MULTISCALE MODEL. SIMUL.
Vol. 7, No. 3, pp. 1348–1386

TIME-REVERSAL SUPERRESOLUTION IN RANDOM WAVEGUIDES*

CHRISTOPHE GOMEZ†

Abstract. In this paper we analyze a time-reversal experiment in a random waveguide. We use asymptotic analysis based on a separation of scales technique. We derive an infinite-dimensional coupled power equation that we analyze in the high-frequency regime. We use this approximation to compute the transverse profile of the refocused field and show that randomness enhances spatial refocusing beyond the diffraction limit; that is, the focal spot is smaller than the carrier wavelength over two. In this experiment the random heterogeneities, in the near-field region of the source, play a primary role.

Key words. acoustic waveguides, random media, asymptotic analysis

AMS subject classifications. 76B15, 35Q99, 60F05

DOI. 10.1137/080719492

Introduction. Time-reversal experiments have been intensively analyzed experimentally and theoretically. This experiment is in two steps. In the first step, a source sends a pulse into a medium. The wave propagates and is recorded by a device called a time-reversal mirror. A time-reversal mirror is a device that can receive a signal, record it, and resend it time-reversed into the medium. In the second step, the wave emitted by the time-reversal mirror has the property of refocusing near the original source location, and it has been observed that random inhomogeneities enhance refocusing [6, 7]. Time-reversal refocusing in one-dimensional media has been studied in [5, 9]. In three-dimensional randomly layered media [10], in the paraxial approximation [4, 3, 18], and in random waveguides [11, 9], it has been shown that the focal spot can be smaller than the Rayleigh resolution formula $\lambda L/D$ (where λ is the carrier wavelength, L is the propagation distance, and D is the mirror diameter). However, the focal spot is still larger than the diffraction limit $\lambda/2$.

Mathias Fink and his group at ESPCI have proposed an approach to obtaining a *superresolution* effect, that is to refocus beyond the diffraction limit, with a far-field time-reversal mirror [15]. This approach consists in adding a random distribution of scatterers in the vicinity of the source. The proposed physical explanation is as follows. The small-scale features (position and shape) of the source are carried by high evanescent modes, and these modes decay exponentially fast with the propagation distance, so that this information is usually not transmitted up to the time-reversal mirror, which is located in the far field. The random medium located around the source location permits the conversion evanescent modes into propagating modes. In other words, the inhomogeneities of the random slab induce mode coupling, so that the information on small scales of the source is transferred to the propagating modes and reaches the time-reversal mirror. During the time-reversal experiment these modes are regenerated in the vicinity of the source from the backpropagated propagating modes, and therefore they can participate in the refocusing process. An application

*Received by the editors March 27, 2008; accepted for publication (in revised form) November 5, 2008; published electronically March 6, 2009.
http://www.siam.org/journals/mms/7-3/71949.html
†Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 7, 2 Place Jussieu, 75251 Paris Cedex 5, France (chgomez@math.jussieu.fr).

of this result to wireless communication is presented in [15].

Throughout this paper, even though the work of Fink and his group was on time reversal of electromagnetic waves, we consider a two-dimensional acoustic waveguide model. The main goal of this paper is to present a mathematical proof that the focal spot can indeed be smaller than the diffraction limit. Before the mathematical analysis, we give some physical explanations to describe the important phenomena induced by the insertion of a section in the vicinity of the source for a long waveguide. First, the case of a waveguide with homogeneous speed of propagation c_0 (see Figure 1(a)) is well known; see, for instance, [9], where the authors obtain the classical diffraction limit. Namely, the focal spot has radius equal to the carrier wavelength over two. In this case, the small-scale features (position and shape) of the source are carried by high evanescent modes that decay exponentially fast with the propagation distance. Consequently, these modes do not reach the time-reversal mirror, which is located in the far field. Only low modes are recorded by the time-reversal mirror. In the second step of the time-reversal experiment, the mirror sends back the recorded low modes that carry only the large-scale features of the original source. This loss of information is responsible for the diffraction-limited transverse profile computed in Proposition 3. In what follows, we refer to high or low modes relatively to a waveguide with homogeneous speed of propagation c_0. Experiments have shown that the situation changes dramatically when a section of medium with low speed of propagation $c_1 \ll c_0$ is inserted in the vicinity of the source. In this paper, we will compare the two following cases with the homogeneous case.

First, we assume that a homogeneous section is inserted in the vicinity of the source, as illustrated in Figure 1(b), such that some high modes of the previous case are propagating modes in this first section. However, we assume that the major part of the waveguide has speed of propagation c_0 so the high modes and the small-scale features of the source do not reach the time-reversal mirror. Therefore, as in the homogeneous case, only low modes are recorded by the time-reversal mirror and the small-scale features of the source are lost. The transverse profile obtain in this case is described in Proposition 2.

Second, if the additional section is randomly perturbed, then coupling mechanisms, between propagating modes of the first section, allow small-scale features of the source, which are carried by the high modes, to be transferred to low modes. Even if the high modes do not propagate over large distances in the second part of the waveguide and are not recorded by the time-reversal mirror, some of the small-scale features of the source reach the time-reversal mirror since they are carried by the low modes which are recorded by the time-reversal mirror. This fact is illustrated in Figure 1(c). These low modes, time-reversed, will come back to the randomly perturbed section in the second step of the time-reversal experiment, and by coupling mechanisms they will regenerate high modes with the small-scale features of the source. This regeneration of small-scale features of the source is responsible for the superresolution described in Proposition 4.

The organization of this paper is as follows: In the first section, we describe the waveguide model that we consider for the experiment. In section 2, we reduce the study of the wave propagation in the random section to the study of a system of differential equations with random coefficients by using a modal decomposition. Moreover, we introduce some assumptions needed for the study of the time-reversal process. In section 3, we state the asymptotic results that we will use in the following section. In section 4, we consider the time-reversal experiment in the random waveguide pre-

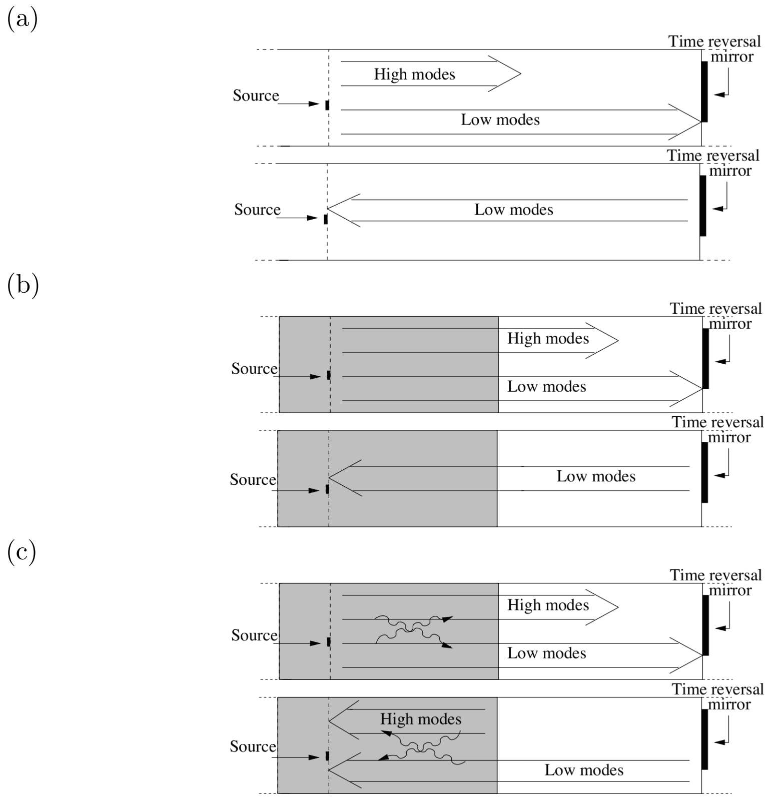


FIG. 1. *Representation of modes propagation in the time reversal experiment. In* (a) *we represent a homogeneous waveguide, in* (b) *we add a homogeneous section with low speed propagation, and in* (c) *we add a randomly heterogeneous section with low background speed of propagation.*

sented in section 1. We analyze the refocused field to emphasize the superresolution effect and show the statistical stability. Finally, the appendix is devoted to the proofs of the theorems stated in section 3.

1. Waveguide model. We consider a two-dimensional linear acoustic wave model. The conservation equations of mass and linear momentum are given by

$$\begin{aligned} \rho^\epsilon(x,z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{F}^\epsilon, \\ \frac{1}{K^\epsilon(x,z)}\frac{\partial p}{\partial t} + \nabla.\mathbf{u} &= 0, \end{aligned} \tag{1.1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ^ϵ is the density of the medium, K^ϵ is the bulk modulus, and the source is modeled by the forcing term $\mathbf{F}^\epsilon(t,x,z)$. The third coordinate z represents the propagation axis along the waveguide. The transverse section of the waveguide is a bounded interval denoted by $[0,d]$, with $d>0$ and $x\in[0,d]$ representing the transverse coordinate. We assume that the medium parameters are given by

$$\frac{1}{K^\epsilon(x,z)} = \begin{cases} \epsilon^{2\alpha_K}\frac{1}{K}\left(1+\sqrt{\epsilon}V\left(x,\frac{z}{\epsilon^\alpha}\right)\right) & \text{if } x\in(0,d),\ z\in[0,L/\epsilon^{1-\alpha}], \\ \epsilon^{2\alpha_K}\frac{1}{K} & \text{if } x\in(0,d),\ z\in(-\infty,0) \\ \frac{1}{K} & \text{if } x\in(0,d),\ z\in(L/\epsilon^{1-\alpha},+\infty), \end{cases}$$

$$\rho^\epsilon(x,z) = \begin{cases} \epsilon^{-2\alpha_\rho}\bar{\rho} & \text{if } x\in(0,d),\ z\in(-\infty,L/\epsilon^{1-\alpha}], \\ \bar{\rho} & \text{if } x\in(0,d),\ z\in(L/\epsilon^{1-\alpha},+\infty), \end{cases}$$

where α_ρ and α_K are such that $\alpha_\rho-\alpha_K=\alpha\in(0,1]$. In what follows, we will see that the important parameter is α, because it determines the order of the sound speed of the first section. This configuration means that the order of the sound speed of the section $(-\infty,L/\epsilon^{1-\alpha})$ is small compared to that of the section $(L/\epsilon^{1-\alpha},+\infty)$. The first section can represent a solid with random inhomogeneities, and the second can represent a homogeneous gas or liquid. The case $\alpha=0$ is equivalent to that studied in [11] and [9, Chapter 20], in which no superresolution effect can be detected. The parameter α represents a possible configuration of the waveguide model, but we will see in Theorem 1 that the set of possible configurations to which we will apply an asymptotic analysis is more restricted.

