Branching random walks and Minkowski sum of random walks

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Abstract

We show that the range of a critical branching random walk conditioned to survive forever and the Minkowski sum of two independent simple random walk ranges are *intersection-equivalent* in any dimension $d \geq 5$, in the sense that they hit any finite set with comparable probability, as their common starting point is sufficiently far away from the set to be hit. Furthermore, we extend a discrete version of Kesten, Spitzer and Whitman's result on the law of large numbers for the volume of a *Wiener sausage*. Here, the sausage is made of the Minkowski sum of N independent simple random walk ranges in $\mathbb{Z}^d$, with $d \geq 2N+1$, and of a finite set $A \subset \mathbb{Z}^d$. When properly normalised the volume of the sausage converges to a quantity equivalent to the capacity of A with respect to the kernel $k(x,y) = (1+\|x-y\|)^{2N-d}$. As a consequence, we establish a new relation between capacity and *branching capacity*.

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1 Introduction

In this paper we establish similarities between the typical behaviour of two multi-parameter processes whose Green's functions are comparable: the Minkowski sum of two independent random walks and the infinite invariant critical branching random walk. Both processes are considered in the transient regime on $\mathbb{Z}^d$, that is when $d \geq 5$. The analogy holds first at the level of the volume of their *Wiener sausages* associated with any set $A \subset \mathbb{Z}^d$. More precisely, the Wiener sausage (of a trajectory of the process) is obtained as we *roll a finite set* $A \subset \mathbb{Z}^d$ over the trajectory. Secondly, the analogy holds for hitting times *from infinity*, showing some form of intersection-equivalence, a notion first discussed by Benjamini, Pemantle and Peres [BPP95]. We then consider the Minkowski sum of N independent random walks, both in terms of their Wiener sausages, and then in terms of their hitting times. Finally, in the critical dimension $d = 4$, we provide a law of large numbers result for the capacity of a discrete Wiener sausage.

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The models. We first consider a so-called infinite invariant tree $\mathcal{T}$, whose one-sided version was introduced by Le Gall and Lin [LGL16] as a natural tool for studying the range of critical branching random walks on $\mathbb{Z}^d$, and whose two-sided version was later defined by Zhu [Zhu18] and Bai and Wan [BW22] respectively with the goal of defining branching interlacements and for studying the capacity of critical branching random walk ranges. Interestingly, the genealogical structure of this tree is an instance of the invariant sin-tree introduced earlier by Aldous [Ald91], and appears as a local limit of critical Bienaymé-Galton-Watson trees, conditioned to have a size going to infinity. Also more importantly for us, and as we shall see in a moment, a one-sided version of $\mathcal{T}$ appears to be a crucial ingredient for defining a notion of capacity for critical branching random walks, called branching capacity [Zhu16].

The tree $\mathcal{T}$ is a labelled ordered (or plane) tree made of a *spine*, that is a semi-infinite line of nodes $(\varnothing, u_1, \dots)$, to which are attached independent critical trees as follows. Independently, each node u_i for $i > 0$ draws a random number of children Z_i with size-biased distribution μ_{sb} (defined by $\mu_{\text{sb}}(i) = i\mu(i)$), identifying one uniformly at random with u_{i+1}, and thus partitioning the $Z_i - 1$ other children as left and right on each side of the spine, and letting each of them in turn produce a critical Bienaymé-Galton-Watson tree with reproduction law μ. Furthermore, the root draws Z_0 children, where Z_0 is distributed according to $\widetilde{\mu}(i) = \mu(i-1)$ (for $i \geq 1$) instead of μ_{sb}. Its first child is identified with u_1, and the remaining $Z_0 - 1$ produce in turn and independently μ-critical trees. The root is assigned label 0 and, using a clockwise depth-first search algorithm from the root, we label vertices on the right of the spine with positive labels. Using a counter-clockwise depth-first search, we label vertices on the left of the spine with negative labels, see Figure 1. For each $n \in \mathbb{Z}$, we denote by $\mathcal{T}(n)$ the vertex with label n. The set of vertices with nonnegative labels is called the future of $\mathcal{T}$, and is denoted by $\mathcal{T}_+$, while the set of vertices with negative labels is called the past of $\mathcal{T}$, and is denoted by $\mathcal{T}_-$.

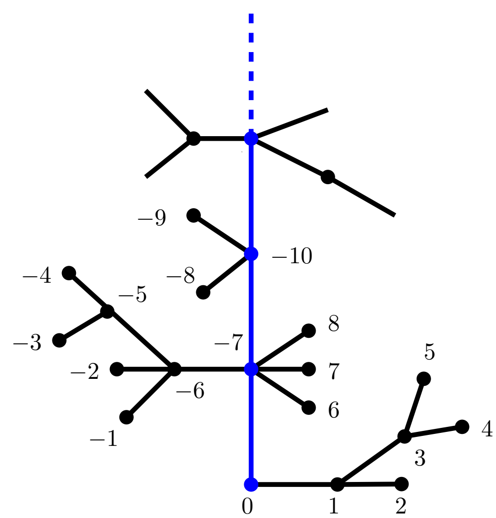

Figure 1: An infinite tree $\mathcal{T}$, with the spine in blue.

Then we consider a random walk $(S_u)_{u\in\mathcal{T}}$, indexed by $\mathcal{T}$ by assigning independent increments to the edges of the tree, whose common law is taken for simplicity to be the uniform distribution on the neighbours of the origin. Given $x \in \mathbb{Z}^d$, the value S_u of the walk starting from x at a vertex $u \in \mathcal{T}$ is obtained by summing the increments along the edges on the shortest path joining u to the root, and adding x to the result. In particular, the value of the walk at the root is x. We next denote by $\mathcal{T}^x = \{S_u : u \in \mathcal{T}\}$, its range and define similarly $\mathcal{T}^x_- = \{S_u : u \in \mathcal{T}_-\}$ (respectively $\mathcal{T}^x_+ = \{S_u : u \in \mathcal{T}_+\}$) its range in the past (respectively in the future).

A second model of interest here is the *Minkowski sum* of two random walk ranges. We recall that the Minkowski sum of two subsets $A, B \subset \mathbb{Z}^d$, is defined to be $A + B = \{a + b : a \in A, b \in B\}$. Let $(X_n)_{n\geq 0}$ and $(\widetilde{X}_n)_{n\geq 0}$ be two independent simple random walks on $\mathbb{Z}^d$. We denote by $\mathcal{R}_\infty = \{X_n : n \geq 0\}$ and $\widetilde{\mathcal{R}}_\infty = \{\widetilde{X}_n : n \geq 0\}$ their respective ranges. In this paper we study their Minkowski sum which is simply $\mathcal{R}_\infty + \widetilde{\mathcal{R}}_\infty$. Thus, intuitively speaking one rolls on the support of one walk the support of another independent walk, obtaining a *sausage*.

Wiener sausage. A celebrated result of Kesten, Spitzer and Whitman [IMcK74, p. 252] (see also [Sp64]) concerns the volume of a Wiener sausage obtained as we roll a compact set, say $A \subset \mathbb{R}^d$ with $d \geq 3$, over the trajectory of a transient Brownian trajectory. As we run the sausage over a time period of length t, and divide the volume of the sausage by t, the ratio converges to the electrostatic capacity of the set A. In our discrete setting their result reads as follows. Given a simple random walk $(X_n)_{n\geq 0}$, define its range in the time window $[a, b]$, with $0 \leq a \leq b \leq \infty$, by $\mathcal{R}[a, b] = \{X_a, \ldots, X_b\}$, with the short-hand notation $\mathcal{R}_n = \mathcal{R}[0, n]$. Then almost surely, for any finite set $A \subset \mathbb{Z}^d$, with $d \geq 3$,

$$\lim_{n\to\infty} \frac{|\mathcal{R}_n + A|}{n} = \mathrm{Cap}(A). \tag{1.1}$$

The limiting functional $\mathrm{Cap}(A)$ turns out to be the discrete capacity of A. It is linked with the Green's function through an *energy*. Indeed, for a kernel $k : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$, and a probability measure ν on A, the k-energy is defined to be

$$\mathcal{E}_k(\nu) := \sum_{x\in A} \sum_{y\in A} k(x, y)\nu(x)\nu(y). \tag{1.2}$$

The capacity of the set A is defined as

$$\mathrm{Cap}(A) := \Big(\inf \Big\{\mathcal{E}_g(\nu) : \ \nu \text{ probability measure on } A\Big\}\Big)^{-1}, \tag{1.3}$$

where g is the Green's function of a simple random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}^d$, defined as

$$g(x, y) := \mathbb{E}_x\Big[\sum_{n\geq 0} \mathbf{1}(X_n = y)\Big],$$

with $\mathbb{E}_x$ denoting expectation with respect to a walk starting from x. We recall that $g(x, y) = g(0, y - x)$, which we shall also write as $g(y - x)$. As Spitzer observed later [Sp73], (1.1) follows directly from Kingman's subadditive ergodic theorem and a last exit decomposition. In [Zhu16] it was shown that, when $d \geq 5$, the hitting probability of a finite set, say $A \subset \mathbb{Z}^d$, by the past tree $\mathcal{T}_-$ appropriately normalised has a limit, called *the branching capacity* of A, and denoted by $\mathrm{BCap}(A)$. More precisely,

$$\mathrm{BCap}(A) := \lim_{\|x\|\to\infty} \frac{2/\sigma^2}{g * g(x)} \mathbb{P}\big(\mathcal{T}_-^x \cap A \neq \emptyset\big). \tag{1.4}$$

Furthermore, it was shown in [ASS23] that the branching capacity is comparable to a capacity corresponding to the kernel $g*g$ in the following sense: there exists a positive constant C depending on the variance of the offspring distribution μ, so that for all finite sets $A \subset \mathbb{Z}^d$, we have

$$\frac{1}{C} \cdot \mathrm{BCap}(A)^{-1} \leq \inf \Big\{\mathcal{E}_{g*g}(\nu) : \ \nu \text{ probability measure on } A\Big\} \leq C \cdot \mathrm{BCap}(A)^{-1}. \tag{1.5}$$

Let $\mathcal{A}$ be an arbitrary set. For any two functions $f, h : \mathcal{A} \to \mathbb{R}_+$ we write $f \asymp h$ if the ratio $f(a)/h(a)$ is bounded both from above and below by positive constants uniformly over all $a \in \mathcal{A}$.

Our first result extends the result by Kesten, Spitzer and Whitman to tree-indexed random walks and additive random walks, thus revealing similarities between these two processes. Fix an offspring distribution μ with mean one and finite variance and consider the associated infinite tree $\mathcal{T}$ defined above, together with the walk $(S_u)_{u\in\mathcal{T}}$ indexed by $\mathcal{T}$. Then, for $n \geq 0$, define $\mathcal{T}_n^0 = \{S_u : u = \mathcal{T}(0), \dots, \mathcal{T}(n)\}$, its range when restricted to the first $n+1$ vertices of $\mathcal{T}$, and starting from the origin.

