_t^W)_t \geq 0$ the natural filtration of the radial process. We know that $(I_t, t \geq 0)$ is an $(\mathcal{F}_t^W)_{t\geq 0}$-Markov process. We will prove that it is also an $(\mathcal{F}_t)_{t\geq 0}$-Markov process. Indeed since it is a.s. continuous and equal to 0 at time 0, this will imply the independence with w. We know that $Y^{W,T}$ is an $(\mathcal{F}_t)_{t\geq 0}$-Markov process, since it is the projection of Y^T. We denote by $P^{W,T}$ the semigroup of $Y^{W,T}$, and by Q^W the semigroup of $(I_t, t \geq 0)$. In particular we already know that for $t \geq 0$ and any continuous and bounded function f, $P_t^{W,T} f$ converges simply to $Q_t^W f$ when $T \to \infty$. For $s < t$, and f and g continuous and bounded functions, we have $\mathbb{E}[f(Y_t^{W,T})g(Y_s^T)] = \mathbb{E}[P_{t-s}^{W,T} f(Y_s^{W,T})g(Y_s^T)]$. For a suitable subsequence of T, the first term tends to $\mathbb{E}[f(I_t)g(\nu I_s)]$, and the second term tends to $\mathbb{E}[Q_{t-s}^W f(I_s)g(\nu I_s)]$, which implies the desired result. This finishes the proof of the theorem. □

We can now prove a generalization of a result of Bougerol and Jeulin [5]. Let $(R_t^{0,T}, 0 \leq t \leq 1)$ be the normalized HO-bridge of length T around 0. It is defined for $t \geq 0$ by

$$R_t^{0,T} = \frac{1}{\sqrt{T}} X_t^{0,T}.$$

Theorem 7.2 *Let $k > 0$. When $T \to \infty$, the process $(R_t^{0,T}, 0 \leq t \leq 1)$ converges in distribution in $\mathbb{D}(\mathbb{R}^+, \mathfrak{a})$ to the bridge $(I_t^{\{*,0,0,1\}}, 0 \leq t \leq 1)$ of length 1 associated to $(I_t^*, 0 \leq t \leq 1)$.*

Proof of the theorem: Here again we can follow the same proof as in [5]. We just need to take care that the estimate of the heat kernel in Proposition 5.3 in [28], is uniform when y lies in some compact of $\mathfrak{a}_+$. But this results directly from the proof of this proposition. $\square$

Remark 7.2 We can define similarly the normalized radial HO-bridge around any $a \in \overline{\mathfrak{a}_+}$. With the same proof, we can also prove that it converges to the bridge $(I_t^{\{0,0,1\}}, 0 \leq t \leq 1)$ of length 1 associated to the intrinsic Brownian motion starting from 0. Let us just notice that in dimension 1 this is the bridge of a Bessel-3, which is also the normalized Brownian excursion. Thus we do recover the result of [6].

Acknowledgments: I very much thank my advisors Jean-Philippe Anker and Philippe Bougerol for their help and many suggestions. I wish to thank also Marc Yor for his comments of a previous version, and Oleksandr Chybiryakov and Emmanuel Cépa for fruitful discussions.

References

[1] **Anker J-Ph., Bougerol Ph., Jeulin T.:** *The infinite Brownian loop on a symmetric space*, Rev. Mat. Iberoamericana 18 (2002), 41-97.

[2] **Babillot M.:** *A probabilistic approach to heat diffusion on symmetric spaces*, J. Theoret. Probab. 7 (1994), 599 – 607.

[3] **Biane Ph.:** *Quelques propriétés du mouvement Brownien dans un cône*, Stochastic Processes. Appl. 53 (1994), 233 – 240.

[4] **Biane Ph., Bougerol Ph., O'Connell N.:** *Littelmann paths and Brownian paths*, Duke Math. J. 130 (2005), 127-167.

[5] **Bougerol Ph., Jeulin T.:** *Brownian bridge on Riemannian symmetric spaces*, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 785–790.

[6] **Bougerol Ph., Jeulin T.:** *Brownian bridge on hyperbolic spaces and on homogeneous trees*, Probab. Th. Rel. Fields 115 (1999), 95-120.

[7] **Bourbaki N.:** *Groupes et algèbres de Lie*, Ch. 4-6, Hermann, Paris, (1968); Masson, Paris, (1981).

[8] **Cépa E.:** *Equations différentielles stochastiques multivoques*, Sém. Probab. XXIX (1995), 86-107.

[9] **Cépa E., Lépingle D.:** *Brownian particles with electrostatic repulsion on the circle : Dyson's model for unitary random matrices revisited*, ESAIM Probab. Statist. 5 (2001), 203-224.

[10] **Cherednik I.:** *A unification of Knizhnik-Zamolodchnikov equations and Dunkl operators via affine Hecke algebras*, Invent. Math. 106 (1991), 411-432.

[11] **Chybiryakov O.:** *Skew-product representations of multidimensional Dunkl Markov processes*, submitted (2005).

[12] **Chybiryakov O.:** *Radial Dunkl Markov processes, local times and a multidimensional extension of Lévy's equivalence*, in preparation.

[13] **Ethier N., Kurtz G.:** *Markov processes. Characterization and convergence*, Wiley Series Probab. Math. Stat. (1986).

[14] **Fukushima M., Oshima Y., Takeda M.:** *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Stud. in Math. 19 (1994).

[15] **Gallardo L., Yor M.:** *Some remarkable properties of the Dunkl martingales*, to appear in Sém. Probab. XXXVIII, dedicated to Paul André Meyer. Springer (2005).

[16] **Gallardo L., Yor M.:** *Some new examples of Markov processes which enjoys the time-inversion property*, Probab. Th. Rel. Fields 132 (2005), 150-162.

[17] **Gallardo L., Yor M.:** *A chaotic representation property of the multidimensional Dunkl processes*, to appear in Ann. Prob.

[18] **Heckman G. J., Opdam E. M.:** *Root systems and hypergeometric functions I*, Compositio Math. 64 (1987), 329 – 352.

[19] **Jacod J.:** *Calcul stochastique et problèmes de martingales*, Lect. Notes Math. 714, Springer, Berlin Heidelberg New York (1979)

[20] **Jacod J., Shiryaev A. N.:** *Limit theorems for stochastic processes*, Springer, Berlin Heidelberg New York (1987).

[21] **De Jeu M.F.E.:** *The Dunkl transform*, Invent. Math. 113 (1993), 147 – 162.

[22] **De Jeu M.F.E.:** *Paley-Wiener theorems for the Dunkl transform*, available on arxiv math.CA/0404439.

[23] **Meyer P. A.:** *Intégrales stochastiques.* Sém. de Probab. I, Lect. Notes Math. 39, Springer-Verlag (1967).

[24] **Opdam E. M.:** *Harmonic analysis for certain representations of graded Hecke algebras*, Acta. Math. 175 (1995), 75 – 121.

[25] **Revuz D., Yor M.:** *Continuous martingales and Brownian motion*, Springer-Verlag, third ed. (1999).

[26] **Rösler M.:** *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Mah. Phys. 192 (3), (1998), 519 – 542.

[27] **Rösler M., Voit M.:** *Markov processes related with Dunkl operators*, Adv. Appl. Math. 21 (1998), no.4, 575 – 643.

[28] **Schapira Br.:** *Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz space, heat kernel*, in preparation.