We consider a source that emits a signal in the z-direction with carrier frequency ω_0. The source is localized in the plane $z=0$.

$$\mathbf{F}^\epsilon(t,x,z)=f^\epsilon(t)\Psi(x)\delta(z)\mathbf{e}_z,\ \text{ where } f^\epsilon(t)=\frac{1}{2\epsilon^\alpha}f(\epsilon^p t)e^{-i\omega_0 t} \text{ with } p\in(0,1), \tag{1.2}$$

$\Psi(x)$ is the transverse profile of the source, and $\mathbf{e}_z$ is the unit vector pointing in the z-direction. The source amplitude is large, of order $1/\epsilon^\alpha$, because transmission coefficients at the interface $z=L/\epsilon^{1-\alpha}$ are small, of order $\epsilon^{\alpha/2}$. We will see that the transmission fcoefficients can be made of order one by inserting a quarter wavelength plate. We shall discuss this in section 4.6. Note that the condition $p>0$ simplifies the algebra, and the condition $p<1$ corresponds to the broadband case and ensures the statistical stability property discussed in section 4.5. In the configuration (1.2), the relative bandwidth is of order ϵ^p, and the carrier wavelength is of order ϵ^α in the $(-\infty,L/\epsilon^{1-\alpha})$ section and of order one in $(L/\epsilon^{1-\alpha},+\infty)$.

The random process $(V(x,z),x\in[0,d],z\geq 0)$ is a continuous real-valued zero-mean stationary Gaussian field with a covariance function given by

$$\mathbb{E}\left[V(x,t)V(y,s)\right]=\gamma(x,y)e^{-a|t-s|},$$

where $a>0$ and $\gamma:[0,d]\times[0,d]\to\mathbb{R}$ is a continuous function that is the kernel of a nonnegative operator. Using standard properties of Gaussian processes, we can state the following results [1]. Let $\mathcal{F}_z=\sigma(V(x,s),x\in[0,d],s\leq z)$ be the σ-algebra generated by $(V(x,s),x\in[0,d],s\leq z)$. We have the Markov property

$$\left(V(x,z+h),x\in[0,d]|\mathcal{F}_z\right)=\left(V(x,z+h),x\in[0,d]|\sigma(V(x,z),x\in[0,d])\right),$$

where the equality holds in law, and this law is the one of a Gaussian field with mean

$$\mathbb{E}\left[V(x,z+h)|\mathcal{F}_z\right]=e^{-ah}V(x,z)$$

and covariance $\gamma(x,y)\left(1-e^{-2ah}\right)$. Moreover, we will use the following two properties. $\forall T>0$,

$$\sqrt{\epsilon}\sup_{z\in[0,T]}\sup_{x\in[0,d]}\left|V\left(x,\frac{z}{\epsilon}\right)\right| \underset{\epsilon\to 0}{\longrightarrow} 0 \quad \text{a.s.} \tag{1.3}$$

and $\forall n \in \mathbb{N}^*$,

$$\mathbb{E}\left[\sup_{x\in[0,d]}\left|V\left(x,\frac{z}{\epsilon}\right)\right|^n\right] = \mathbb{E}\left[\sup_{x\in[0,d]}|V(x,0)|^n\right] < +\infty. \tag{1.4}$$

We can remark that the process V is unbounded. This fact implies that the bulk modulus can take negative values. However, to avoid this situation, we can work on the event

$$\left(\forall (x,z)\in[0,d]\times[0,L],\, 1+\sqrt{\epsilon}V\left(x,\frac{z}{\epsilon^\alpha}\right)>0\right),$$

since by the property (1.3)

$$\begin{aligned}&\lim_{\epsilon\to 0}\mathbb{P}\left(\exists (x,z)\in[0,d]\times[0,L]:\, 1+\sqrt{\epsilon}V\left(x,\frac{z}{\epsilon^\alpha}\right)\leq 0\right)\\ &\quad\leq \lim_{\epsilon\to 0}\mathbb{P}\left(\sqrt{\epsilon^\alpha}\sup_{z\in[0,L]}\sup_{x\in[0,d]}\left|V\left(x,\frac{z}{\epsilon^\alpha}\right)\right|\geq\frac{1}{\sqrt{\epsilon^{1-\alpha}}}\right)=0.\end{aligned}$$

2. Waveguide propagation.

2.1. Propagation in homogeneous waveguides. In this section, we assume that the medium parameters are given by

$$\rho^\epsilon(x,z)=\frac{\bar{\rho}}{\epsilon^{2\alpha_\rho}} \text{ and } K^\epsilon(x,z)=\frac{\bar{K}}{\epsilon^{2\alpha_K}} \quad \forall (x,z)\in(0,d)\times\mathbb{R}.$$

From the conservation equations (1.1), we can derive the wave equation for the pressure field,

$$\Delta p - \frac{1}{c^{\epsilon 2}}\frac{\partial^2 p}{\partial t^2} = \nabla.\mathbf{F}^\epsilon,$$

where $c^\epsilon = \epsilon^\alpha\sqrt{\frac{\bar{K}}{\bar{\rho}}} = \epsilon^\alpha c$ and $\Delta = \partial_x^2 + \partial_z^2$. We consider Dirichlet boundary conditions

$$p(t,0,z)=p(t,d,z)=0 \quad \forall (t,z)\in\mathbb{R}_+\times\mathbb{R}.$$

Here, the Fourier transform and the inverse Fourier transform, with respect to time, are defined by

$$\widehat{f}(\omega)=\int f(t)e^{i\omega t}dt, \quad f(t)=\frac{1}{2\pi}\int\widehat{f}(\omega)e^{-i\omega t}d\omega.$$

In the half-space $z>0$ (resp., $z<0$), taking the Fourier transform in (1.1), we get that $\widehat{p}(\omega,x,z)$ satisfies the time harmonic wave equation without source term

$$\partial_z^2\widehat{p}(\omega,x,z)+\partial_x^2\widehat{p}(\omega,x,z)+\frac{k^2(\omega)}{\epsilon^{2\alpha}}\widehat{p}(\omega,x,z)=0,$$

with Dirichlet boundary conditions. Here, $k(\omega)=\frac{\omega}{c}$. We can decompose this solution in a spectral basis of $L^2(0,d)$, which can be chosen as the set of eigenfunctions $(\phi_j(x))_{j\geq 1}$ of $-\partial_x^2$

$$-\partial_x^2\phi_j(x)=\lambda_j\phi_j(x) \quad \text{and} \quad \int_0^d\phi_j(x)\phi_l(x)dx=\delta_{jl} \quad \forall j,l\geq 1,$$

where δ_{jl} denotes the Kronecker symbol. This family is given by

$$\phi_j(x) = \sqrt{\frac{2}{d}} \sin\left(\frac{j\pi}{d}x\right) \text{ with } \lambda_j = \frac{j^2\pi^2}{d^2} \text{ for } j \geq 1$$

and corresponds to the basis of the unperturbed waveguide. Thus, we can write

$$\widehat{p}(\omega, x, z) = \sum_{j\geq 1} \widehat{p}_j(\omega, z)\phi_j(x). \tag{2.1}$$

This implies that $\forall j \geq 1$, $\widehat{p}_j(\omega, z)$ satisfies the differential equation

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega, z) + \left(\frac{k^2(\omega)}{\epsilon^{2\alpha}} - \lambda_j\right)\widehat{p}_j(\omega, z) = 0. \tag{2.2}$$

For each frequency ω,

$$\epsilon^{2\alpha}\lambda_{N_\epsilon(\omega)} \leq k^2(\omega) < \epsilon^{2\alpha}\lambda_{N_\epsilon(\omega)+1}$$

with $N_\epsilon(\omega) = [\frac{k(\omega)d}{\pi\epsilon^\alpha}]$. There are two cases. First, for $j \leq N_\epsilon(\omega)$, these modes represent the propagating modes, and we define the associated modal wavenumbers by

$$\beta_j^\epsilon(\omega) = \sqrt{\frac{k^2(\omega)}{\epsilon^{2\alpha}} - \lambda_j}.$$

Second, for $j > N_\epsilon(\omega)$, these modes represent evanescent modes, and in this case we define the modal wavenumbers by

$$\beta_j^\epsilon(\omega) = \sqrt{\lambda_j - \frac{k^2(\omega)}{\epsilon^{2\alpha}}}.$$

Finally, using (2.2) and (2.1), the pressure field can be written as an expansion over the complete set of modes

$$\begin{aligned} \widehat{p}(\omega, x, z) &= \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega)z}\phi_j(x) + \sum_{j\geq N_\epsilon(\omega)+1} \frac{\widehat{c}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-\beta_j^\epsilon(\omega)z}\phi_j(x)\right]\mathbf{1}_{(0,+\infty)}(z) \\ &+ \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{b}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-i\beta_j^\epsilon(\omega)z}\phi_j(x) + \sum_{j\geq N_\epsilon(\omega)+1} \frac{\widehat{d}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{\beta_j^\epsilon(\omega)z}\phi_j(x)\right]\mathbf{1}_{(-\infty,0)}(z), \end{aligned} \tag{2.3}$$

where $\widehat{a}_{j,0}^\epsilon(\omega)$ (resp., $\widehat{b}_{j,0}^\epsilon(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating in the right half-space $z > 0$ (resp., left half-space $z < 0$), and $\widehat{c}_{j,0}^\epsilon(\omega)$ (resp., $\widehat{d}_{j,0}^\epsilon(\omega)$) is the amplitude of the jth right-going (resp., left-going) evanescent mode in the right half-space $z > 0$ (resp., left half-space $z < 0$). We recall that the source is located in the plane $z = 0$ with the transverse profile $\Psi(x)$.