Theorem 1.1. *Fix $d \geq 5$. Let X and $\widetilde{X}$ be two independent simple random walks on $\mathbb{Z}^d$ and let $\mathcal{T}_n^0$ be as above. Let A be a finite subset of $\mathbb{Z}^d$. Then the following limits hold almost surely,*

$$\lim_{n\to\infty} \frac{|\mathcal{T}_n^0 + A|}{n} \asymp \lim_{n\to\infty} \frac{|\mathcal{R}_n + \widetilde{\mathcal{R}}_n + A|}{n^2}, \tag{1.6}$$

with the implied constants only depending on the variance of μ. Moreover,

$$\lim_{n\to\infty} \frac{|\mathcal{T}_n^0 + A|}{n} = \mathrm{BCap}(A), \tag{1.7}$$

and

$$\lim_{n\to\infty} \frac{|\mathcal{R}_n + \widetilde{\mathcal{R}}_n + A|}{n^2} = \lim_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{R}_n + A)}{n}. \tag{1.8}$$

Theorem 1.1 states that the two limits in (1.6) exist, and that they are comparable. To the best of our knowledge, the observation that the limits are comparable is new, as well as the characterisation of the limits in terms of the branching capacity. Note that (1.8) generalises a law of large numbers result for $\mathrm{Cap}(\mathcal{R}_n)$ proved by Jain and Orey [JO69] with the limit being positive if and only if $d \geq 5$. We show in (2.8) a similar result where the branching capacity is considered instead of the capacity. In Lemma 2.4 we give a representation of the limit in terms of escape probabilities in analogy with the classical formula for capacity. We finally note that (1.7) generalises a recent result of Le Gall and Lin [LGL16] where they treat the case $A = \{0\}$. More precisely, Le Gall and Lin study a critical BRW conditioned on having exactly n nodes and take the limit as $n \to \infty$. In order to establish limit laws for the volume of its range, they had to introduce the infinite invariant tree we described above as its invariance under the shift of labels makes the corresponding law of large numbers a simple application of Kingman's subadditive ergodic theorem. Here, our starting point is directly the infinite invariant tree, and the interested reader can check [LGL16] for linking the conditioned process and the infinite invariant one. Note that in the case $A = \{0\}$, Le Gall and Lin provide a dual formula for $\mathrm{Bcap}(\{0\})$ in Theorem 4 of [LGL16]. This formula was later generalised by Zhu in [Zhu16], where he defined the branching capacity of any finite set $A \subset \mathbb{Z}^d$, as in (1.4).

Concerning the proof of Theorem 1.1, let us say that (1.7) follows from Kingman's subadditive ergodic theorem, together with a last exit-decomposition, exactly as for (1.1). Likewise (1.8) also simply follows from (1.1) and a multidimensional subadditive ergodic theorem, for which we refer to Section 2.2 for a precise statement. In fact, the main observation is (1.6). The strategy is not to compare both terms directly here, but rather to show that they are both comparable to a third quantity, the $(d-4)$-capacity of A, which will be introduced later, in (1.10). Indeed, the fact that the left-hand side of (1.6) is comparable to this third quantity follows from (1.7) and (1.5). Hence, the main novelty here is to show that the right-hand side of (1.6), or equivalently of (1.8), is comparable to the $(d-4)$-capacity of A. For this we first use that the capacity of a set is related to the probability of hitting this set by a simple random walk starting from far away. Hence after some elementary computation (see Proposition 1.6), we reduce the whole problem to that of estimating hitting probabilities of a set A by a sum of two walks, which is the object of our second result below.

Hitting probabilities. Our second result shows that, in the terminology of [P96], $\mathcal{T}_-^x$ and $x + \mathcal{R}_\infty + \widetilde{\mathcal{R}}_\infty$ are *intersection-equivalent* when x is away from the set to be hit. For a finite set $A \subset \mathbb{Z}^d$, we define its diameter as

$$\mathrm{diam}(A) = 1 + \max\{\|x-y\| : x, y \in A\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Theorem 1.2. *Assume $d \geq 5$ and let μ be an offspring distribution with mean one and finite variance. There exists a positive constant C so that the following holds. Let $\mathcal{R}_\infty$ and $\widetilde{\mathcal{R}}_\infty$ be two independent simple random walk ranges in $\mathbb{Z}^d$. Then for any finite set $A \subset \mathbb{Z}^d$, containing the origin, and any $x \in \mathbb{Z}^d$, satisfying $\|x\| \geq 2 \cdot \mathrm{diam}(A)$,*

$$\frac{1}{C} \cdot \mathbb{P}\big(\mathcal{T}_-^x \cap A \neq \emptyset\big) \leq \mathbb{P}\Big((x + \mathcal{R}_\infty + \widetilde{\mathcal{R}}_\infty) \cap A \neq \emptyset\Big) \leq C \cdot \mathbb{P}\big(\mathcal{T}_-^x \cap A \neq \emptyset\big). \tag{1.9}$$

Note that Benjamini, Pemantle and Peres [BPP95] focus on the Martin capacity for a general transient Markov chain on a countable state space. The Martin capacity is associated with the so-called Martin Kernel $k(x,y) = g(x,y)/g(\rho,y)$, where ρ is the starting site of the chain and g is its Green's function. Their main result states that the hitting probability of a set A is within a constant factor of two equal to the Martin capacity of A; in particular two Markov chains with comparable Green's functions are intersection equivalent. While here both the random walk indexed by $\mathcal{T}_-$, and the sum of two walks, have comparable Green's function, the main difficulty in proving Theorem 1.2 is of course the lack of Markov property: for branching random walks or for additive random walks, as Salisbury explains in [S96] *"the difficulty is that there may be no first hitting time at which any kind of Markov property applies"*. Fitzsimmons and Salisbury [FS89, S96] using remarkable ideas managed to deal with multivariate processes. Here we shall adapt proofs of Khoshnevisan and Shi [KS99], inspired by [S96], which provide estimates on the probability that a Brownian sheet hits a distant compact set $A \subset \mathbb{R}^d$, in terms of some appropriate (continuous) γ-capacity of A, see (1.10) below for a definition in the discrete setting. On the other hand, as already mentioned, the fact that the hitting probability of A by a random walk indexed by $\mathcal{T}_-$ is also comparable to the $(d-4)$-capacity of A was obtained independently earlier, see (1.4) and (1.5).

Sum of N random walks. One of the achievements of potential theory for Markov processes (in the continuous setting) is to establish necessary and sufficient conditions for a transient process to hit a set. Morally, the set should support a probability measure whose g-energy is finite, where g is the Green's function. Fitzsimmons and Salisbury in [FS89] managed to build such a measure for additive Markov processes using random times which are not stopping times. Their proof requires estimates on the first two moments of the local times and their approach is adapted to tackle a sum of N walks, as we shall see later in Lemma 3.1.

We now define the γ-capacity, denoted Cap_γ, for any $\gamma > 0$, by replacing g in (1.3) by the kernel $k_\gamma(x,y) = (1 + \|x-y\|)^{-\gamma}$. More precisely, for a finite and nonempty subset $A \subset \mathbb{Z}^d$, we define

$$\frac{1}{\mathrm{Cap}_\gamma(A)} = \inf\Big\{\mathcal{E}_{k_\gamma}(\nu) : \ \nu \text{ probability measure on } A\Big\}. \tag{1.10}$$

Our third result is a natural generalisation of Theorem 1.1.

Theorem 1.3. *Let $N \geq 1$, and $d \geq 2N+1$. Let $(\mathcal{R}_\infty^i)_{i=1,\dots,N}$ be N independent simple random walk ranges on $\mathbb{Z}^d$. There exists a positive set-function f_N, such that for any finite and nonempty*

$A \subset \mathbb{Z}^d$, almost surely

$$f_N(A) := \lim_{n\to\infty} \frac{\left|\mathcal{R}_n^1 + \cdots + \mathcal{R}_n^N + A\right|}{n^N} \asymp \mathrm{Cap}_{d-2N}(A). \tag{1.11}$$

Furthermore, the following limit exists almost surely and satisfies

$$\lim_{n\to\infty} \frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n^1 + A)}{n} \asymp \mathrm{Cap}_{d-2N}(A). \tag{1.12}$$

Remark 1.4. In Section 2.2 we give a dual representation of $f_N(A)$ in terms of escape probabilities for sums of walks analogously to the case of capacity for $N = 1$.

As in the case $N = 2$, once (1.12) is established, (1.11) follows from a multidimensional subadditive ergodic theorem, hence the whole matter is here again to prove (1.12), which will be obtained as a consequence of our next two results.

First, we relate hitting probabilities of a set by the sum of N walks to its $(d-2N)$-capacity. This is where we use the ideas of Fitzsimmons and Salisbury mentionned above.

Theorem 1.5. *Let $N \geq 1$ and $d \geq 1+2N$. Let $(\mathcal{R}_\infty^i)_{i=1,\dots,N}$, be N independent simple random walk ranges in $\mathbb{Z}^d$. There exists a positive constant C, such that for any finite set $A \subset \mathbb{Z}^d$, containing the origin, and any $x \in \mathbb{Z}^d$, with $\|x\| \geq 2 \cdot \mathrm{diam}(A)$,*

$$\frac{1}{C} \cdot \frac{\mathrm{Cap}_{d-2N}(A)}{\|x\|^{d-2N}} \leq \mathbb{P}\Big((x + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^N) \cap A \neq \emptyset\Big) \leq C \cdot \frac{\mathrm{Cap}_{d-2N}(A)}{\|x\|^{d-2N}}. \tag{1.13}$$

Note that the case $N = 1$ of Theorem 1.5 is well-known, see e.g. [L91] or [BPP95] for a more precise result. In addition to Theorem 1.5 we have the following proposition, which makes the link between the probability of hitting a set A by a sum of ranges, and the ergodic limit appearing in (1.12) of Theorem 1.3.

Proposition 1.6. *Let $N \geq 2$ and $d \geq 2N + 1$. Suppose that $\mathcal{R}^1, \dots, \mathcal{R}^N$, are N independent simple random walk ranges in $\mathbb{Z}^d$. There exist positive constants c_1 and c_2, such that for any finite nonempty set $A \subset \mathbb{Z}^d$,*

$$\liminf_{\|x\|\to\infty} \|x\|^{d-2N} \cdot \mathbb{P}\Big((x + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^N) \cap A \neq \emptyset\Big) \geq c_1 \cdot \lim_{n\to\infty} \frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n^1 + A)}{n}, \tag{1.14}$$

and

$$\limsup_{\|x\|\to\infty} \|x\|^{d-2N} \cdot \mathbb{P}\Big((x + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^N) \cap A \neq \emptyset\Big) \leq c_2 \cdot \lim_{n\to\infty} \frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n^1 + A)}{n}. \tag{1.15}$$

Observe that (1.12) indeed follows from Theorem 1.5 and Proposition 1.6. The proof of Proposition 1.6 follows from the fact that the capacity of a set is related to its hitting probability by a walk starting from far away. The whole point here is to make sure that one can exchange limits, which is done by a careful, though somewhat standard, decomposition of a random walk trajectory into excursions.

Critical dimension four. We now briefly discuss the case of dimension four which is critical for the capacity of the range. By analogy with the case of the volume in dimension two, first considered in Spitzer's original paper [Sp64], and then by Le Gall [LG90] and Port [Port65], one can expect that in the asymptotic development of $\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n + A)]$, only the second order term should depend on A (and be related to a properly defined notion of branching capacity). Here we do not pursue such a precise result, but notice that indeed the first order term does not depend on A.

Proposition 1.7. *Let $\mathcal{R}$ be a simple random walk range in $\mathbb{Z}^4$. Then for any finite and nonempty set $A \subset \mathbb{Z}^4$, one has*

$$\lim_{n\to\infty} \frac{\log n}{n} \cdot \mathbb{E}[\mathrm{Cap}(\mathcal{R}_n + A)] = \frac{\pi^2}{8}. \tag{1.16}$$

To prove the proposition above we use key ideas from Lawler's book [L91]: the relationship between capacity and Green's function in Theorem 3.6.1, and the estimates from Section 3.4 in [L91]. It was proved in [ASS19] that when $A = \{0\}$, then $(\log n)\mathrm{Cap}(\mathcal{R}_n + A)/n$ converges to $\pi^2/8$ almost surely as $n \to \infty$. The same argument as in [ASS19] can be used to prove almost sure convergence also in the case when A is a general finite set, see Remark 6.4. However, a central limit theorem is missing in the general case.

Notation. We will use the notation $f \gtrsim g$ if there exists a positive constant c, such that $f \geq cg$, and $f \lesssim g$ (or sometimes $f = \mathcal{O}(g)$) if $g \gtrsim f$. We also use the standard notation $o(1)$ for a quantity which converges to 0 as the parameter n goes to infinity. For $x \in \mathbb{Z}^d$, and $r \geq 0$ let $B(x, r) = \{y \in \mathbb{Z}^d : \|y - x\| \leq r\}$, the Euclidean ball of radius r. We write $\partial\Lambda$ for the inner boundary of a set $\Lambda \subseteq \mathbb{Z}^d$, i.e. the set of points in Λ having at least one neighbor in Λ^c.