Substituting (2.3) into

$$\partial_z^2\widehat{p}(\omega, x, z) + \partial_x^2\widehat{p}(\omega, x, z) + \frac{k^2(\omega)}{\epsilon^{2\alpha}}\widehat{p}(\omega, x, z) = \widehat{f}(\omega)\Psi(x)\delta'(z), \tag{2.4}$$

multiplying by $\phi_j(x)$, and integrating over $(0,d)$ permit us to express the mode amplitudes

$$\widehat{a}^{\epsilon}_{j,0}(\omega) = -\widehat{b}^{\epsilon}_{j,0}(\omega) = \frac{\sqrt{\beta^{\epsilon}_j(\omega)}}{4\epsilon^{\alpha+p}}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)\theta_j,$$
$$\widehat{c}_{j,0}(\omega) = -\widehat{d}_{j,0}(\omega) = -\frac{\sqrt{\beta^{\epsilon}_j(\omega)}}{4\epsilon^{\alpha+p}}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)\theta_j,$$

where $\forall j \geq 1$,

$$\theta_j = \langle \Psi, \phi_j \rangle_{L^2(0,d)} = \int_0^d \Psi(x)\phi_j(x)dx.$$

2.2. Mode coupling in random waveguides. In this section, we study the expansion of $\widehat{p}(\omega,x,z)$ when a random section $z \in [0, L/\epsilon^{1-\alpha}]$ is inserted between two homogeneous waveguides:

$$\frac{1}{K^{\epsilon}(x,z)} = \begin{cases} \epsilon^{2\alpha_K}\frac{1}{K}\left(1+\sqrt{\epsilon}V\left(x,\frac{z}{\epsilon^{\alpha}}\right)\right) & \text{if } x\in(0,d),\ z\in[0,L/\epsilon^{1-\alpha}], \\ \epsilon^{2\alpha_K}\frac{1}{K} & \text{if } x\in(0,d),\ z\in(-\infty,0) \\ \frac{1}{K} & \text{if } x\in(0,d),\ z\in(L/\epsilon^{1-\alpha},+\infty), \end{cases}$$
$$\rho^{\epsilon}(x,z) = \begin{cases} \epsilon^{-2\alpha_\rho}\bar{\rho} & \text{if } x\in(0,d),\ z\in(-\infty, L/\epsilon^{1-\alpha}] \\ \bar{\rho} & \text{if } x\in(0,d),\ z\in(L/\epsilon^{1-\alpha},+\infty). \end{cases}$$

In this region, the pressure field can be decomposed on the basis of eigenmodes of the unperturbed waveguide

$$\widehat{p}(\omega,x,z) = \sum_{j=1}^{N_\epsilon(\omega)} \widehat{p}_j(\omega,z)\phi_j(x) + \sum_{j>N_\epsilon(\omega)} \widehat{q}_j(\omega,z)\phi_j(x).$$

Evanescent modes correspond to $j > N_\epsilon(\omega)$, and $N_\epsilon(\omega)$ goes to $+\infty$ as $\epsilon \searrow 0$. Therefore, we will neglect the modes $j > N_\epsilon(\omega)$. Note that it could be possible to incorporate the modes $j > N_\epsilon(\omega)$ using the method described in [9, Chapter 20], but this would lead to complicated algebra without modifying the overall result. Indeed, we will check a posteriori that the mode decomposition of the wave is supported by a number of modes of order one as $\epsilon \searrow 0$. Consequently, we will consider in what follows the decomposition

$$\widehat{p}(\omega,x,z) = \sum_{j=1}^{N_\epsilon(\omega)} \widehat{p}_j(\omega,z)\phi_j(x),$$

where $\widehat{p}_j(\omega,z)$ satisfies

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega,z) + \beta^{\epsilon}_j(\omega)^2\widehat{p}_j(\omega,z) + \epsilon^{\frac{1}{2}-2\alpha}k^2(\omega)\sum_{l=1}^{N_\epsilon(\omega)} C_{jl}\left(\frac{z}{\epsilon^{\alpha}}\right)\widehat{p}_l(\omega,z) = 0 \tag{2.5}$$

with

$$C_{jl}(z) = \langle \phi_j, \phi_l V(.,z)\rangle_{L^2(0,d)} = \int_0^d \phi_j(x)\phi_l(x)V(x,z)dx \quad \forall (j,l)\in\{1,\dots,N_\epsilon(\omega)\}^2.$$

Note that $\forall (j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$, the coefficient C_{jl} represents the coupling between the jth propagating mode with the lth propagating mode. Next, we introduce the amplitudes of the generalized right- and left-going modes $\widehat{a}_j(\omega,z)$ and $\widehat{b}_j(\omega,z)$ for $j \in \{1,\dots,N_\epsilon(\omega)\}$. They are given by

$$\widehat{p}_j(\omega,z) = \frac{1}{\sqrt{\beta_j^\epsilon(\omega)}}\left(\widehat{a}_j(\omega,z)e^{i\beta_j^\epsilon(\omega)z} + \widehat{b}_j(\omega,z)e^{-i\beta_j^\epsilon(\omega)z}\right),$$
$$\frac{d}{dz}\widehat{p}_j(\omega,z) = i\sqrt{\beta_j^\epsilon(\omega)}\left(\widehat{a}_j(\omega,z)e^{i\beta_j^\epsilon(\omega)z} - \widehat{b}_j(\omega,z)e^{-i\beta_j^\epsilon(\omega)z}\right).$$

In the absence of random perturbation, these amplitudes are constant. In the presence of random perturbations, we obtain from (2.5) the coupled mode equation

$$\frac{d}{dz}\widehat{a}_j = \epsilon^{\frac{1}{2}-\alpha}\frac{ik^2}{2}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon^\alpha}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l e^{i\left(\beta_l^\epsilon-\beta_j^\epsilon\right)z} + \widehat{b}_l e^{-i\left(\beta_l^\epsilon+\beta_j^\epsilon\right)z}\right),$$
$$\frac{d}{dz}\widehat{b}_j = -\epsilon^{\frac{1}{2}-\alpha}\frac{ik^2}{2}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon^\alpha}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l e^{i\left(\beta_l^\epsilon+\beta_j^\epsilon\right)z} + \widehat{b}_l e^{-i\left(\beta_l^\epsilon-\beta_j^\epsilon\right)z}\right)$$

$\forall j \in \{1,\dots,N_\epsilon(\omega)\}$.

Let us define the rescaled processes

$$\widehat{a}_j^\epsilon(\omega,z) = \widehat{a}_j\left(\omega,\frac{z}{\epsilon^{1-\alpha}}\right) \text{ and } \widehat{b}_j^\epsilon(\omega,z) = \widehat{b}_j\left(\omega,\frac{z}{\epsilon^{1-\alpha}}\right) \text{ for } z\in(0,L),$$

$\forall j \in \{1,\dots,N_\epsilon(\omega)\}$. These scalings correspond to the size of the random section $(0, L/\epsilon^{1-\alpha})$. They satisfy the rescaled coupled mode equation

$$\frac{d}{dz}\widehat{a}_j^\epsilon = \frac{ik^2}{2\sqrt{\epsilon}}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l^\epsilon e^{i\epsilon^\alpha\left(\beta_l^\epsilon-\beta_j^\epsilon\right)\frac{z}{\epsilon}} + \widehat{b}_l^\epsilon e^{-i\epsilon^\alpha\left(\beta_l^\epsilon+\beta_j^\epsilon\right)\frac{z}{\epsilon}}\right),$$
$$\frac{d}{dz}\widehat{b}_j^\epsilon = -\frac{ik^2}{2\sqrt{\epsilon}}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l^\epsilon e^{i\epsilon^\alpha\left(\beta_l^\epsilon+\beta_j^\epsilon\right)\frac{z}{\epsilon}} + \widehat{b}_l^\epsilon e^{-i\epsilon^\alpha\left(\beta_l^\epsilon-\beta_j^\epsilon\right)\frac{z}{\epsilon}}\right). \tag{2.6}$$

This system is endowed with the boundary conditions $\forall j \in \{1,\dots,N_\epsilon(\omega)\}$,

$$\widehat{a}_j^\epsilon(\omega,0) = \widehat{a}_{j,0}^\epsilon(\omega) \text{ and } \widehat{b}_j^\epsilon(\omega,L) = 0.$$

Note that $\forall j \in \{1,\dots,N_\epsilon(\omega)\}$, $\widehat{a}_{j,0}^\epsilon(\omega)$ represents the initial amplitude of the jth propagating mode generated by the source at $z = 0^+$. The second condition means that no wave comes from the right. We can rewrite (2.6) in a vector-matrix form as

$$\frac{d}{dz}X^\epsilon = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^\epsilon\left(\frac{z}{\epsilon}\right)X^\epsilon,$$

where

$$X^\epsilon(z) = \begin{bmatrix}\widehat{a}^\epsilon(\omega,z)\\ \widehat{b}^\epsilon(\omega,z)\end{bmatrix}, \quad \mathbf{H}^\epsilon(z) = \begin{bmatrix}\mathbf{H}^{a,\epsilon}(z) & \mathbf{H}^{b,\epsilon}(z)\\ \overline{\mathbf{H}^{b,\epsilon}}(z) & \overline{\mathbf{H}^{a,\epsilon}}(z)\end{bmatrix},$$

and $\forall(j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$,

$$\mathbf{H}^{a,\epsilon}_{jl}(z) = \frac{ik^2(\omega)}{2} \frac{C_{jl}(z)}{\epsilon^\alpha \sqrt{\beta^\epsilon_j(\omega)\beta^\epsilon_l(\omega)}} e^{i\epsilon^\alpha\left(\beta^\epsilon_l(\omega)-\beta^\epsilon_j(\omega)\right)z},$$

$$\mathbf{H}^{b,\epsilon}_{jl}(z) = \frac{ik^2(\omega)}{2} \frac{C_{jl}(z)}{\epsilon^\alpha \sqrt{\beta^\epsilon_j(\omega)\beta^\epsilon_l(\omega)}} e^{-i\epsilon^\alpha\left(\beta^\epsilon_l(\omega)+\beta^\epsilon_j(\omega)\right)z}.$$

Now, we introduce the propagator matrix $\mathbf{P}^\epsilon(\omega,z)$, that is, the $2N_\epsilon(\omega) \times 2N_\epsilon(\omega)$ matrix solution of the differential equation

$$\frac{d}{dz}\mathbf{P}^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^\epsilon\left(\frac{z}{\epsilon}\right)\mathbf{P}^\epsilon(\omega,z) \quad \text{with} \quad \mathbf{P}^\epsilon(\omega,0) = \mathbf{I}.$$

This relation implies

$$\begin{bmatrix} \widehat{a}^\epsilon(\omega,z) \\ \widehat{b}^\epsilon(\omega,z) \end{bmatrix} = \mathbf{P}^\epsilon(\omega,z) \begin{bmatrix} \widehat{a}^\epsilon(\omega,0) \\ \widehat{b}^\epsilon(\omega,0) \end{bmatrix},$$

and the symmetry of $\mathbf{H}^\epsilon(z)$ gives a particular form of the propagator:

$$\mathbf{P}^\epsilon(\omega,z) = \begin{bmatrix} \mathbf{P}^a_\epsilon(\omega,z) & \mathbf{P}^b_\epsilon(\omega,z) \\ \overline{\mathbf{P}^b_\epsilon}(\omega,z) & \overline{\mathbf{P}^a_\epsilon}(\omega,z) \end{bmatrix},$$

where $\mathbf{P}^a_\epsilon(\omega,z)$ and $\mathbf{P}^b_\epsilon(\omega,z)$ are $N_\epsilon(\omega)\times N_\epsilon(\omega)$ matrices which represent, respectively, the coupling between right-going modes and the coupling between right-going and left-going modes.