The paper is organised as follows. In Section 2, we gather known results from ergodic theory that we apply to trees and sums of walks. We then provide an expression for $f_N(A)$ from Theorem 1.3, and show that it is positive when $d \geq 2N + 1$. Finally, we recall why γ-capacities are sub-additive. In Section 3 we prove an estimate on the second moment of local times, Lemma 3.1, which is an important ingredient for the proof of Theorem 1.5 given in Section 4, together with the proof of Theorem 1.2. In Section 5 we prove Proposition 1.6 using Theorem 1.5 and in Section 6 we focus on the 4-dimensional case and give the proof of Proposition 1.7. Finally, in Section 7, we gather related open problems.

2 Subadditive functionals & Ergodic theorems

In this section we show the existence of three ergodic limits, namely the limits in (1.6), (1.11) and (1.12) holding almost surely. We start in the next section by recalling some results about γ-capacities. Then in Section 2.2 we recall a multi-parameter extension of the subadditive ergodic theorem and then deduce that the limit in (1.11) exists almost surely. Then in Section 2.3 we apply it to functionals on trees.

2.1 γ-Capacities

In this section we collect some results about γ-capacities. In particular the existence of the limit in (1.12) directly follows from Kingman's subadditive theorem [K73] and the subadditivity of γ-capacities, which we recall now.

Claim 2.1. *Let $\gamma > 0$. Then for any finite sets $A, B \subseteq \mathbb{Z}^d$ we have*

$$\mathrm{Cap}_\gamma(A \cup B) \leq \mathrm{Cap}_\gamma(A) + \mathrm{Cap}_\gamma(B).$$

Proof. First notice that Cap_γ is increasing for inclusion, i.e. if $A \subseteq B$, then $\mathrm{Cap}_\gamma(A) \le \mathrm{Cap}_\gamma(B)$, since a probability measure on A is also a probability measure on B. It thus suffices to prove sub-additivity for disjoint subsets. Now consider A and B two disjoint subsets of $\mathbb{Z}^d$, and let ν be a probability measure on $A \cup B$. Let $\alpha = \sum_{x\in A} \nu(x)$. Then it is easy to see that

$$\sum_{x,y\in A\cup B} (1 + \|x - y\|)^{-\gamma}\nu(x)\nu(y) \ge \frac{\alpha^2}{\mathrm{Cap}_\gamma(A)} + \frac{(1-\alpha)^2}{\mathrm{Cap}_\gamma(B)}.$$

Indeed, this is trivially true if $\alpha \in \{0, 1\}$, while otherwise the restriction of ν/α to A is a probability measure on A, and the restriction of $\frac{\nu}{1-\alpha}$ to B is a probability measure on B. Taking the infimum over all ν on the left hand side yields

$$\frac{1}{\mathrm{Cap}_\gamma(A \cup B)} \ge \inf_{\alpha\in[0,1]} \left\{ \frac{\alpha^2}{\mathrm{Cap}_\gamma(A)} + \frac{(1-\alpha)^2}{\mathrm{Cap}_\gamma(B)} \right\}.$$

Now observe that for any $x, y > 0$, and any $\alpha \in [0, 1]$,

$$\frac{\alpha^2}{x} + \frac{(1-\alpha)^2}{y} \ge \frac{1}{x+y},$$

which proves well that $\mathrm{Cap}_\gamma(A \cup B) \le \mathrm{Cap}_\gamma(A) + \mathrm{Cap}_\gamma(B)$ and finishes the proof. □

Lemma 2.2. *Let $d \ge 3$, $\gamma > 2$ and $A \subseteq \mathbb{Z}^d$, be a finite set. Let $\mathcal{R}$ be the range of a simple random walk in $\mathbb{Z}^d$. Then we have almost surely*

$$\lim_{n\to\infty} \frac{\mathrm{Cap}_\gamma(\mathcal{R}_n + A)}{n} = \inf_{n\ge 1} \frac{\mathbb{E}\big[\mathrm{Cap}_\gamma(\mathcal{R}_n + A)\big]}{n} \gtrsim \mathrm{Cap}_{\gamma-2}(A).$$

Proof. Let μ be a probability measure on A. Given $n \ge 0$, and $x \in \mathbb{Z}^d$, define

$$\ell_n(x) = \mathbb{E}\left[\sum_{k=0}^{n} \mathbf{1}(X_k = x)\right],$$

and consider the probability measure ν_n on $\mathcal{R}_n + A$ given by

$$\nu_n(z) = \sum_{x\in\mathcal{R}_n} \sum_{a\in A} \mathbf{1}(x + a = z) \cdot \frac{\ell_n(x)}{n+1} \cdot \mu(a), \qquad \text{for all } z \in \mathcal{R}_n + A.$$

Note that this is indeed a probability measure on $\mathcal{R}_n + A$. Then we have for any $n \ge 1$

$$\begin{aligned} \mathrm{Cap}_\gamma(\mathcal{R}_n + A) &\ge \frac{1}{\sum_{x,y\in\mathbb{Z}^d} (1 + \|x - y\|)^{-\gamma}\nu_n(x)\nu_n(y)} \\ &= \frac{(n+1)^2}{\sum_{a,b\in A} \sum_{x,y\in\mathbb{Z}^d} (1 + \|a - b + x - y\|)^{-\gamma}\mu(a)\mu(b)\ell_n(x)\ell_n(y)}. \end{aligned} \tag{2.1}$$

By Jensen's inequality we get

$$\mathbb{E}\big[\mathrm{Cap}_\gamma(\mathcal{R}_n + A)\big] \ge \frac{(n+1)^2}{\sum_{a,b\in A} \sum_{x,y\in\mathbb{Z}^d} (1 + \|a - b + x - y\|)^{-\gamma}\mu(a)\mu(b)\mathbb{E}[\ell_n(x)\ell_n(y)]}.$$

Let $g_n(x) = \mathbb{E}[\ell_n(x)]$, and $g(x) = g_\infty(x)$. Then the Markov property and the symmetry of the random walk steps yield,

$$\begin{aligned}\mathbb{E}[\ell_n(x)\ell_n(y)] &\le \sum_{k\le k'\le n} (\mathbb{P}(X_k = x, X_{k'} = y) + \mathbb{P}(X_k = y, X_{k'} = x)) \\ &\le \sum_{k=0}^{n}\sum_{k'=k}^{\infty} (\mathbb{P}(X_k = x) + \mathbb{P}(X_k = y)) \cdot \mathbb{P}(X_{k'-k} = y - x) \\ &= (g_n(x) + g_n(y))g(x-y).\end{aligned}$$

Therefore, letting $k_\gamma(u) = (1 + \|u\|)^{-\gamma}$, we get that for all $a, b \in A$,

$$\sum_{x,y\in\mathbb{Z}^d} k_\gamma(a-b+x-y)\cdot\mathbb{E}[\ell_n(x)\ell_n(y)] \lesssim (k_\gamma * g(a-b))\cdot\sum_{x\in\mathbb{Z}^d} g_n(x) = (n+1)\cdot(k_\gamma * g(a-b)).$$

Plugging this into (2.1) we get

$$\mathrm{Cap}_\gamma(\mathcal{R}_n + A) \gtrsim \frac{n+1}{\sum_{a,b\in A} k_\gamma * g(a-b)\mu(a)\mu(b)}.$$

Now we claim that as soon as $\gamma > 2$, and $d \ge 3$, then $k_\gamma * g \asymp k_{\gamma-2}$. Indeed, recall that $g(u) \asymp (1 + \|u\|)^{2-d}$, and thus for $u \neq 0$,

$$\begin{aligned}k_\gamma * g(u) &= \sum_{v\in\mathbb{Z}^d} k_\gamma(u-v)g(v) \\ &\asymp \frac{1}{\|u\|^{d-2}} \sum_{v\in B(u,\|u\|/2)} \frac{1}{1+\|v-u\|^\gamma} + \frac{1}{\|u\|^\gamma}\sum_{v\in B(0,2\|u\|)} \frac{1}{\|v\|^{d-2}} + \sum_{\|v\|>2\|u\|} \frac{1}{\|v\|^{\gamma+d-2}} \\ &\asymp \frac{1}{\|u\|^{\gamma-2}},\end{aligned}$$

proving our claim. Therefore taking the infimum above over all probability measures μ on A, we get that almost surely,

$$\lim_{n\to\infty} \frac{\mathrm{Cap}_\gamma(\mathcal{R}_n + A)}{n} = \inf_{n\ge1} \frac{\mathbb{E}\big[\mathrm{Cap}_\gamma(\mathcal{R}_n + A)\big]}{n} \gtrsim \mathrm{Cap}_{\gamma-2}(A), \tag{2.2}$$

where the existence of the limit and the first equality both follow from Claim 2.1 and Kingman's subadditive ergodic theorem [K73, Sp73]. This concludes the proof of the lemma. □

2.2 Multiparameter subadditive ergodic theorem

We start by recalling a multi-parameter extension of the subadditive ergodic theorem due to Akcoglu and Krengel. We denote by $\mathcal{U}_N$ the set of all N-dimensional rectangles of $\mathbb{N}^N$, i.e. sets of the form $\prod_{i=1}^N \{n_i, \dots, m_i\}$, with $0 \le n_i \le m_i$ for all $i \le N$.

Theorem 2.3 (Akcoglu–Krengel [AK81])**.** *Let $N \ge 1$, and $(L(U))_{U\in\mathcal{U}_N}$ be a sequence of real-valued random variables, satisfying the following properties:*

(i) (Stationarity) For any k, any $U_1, \dots, U_k \in \mathcal{U}_N$, and any $u \in \mathbb{N}^N$, the joint distribution of $(L(u + U_1), \dots, L(u + U_k))$ is the same as that of $(L(U_1), \dots, L(U_k))$.

(ii) *(Subadditivity) Given any disjoint rectangles* $U_1, \dots, U_k$*, such that* $\cup_{i=1}^k U_i \in \mathcal{U}_N$*, one has* $L(\cup_{i\le k} U_i) \le \sum_{i\le k} L(U_i)$.

(iii) *(Integrability) The random variables* $L(U)$ *are integrable for all* $U \in \mathcal{U}_N$.

(iv) *(Boundedness in mean) One has* $\sup_{U\in\mathcal{U}_N} \mathbb{E}\big[|L(U)|\big]/|U| < \infty$.

Then there exists a random variable Γ*, such that almost surely*

$$\lim_{n\to\infty} \frac{L(\{0,\dots,n\}^N)}{n^N} = \Gamma,$$

and furthermore,

$$\inf_{n_1,\dots,n_N\ge 1} \frac{\mathbb{E}\big[L\big(\prod_{i=1}^N\{0,\dots,n_i\}\big)\big]}{n_1\dots n_N} = \lim_{n_1,\dots,n_N\to\infty} \frac{\mathbb{E}\big[L\big(\prod_{i=1}^N\{0,\dots,n_i\}\big)\big]}{n_1\dots n_N}. \tag{2.3}$$

Indeed, the existence of the limit Γ follows from Theorem (2.4) in [AK81], and (2.3) can be proved as Lemma (3.4) there. Using this we now explain the existence of the limit in (1.11).