2.3. Band-limiting idealization and forward scattering approximation. In this section, we introduce a band-limiting idealization hypothesis in which the power spectral density of the random fluctuations is assumed to be limited in both the transverse and the longitudinal directions. This hypothesis simplifies the study of the time-reversal process. Note that $\forall j \geq 1$ and $z \in [0,+\infty)$, we have

$$\begin{aligned} \mathbb{E}[C_{jl}(z)^2] &= \int_0^d \int_0^d \gamma(x,y)\phi_j(x)\phi_l(x)\phi_j(y)\phi_l(y)dxdy \\ &= S(j-l,j-l) + S(j+l,j+l) - S(j-l,j+l) - S(j+l,j-l), \end{aligned}$$

where

$$S(a,b) = \frac{4}{d^2}\int_0^d \int_0^d \gamma(x,y)\cos\left(a\frac{\pi}{d}x\right)\cos\left(b\frac{\pi}{d}y\right)dxdy.$$

We assume that the support of S lies in the square $\left[-\frac{3}{2},\frac{3}{2}\right] \times \left[-\frac{3}{2},\frac{3}{2}\right]$. Our compact support hypothesis implies

$$C_{jl}(z) = 0 \quad \text{if } |j-l| > 1,$$

which is tantamount to a nearest neighbor coupling. More precisely, this assumption implies that $\forall(j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$, the jth mode amplitude can exchange information with the lth amplitude mode if they are direct neighbors, that is, if they satisfy $|j-l| \leq 1$.

The same proof as the one in section 5.1 shows that $\mathbf{P}^\epsilon$ converges in law. The limit processes of $\mathbf{P}^a_\epsilon$ and $\mathbf{P}^b_\epsilon$, as $\epsilon \to 0$, are coupled through the coefficients

$$\int_0^{+\infty} \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos(2k(\omega)z)dz,$$

because of the factor $e^{\pm i\epsilon\left(\beta^\epsilon_l(\omega)+\beta^\epsilon_j(\omega)\right)z}$ in $\mathbf{H}^{b,\epsilon}_{jl}(z)$ and the fact that $\forall j \geq 1$,

$$\lim_{\epsilon \to 0} \epsilon^\alpha \beta^\epsilon_j(\omega) = k(\omega). \tag{2.7}$$

We assume that the power spectral density of the process V, i.e., the Fourier transform of its z-autocorrelation function, possesses a cut-off wavenumber strictly less than $2k(\omega)$. In other words, we consider the case where

$$\int_0^{+\infty} \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos(2k(\omega)z)dz = 0 \quad \forall j, l \geq 1.$$

Consequently, the limit coupling between $\mathbf{P}^a_\epsilon(\omega, z)$ and $\mathbf{P}^b_\epsilon(\omega, z)$ becomes zero. Moreover, the initial condition $\mathbf{P}^b_\epsilon(\omega, 0) = \mathbf{0}$ implies that $\mathbf{P}^b_\epsilon$ converges to $\mathbf{0}$. In this forward scattering approximation, we can neglect the left-going propagating modes in the asymptotic $\epsilon \to 0$. With this assumption, one can consider the simplified coupled amplitude equation given by

$$\frac{d}{dz}\widehat{a}^\epsilon(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{a,\epsilon}\left(\frac{z}{\epsilon}\right)\widehat{a}^\epsilon(\omega, z) \quad \text{with} \quad \widehat{a}^\epsilon(\omega, 0) = \widehat{a}_0(\omega).$$

Finally, we introduce the transfer matrix $\mathbf{T}^\epsilon(\omega, z)$, which is the $N_\epsilon(\omega) \times N_\epsilon(\omega)$ matrix solution of

$$\frac{d}{dz}\mathbf{T}^\epsilon(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{a,\epsilon}\left(\frac{z}{\epsilon}\right)\mathbf{T}^\epsilon(\omega, z) \quad \text{with} \quad \mathbf{T}^\epsilon(\omega, 0) = \mathbf{I}. \tag{2.8}$$

From this equation, one can check that the transfer matrix $\mathbf{T}^\epsilon(\omega, z)$ is unitary since $\mathbf{H}^{a,\epsilon}$ is skew-Hermitian.

3. The coupled mode process. This section presents the theoretical results needed in this paper. In [11] and [13], the authors used the theorem stated in [17] since the number of propagating modes was fixed, but this is not the case in our configuration. The first result concerns the diffusion-approximation for a solution of an ordinary differential equation with random coefficients. This result is a version of that stated in [17], where the dimension of the system is fixed, adapted to the case where the dimension of the system goes to infinity in the asymptotic $\epsilon \searrow 0$. The second result, which follows from Theorem 1, is about the asymptotic behavior of the expectation of the product of two transfer coefficients. These two results will be used in the following section to compute the refocused pulse in the asymptotic regime $\epsilon \searrow 0$. The third result concerns the high-frequency approximation to the coupled power equation obtained in Proposition 1. Using a probabilistic representation of solutions of this equation, we establish a convergence in law to a continuous diffusion process. From Theorem 2, we give the high-frequency approximation to the coupled power equation that will allow us to compute the transverse profile of the refocused pulse and show that randomness enhances spatial refocusing beyond the diffraction limit.

Let $\mathcal{H} = l^2(E,\mathbb{C})$, with $E = (\mathbb{N}^*)^2$, equipped with the inner product be defined by

$$\forall(\lambda,\mu) \in \mathcal{H}\times\mathcal{H}, \quad \langle\lambda,\mu\rangle_{\mathcal{H}} = \sum_{j,m\geq 1} \lambda_{jm}\overline{\mu_{jm}}.$$

Let us fix $(l,n) \in (\mathbb{N}^*)^2$ and consider

$$U^\epsilon_{jm}(\omega,z) = \overline{T^\epsilon_{jl}(\omega,z)}T^\epsilon_{mn}(\omega,z),$$

which is an $\mathcal{H}$-valued process such that $\|\mathbf{U}^\epsilon(\omega,z)\|_{\mathcal{H}} = 1$ $\forall z \geq 0$. Note that we have dropped the indexes l and n in the previous definition because they do not play any role in (2.8).

THEOREM 1. *For $\alpha \in (0,1/4)$, the family of processes $(\boldsymbol{U}^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ converges in distribution in $\mathcal{C}([0,+\infty),\mathcal{H})$ to a limit denoted by $\boldsymbol{U}(\omega,.)$. This limit satisfies the infinite-dimensional stochastic differential equation*

$$d\boldsymbol{U}(\omega,z) = J(\boldsymbol{U}(\omega,z))dz + \psi_1(\boldsymbol{U}(\omega,z))(dB^1_z) + \psi_2(\boldsymbol{U}(\omega,z))(dB^2_z),$$

where $(B^\eta_{jm})_{\substack{\eta=1,2\\ j,m\geq 1}}$ is a family of independent one-dimensional standard Brownian motions and

$$\begin{aligned}
J(u)_{jm} &= \Lambda\left[(u_{j+1j+1}\delta_{jm} - u_{jm}) + (u_{j-1j-1}\delta_{jm} - u_{jm})\right]\\
\psi_1(u)(\lambda)_{jm} &= \sqrt{\frac{\Lambda}{2}}(u_{j+1m}\lambda_{jj+1} - u_{j-1m}\lambda_{j-1j} + u_{jm+1}\lambda_{mm+1} - u_{jm-1}\lambda_{m-1m}),\\
\psi_2(u)(\lambda)_{jm} &= i\sqrt{\frac{\Lambda}{2}}(-u_{j+1m}\lambda_{jj+1} - u_{j-1m}\lambda_{j-1j} + u_{jm+1}\lambda_{mm+1} + u_{jm-1}\lambda_{m-1m})
\end{aligned}$$

$\forall(u,\lambda) \in \mathcal{H}\times l^2(E,\mathbb{R})$, with $\Lambda = \frac{k^2(\omega)}{2a}S(1,1)$. We use the convention $(y_{0,m})_{m\geq 1} = (y_{j,0})_{j\geq 1} = 0$ for $y \in \mathcal{H}$.

This theorem gives the asymptotic behavior of the statistical properties of the matrix $(U^\epsilon_{jm})_{j,m}$ in terms is of the diffusion model given by the infinite-dimensional stochastic differential equation.

The proof of this theorem, given in the appendix, is based on a martingale approach using the perturbed-test-function method. In a first step we show the tightness of the process, and in a second step we characterize all the accumulation points by mean of a well-posed martingale problem in a Hilbert space.

PROPOSITION 1.

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathbb{E}\left[\overline{T^\epsilon_{jj}(\omega,L)}T^\epsilon_{mm}(\omega,L)\right] &= \mathbb{E}\left[U_{jm}(\omega,L)\right] = \begin{cases} e^{-\Lambda L} & \textit{if } j\neq m \textit{ and } j=1 \textit{ or } m=1,\\ e^{-2\Lambda L} & \textit{if } j\neq m\neq 1,\end{cases}\\
\lim_{\epsilon\to 0}\mathbb{E}\left[\overline{T^\epsilon_{jl}(\omega,L)}T^\epsilon_{jl}(\omega,L)\right] &= \mathbb{E}\left[U_{jj}(\omega,L)\right] = \mathcal{T}^l_j(\omega,L),\\
\lim_{\epsilon\to 0}\mathbb{E}\left[\overline{T^\epsilon_{jl}(\omega,L)}T^\epsilon_{mn}(\omega,L)\right] &= \mathbb{E}\left[U_{jm}(\omega,L)\right] = 0 \textit{ in the other cases,}
\end{aligned}$$

where $\left(\mathcal{T}^l_j(\omega,z)\right)_{j\geq 1}$ is the solution of the differential equation, called the coupled power equation,

$$\begin{aligned}
\frac{d}{dz}\mathcal{T}^l_j(\omega,z) &= \Lambda(\omega)\left[\mathcal{T}^l_{j+1}(\omega,z) + \mathcal{T}^l_{j-1}(\omega,z) - 2\mathcal{T}^l_j(\omega,z)\right], \quad j\geq 1,\\
\frac{d}{dz}\mathcal{T}^l_1(\omega,z) &= \Lambda(\omega)\left[\mathcal{T}^l_2(\omega,z) - \mathcal{T}^l_1(\omega,z)\right],
\end{aligned}$$

with $\mathcal{T}_j^l(\omega, 0) = \delta_{jl}$.

This equation represents the transfer of energy between propagating modes, and Λ is the energy transport coefficient. We are interested in studying this equation in the high-frequency regime, that is, when $\omega \gg 1$. To this end we take a probabilistic representation of this equation. We introduce the jump Markov process $(X_t)_{t\geq 0}$ whose state space is $\mathbb{N}^*$ and whose infinitesimal generator is

$$\mathcal{L}_X\varphi(j) = \Lambda(\omega)(\varphi(j+1) + \varphi(j-1) - 2\varphi(j)), \quad j \geq 2,$$
$$\mathcal{L}_X\varphi(1) = \Lambda(\omega)(\varphi(2) - \varphi(1)).$$

We get

$$\mathcal{T}_j^l(\omega, L) = \mathbb{P}(X_L = j | X_0 = l) = \mathbb{P}\left(\frac{X_L}{N} = \frac{j}{N}\middle|\frac{X_0}{N} = \frac{l}{N}\right),$$

where $N(\omega) = \left[\frac{\omega d}{\pi \bar{c}}\right]$ is the number of propagations in the homogeneous part $(L/\epsilon^{1-\alpha}, +\infty)$ of the waveguide model. The interest of the last equality will be justified by the following theorem and in the following section, when, in the high frequency regime, we will compute the transverse profile of the refocused pulse.

THEOREM 2. *Let* $t \geq 0$ *and* $(l(N))_{N\geq 1}$ *be a sequence with values in* $\mathbb{N}^*$ *such that* $\lim_{N\to+\infty} l(N)/N = y$. *Denote by* $\mathbb{P}^{N,t}_{l(N)/N}$ *the law of* X_t/N *starting from* $l(N)/N$. *Then,* $(\mathbb{P}^{N,t}_{l(N)/N})_N$ *converges weakly to the law of* $|\sigma B_t + y|$, *where* $(B_t)_{t\geq 0}$ *is a real standard Brownian motion and* $\sigma^2 = \frac{\pi^2}{d^2 a} S(1,1)$.

This theorem is a continuum approximation in the limit of a large number of propagating modes. From this theorem, we can derive the high-frequency approximation to the coupled power equation. We can consider $(\mathcal{T}^l(\omega, L))_{l\geq 1}$ as a family of probability measures on $\mathbb{R}_+$. Using the previous theorem, we can show that for all sequences $(l(N))_N$, with values in $\mathbb{N}^*$, such that $(l(N)/N)_N$ converges to $y \in \mathbb{R}_+$, we have that $(\mathcal{T}^{l(N)}(\omega, L))_N$ converges weakly to $\mathcal{W}(L, y, y')dy'$. In another words, $\forall\varphi$ bounded continuous functions

$$\mathcal{T}_\varphi^{l(N)}(\omega, L) = \mathbb{E}\left[\varphi\left(\frac{X_t}{N}\right)\middle|\frac{X_0}{N} = \frac{l(N)}{N}\right] \underset{N\to+\infty}{\longrightarrow} \int_{\mathbb{R}_+} \varphi(y')\mathcal{W}(L, y, y')dy',$$

where, $\forall t > 0$ and $(y, y') \in (\mathbb{R}_+)^2$,

$$\frac{\partial}{\partial t}\mathcal{W}(t, y, y') = \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\mathcal{W}(t, y, y'),$$

with

$$\frac{\partial}{\partial y}\mathcal{W}(t, 0, y') = 0 \text{ and } \mathcal{W}(0, y, y') = \delta(y - y').$$

Consequently, for all N large enough or in the high-frequency regime, $\mathcal{T}_\varphi^l(\omega, L)$ can be approximated by $\int_{\mathbb{R}_+} \varphi(y')\mathcal{W}\left(L, \frac{l}{N}, y'\right) dy'$ in the sense that the difference converges to 0.

This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between propagating modes. In our case, the diffusion model of the coupled power equation takes a particularly simple form; it is the heat equation with a reflecting barrier. More details about this approximation and another model of diffusion in a different model of waveguide can be found in [13].

4. Time reversal in a waveguide.

4.1. First step of the time-reversal experiment. In the first step of the experiment, a source sends a pulse into the medium, and the wave propagates and is recorded by the time-reversal mirror. In this section we obtain the integral representation of the wave recorded by the time-reversal mirror.

A source is located in the plane $z = 0$ and emits a pulse $f^\epsilon(t)$ of the form (1.2),

$$f^\epsilon(t) = \frac{1}{2\epsilon^\alpha} f(\epsilon^p t) e^{-i\omega_0 t} \text{ with } p \in (0,1).$$

A time-reversal mirror is located in the plane $z = L_M/\epsilon^{1-\alpha}$, it occupies the transverse subdomain $\mathcal{D}_M \subset [0,d]$, and in the first step of the experiment the time-reversal mirror plays the role of a receiving array. The transmitted wave is recorded for a time interval $\left[\frac{t_0}{\epsilon}, \frac{t_1}{\epsilon}\right]$ at the time-reversal mirror and is re-emitted time-reversed into the waveguide toward the source. We have chosen such a time window because it is of the order of the total travel time of the section. We recall that the distance of propagation is of order $1/\epsilon^{1-\alpha}$ and the sound speed is of order ϵ^α in $(-\infty, L/\epsilon^{1-\alpha})$.

The Fourier transform of the pressure field at the end of the random section $[0, L/\epsilon^{1-\alpha}]$ is given by

$$\widehat{p}_{tr}^{0,\epsilon}\left(\omega, x, \frac{L^-}{\epsilon^{1-\alpha}}\right) = \sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_j^\epsilon(\omega, L)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega)\frac{L}{\epsilon^{1-\alpha}}} \phi_j(x).$$

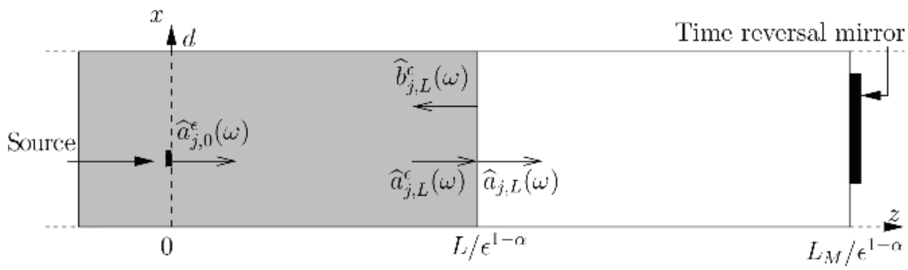
Jumps of the medium parameters at $z = L/\epsilon^{1-\alpha}$ imply that the incoming pulse produces a reflected and a transmitted field. The modal decomposition obtained in section 2.1 for the first part of the waveguide can be obtained in the same way for the second part with $\epsilon = 1$. The decomposition over the eigenmodes gives

$$\begin{aligned}
\widehat{p}_{tr}^{L,\epsilon}(\omega, x, z) = &\Bigg[\sum_{j=1}^{N(\omega)} \frac{\widehat{a}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{b}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) \\
&+ \sum_{j=N(\omega)+1}^{N_\epsilon(\omega)} \frac{\widehat{c}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{-\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{d}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x)\Bigg] \mathbf{1}_{(L/\epsilon^{1-\alpha}, +\infty)}(z) \\
&+ \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_{j,L}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{b}_{j,L}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-i\beta_j^\epsilon(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x)\right] \mathbf{1}_{(0, L/\epsilon^{1-\alpha})}(z),
\end{aligned} \tag{4.1}$$

where $\widehat{a}_{j,L}(\omega)$ (resp., $\widehat{b}_{j,L}(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating, and $\widehat{c}_{j,L}(\omega)$ (resp., $\widehat{d}_{j,L}(\omega)$) is the amplitude of the jth right-going (resp., left-going) evanescent mode in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$. Moreover, $\widehat{a}_{j,L}^\epsilon(\omega)$ (resp., $\widehat{b}_{j,L}^\epsilon(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating in the section $(0, L/\epsilon^{1-\alpha})$. Note that we have kept the evanescent modes $j > N(\omega)$ in the expression (4.1) because $N(\omega)$ is of order one.

From the continuity of the pressure and velocity fields, we get $\forall j \in \{1, \dots, N(\omega)\}$

$$\begin{bmatrix} \widehat{a}_{j,L}(\omega) \\ \widehat{b}_{j,L}(\omega) \end{bmatrix} = \begin{bmatrix} r_j^{\epsilon,+} & r_j^{\epsilon,-} \\ r_j^{\epsilon,-} & r_j^{\epsilon,+} \end{bmatrix} \begin{bmatrix} \widehat{a}_{j,L}^\epsilon(\omega) \\ \widehat{b}_{j,L}^\epsilon(\omega) \end{bmatrix}, \text{where } r_j^{\epsilon,\pm} = \frac{1}{2}\left[\sqrt{\frac{\beta_j(\omega)}{\beta_j^\epsilon(\omega)}} \pm \sqrt{\frac{\beta_j^\epsilon(\omega)}{\beta_j(\omega)}}\right],$$



and $\forall j \in \{N(\omega)+1,\dots,N_\epsilon(\omega)\}$

$$\begin{bmatrix}\widehat{c}_{j,L}(\omega)\\ \widehat{d}_{j,L}(\omega)\end{bmatrix} = \begin{bmatrix} r^{\epsilon,i}_j & \overline{r^{\epsilon,i}_j}\\ \overline{r^{\epsilon,i}_j} & r^{\epsilon,i}_j\end{bmatrix}\begin{bmatrix}\widehat{a}^{\epsilon}_{j,L}(\omega)\\ \widehat{b}^{\epsilon}_{j,L}(\omega)\end{bmatrix}, \text{where } r^{\epsilon,i}_j = \frac{1}{2}\left[\sqrt{\frac{\beta_j(\omega)}{\beta^\epsilon_j(\omega)}} - i\sqrt{\frac{\beta^\epsilon_j(\omega)}{\beta_j(\omega)}}\right]$$

with

$$\widehat{a}^{\epsilon}_{j,L}(\omega) = \widehat{a}^{\epsilon}_{j}(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}},\ \widehat{b}_{j,L}(\omega) = 0, \text{ and } \widehat{d}_{j,L}(\omega) = 0.$$

The two last conditions mean that no wave comes from the right. In fact, in the first part of the experiment the time-reversal mirror records the signal and does not produce reflected waves. Solving these equations allows us to express the transmitted and the reflected coefficients. Consequently, $\forall j \in \{1,\dots,N(\omega)\}$, we have

$$\widehat{a}_{j,L}(\omega) = \tau^{\epsilon,+}_j(\omega)\widehat{a}^{\epsilon}_{j}(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} \text{ and } \widehat{b}^{\epsilon}_{j,L}(\omega) = -\frac{r^{\epsilon,-}_j}{r^{\epsilon,+}_j}\widehat{a}^{\epsilon}_{j}(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}},$$

where

$$\tau^{\epsilon,+}_j(\omega) = \frac{1}{r^{\epsilon,+}_j(\omega)} \tag{4.2}$$

is the transmission coefficient of the interface $z = L/\epsilon^{1-\alpha}$, and $\forall j \in \{N+1,\dots,N_\epsilon(\omega)\}$

$$\widehat{c}_{j,L}(\omega) = -\frac{i}{r^{\epsilon,i}_j}\widehat{a}^{\epsilon}_{j}(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} \text{ and } \widehat{b}^{\epsilon}_{j,L}(\omega) = -\frac{\overline{r^{\epsilon,i}_j}}{r^{\epsilon,i}_j}e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}}.$$

We can remark that $\forall j \in \{1,\dots,N(\omega)\}$, the transmission coefficients $\tau^{\epsilon,+}_j(\omega)$, which are defined by (4.2), are of order $\epsilon^{\alpha/2}$. We recall that we have taken a source amplitude of order $1/\epsilon^{\alpha}$ in (1.2). This fact will allow us to have, after the second step of the time-reversal experiment, a refocused pulse of order one. However, we recall that we will see, in section 4.6, that the transmission coefficients can be made of order one by inserting a quarter wavelength plate.