First, by the elementary exclusion-inclusion formula for the volume of the Minkowski sum we get for A, B finite subsets of $\mathbb{Z}^d$,

$$|\mathcal{R} + (A\cup B)| = |(\mathcal{R}+A)\cup(\mathcal{R}+B)| = |\mathcal{R}+A| + |\mathcal{R}+B| - |(\mathcal{R}+A)\cap(\mathcal{R}+B)|.$$

Since, $\mathcal{R} + (A\cap B) \subset (\mathcal{R}+A)\cap(\mathcal{R}+B)$, we have a strong form of subadditivity

$$|\mathcal{R} + (A\cup B)| + |\mathcal{R} + (A\cap B)| \le |\mathcal{R}+A| + |\mathcal{R}+B|. \tag{2.4}$$

Now clearly, for any fixed $A \subset \mathbb{Z}^d$ and $\mathcal{R}^1, \dots, \mathcal{R}^N$ independent simple random walk ranges, the process defined by

$$L\Big(\prod_{i=1}^N \{n_i,\dots,m_i\}\Big) = \Big|\mathcal{R}^1[n_1,m_1] + \dots + \mathcal{R}^N[n_N,m_N] + A\Big|,$$

satisfies all the hypotheses of the previous theorem (in particular stationarity follows from the fact that for any $k \ge 0$ and any $n \le m$, $\mathcal{R}[k+n, k+m]$ has the same law as $X_k + \mathcal{R}[n,m]$), and hence we get the almost sure existence of the limit below for any finite set A,

$$f_N(A) := \lim_{n\to\infty} \frac{\Big|\mathcal{R}_n^1 + \dots + \mathcal{R}_n^N + A\Big|}{n^N}.$$

Furthermore, an immediate application of Kolmogorov's $0-1$ law implies that $f_N(A)$ is almost surely constant. Then an application of the dominated convergence theorem and (2.3) yield

$$f_N(A) = \lim_{n_1,\dots,n_N\to\infty} \frac{\mathbb{E}\Big[\Big|\mathcal{R}_{n_1}^1 + \dots + \mathcal{R}_{n_N}^N + A\Big|\Big]}{n_1\dots n_N}.$$

Using (2.4) we deduce that f_N also satisfies the strong subadditivity property

$$f_N(A\cup B) + f_N(A\cap B) \le f_N(A) + f_N(B).$$

Therefore almost surely as well,

$$\lim_{n\to\infty} \frac{f_{N-1}(\mathcal{R}_n + A)}{n} = \lim_{n\to\infty} \frac{\mathbb{E}\big[f_{N-1}(\mathcal{R}_n + A)\big]}{n}.$$

Applying twice the dominated convergence theorem yields

$$\begin{aligned} f_N(A) &= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{m\to\infty} \frac{\mathbb{E}\left[\left|\mathcal{R}_m^1 + \cdots + \mathcal{R}_m^{N-1} + \mathcal{R}_n^N + A\right|\right]}{m^{N-1}} \\ &= \lim_{n\geq 1} \frac{\mathbb{E}[f_{N-1}(\mathcal{R}_n^N + A)]}{n} = \lim_{n\to\infty} \frac{f_{N-1}(\mathcal{R}_n^N + A)}{n}. \end{aligned} \tag{2.5}$$

In the next result we give an expression for the set-function $f_N(A)$ generalising the expression for capacity in the case when $N = 1$. We give the proof in Section 2.4.

Lemma 2.4. *Let N be an integer, and consider dimension $d \geq 2N + 1$. Let $\mathcal{R}^1, \ldots, \mathcal{R}^N$ be independent ranges of double-sided simple random walks in $\mathbb{Z}^d$. Then,*

$$f_N(A) = \sum_{a\in A} \mathbb{P}\left(\bigcap_{i=1}^{N} \big\{ (\mathcal{R}^i(0,\infty) + \sum_{i<j\leq N} \mathcal{R}^j(-\infty,\infty) + a) \cap A = \emptyset \big\} \right). \tag{2.6}$$

2.3 Functionals on trees

A fundamental property of the infinite invariant tree is its invariance in law after applying the shift on the labels. This fact was first observed by Le Gall and Lin [LGL16] on the restriction of the tree to $\mathcal{T}_+$, and then on the full tree independently by Zhu [Zhu18] and Bai and Wan [BW22]. The infinite tree has an invariant product measure, and the shift is actually a reversible map. Le Gall and Lin introduced the infinite tree to be able to use ergodic theory to prove asymptotics for the size of the range of the first n labelled sites of the invariant branching random walk. They then transferred their result to critical trees conditioned on having total population n, as n goes to infinity. Here, for simplicity, we only discuss $\mathcal{T}_n^0$, and the reader is referred to [LGL16] to transfer the results to critical branching random walks conditioned to have population n.

In fact [LGL16] shows that $|\mathcal{T}_n^0|/n$ converges almost surely to the probability that a walk indexed by $\mathcal{T}_+$ avoids the origin (except at the root), simply as a consequence of Kingman's sub-additive ergodic theorem. By following the same argument, together with a last exit decomposition, exactly as for (1.1), we obtain that for any finite set $A \subset \mathbb{Z}^d$,

$$\frac{|\mathcal{T}_n^0 + A|}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} \sum_{x\in A} \mathbb{P}(\mathcal{T}_{++}^x \cap A = \emptyset),$$

where $\mathcal{T}_{++} = \mathcal{T}_+ \setminus \{\varnothing\}$ (and $\mathcal{T}_{++}^x$ is the restriction of the range to $\mathcal{T}_{++}$). This latter expression turns out to be the branching capacity of A. Indeed, Zhu [Zhu16, Proposition 8.1] showed that

$$\mathrm{BCap}(A) = \sum_{x\in A} \mathbb{P}(\mathcal{T}_-^x \cap A = \emptyset) = \sum_{x\in A} \mathbb{P}(\mathcal{T}_{++}^x \cap A = \emptyset). \tag{2.7}$$

Thus, the original part in Theorem 1.1 is to make the link with the set-function $f_2(A)$ obtained with two independent random walks from (1.11).

Now let us mention some natural extensions of our results. We can indeed deduce that also $\mathrm{cap}(\mathcal{T}_n^0 + A)/n$ converges in probability, and furthermore that the limit is of order $\mathrm{Cap}_{d-6}(A)$, when $d \geq 7$. Let us just explain the proof in this case. First, applying twice the multi-parameter ergodic theorem, Theorem 2.3, and using Theorems 1.1 and 1.3, we get that almost surely

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{T}_n^0 + A)}{n} &= \lim_{n\to\infty} \frac{|\mathcal{R}_n + \mathcal{T}_n^0 + A|}{n^2} = \lim_{n\to\infty} \frac{\mathrm{BCap}(\mathcal{R}_n + A)}{n} \\ &\asymp \lim_{n\to\infty} \frac{f_2(\mathcal{R}_n + A)}{n} = \lim_{n\to\infty} \frac{|\mathcal{R}_n^1 + \mathcal{R}_n^2 + \mathcal{R}_n^3 + A|}{n^3} = f_3(A) \asymp \mathrm{Cap}_{d-6}(A),\end{aligned} \tag{2.8}$$

where $\mathcal{R}_n$ and $(\mathcal{R}_n^i)_{i=1,2,3}$ are independent ranges of simple random walks, independent of $\mathcal{T}^0$.

2.4 Dual representation for f_N

In this section we give the proof of Lemma 2.4 which again makes use of Theorem 2.3.

Proof of Lemma 2.4. We can write

$$\begin{aligned}|\mathcal{R}_n^1 + \ldots + \mathcal{R}_n^N + A| &= \sum_{i_1 \leq n} \sum_{x \in \mathcal{R}_n^2 + \ldots + \mathcal{R}_n^N + A} \mathbf{1}(X_{i_1}^1 + x \notin \cup_{j > i_1}(X_j^1 + \mathcal{R}_n^2 + \ldots + \mathcal{R}_n^N + A)) \\ &= \sum_{i_1, i_2 \leq n} \sum_{x \in \mathcal{R}_n^3 + \ldots + \mathcal{R}_n^N + A} \mathbf{1}(X_{i_1}^1 + X_{i_2}^2 + x \notin \cup_{j > i_1}(X_j^1 + \mathcal{R}_n^2 + \ldots + \mathcal{R}_n^N + A)) \\ &\qquad \times \mathbf{1}(X_{i_2}^2 + x \notin \cup_{j_2 > i_2}(X_{j_2}^2 + \mathcal{R}_n^3 + \ldots + \mathcal{R}_n^N + A)).\end{aligned}$$

Iterating this, we obtain

$$\begin{aligned}&|\mathcal{R}_n^1 + \ldots + \mathcal{R}_n^N + A| \\ &= \sum_{a \in A} \sum_{i_1, \ldots, i_N \leq n} \mathbf{1}(X_{i_N}^N + a \notin \cup_{j > i_N}(X_j^N + A)) \times \mathbf{1}(X_{i_{N-1}}^{N-1} + X_{i_N}^N + a \notin \cup_{j > i_{N-1}}(X_j^{N-1} + \mathcal{R}_n^N + A)) \\ &\qquad \times \cdots \times \mathbf{1}(X_{i_1}^1 + \ldots + X_{i_N}^N + a \notin \cup_{j > i_1}(X_j^1 + \mathcal{R}_n^2 + \ldots + \mathcal{R}_n^N + A)).\end{aligned}$$

For all $i \in \{1, \ldots, N\}$ and $k \leq n$, we set

$$\widehat{\mathcal{R}}^i(-k, n-k) = X_k^i - \mathcal{R}_n^i \quad \text{and} \quad \widehat{\mathcal{R}}^i(0, n-k) = X_k^i - \mathcal{R}^i[k+1, n].$$

Then we can rewrite the expression above as

$$\begin{aligned}&|\mathcal{R}_n^1 + \ldots + \mathcal{R}_n^N + A| \\ &= \sum_{a \in A} \sum_{i_1, \ldots, i_N \leq n} \mathbf{1}((\widehat{\mathcal{R}}^N(0, n - i_N) + a) \cap A = \emptyset) \mathbf{1}((\widehat{\mathcal{R}}^{N-1}(0, n - i_{N-1}) + \widehat{\mathcal{R}}^N(-i_N, n - i_N) + a) \cap A = \emptyset) \\ &\qquad \times \cdots \times \mathbf{1}((\widehat{\mathcal{R}}^1(0, n - i_1) + \widehat{\mathcal{R}}^2(-i_2, n - i_2) + \cdots \widehat{\mathcal{R}}^N(-i_N, n - i_N) + a) \cap A = \emptyset).\end{aligned}$$

Restricting the sum above over all $i_1, \ldots, i_N \in (\log n, n - \log n)$, dividing through by n^N and applying Theorem 2.3 we deduce that almost surely as $n \to \infty$

$$\frac{1}{n^N} \cdot |\mathcal{R}_n^1 + \ldots + \mathcal{R}_n^N + A| \to \sum_{a \in A} \mathbb{P}\left(\bigcap_{i=1}^N \big\{ (\mathcal{R}^i[1, \infty) + \sum_{i < j \leq N} \mathcal{R}^j(-\infty, \infty) + a) \cap A = \emptyset \big\} \right),$$

where $\mathcal{R}(-\infty, \infty)$ corresponds to the range of a double-sided simple random walk. This now concludes the proof. □

3 Preliminaries on local times

Our goal in this section is to prove Lemma 3.1 below, which provides second moment estimates for the local times of the sum of N independent walks. Fix $N \geq 1$, and consider $X^1, \dots, X^N$, a sequence of N simple random walks, all starting from the origin. Then given any $x, z \in \mathbb{Z}^d$, set

$$\ell_{z+X^1+\cdots+X^N}(x) := \sum_{t_1,\dots,t_N} \mathbf{1}\big(z + X^1_{t_1} + \cdots + X^N_{t_N} = x\big). \tag{3.1}$$

Let G_N be the N-th convolution power of the simple random walk's Green's function g (so that $G_N(x-z) = \mathbb{E}\big[\ell_{z+X^1+\cdots+X^N}(x)\big]$). Our second moment estimate reads as follows.

Lemma 3.1. Let $N \geq 1$ and $d \geq 2N+1$. There exists a constant $C = C(d) > 0$, such that for any $z, a, b \in \mathbb{Z}^d$, with $\|z\| \geq 2\max(\|a\|, \|b\|)$, we have

$$\mathbb{E}\big[\ell_{z+X^1+\cdots+X^N}(a)\ell_{z+X^1+\cdots+X^N}(b)\big] \leq C\Big(G_N(z-a) + G_N(z-b)\Big) \cdot G_N(a-b). \tag{3.2}$$

Before we move to the proof of this lemma, let us recall that $g(x) \asymp \frac{1}{(1+\|x\|)^{d-2}}$, which by an immediate induction shows that for any $k \in \{1, \dots, N\}$ (and as long as $d \geq 1 + 2N$), one has

$$G_k(x) \asymp \frac{1}{(1+\|x\|)^{d-2k}}. \tag{3.3}$$

Proof of Lemma 3.1

Let $k, \ell, m \in \{1, \dots, N\}$, and for $a, b, z \in \mathbb{Z}^d$, let

$$F_{k,\ell,m}(z,a,b) = \sum_w G_k(z-b+w)G_\ell(w)G_m(a-b+w).$$

The first step towards the proof of Lemma 3.1 is the following claim.

Claim 3.2. One has

$$\sum_{y,y'} (g(y)g(y-y') + g(y')g(y-y'))F_{k,\ell,m}(z, a+y, b+y') = F_{k+1,\ell+1,m}(z,a,b) + F_{k+1,\ell,m+1}(z,a,b).$$

Proof. First of all we notice that $F_{k,\ell,m}(z,a,b) = F_{k,m,\ell}(z,b,a)$. Therefore it suffices to prove that

$$\sum_{y,y'} g(y)g(y-y')F_{k,\ell,m}(z,a+y,b+y') = F_{k+1,\ell+1,m}(z,a,b).$$

We have

$$\begin{aligned}
&\sum_{y,y'} g(y)g(y-y')F_{k,\ell,m}(z,a+y,b+y') \\
&= \sum_{y,y',w} g(y)g(y-y')G_k(z-b-y'+w)G_\ell(w)G_m(a+y-b-y'+w) \\
&= \sum_{y,u,w} g(y)g(u)G_k(z-b+u-y+w)G_\ell(w)G_m(a-b+u+w)
\end{aligned}$$

$$= \sum_{u,w} g(u)G_\ell(w)G_m(a-b+u+w)G_{k+1}(z-b+u+w)$$
$$= \sum_{u,v} g(u)G_\ell(v-u)G_m(a-b+v)G_{k+1}(z-b+v)$$
$$= \sum_{v} G_{\ell+1}(v)G_m(a-b+v)G_{k+1}(z-b+v) = F_{k+1,\ell+1,m}(z,a,b)$$

and this completes the proof. □

For $z, a, b \in \mathbb{Z}^d$, define now

$$V_N(z,a,b) = \mathbb{E}\big[\ell_{z+X^1+\cdots+X^N}(a)\ell_{z+X^1+\cdots+X^N}(b)\big] .$$

The next step is obtained by a simple induction on $N \geq 1$.

Lemma 3.3. One has for any $z, a, b \in \mathbb{Z}^d$,

$$V_N(z,a,b) \leq \; G_N(z-b)G_N(a-b) + \sum_{k=1}^{N-1} \binom{N-1}{k-1} F_{N,N-k,k}(z,a,b)$$
$$+ \; G_N(z-a)G_N(a-b) + \sum_{k=1}^{N-1} \binom{N-1}{k-1} F_{N,N-k,k}(z,b,a),$$

with the convention that the two sums are zero when $N = 1$.

Proof. Defining,

$$W_N(z,a,b) = G_N(z-b)G_N(a-b) + \sum_{k=1}^{N-1} \binom{N-1}{k-1} F_{N,N-k,k}(z,a,b),$$

the statement of the lemma then becomes

$$V_N(z,a,b) \leq \; W_N(z,a,b) + W_N(z,b,a). \tag{3.4}$$

We will prove this by induction on N. For $N = 1$ we get

$$V_N(z,a,b) \leq \mathbb{E}\Big[\sum_{s,t\in\mathbb{N}} \mathbf{1}(z+X_t = a)\cdot \mathbf{1}(z+X_{s+t} = b)\Big] + \mathbb{E}\Big[\sum_{s,t\in\mathbb{N}} \mathbf{1}(z+X_{s+t} = a)\cdot \mathbf{1}(z+X_t = b)\Big]$$
$$= g(z-a)g(a-b) + g(z-b)g(a-b),$$

and hence (3.4) holds for $N = 1$. Suppose now that (3.4) holds for N. We will establish it also for $N+1$. Summing over all the possible locations of the $(N+1)$-st walk and using the induction hypothesis gives

$$V_{N+1}(z,a,b) \;\leq\; \sum_{y,y'} (g(y)g(y-y') + g(y')g(y-y'))V_N(z,a+y,b+y')$$
$$\leq\; \sum_{y,y'} (g(y)g(y-y') + g(y')g(y-y'))\cdot \big(W_N(z,a+y,b+y') + W_N(z,b+y',a+y)\big).$$

To simplify notation we let A be the operator given by

$$Af(z,a,b) = \sum_{y,y'} (g(y)g(y-y') + g(y')g(y-y'))f(z,a+y,b+y')$$

for any function f. To prove the lemma it thus suffices to show that

$$AW_N(z,a,b) = W_{N+1}(z,a,b). \tag{3.5}$$

Note first that

$$\sum_{y,y'} g(y')g(y-y')G_N(z-b-y')G_N(a-b+y-y') = G_{N+1}(z-b)G_{N+1}(a-b),$$

which gives the first term in $W_{N+1}(z,a,b)$. Note also that

$$\sum_{y,y'} g(y)g(y-y')G_N(z-b-y')G_N(a-b+y-y') = F_{N+1,1,N}(z,a,b).$$

By Claim 3.2 we obtain

$$AF_{N,1,N-1}(z,a,b) = F_{N+1,1,N}(z,a,b) + F_{N+1,2,N-1}(z,a,b),$$

which shows that the coefficient of the term $F_{N+1,1,N}(z,a,b)$ in $AW_N(z,a,b)$ is given by $1+\binom{N-1}{N-2}$ which is equal to $\binom{N}{N-1}$. Thus the term $F_{N+1,1,N}(z,a,b)$ appears with the same coefficient in both $W_{N+1}(z,a,b)$ and $AW_N(z,a,b)$. Let $k\in\{2,\dots,N-1\}$. Using Claim 3.2 again we get that

$$AF_{N,N-k,k}(z,a,b) = F_{N+1,N+1-k,k}(z,a,b) + F_{N+1,N-k,k+1}(z,a,b) \quad \text{and}$$
$$AF_{N,N+1-k,k-1}(z,a,b) = F_{N+1,N+1-k,k}(z,a,b) + F_{N+1,N+2-k,k-1}(z,a,b).$$

We thus see that for $k\in\{2,\dots,N-1\}$ the coefficient of the term $F_{N+1,N+1-k,k}(z,a,b)$ in $AW_N(z,a,b)$ is equal to

$$\binom{N-1}{k-1} + \binom{N-1}{k-2} = \binom{N}{k-1},$$

which is the same as its coefficient in $W_{N+1}(z,a,b)$. Using Claim 3.2 for a last time we see that the term $F_{N+1,N,1}(z,a,b)$ is one of the two terms of $AF_{N,N-1,1}(z,a,b)$, and hence its coefficient in $AW_N(z,a,b)$ must be $\binom{N}{0} = \binom{N-1}{0} = 1$. Therefore, we see that the coefficients of all the terms appearing in $AW_N(z,a,b)$ and $W_{N+1}(z,a,b)$ are equal and this completes the proof of (3.5). □

Finally we shall need the following claim.

Claim 3.4. Let $N\ge 1$ and $d>2N$. There exists $C>0$, such that for all $k\in\{1,\dots,N-1\}$, and all $z,a,b\in\mathbb{Z}^d$, with $\|z\|\ge 2\max(\|a\|,\|b\|)$,

$$F_{N,N-k,k}(z,a,b) \le C\cdot G_N(z)G_N(a-b).$$

Proof. First of all note that for all ℓ,m, such that $\ell+m\le N$,

$$G_\ell * G_m = G_{\ell+m}.$$

Moreover, a change of variables gives

$$F_{N,N-k,k}(z,a,b) = \sum_{u\in\mathbb{Z}^d} G_N(u)G_{N-k}(u+b-z)G_k(u+a-z).$$

We then have for $\|z\|\ge 2\max(\|a\|,\|b\|)$, using (3.3),

$$\sum_{\|u\|\ge\|z\|/4} G_N(u)G_{N-k}(u+b-z)G_k(u+a-z)$$

$$\lesssim G_N(z)\sum_u G_{N-k}(u+b-z)G_k(u+a-z) = G_N(z)G_N(a-b).$$

On the other hand, if $\|u\| \le \|z\|/4$, since we also have $\|z\| \ge 2\max(\|a\|,\|b\|)$, one has $\|u+b-z\| \asymp \|z\|$ and $\|u+a-z\| \asymp \|z\|$. Therefore, using again (3.3),

$$\sum_{\|u\|\le\|z\|/4} G_N(u)G_{N-k}(u+b-z)G_k(u+a-z)$$
$$\lesssim G_{N-k}(z)G_k(z)\sum_{\|u\|\le\|z\|/4} G_N(u) \asymp \frac{\|z\|^{2N}}{1+\|z\|^{2d-2N}} \asymp G_N(z)^2 \lesssim G_N(z)G_N(a-b)$$

and this finishes the proof. □

Lemma 3.1 now follows from a combination of Lemma 3.3 and Claim 3.4. □

4 Hitting probabilities and capacities

In this section we give the proof of Theorems 1.2 and 1.5. We start by giving the proof of Theorem 1.2 assuming Theorem 1.5 and then give the proof of the latter for which we mainly follow the arguments of [KS99], see also [K03], which extends the approach of [FS89].

Proof of Theorem 1.2. The proof follows from the combination of four distinct observations: (i) Theorem 1.5 with $N=2$, (ii) the hitting time asymptotics for the infinite invariant tree (1.4), (iii) the fact that $\mathrm{BCap}(A) \asymp \mathrm{Cap}_{d-4}(A)$ proved in [ASS23], and finally (iv) the asymptotic (3.3) with $k=2$. □

The rest of this section is devoted to the proof of Theorem 1.5.

Let $X^1,\dots,X^N$ be i.i.d. simple random walks on $\mathbb{Z}^d$ started from 0 with ranges $\mathcal{R}^1_\infty,\dots,\mathcal{R}^N_\infty$ respectively. For $\gamma>0$, recall that we defined $k_\gamma(x,y) = (1+\|y-x\|)^{-\gamma}$, and now given a probability measure ν on A, to lighten notation we set $\mathcal{E}_\gamma(\nu) = \mathcal{E}_{k_\gamma}(\nu)$, where $\mathcal{E}_{k_\gamma}$ is defined by (1.2).

Lower Bound. It suffices to prove that if ν is a probability measure on A, then

$$\frac{\mathbb{P}\big((x+\mathcal{R}^1_\infty+\cdots+\mathcal{R}^N_\infty)\cap A \neq \emptyset\big)}{G_N(z)} \gtrsim \frac{1}{\mathcal{E}_{d-2N}(\nu)}, \tag{4.1}$$

with an implicit constant that is independent of ν. To this end, for any probability measure ν with support on A, let

$$Z_\nu = \sum_{a\in A}\nu(a)\cdot \ell_{x+X^1+\cdots+X^N}(a).$$

Then it is immediate that

$$\mathbb{P}\big((x+\mathcal{R}^1_\infty+\cdots+\mathcal{R}^N_\infty)\cap A\neq\emptyset\big) \ge \mathbb{P}(Z_\nu>0) \ge \frac{(\mathbb{E}[Z_\nu])^2}{\mathbb{E}[Z_\nu^2]}, \tag{4.2}$$

where for the last inequality we used the Paley-Zygmund inequality. For the first moment of Z_ν, we have for any x with $\|x\| \geq 2 \cdot \mathrm{diam}(A)$, using (3.3),

$$\mathbb{E}[Z_\nu] = \sum_{a \in A} \nu(a) G_N(x-a) \gtrsim G_N(x).$$

For the second moment, by Lemma 3.1 we have for x with $\|x\| \geq 2 \cdot \mathrm{diam}(A)$,

$$\mathbb{E}\big[Z_\nu^2\big] = \sum_{a,b \in A} \nu(a)\nu(b) V_N(x,a,b) \lesssim G_N(x) \sum_{a,b\in A} \nu(a)\nu(b) G_N(a-b) \lesssim G_N(x) \mathcal{E}_{d-2N}(\nu). \tag{4.3}$$

Plugging these two bounds into (4.2) yields (4.1).