The reflected wave produced at the interface $z = L/\epsilon^{1-\alpha}$ does not reach the time-reversal mirror. Moreover, $L_M/\epsilon^{1-\alpha}$ is sufficiently large so that one can assume that the evanescent modes, that is, the jth right-going modes for $j \in \{N(\omega)+1,\dots,N_\epsilon(\omega)\}$ in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$ which decrease exponentially fast, do not reach the time-reversal mirror either. Therefore, only the transmitted propagating wave

$$p^\epsilon_{tr}\left(\frac{t}{\epsilon}, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) = \frac{1}{2\pi}\int\sum_{j=1}^{N(\omega)}\frac{\widehat{a}^{\epsilon}_{j}(\omega, L)}{\sqrt{\beta_j(\omega)}}\tau^{\epsilon,+}_j(\omega)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}}e^{i\beta_j(\omega)\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}\phi_j(x)e^{-i\omega\frac{t}{\epsilon}}d\omega \tag{4.3}$$

is recorded by the time-reversal mirror.

4.2. Second step of the time-reversal experiment. In the second step of the time-reversal experiment, the time-reversal mirror plays the role of a source array, and the flipped signal is transmitted back. This source is given by

$$\mathbf{F}^{\epsilon}_{TR}(t,x,z) = -f^{\epsilon}_{TR}(t,x)\delta(z - L_M/\epsilon^{1-\alpha})\mathbf{e}_z,$$

with

$$f^{\epsilon}_{TR}(t,x) = p^{\epsilon}_{tr}\left(\frac{t_1}{\epsilon} - t, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) G_1(t_1 - \epsilon t) G_2(x),$$

and

$$G_1(t) = \mathbf{1}_{[t_0,t_1]}(t) \quad \text{and} \quad G_2(x) = \mathbf{1}_{\mathcal{D}_M}(x).$$

In this paper we are interested in the spatial effects of the refocalization, so we will assume that we record the field for all time at the time reversal mirror, i.e.,

$$f^{\epsilon}_{TR}(t,x) = p^{\epsilon}_{tr}\left(\frac{t_1}{\epsilon} - t, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) G_2(x). \tag{4.4}$$

We study the propagation from $z = L_M/\epsilon^{1-\alpha}$ to $z = 0$. The decomposition on the eigenmodes gives

$$\widehat{p}^{L_M,\epsilon}_{TR}\left(\omega, x, \frac{z}{\epsilon^{1-\alpha}}\right) = \sum_{m=1}^{N(\omega)} \frac{\widehat{b}_{m,L_M}(\omega)}{\sqrt{\beta_m(\omega)}} e^{-i\beta_m(\omega)\left(\frac{z-L_M}{\epsilon^{1-\alpha}}\right)}\phi_m(x)$$

in the homogeneous part of the waveguide, with

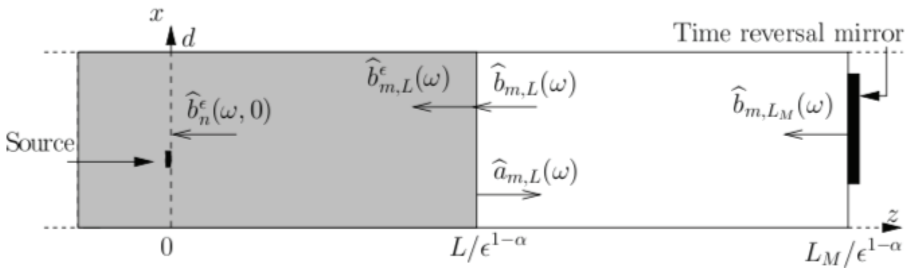
$$\widehat{b}_{m,L_M}(\omega) = \frac{\sqrt{\beta_m(\omega)}}{2}\int_0^d \widehat{f}^{\epsilon}_{TR}(\omega,x)\phi_m(x)dx, \tag{4.5}$$

where

$$\widehat{f}^{\epsilon}_{TR}(\omega,x) = \sum_{j=1}^{N(\omega)} \overline{\frac{\widehat{a}^{\epsilon}_j(\omega,L)}{\sqrt{\beta_j(\omega)}}\tau^{\epsilon,+}_j(\omega)} e^{-i\beta^{\epsilon}_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} e^{-i\beta_j(\omega)\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}\phi_j(x) e^{i\omega\frac{t_1}{\epsilon}}, \tag{4.6}$$

and $\widehat{b}_{m,L_M}(\omega) = 0$ for $m > N$. We are now interested in the refocused pulse near the source location. The transmission through the interface $z = L/\epsilon^{1-\alpha}$ and the back propagation in the random section are treated in the same way as the first step of the time-reversal experiment. The eigenmode decomposition at the interface $z = L/\epsilon^{1-\alpha}$ is given by

$$\begin{aligned}\widehat{p}^{L,\epsilon}_{TR}(\omega,x,z) = &\left[\sum_{m=1}^{N(\omega)} \frac{\widehat{a}_{m,L}(\omega)}{\sqrt{\beta_m(\omega)}} e^{i\beta_m(\omega)\left(z-\frac{L}{\epsilon^{1-\alpha}}\right)}\phi_m(x)\right.\\ &\left.+ \frac{\widehat{b}_{m,L}(\omega)}{\sqrt{\beta_m(\omega)}} e^{-i\beta_m(\omega)\left(z-\frac{L}{\epsilon^{1-\alpha}}\right)}\phi_m(x)\right]\mathbf{1}_{(L/\epsilon^{1-\alpha},+\infty)}(z)\\ &+\left[\sum_{m=1}^{N_\epsilon(\omega)} \frac{\widehat{a}^{\epsilon}_{m,L}(\omega)}{\sqrt{\beta^{\epsilon}_m(\omega)}} e^{i\beta^{\epsilon}_m(\omega)\left(z-\frac{L}{\epsilon^{1-\alpha}}\right)}\phi_m(x)\right.\\ &\left.+ \frac{\widehat{b}^{\epsilon}_{m,L}(\omega)}{\sqrt{\beta^{\epsilon}_m(\omega)}} e^{-i\beta^{\epsilon}_m(\omega)\left(z-\frac{L}{\epsilon^{1-\alpha}}\right)}\phi_m(x)\right]\mathbf{1}_{(0,L/\epsilon^{1-\alpha})}(z),\end{aligned} \tag{4.7}$$



where $\widehat{a}_{m,L}(\omega)$ (resp., $\widehat{b}_{m,L}(\omega)$) is the amplitude of the mth right-going (resp., left-going) mode propagating in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$, and $\widehat{a}^\epsilon_{m,L}(\omega)$ (resp., $\widehat{b}^\epsilon_{m,L}(\omega)$) is the amplitude of the mth right-going (resp., left-going) mode propagating in the section $(0, L/\epsilon^{1-\alpha})$.

From the continuity of the pressure and velocity fields, we get $\forall m \in \{1, \dots, N_\epsilon(\omega)\}$

$$\begin{bmatrix} \widehat{a}_{m,L}(\omega) \\ \widehat{b}_{m,L}(\omega) \end{bmatrix} = \begin{bmatrix} r^{\epsilon,+}_m & r^{\epsilon,-}_m \\ r^{\epsilon,-}_m & r^{\epsilon,+}_m \end{bmatrix} \begin{bmatrix} \widehat{a}^\epsilon_{m,L}(\omega) \\ \widehat{b}^\epsilon_{m,L}(\omega) \end{bmatrix}.$$

However, the source emits only $N(\omega)$ propagating modes; therefore, $\widehat{a}_{m,L}(\omega) = \widehat{b}_{m,L}(\omega) = 0$ for $m > N(\omega)$ and for $m \leq N(\omega)$

$$\widehat{a}^\epsilon_{m,L}(\omega) = 0 \text{ and } \widehat{b}_{m,L}(\omega) = \widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)}.$$

The first condition means that no wave comes from the left in this forward approximation that we are considering. Solving this equation permits us to express the transmitted and the reflected coefficients. $\forall m \in \{1, \dots, N(\omega)\}$,

$$\widehat{a}_{m,L}(\omega) = \frac{r^{\epsilon,-}_m}{r^{\epsilon,+}_m} \widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)}, \quad \widehat{b}^\epsilon_{m,L}(\omega) = \tau^{\epsilon,+}_m(\omega) \widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)},$$

where $\tau^{\epsilon,+}_m(\omega) = \frac{1}{r^{\epsilon,+}_m(\omega)}$ and $\widehat{b}^\epsilon_{m,L}(\omega) = 0 \; \forall m \in \{N(\omega)+1, \dots, N_\epsilon(\omega)\}$. Thus, we have obtained the expression of the boundary conditions at the plane $z = L/\epsilon^{1-\alpha}$. Now, we are interested in the back propagation through the random section from $z = L/\epsilon^{1-\alpha}$ to $z = 0$:

$$\widehat{p}^\epsilon_{TR}(\omega, x, 0) = \sum_{n=1}^{N_\epsilon(\omega)} \frac{\widehat{b}^\epsilon_n(\omega, 0)}{\sqrt{\beta^\epsilon_n(\omega)}} \phi_n(x).$$

Since the transfer matrix $\mathbf{T}^\epsilon(\omega, z)$ is unitary,

$$\begin{aligned} \widehat{b}^\epsilon_n(\omega, 0) &= \sum_{m=1}^{N(\omega)} T^\epsilon_{mn}(\omega, L) \widehat{b}^\epsilon_m(\omega, L) e^{i\beta^\epsilon_m(\omega)\frac{L}{\epsilon^{1-\alpha}}} \\ &= \sum_{m=1}^{N(\omega)} T^\epsilon_{mn}(\omega, L) \tau^{\epsilon,+}_m(\omega) \widehat{b}_{m,L_M} e^{i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)} e^{i\beta^\epsilon_m(\omega)\frac{L}{\epsilon^{1-\alpha}}}, \end{aligned}$$

and using (4.3), (4.4), (4.5), and (4.6) we get

$$\widehat{b}_{m,L_M}(\omega) = \frac{1}{8\epsilon^p} \sum_{j=1}^{N(\omega)} \sum_{l=1}^{N_\epsilon(\omega)} \sqrt{\frac{\beta_l^\epsilon(\omega)\beta_m(\omega)}{\beta_j(\omega)}} M_{mj}\theta_l \overline{\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)} \overline{T_{jl}^\epsilon(\omega, L)}$$
$$\times \frac{1}{\epsilon^\alpha} \overline{\tau_j^{\epsilon,+}(\omega)} \tau_m^{\epsilon,+}(\omega) e^{-i\beta_j^\epsilon(\omega)\frac{L}{\epsilon^{1-\alpha}}} e^{-i\beta_j(\omega)\left(\frac{L_M - L}{\epsilon^{1-\alpha}}\right)} e^{i\omega\frac{t_1}{\epsilon}},$$

where

$$M_{jl} = \int_0^d G_2(x)\phi_j(x)\phi_l(x).$$

The matrix (M_{jl}) represents the coupling produced by the time-reversal mirror between the propagating modes during the two steps of the time-reversal experiment. We recall that $\widehat{b}_{m,L_M}$ is the projection over the mth propagating mode for the Fourier transform of the time-reversed signal recorded by the time-reversal mirror. Therefore, the refocused pulse is

$$\begin{aligned} p_{TR}^\epsilon\left(\frac{t}{\epsilon}, x, 0\right) &= \frac{1}{16\pi\epsilon^p} \int \sum_{j,m=1}^{N(\omega)} \sum_{l,n=1}^{N_\epsilon(\omega)} \sqrt{\frac{\beta_l^\epsilon(\omega)\beta_m(\omega)}{\beta_j(\omega)\beta_n^\epsilon(\omega)}} M_{mj}\theta_l\phi_n(x) \\ &\quad \times \overline{T_{jl}^\epsilon(\omega, L)} T_{mn}^\epsilon(\omega, L) \frac{1}{\epsilon^\alpha} \overline{\tau_j^{\epsilon,+}(\omega)} \tau_m^{\epsilon,+}(\omega) \overline{\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)} \\ &\quad \times e^{i(\beta_m(\omega)-\beta_j(\omega))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)} e^{i(\beta_m^\epsilon(\omega)-\beta_j^\epsilon(\omega))\frac{L}{\epsilon^{1-\alpha}}} e^{i\omega\frac{t_1-t}{\epsilon}} d\omega. \end{aligned} \tag{4.8}$$

Now, we make the change of variable $\omega = \omega_0 + \epsilon^p h$. Consequently, (4.8) becomes

$$\begin{aligned} & e^{-i\omega_0\frac{t_1-t}{\epsilon}} p_{TR}^\epsilon\left(\frac{t}{\epsilon}, x, 0\right) \\ &= \frac{1}{16\pi} \int \sum_{j,m=1}^{N(\omega_0+\epsilon^p h)} \sum_{l,n=1}^{N_\epsilon(\omega_0+\epsilon^p h)} \sqrt{\frac{\beta_l^\epsilon(\omega_0+\epsilon^p h)\beta_m(\omega_0+\epsilon^p h)}{\beta_j(\omega_0+\epsilon^p h)\beta_n^\epsilon(\omega_0+\epsilon^p h)}} \\ &\times \overline{T_{jl}^\epsilon(\omega_0+\epsilon^p h, L)} T_{mn}^\epsilon(\omega_0+\epsilon^p h, L) \frac{1}{\epsilon^\alpha} \overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)} \tau_m^{\epsilon,+}(\omega_0+\epsilon^p h) \\ &\times M_{mj}\theta_l\phi_n(x)\overline{\widehat{f}(h)} e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)} \\ &\times e^{i(\beta_m^\epsilon(\omega_0+\epsilon^p h)-\beta_j^\epsilon(\omega_0+\epsilon^p h))\frac{L}{\epsilon^{1-\alpha}}} e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)} dh. \end{aligned} \tag{4.9}$$

In what follows, we consider the following:

1. A source with transverse profile of the form

$$\forall x \in [0, d], \quad \Psi(x) = \sum_{l=1}^{\zeta} \phi_l(x_0)\phi_l(x),$$

where we assume that $\zeta \gg N(\omega_0)$. Then, $\theta_l = \phi_l(x_0)$ for $l \in \{1, \dots, \zeta\}$ and $\theta_l = 0$ for $l \geq \zeta + 1$. This profile is an approximation of a Dirac distribution at x_0, which models a point source at x_0.

2. A time-reversal mirror of the form $\mathcal{D}_M = [d_1, d_2]$ with

$$d_2 = d_M + \lambda_0^{\alpha_M} \tilde{d}_2 \text{ and } d_1 = d_M - \lambda_0^{\alpha_M} \tilde{d}_1,$$

where $d_M \in (0, d)$, $(\tilde{d}_2, \tilde{d}_1) \in (0, +\infty)^2$, and $\alpha_M \in [0, 1]$. The time-reversal coupling matrix is given by

$$\begin{aligned} M_{jl} = \frac{d_2 - d_1}{d} \Bigg[&\cos\left((j-l)\left(\frac{d_2+d_1}{2d}\right)\pi\right) \operatorname{sinc}\left((j-l)\left(\frac{d_2-d_1}{2d}\right)\pi\right) \\ &- \cos\left((j+l)\left(\frac{d_2+d_1}{2d}\right)\pi\right) \operatorname{sinc}\left((j+l)\left(\frac{d_2-d_1}{2d}\right)\pi\right)\Bigg]. \end{aligned}$$

The parameter α_M represents the order of the magnitude of the size of the mirror with respect to the wavelength. In fact, we will see that the size of the mirror plays a role in the homogeneous case only when it is of the order the carrier wavelength $\lambda_0 = 2\pi c/\omega_0$.

Moreover, we will study the spatial profile of the refocused pulse in the continuum limit $N(\omega_0) \gg 1$, that is, in our case, in the high-frequency regime $\omega_0 \nearrow +\infty$. However, we know that the main focal spot must be of order λ_0, which tends to 0 in this continuum limit. Therefore, we will study the spatial profile in a window of size λ_0 centered around x_0.

4.3. Homogeneous waveguide. Here we examine the homogeneous case, that is, the case in which the section $[0, L/\epsilon^{1-\alpha}]$ has homogeneous parameters $\bar{K}/\epsilon^{2\alpha_K}$ and $\bar{\rho}/\epsilon^{2\alpha_\rho}$. In these conditions we have $T^\epsilon_{jl}(\omega, z) = \delta_{jl}$. We recall that the continuum limit $N(\omega_0) \gg 1$ is achieved in the high-frequency regime $\omega_0 \nearrow +\infty$ and the carrier wavelength is given by $\lambda_0 = 2\pi c/\omega_0$.

PROPOSITION 2. *The refocused field is given by*

$$\begin{aligned} \lim_{\epsilon \to 0} e^{i\omega_0 \frac{t}{\epsilon^p}} p^\epsilon_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, x, 0\right) &= \frac{1}{2} \sum_{j=1}^{N(\omega_0)} \frac{\beta_j(\omega_0)}{k(\omega_0)} M_{jj} \phi_j(x_0)\phi_j(x) f(-t) \\ &= H^{\alpha_M}_{x_0}(\omega_0, x) f(-t). \end{aligned}$$

For $\alpha_M \in [0, 1)$, the transverse profile of the refocused pulse in the continuum limit is given by

$$\lim_{\omega_0 \to +\infty} \frac{\lambda_0^{1-\alpha_M}}{2} H^{\alpha_M}_{x_0}(\omega_0, x_0 + \lambda_0 \tilde{x}) = \frac{\tilde{d}_2 - \tilde{d}_1}{d} \int_0^1 \sqrt{1-u^2} \cos(2\pi \tilde{x} u)\, du. \tag{4.10}$$

Proof. First, we have $\forall p \in (0, 1)$ and $\forall \alpha \in (0, 1]$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^\alpha} \overline{\tau^{\epsilon,+}_j(\omega_0 + \epsilon^p h)} \tau^{\epsilon,+}_m(\omega_0 + \epsilon^p h) = 4 \frac{\sqrt{\beta_j(\omega_0)\beta_m(\omega_0)}}{k(\omega_0)}. \tag{4.11}$$

Second, we will fix the parameters p and α in order to give, for illustration, a simpler proof. Let $p = 1/2$ and $\alpha = 1/6$. These two values allow us to have a not too long truncated expansion (4.12); then the refocused field is given by the deterministic expression, for $\epsilon \ll 1$,

$$p^\epsilon_{TR}\left(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^{1/2}},x,0\right)\simeq\frac{1}{2}\sum_{j,m=1}^{N(\omega_0)}\frac{\beta_m(\omega_0)}{k(\omega_0)}M_{mj}\theta_j\phi_m(x)e^{i(\beta_m(\omega_0)-\beta_j(\omega_0))\left(\frac{L_M-L}{\epsilon^{5/6}}\right)}$$
$$\times\, e^{i(\beta^\epsilon_m(\omega_0)-\beta^\epsilon_j(\omega_0))\frac{L}{\epsilon^{5/6}}}e^{-i\omega_0\frac{t}{\epsilon^{1/2}}}$$
$$\times\, f\left(\left[\epsilon^{1/6}(\beta'_m(\omega_0)-\beta'_j(\omega_0))(L_M-L)+\epsilon^{1/3}(m^2-j^2)\frac{\pi^2c^2}{\omega_0^2d^2}\frac{L}{2c}\right]\frac{1}{\epsilon^{1/2}}-t\right),$$

since

$$\begin{aligned}\left(\beta^\epsilon_m(\omega_0+\epsilon^{1/2}h)-\beta^\epsilon_j(\omega_0+\epsilon^{1/2}h)\right)\frac{L}{\epsilon^{1-\alpha}}&=\left(\beta^\epsilon_m(\omega_0)-\beta^\epsilon_j(\omega_0)\right)\frac{L}{\epsilon^{5/6}}\\&\quad+\frac{h}{2c}(m^2-j^2)\frac{\pi^2c^2}{\omega_0^2d^2}\frac{L}{\epsilon^{1/6}}+o(1).\end{aligned}\tag{4.12}$$

Finally, the transverse profile is given by

$$\begin{aligned}\frac{d}{\tilde d_2-\tilde d_1}\frac{\lambda_0^{1-\alpha_M}}{2}H^{\alpha_M}_{x_0}(\omega_0,x_0+\lambda_0\tilde x)&=\frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\frac{j^2}{N^2}}\cos\left(2\pi\tilde x\frac{j}{N}\right)\\&\quad+\frac{\lambda_0}{2d}\sum_{j=1}^{N}\frac{\beta_j(\omega_0)}{k(\omega_0)}\cos\left(j\frac{\pi}{d}(2x_0+\lambda_0\tilde x)\right)\\&\quad-\frac{d}{\tilde d_2-\tilde d_1}\frac{\lambda_0^{1-\alpha_M}}{2}\sum_{j=1}^{N}\frac{\beta_j(\omega_0)}{j\pi k(\omega_0)}\phi_j(x_0)\phi_j(x_0+\lambda_0\tilde x)\\&\quad\times\cos\left(j\pi\left(\frac{d_2+d_1}{d}\right)\right)\sin\left(j\pi\left(\frac{d_2-d_1}{d}\right)\right)\\&\quad+o(1).\end{aligned}$$