Upper Bound. We define N random times, which are not stopping times. First,

$$T_1 = \inf\{t_1 \geq 0 : \exists\ t_2, \dots, t_N \ \text{ s.t. } x + X^1_{t_1} + \ldots + X^N_{t_N} \in A\},$$

and then inductively for $i = 2, \dots, N$,

$$T_i = \inf\{t_i \geq 0 : \exists\ t_{i+1}, \dots, t_N \ \text{ s.t. } x + X^1_{T_1} + \ldots + X^{i-1}_{T_{i-1}} + X^i_{t_i} + \ldots + X^N_{t_N} \in A\}.$$

We next define a probability measure μ on A by setting for $a \in A$

$$\mu(a) = \mathbb{P}\big(x + X^1_{T_1} + \ldots + X^N_{T_N} = a \bigm| T_1 < \infty\big)\,.$$

It then suffices to prove that for $\|x\| \geq 2 \cdot \mathrm{diam}(A)$,

$$\frac{\mathbb{P}(T_1 < \infty)}{G_N(x)} \lesssim \frac{1}{\mathcal{E}_{d-2N}(\mu)}. \tag{4.4}$$

To this end we define the variable $Z_\mu = \sum_{a\in A} \mu(a) \ell_{x+X^1+\dots+X^N}(a)$, and if $(\mathcal{F}^i_n)_{n\geq 0}$ stands for the natural filtration of the walk X^i, then we define the multi-parameter process

$$\begin{aligned} &M(t_1,\dots,t_N) = \mathbb{E}\big[Z_\mu \bigm| \mathcal{F}^1_{t_1} \vee \ldots \vee \mathcal{F}^N_{t_N}\big] \\ &= \sum_{a\in A} \mu(a) \sum_{s_1,\dots,s_N} \sum_{x_1,\dots,x_{N-1}} \mathbb{P}\big(X^1_{s_1} = x_1 \bigm| \mathcal{F}^1_{t_1}\big) \cdots \mathbb{P}\big(X^N_{s_N} = a - x - x_1 - \ldots - x_{N-1} \bigm| \mathcal{F}^N_{t_N}\big)\,. \end{aligned} \tag{4.5}$$

We then have almost surely for all $t_1, \dots, t_N \in \mathbb{N}$

$$\begin{aligned} &M(t_1,\dots,t_N) \\ &\geq \sum_{a\in A} \mu(a) \sum_{s_1 \geq t_1, \dots, s_N \geq t_N} \sum_{x_1,\dots,x_{N-1} \in \mathbb{Z}^d} \mathbb{P}\big(X^1_{s_1} = x_1 \bigm| \mathcal{F}^1_{t_1}\big) \cdots \mathbb{P}\big(X^N_{s_N} = a - x - x_1 - \ldots - x_{N-1} \bigm| \mathcal{F}^N_{t_N}\big) \\ &= \sum_{a\in A} \mu(a) \sum_{x_1,\dots,x_{N-1}\in\mathbb{Z}^d} g(X^1_{t_1} - x_1) \cdots g(X^{N-1}_{t_{N-1}} - x_{N-1}) g(x + x_1 + \ldots + x_{N-1} + X^N_{t_N} - a) \\ &= \sum_{a\in A} \mu(a) G_N(x + X^1_{t_1} + \ldots + X^N_{t_N} - a). \end{aligned}$$

Therefore, almost surely we get

$$\sup_{t_1,\dots,t_N} M(t_1,\dots,t_N) \ \geq\ \mathbf{1}(T_1 < \infty) \sum_{a\in A} \mu(a) G_N(x + X^1_{T_1} + \ldots + X^N_{T_N} - a),$$

and hence squaring both sides and taking expectations we obtain

$$\mathbb{E}\Big[\sup_{t_1,\dots,t_N} M^2(t_1,\dots,t_N)\Big] \geq \mathbb{E}\left[\left(\sum_{a\in A}\mu(a)G_N(x+X^1_{T_1}+\ldots+X^N_{T_N}-a)\right)^2 \,\middle|\, T_1<\infty\right]\cdot\mathbb{P}(T_1<\infty)$$
$$=\sum_{b\in A}\mu(b)\left(\sum_{a\in A}\mu(a)G_N(b-a)\right)^2\cdot\mathbb{P}(T_1<\infty)\gtrsim(\mathcal{E}_{d-2N}(\mu))^2\cdot\mathbb{P}(T_1<\infty)\,,$$

where in the last step we used the Cauchy-Schwarz inequality and (3.3).

By monotone convergence it then suffices to prove that for any $u_1,\dots,u_N$,

$$\mathbb{E}\left[\sup_{t_1\leq u_1,\dots,t_N\leq u_N} M^2(t_1,\dots,t_N)\right]\lesssim G_N(z)\mathcal{E}_{d-2N}(\mu), \tag{4.6}$$

with an implicit constant that is independent of $u_1,\dots,u_N$. This together with the inequality above would conclude the proof of (4.4).

Using the product expression for M from (4.5), it is easy to check that the process

$$t_1\mapsto\sup_{t_2\leq u_2,\dots,t_N\leq u_N} M(t_1,t_2,\dots,t_N),$$

is a non-negative submartingale with respect to the filtration $(\mathcal{F}^1_{t_1}\vee\mathcal{F}^2_{u_2}\vee\dots\vee\mathcal{F}^N_{u_N})_{t_1\geq 0}$. Applying Doob's L^2-inequality we then deduce

$$\mathbb{E}\left[\sup_{t_1\leq u_1}\left(\sup_{t_2\leq u_2,\dots,t_N\leq u_N} M^2(t_1,t_2,\dots,t_N)\right)\right]\leq 4\cdot\mathbb{E}\left[\sup_{t_2\leq u_2,\dots,t_N\leq u_N} M^2(u_1,t_2,\dots,t_N)\right].$$

Repeating the same argument N times, gives

$$\mathbb{E}\Big[\sup_{t_1\leq u_1,\dots,t_N\leq u_N} M^2(t_1,\dots,t_N)\Big]\leq 4^N\cdot\mathbb{E}\big[M^2(u_1,\dots,u_N)\big]\leq 4^N\cdot\mathbb{E}[Z_\mu^2],$$

where for the final inequality we used Jensen's inequality. By (4.3) for x with $\|x\|\geq 2\cdot\mathrm{diam}(A)$, we get

$$\mathbb{E}\big[Z_\mu^2\big]\lesssim G_N(z)\mathcal{E}_{d-2N}(\mu).$$

Altogether this proves (4.6) and thus completes the proof.

5 Hitting probabilities and ergodic limits

In this section, we prove Proposition 1.6. The proof is divided in two parts. First we prove (1.14) in Section 5.1, which is the easiest direction, and then (1.15) in Section 5.2, which is slightly more demanding. We note that (1.15) follows in fact from a combination of Theorem 1.5 and Lemma 2.2, but we provide here another direct proof, as it might be of independent interest.

5.1 Proof of (1.14).

Let $N \geq 2$, and assume that $d \geq 2N+1$. Recall the definition of the functions G_N from the beginning of Section 3. Since $\mathcal{R}_\infty$ and $-\mathcal{R}_\infty$ have the same law, it amounts to proving that

$$\liminf_{\|z\|\to\infty} \frac{\mathbb{P}\big((z+\mathcal{R}_\infty^1+\dots+\mathcal{R}_\infty^{N-1})\cap(\mathcal{R}_\infty+A)\neq\emptyset\big)}{G_N(z)} \gtrsim \widehat{f}_N(A),$$

where $\mathcal{R}_\infty$ is the range of a random walk $(X_k)_{k\geq 0}$, which is independent of $\mathcal{R}_\infty^1,\dots,\mathcal{R}_\infty^{N-1}$, and

$$\widehat{f}_N(A) = \lim_{n\to\infty} \frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n^1+A)}{n}.$$

(The existence of the limit here follows from Lemma 2.2.) Let $\varepsilon>0$ and let $\tau_r = \inf\{k\geq 0 : X_k \notin B(0,r)\}$. Then we get for $\|z\|$ large enough, using Theorem 1.5 for the second inequality,

$$\begin{aligned}
&\frac{\mathbb{P}\big((z+\mathcal{R}_\infty^1+\dots+\mathcal{R}_\infty^{N-1})\cap(\mathcal{R}_\infty+A)\neq\emptyset\big)}{G_N(z)}\\
&\geq \frac{\mathbb{P}\big((z+\mathcal{R}_\infty^1+\dots+\mathcal{R}_\infty^{N-1})\cap(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\neq\emptyset, \tau_{\|z\|/2}>\varepsilon\|z\|^2\big)}{G_N(z)}\\
&\gtrsim \frac{G_{N-1}(z)}{G_N(z)}\cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\cdot \mathbf{1}(\tau_{\|z\|/2}>\varepsilon\|z\|^2)\Big]\\
&\gtrsim \frac{1}{\|z\|^2}\cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\Big]\\
&\quad - \frac{1}{\|z\|^2}\cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\cdot \mathbf{1}(\tau_{\|z\|/2}\leq\varepsilon\|z\|^2)\Big].
\end{aligned}$$

As $\|z\|\to\infty$ we have that

$$\frac{1}{\|z\|^2}\cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\Big] \to \varepsilon \widehat{f}_N(A).$$

Indeed, the convergence holds in L^1 since the sequence $(\frac{\mathrm{Cap}_\gamma(\mathcal{R}_n+A)}{n})_n$ is uniformly bounded by some deterministic constant, for any $\gamma>0$. Furthermore, by Cauchy-Schwarz we get

$$\begin{aligned}
&\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\cdot \mathbf{1}(\tau_{\|z\|/2}\leq\varepsilon\|z\|^2)\Big]\\
&\qquad\leq \sqrt{\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)^2\Big]\,\mathbb{P}\big(\tau_{\|z\|/2}\leq\varepsilon\|z\|^2\big)}.
\end{aligned}$$

By a standard random walk estimate we get for a positive constant c that

$$\mathbb{P}\big(\tau_{\|z\|/2}\leq\varepsilon\|z\|^2\big)\leq \exp(-c/\varepsilon).$$

Using again that $(\frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n+A)}{n})_n$ is bounded we get that it also converges to $\widehat{f}_N(A)$ in L^2. Hence this gives for $\|z\|$ sufficiently large

$$\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)^2\Big]\leq 2\|z\|^4\cdot \widehat{f}_N(A)^2,$$

using also that $\widehat{f}_N(A)$ is positive (since $d\geq 2N+1$). Therefore we get

$$\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\varepsilon\|z\|^2]+A)\cdot \mathbf{1}(\tau_{\|z\|/2}\leq\varepsilon\|z\|^2)\Big]\lesssim \|z\|^2\widehat{f}_N(A)\exp(-c/(2\varepsilon)).$$

Putting everything together now gives that for $\|z\|$ sufficiently large

$$\frac{\mathbb{P}\big((z+\mathcal{R}_\infty^1+\dots+\mathcal{R}_\infty^{N-1})\cap(\mathcal{R}_\infty+A)\neq\emptyset\big)}{G_N(z)}\gtrsim \varepsilon\widehat{f}_N(A)-\exp(-c/(2\varepsilon))\widehat{f}_N(A)\gtrsim \widehat{f}_N(A),$$

by taking ε sufficiently small. This finishes the proof. □

5.2 Proof of (1.15).

We use here the same notation as in Section 5.1. We define for $i \in \mathbb{Z}$, $r_i = 2^i\|z\|$, and let I be the maximal index such that $r_{-I} \geq 4\,\mathrm{diam}(A)$. Now for $i \geq -I$, define

$$\mathcal{B}_i = \partial B(z, r_{i+1}) \cup \partial B(z, r_{i-1}).$$

and for $i \geq -I$, and $k \geq 0$, let

$$\tau_i^k = \inf\Big\{n \geq \sigma_i^{k-1} : X_n \in \partial B(z, r_i)\Big\}, \quad \text{and} \quad \sigma_i^k = \inf\Big\{n \geq \tau_i^k : X_n \in \mathcal{B}_i\Big\},$$

with the convention $\sigma_i^{-1} = 0$. To simplify notation we will also write $\tau_i = \tau_i^0$ and $\sigma_i = \sigma_i^0$, for $i \geq -I$. Note that by definition one has $\tau_0 = 0$. Then let for $i \geq -I$,

$$\mathcal{R}_{(i)} = \bigcup_{k\geq 0} \mathcal{R}[\tau_i^k, \sigma_i^k].$$

Observe that on the event $\{\tau_{-I} = \infty\}$, one has

$$\mathcal{R}_\infty = \bigcup_{i\geq -I} \mathcal{R}_{(i)}.$$

Note that, at least if $\|z\|$ is large enough, one has for $i \geq I$, $\mathcal{R}_{(i)} + A \subset B(z, r_{i+2}) \setminus B(z, r_{i-2})$. In particular it is always possible to split $\mathcal{R}_{(i)} + A$ into a finite number (independent of z and A) of pieces with diameter at most $r_i/8$ each. Then using Theorem 1.5 and a union bound, we obtain

$$\mathbb{P}\big((z + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^{N-1}) \cap (\mathcal{R}_{(i)} + A) \neq \emptyset\big) \lesssim G_{N-1}(r_i) \cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_{(i)} + A)\Big].$$

Using another union bound and the above we then get, with $g(r) = r^{2-d}$ for $r > 0$,

$$\begin{aligned}
&\mathbb{P}\big((z + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^{N-1}) \cap (\mathcal{R}_\infty + A) \neq \emptyset\big) \\
&\leq \mathbb{P}(\tau_{-I} < \infty) + \sum_{i=-I}^{\infty} \mathbb{P}\big((z + \mathcal{R}_\infty^1 + \cdots + \mathcal{R}_\infty^{N-1}) \cap (\mathcal{R}_{(i)} + A) \neq \emptyset\big) \\
&\lesssim \frac{g(z)}{g(\mathrm{diam}(A))} + \sum_{i=-I}^{\infty} G_{N-1}(r_i) \cdot \mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_{(i)} + A)\Big], \qquad (5.1)
\end{aligned}$$

where for the last inequality, we used the well-known fact that for a simple random walk starting from z, the probability to hit a ball $B(0, r)$, with $r < \|z\|/2$, is of order $g(z)/g(r)$. Now one has for any $-I \leq i < 0$, using the transience of the walk, and recalling that $\mathbb{E}_x$ denotes the expectation with respect to the law of a simple random walk starting from x,

$$\begin{aligned}
\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_{(i)} + A)\Big] &\leq \mathbb{P}(\tau_i < \infty) \cdot \sup_{x\in\partial B(z,r_i)} \mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_{(i)} + A)\Big] \\
&\lesssim \frac{g(z)}{g(r_i)} \cdot \sup_{x\in\partial B(z,r_i)} \Big(\sum_{k\geq 0} \mathbb{P}_x(\tau_i^k < \infty)\Big) \cdot \mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0, \sigma_i] + A)\Big] \\
&\lesssim \frac{g(z)}{g(r_i)} \cdot \sup_{x\in\partial B(z,r_i)} \mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0, \sigma_i] + A)\Big]. \qquad (5.2)
\end{aligned}$$

Likewise for any $i \geq 0$, one has

$$\mathbb{E}\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_{(i)} + A)\Big] \lesssim \sup_{x\in\partial B(z,r_i)} \mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0, \sigma_i] + A)\Big]. \qquad (5.3)$$

Now we claim that for any $i \geq -I$, one has

$$\sup_{x\in\partial B(z,r_i)} \mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\sigma_i]+A)\Big] \lesssim \widehat{f}_N(A)\cdot r_i^2. \tag{5.4}$$

Let us postpone the proof of the claim and conclude the proof of (1.15). Plugging (5.4) into (5.2) and (5.3), and using (5.1) we get using that $d \geq 1+2N$,

$$\mathbb{P}\big((z+\mathcal{R}^1_\infty+\dots+\mathcal{R}^{N-1}_\infty)\cap(\mathcal{R}_\infty+A)\neq\emptyset\big) \lesssim \frac{g(z)}{g(\mathrm{diam}(A))} + \widehat{f}_N(A)\cdot G_N(z).$$

Dividing both sides by $G_N(z)$, and letting $\|z\|\to\infty$ concludes the proof of (1.15).

Thus it only remains to prove the claim (5.4). For this one can just write, using monotonicity of γ-capacities, for any $x\in\partial B(z,r_i)$, and some constant $c>0$,

$$\begin{aligned}
\mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\sigma_i]+A)\Big] &= \sum_{k\geq 0}\mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,\sigma_i]+A)\cdot \mathbf{1}(kr_i^2\leq\sigma_i<(k+1)r_i^2)\Big]\\
&\lesssim \sum_{k\geq 0}\mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,(k+1)r_i^2]+A)\cdot\mathbf{1}(\sigma_i\geq kr_i^2)\Big]\\
&\lesssim \sum_{k\geq 0}\mathbb{E}_x\Big[\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}[0,(k+1)r_i^2]+A)^2\Big]^{1/2}\cdot\mathbb{P}(\sigma_i\geq kr_i^2)^{1/2}\\
&\lesssim \widehat{f}_N(A)\cdot\sum_{k\geq 0} kr_i^2\cdot\exp(-ck)\lesssim \widehat{f}_N(A)\cdot r_i^2,
\end{aligned}$$

as wanted, where for the penultimate bound we used again that since $(\frac{\mathrm{Cap}_{d-2(N-1)}(\mathcal{R}_n+A)}{n})_n$ is bounded, it also converges to $\widehat{f}_N(A)$ in L^2. □

6 Capacity of the sausage in $d=4$

In this section we prove Proposition 1.7. Recall that we assume here that $d=4$. Following the notation of [L91], we set $a_4 = 2/\pi^2$.

Claim 6.1. Fix $M>0$. There exists a positive constant c and n_0 so that for all $n\geq n_0$, if ξ is a geometric random variable of parameter $1/n$, then for all x with $\|x\|\leq M$ we have for all $\varepsilon\in(0,1)$,

$$\mathbb{P}\left(\left|\sum_{k=0}^{\xi} g(X_k-x)-2a_4(\log n)\right|\geq\varepsilon\log n\right)\leq\frac{c}{\varepsilon^2\log n}.$$

Proof. Let

$$\Delta=\sum_{k=0}^{\xi}\big(g(X_k-x)-g(X_k)\big).$$

By the gradient estimate for Green's function (see e.g. Theorem 1.5.5 in [L91]), one has

$$|g(X_k-x)-g(X_k)|\lesssim\frac{1}{1+\|X_k\|^3}.$$

Furthermore, a standard computation gives

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \frac{1}{1+\|X_k\|^3}\right] = \sum_{x\in\mathbb{Z}^4} \frac{g(x)}{1+\|x\|^3} < \infty.$$

Therefore Markov's inequality gives that

$$\mathbb{P}\big(|\Delta| \geq \frac{\varepsilon}{2}\log n\big) \lesssim \frac{1}{\varepsilon \log n}.$$

To conclude the proof we use the concentration results proved by Lawler. Indeed, he shows in Lemma 4.2.1 in [L91] that

$$\mathbb{P}\left(\left|\sum_{k=0}^{\xi} g(X_k) - 2a_4(\log n)\right| \geq \frac{\varepsilon}{2}\log n\right) \lesssim \frac{1}{\varepsilon^2 \log n}.$$

□

We now introduce some notation. Fix a set A and let X be a simple random walk starting from the origin with range $\mathcal{R}$. Let $\widetilde{\mathcal{R}}$ be an independent copy of $\mathcal{R}$. Let ξ_n^ℓ and ξ_n^r be two independent geometric random variables of parameter $1/n$. For every $x \in A$ we set

$$\begin{aligned}
\mathcal{A}_n^x &= \big\{(x+\widetilde{\mathcal{R}}[1,\infty)) \cap (\mathcal{R}[-\xi_n^\ell, \xi_n^r] + A) = \emptyset\big\} \\
e_n^x &= \mathbf{1}(x \notin (\mathcal{R}[1,\xi_n^r] + A)) \\
\mathcal{U}_n^x &= \sum_{y\in A} \sum_{-\xi_n^\ell \leq k \leq \xi_n^r} g(x, X_k + y).
\end{aligned}$$

The next lemma is an extension of a beautiful identity found by Lawler [L91, Theorem 3.6.1], which corresponds to the case $A = \{0\}$. Here we mainly follow the notation and presentation of [BW22, Lemma 2.12].

Lemma 6.2. We have

$$\sum_{x\in A} \mathbb{E}[\mathbf{1}(\mathcal{A}_n^x) \cdot e_n^x \cdot \mathcal{U}_n^x] = |A|.$$

Proof. For every nearest neighbour path $(x_1, \ldots, x_m)$ we define

$$B(m, x_1, \ldots, x_m) = \{\xi_n^\ell + \xi_n^r = m,\ X_{-\xi_n^\ell + k} - X_{-\xi_n^\ell} = x_k,\ \forall\, 1 \leq k \leq m\},$$

and for all $0 \leq j \leq m$ we define

$$B(m, j, x_1, \ldots, x_m) = \{\xi_n^\ell = j,\ \xi_n^r = m - j,\ X_{-\xi_n^\ell + k} - X_{-\xi_n^\ell} = x_k,\ \forall\, 1 \leq k \leq m\}.$$

Using the independence of the increments of the walk and the geometric random variables we then obtain

$$\mathbb{P}(B(m, j, x_1, \ldots, x_m) \mid B(m, x_1, \ldots, x_m)) = \frac{1}{m+1}.$$

Setting $x_0 = 0$, we then have

$$\sum_{x\in A} \mathbb{E}[\mathbf{1}(\mathcal{A}_n^x) \cdot e_n^x \cdot \mathcal{U}_n^x]$$

$$= \sum_{x\in A}\sum_{m=0}^{\infty}\sum_{(x_1,\dots,x_m)} \frac{\mathbb{P}(B(m,x_1,\dots,x_m))}{m+1}\cdot \sum_{k=0}^{m}\sum_{j=0}^{m} \mathbf{1}(x+x_j \notin (\{x_{j+1},\dots,x_m\}+A))$$
$$\times \mathbb{P}\Big((x+x_j+\widetilde{\mathcal{R}}[1,\infty))\cap(\{x_0,x_1,\dots,x_m\}+A)=\emptyset\Big)\sum_{y\in A} g(x+x_j-x_k-y).$$

Recall that the last exit decomposition formula [L91, Proposition 2.4.1(d)] entails that for any finite set $B\subset\mathbb{Z}^d$, and any $b\in B$,

$$1=\sum_{b'\in B} g(b-b')\cdot \mathbb{P}\big((b'+\widetilde{\mathcal{R}}[1,\infty))\cap B=\emptyset\big).$$

Applying this to the set $B=\{x_0,\dots,x_m\}+A$, and $b=x_k+y$, with $y\in A$ and $k\in\{0,\dots,m\}$ fixed, we get

$$1=\sum_{x\in A}\sum_{j=0}^{m}\mathbf{1}(x+x_j\notin(\{x_{j+1},\dots,x_m\}+A))$$
$$\times\mathbb{P}((x+x_j+\widetilde{\mathcal{R}}[1,\infty))\cap(\{x_0,x_1,\dots,x_m\}+A)=\emptyset)\cdot g(x+x_j-x_k-y).$$

Substituting this above we obtain

$$\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x\cdot\mathcal{U}_n^x]=\sum_{m=0}^{\infty}\sum_{(x_1,\dots,x_m)}\mathbb{P}(B(m,x_1,\dots,x_m))\cdot|A|=|A|,$$

and this concludes the proof. □

Lemma 6.3. We have

$$\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x]=(1+o(1))\cdot\frac{|A|}{4a_4\log n}.$$

Proof. We have from Lemma 6.2 that

$$\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x\cdot\mathcal{U}_n^x]=|A|.$$

We now get

$$\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x]=\frac{|A|}{4a_4\log n}+\frac{1}{4a_4\log n}\cdot\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x\cdot(4a_4\log n-\mathcal{U}_n^x)]\,.$$

For every $x\in A$ and $\varepsilon>0$ we let

$$\mathcal{B}_n^x=\{|\mathcal{U}_n^x-\mathbb{E}[\mathcal{U}_n^x]\,|\geq\varepsilon\log n\}.$$

Then we have

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x\cdot|\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x|]\leq\varepsilon\log n\cdot\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x]+\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot|\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x|\cdot\mathbf{1}(\mathcal{B}_n^x)]\,.$$

We now explain that it suffices to prove that

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot|\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x|\cdot\mathbf{1}(\mathcal{B}_n^x)]\lesssim\frac{1}{(\log n)^{1/4}}. \tag{6.1}$$

Indeed, once this is established, then we get

$$\left|\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x]-\frac{|A|}{4a_4(\log n)}\right|\le\varepsilon\sum_{x\in A}\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot e_n^x]+\mathcal{O}\left(\frac{1}{(\log n)^{5/4}}\right),$$

and since this holds for any $\varepsilon>0$, this concludes the proof. So we now turn to prove (6.1). By the Cauchy-Schwarz inequality we obtain

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot|\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x|\cdot\mathbf{1}(\mathcal{B}_n^x)]\le\sqrt{\mathbb{P}(\mathcal{A}_n^x\cap\mathcal{B}_n^x)\cdot\mathbb{E}[(\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x)^2]}\le\sqrt{\mathbb{P}(\mathcal{A}_n^x\cap\mathcal{B}_n^x)\cdot\log n}$$

using Claim 6.1 for the last inequality. It remains to bound the last probability appearing above. To do this we define

$$\mathcal{U}_n^{x,1}=\sum_{k=-\xi_n^\ell}^{0}g(x,X_k)\quad\text{and}\quad\mathcal{U}_n^{x,2}=\sum_{k=0}^{\xi_n^r}g(x,X_k),$$

and also two events for $i=1,2$

$$\mathcal{B}_n^{x,i}=\{|\mathcal{U}_n^{x,i}-2a_4\log n|\ge\varepsilon\log n/4\}.$$

Then it is clear that $\mathcal{B}_n^x\subseteq\mathcal{B}_n^{x,1}\cup\mathcal{B}_n^{x,2}$, at least for n large enough, and we have

$$\begin{aligned}\mathbb{P}(\mathcal{A}_n^x\cap\mathcal{B}_n^x)&\le\mathbb{P}\Big((x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-\xi_n^\ell,0]+A)=\emptyset,\mathcal{B}_n^{x,2}\Big)\\&\quad+\mathbb{P}\Big((x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[0,\xi_n^r]+A)=\emptyset,\mathcal{B}_n^{x,1}\Big)\\&=2\mathbb{P}\Big((x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-\xi_n^\ell,0]+A)=\emptyset\Big)\,\mathbb{P}\big(\mathcal{B}_n^{x,2}\big).\end{aligned}$$

Since $x\in A$ we get

$$\mathbb{P}\Big((x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-\xi_n^\ell,0]+A)=\emptyset\Big)\le\mathbb{P}\Big(\widetilde{\mathcal{R}}[1,\infty)\cap\mathcal{R}[-\xi_n^\ell,0]=\emptyset\Big)\lesssim\frac{1}{\sqrt{\log n}},$$

using Corollary 3.7.1 in [L91] for the last inequality. Moreover, by Claim 6.1 we get that

$$\mathbb{P}\big(\mathcal{B}_n^{x,2}\big)\lesssim\frac{1}{\log n},$$

and hence altogether this gives

$$\mathbb{E}[\mathbf{1}(\mathcal{A}_n^x)\cdot|\mathbb{E}[\mathcal{U}_n^x]-\mathcal{U}_n^x|\cdot\mathbf{1}(\mathcal{B}_n^x)]\lesssim\frac{1}{(\log n)^{1/4}},$$

concluding the proof. □

Proof of Proposition 1.7. Recall that for a finite set $B\subset\mathbb{Z}^d$, one has $\mathrm{cap}(B)=\sum_{b\in B}\mathbb{P}((b+\mathcal{R}[1,\infty))\cap B=\emptyset)$, see [L91, Section 2.2]. We thus have

$$\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n+A)]=\sum_{x\in A}\sum_{j=0}^{n}\mathbb{P}\Big(x\notin(\mathcal{R}[1,n-j]+A),(x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-j,n-j]+A)=\emptyset\Big).$$

We then get the following bounds for $m=n/(\log n)^2$

$$\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n+A)]\ge n\cdot\sum_{x\in A}\mathbb{P}\Big(x\notin(\mathcal{R}[1,n]+A),(x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-n,n]+A)=\emptyset\Big)\ \text{and}$$

$$\mathbb{E}[\mathrm{Cap}(\mathcal{R}_n+A)]\le m\cdot|A|+(n-m)\cdot\sum_{x\in A}\mathbb{P}\Big(x\notin(\mathcal{R}[1,m]+A),(x+\widetilde{\mathcal{R}}[1,\infty))\cap(\mathcal{R}[-m,m]+A)=\emptyset\Big),$$

and we conclude using Lemma 6.3. □

Remark 6.4. As mentioned in the introduction, it would be possible to strengthen the result of the proposition by proving an almost sure convergence, following the arguments of [ASS19]. The idea is to obtain good bounds on the variance of $\mathrm{Cap}(\mathcal{R}_n+A)$, and then use Chebyshev's inequality and the Borel-Cantelli lemma. The variance estimates are based on the fact that the capacity is almost an additive functional, with an error term (called cross-term in [ASS19]) having small second moment. This approach is robust and works as well when we consider $\mathcal{R}_n + A$ for a finite set A.

7 Open problems

We discuss some open problems related to our present analysis.

General γ-capacity. It would be interesting to show that for any $\gamma \in (2, d)$, one has

$$\lim_{n\to\infty} \frac{\mathrm{Cap}_\gamma(\mathcal{R}_n + A)}{n} \asymp \mathrm{Cap}_{\gamma-2}(A).$$

The results of this paper show that it is true when $\gamma = d - 2k$, for some integer k, and Lemma 2.2 proves one direction for general $\gamma > 2$, but the other direction is missing.

Hitting Times. In Theorem 1.5 we established that the probability that a sum of N simple random walks started from z hits a finite set A is of order $G_N(z) \cdot \mathrm{Cap}_{d-2N}(A)$, when $d \geq 2N+1$ and $\|z\| \to \infty$. A natural question is whether the quantity

$$\frac{1}{G_N(z)}\mathbb{P}\big(z + \mathcal{R}^1_\infty + \cdots + \mathcal{R}^N_\infty \cap A \neq \emptyset\big)$$

has a limit as $\|z\| \to \infty$.

Another natural question is whether the analogue of Theorem 1.5 holds for the sum of invariant trees. More precisely, when $d \geq 4N+1$ and A is a finite subset of $\mathbb{Z}^d$, is the quantity

$$\frac{1}{G_{2N}(z)}\mathbb{P}\big(z + \mathcal{T}^1_\infty + \cdots + \mathcal{T}^N_\infty \cap A \neq \emptyset\big)$$

of order $\mathrm{Cap}_{d-4N}(A)$ as $\|z\| \to \infty$? One difficulty here will be to be able to define hitting times of the set A for which we can decouple future and past for each invariant tree.

Tails of local times. In Theorem 1.2 we stated an intersection equivalence between the sum of two simple random walks and an infinite invariant tree. However, the equivalence between these two processes is not expected to hold beyond hitting probabilities. Obtaining tails for the local times of additive walks is an open problem. We expect the local times of the sum of two walks to decay as a stretched exponential when $d \geq 5$ as opposed to an exponential decay in the case of the invariant tree (see [ASS23, Theorem 1.6]).

Critical models. There is a range of critical models for which several questions arise: the Minkowski sum of two simple random walks in $d = 4$, the Minkowski sum of 3 walks in $d = 6$ and so on. Interesting questions include:

(i) The fluctuations of the capacity of a sausage obtained as we roll a finite set over the trajectory of the process.

(ii) The tail of the local times, where we expect a stretched exponential tail. It would be interesting to have a representation of the rate function.

(iii) The *folding phenomenon* for additive walks or trees, and the first estimates we need is an upper bound on the probability to cover a region up to a certain density (measured in a certain space-scale). A typical example of such a folding phenomenon is the event of having a large intersection between two invariant trees in dimension $d \geq 9$, and the approach should follow the analogous problem of intersection of two random walks in $d \geq 5$ studied recently in [AS21].

References

[AK81] M. A. Akcoglu, U. Krengel. Ergodic theorems for superadditive processes. J. Reine Angew. Math. 323 (1981), 53–67.

[Ald91] D. Aldous. Asymptotic fringe distributions for general families of random trees. Ann. Appl. Probab. 1 (1991), 228–266.

[AS21] A. Asselah,; B. Schapira, Large deviations for intersections of random walks. Comm. Pure Appl. Math. 76 (2023), no. 8, 1531–1553.

[ASS19] A. Asselah, B. Schapira, P. Sousi. Capacity of the range of random walk on $\mathbb{Z}^4$. Ann. Probab. 47 (2019), 1447–1497.

[ASS23] A. Asselah, B. Schapira, P. Sousi. Local times and capacity for transient branching random walks. arXiv:2303.17572.

[BW22] T. Bai, Y. Wan. Capacity of the range of tree-indexed random walk. Ann. Appl. Probab. 32 (2022), 1557–1589.

[BPP95] I. Benjamini, R. Pemantle, Y. Peres. Martin capacity for Markov chains. Ann. Probab. 23 (1995), 1332–1346.

[FS89] P. J. Fitzsimmons, T. S. Salisbury. Capacity and energy for multiparameter Markov processes. Ann. Instit. H. Poinc. Probab. Stat. 25 (1989), 325–350.

[IMcK74] K. Ito, H. P. McKean, Jr. Diffusion processes and their sample paths. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[JO69] N. Jain, S. Orey. On the range of random walk. Israel J. Math. 6 (1968), 373–380 (1969).

[K03] D. Khoshnevisan. Intersections of Brownian motions. Expo. Math. 21 (2003), 97–114.

[KS99] D. Khoshnevisan, Z. Shi. Brownian sheet and capacity. Ann. Probab. 27 (1999), 1135–1159.

[K73] J. F. C. Kingman. Subadditive ergodic theory. Ann. Probability 1 (1973), 883–909.

[L91] G. F. Lawler. Intersections of random walks. Second edition, Birkhauser, 1996.

[LG90] J.-F. Le Gall. Wiener sausage and self-intersection local times. J. Funct. Anal. 88 (1990), 299–341.

[LGL16] J.-F. Le Gall, S. Lin. The range of tree-indexed random walk. J. Inst. Math. Jussieu 15 (2016), 271–317.

[P96] Y. Peres. Intersection-equivalence of Brownian paths and certain branching processes. Comm. Math. Phys. 177 (1996), 417–434.

[Port65] S. C. Port. Limit theorems involving capacities for recurrent Markov chains. J. Math. Anal. Appl. 12 (1965), 555–569.

[S96] T. S. Salisbury. Energy, and intersections of Markov chains. Random discrete structures (Minneapolis, MN, 1993), 213–225, IMA Vol. Math. Appl., 76, Springer, New York, 1996.

[Sp64] F. Spitzer. Electrostatic capacity, heat flow, and Brownian motion. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 3 (1964), 110–121.

[Sp73] F. Spitzer. Discussion on Professor Kingman's Paper. Ann. Probab. 1 (1973), 900–909.

[Zhu16] Q. Zhu. On the critical branching random walk I: branching capacity and visiting probability, arXiv:1611.10324,

[Zhu18] Q. Zhu. Branching interlacements and tree-indexed random walks in tori. arXiv:1812.10858.