Using the Abel formula, the second and the third sums on the right are $\mathcal{O}(1)$. This completes the proof of the proposition. ∎

To finish this section, we consider the difference between the previous profile (obtained in the case where the homogeneous section $[0,L/\epsilon^{1-\alpha}]$, with the parameters $\bar K/\epsilon^{2\alpha_K}$ and $\bar\rho/\epsilon^{2\alpha_\rho}$, is present) and the one in which this homogeneous section is missing (that is, the waveguide is homogeneous with parameters $\bar K$ and $\bar\rho$). The second profile is given, in [9, Chapter 20], by

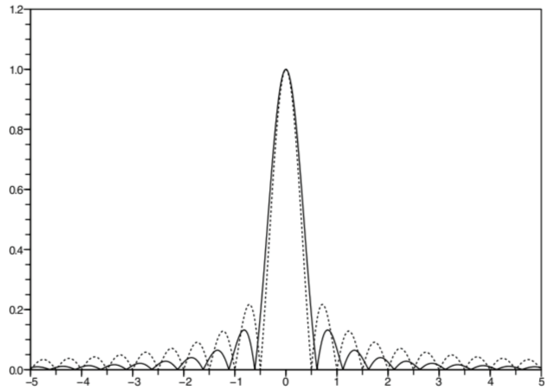
$$H^{\alpha_M}_{x_0,\text{no section}}(\omega_0,x)=\frac{1}{2}\sum_{j=1}^{N(\omega_0)}M_{jj}\phi_j(x_0)\phi_j(x),$$

which we can rewrite in the continuum limit.

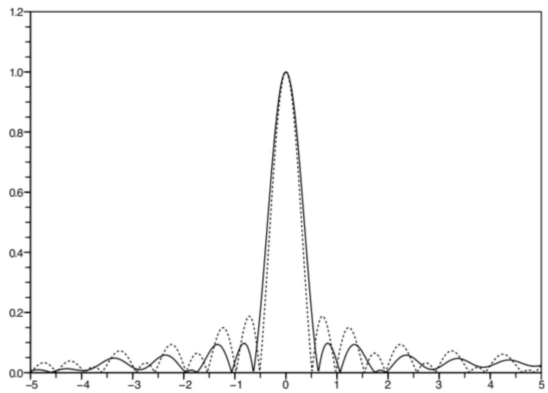
PROPOSITION 3. *For $\alpha_M\in[0,1)$, the spatial profile in the continuum limit is given by*

$$\lim_{\omega_0\to+\infty}\frac{\lambda_0^{1-\alpha_M}}{2}H^{\alpha_M}_{x_0,\text{no section}}(\omega_0,x_0+\lambda_0\tilde x)=\frac{\tilde d_2-\tilde d_1}{d}\text{sinc}(2\pi\tilde x),\tag{4.13}$$

where the sinc *function is defined by* $\text{sinc}(v)=\sin(v)/v$.



(a)



(b)

FIG. 2. *Renormalized modulus of the transverse profiles in the homogeneous waveguide. Here* $d = 10$, $d_M = 6$, $\tilde{d}_2 = 3$, $\tilde{d}_1 = 2$, $\lambda_0 = 0.01$, *and* $x_0 = 6$. *The dashed curve is the transverse profile in the case where the section is missing, and the solid curve is the refocusing profile in the case where we add a homogeneous section. In* (a) *we illustrate the case where* $\alpha_M \in [0, 1)$ *and in* (b) $\alpha_M = 1$.

The formula (4.13) corresponds to the classical diffraction limit with a focal spot of radius $\lambda_0/2$. In Figure 2, we compare, in the homogeneous case, the spatial profile (4.10) in the case where the homogeneous section $[0, L/\epsilon^{1-\alpha}]$ is present with the profile (4.13), where this section is missing. We can see that the main focal spot, in the case where a section is inserted, is larger than the focal spot produced when this section is missing (see Figure 4). The use of this section does not improve the refocalization in the homogeneous case. It is necessary to use an inhomogeneous section to induce mode coupling in order to enhance refocusing, as we will see in the next section.

4.4. Mean refocused field in the random case. Taking the expectation of (4.9), we obtain the mean refocused pulse

$$\mathbb{E}\left[e^{-i\omega_0 \frac{t_1 - t}{\epsilon}} p^{\epsilon}_{TR}\left(\frac{t}{\epsilon}, x, 0\right)\right]$$
$$= \frac{1}{16\pi} \int \sum_{j,m=1}^{N(\omega_0+\epsilon^p h)} \sum_{n=1}^{N_\epsilon(\omega_0+\epsilon^p h)} \sum_{l=1}^{\zeta} \sqrt{\frac{\beta^{\epsilon}_l(\omega_0 + \epsilon^p h)\beta_m(\omega_0 + \epsilon^p h)}{\beta_j(\omega_0 + \epsilon^p h)\beta^{\epsilon}_n(\omega_0 + \epsilon^p h)}}$$

$$\times \mathbb{E}\left[\overline{T^{\epsilon}_{jl}(\omega_0+\epsilon^p h, L)}T^{\epsilon}_{mn}(\omega_0+\epsilon^p h, L)\right] M_{mj}\phi_l(x_0)\phi_n(x)\overline{\widehat{f}(h)}$$
$$\times \frac{1}{\epsilon^{\alpha}}\overline{\tau^{\epsilon,+}_j(\omega_0+\epsilon^p h)}\tau^{\epsilon,+}_m(\omega_0+\epsilon^p h)e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}$$
$$\times\, e^{i(\beta^{\epsilon}_m(\omega_0+\epsilon^p h)-\beta^{\epsilon}_j(\omega_0+\epsilon^p h))\frac{L}{\epsilon^{1-\alpha}}}e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)}dh.$$

We will establish the convergence of the mean refocused pulse in the topological dual $\mathcal{E}'$ equipped with the weak topology, with $\mathcal{E}=\bigcup_{M\geq 1}\mathcal{E}_M$, and where

$$\mathcal{E}_M=\left\{\sum_{j=1}^{M}\mu_j\phi_j, \quad (\mu_j)_j\in\mathbb{R}^M\right\}.$$

$\mathcal{E}_M$ is equipped with the topology induced by $\langle .,.\rangle_{L^2(0,d)}$ and $\mathcal{E}$ with the inductive limit topology. It suffices to study $<\mathbb{E}[e^{i\omega_0\frac{t}{\epsilon^p}}p^{\epsilon}_{TR}(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^p},.,0)],\phi_n>_{L^2(0,d)}$ for $n\in\mathbb{N}^*$. Using Proposition 1, we get

(4.14)
$$\lim_{\zeta\to+\infty}\lim_{\epsilon\to 0}\left\langle\mathbb{E}\left[e^{i\omega_0\frac{t}{\epsilon^p}}p^{\epsilon}_{TR}\left(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^p},.,0\right)\right],\phi_n\right\rangle_{L^2(0,d)}$$
$$=\frac{1}{2}\sum_{j=1}^{N(\omega_0)}\sum_{l\geq 1}\frac{\beta_j(\omega_0)}{k(\omega_0)}\mathcal{T}^l_j(\omega_0,L)M_{jj}\phi_l(x_0)f(-t)\delta_{ln}+\mathcal{O}(N^2e^{-\Lambda L})$$
$$=\left\langle f(-t)H^{\alpha_M}_{x_0}(\omega_0,.),\phi_n\right\rangle_{L^2(0,d)}+\mathcal{O}(N^2e^{-\Lambda L}),$$

where the transverse profile is given by

$$H^{\alpha_M}_{x_0}(\omega_0,x)=\frac{1}{2}\sum_{l\geq 1}\sum_{j=1}^{N(\omega_0)}\frac{\beta_j(\omega_0)}{k(\omega_0)}\mathcal{T}^l_j(\omega_0,L)M_{jj}\phi_l(x)\phi_l(x_0).$$

In the continuum limit, the terms which correspond to $j\neq m$ decay exponentially because of the damping term $e^{-\Lambda L}$ since $\Lambda\simeq N^2\sigma^2/2$.

PROPOSITION 4. *For $\alpha_M\in[0,1]$, in the continuum limit, we have*

$$\lim_{\omega_0\to+\infty}\left(H^{\alpha_M}_{x_0}(\omega_0,.)-\tilde{H}^{\alpha_M}_{x_0}(\omega_0,.)\right)=0$$

in $\mathcal{E}'$, where

(4.15)
$$\lim_{\omega_0\to+\infty}\frac{\lambda_0^{1-\alpha_M}}{2}\tilde{H}^{\alpha_M}_{x_0}(\omega_0,x_0+\lambda_0\tilde{x})=\frac{\tilde{d}_2-\tilde{d}_1}{d}e^{-2L\sigma^2\pi^2\tilde{x}^2}\int_0^1\sqrt{1-u^2}\cos(2\pi\tilde{x}u)\,du.$$

From this proposition, in contrast with Proposition 2 which considers a homogeneous waveguide, the time-reversal coupling matrix does not play any role in the transverse profile of the mean refocused pulse. This result is consistent with those of [9] and [11].

Proof. Let $n\in\mathbb{N}^*$; we have

$$\lim_{\zeta\to+\infty}\left\langle H^{\alpha_M}_{x_0}(\omega_0,.),\phi_n\right\rangle_{L^2(0,d)}=\frac{1}{2}\phi_n(x_0)\sum_{j=1}^{N}\frac{\beta_j(\omega_0)}{k(\omega_0)}M_{jj}\mathcal{T}^n_j(\omega_0,L).$$

Using the probabilistic interpretation of $\mathcal{T}_j^n(\omega_0, L)$ in section 3, we get

$$\sum_{j=1}^N \frac{\beta_j(\omega_0)}{k(\omega_0)} M_{jj} \mathcal{T}_j^n(\omega_0, L) = \frac{d_2 - d_1}{2d} \mathbb{E}\left[\sqrt{1 - \left(\frac{X_L}{N}\right)^2} \mathbf{1}_{\left(\frac{X_L}{N} \in \left\{\frac{1}{N}, \dots, 1\right\}\right)} \middle| \frac{X_0}{N} = \frac{n}{N}\right]$$
$$- \sum_{j=1}^N \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \cos\left(j\pi \frac{d_2 + d_1}{d}\right) \sin\left(j\pi \frac{d_2 - d_1}{d}\right)$$
$$+ o(1).$$

Moreover, using Theorem 2

$$\mathbb{E}\left[\sqrt{1 - \left(\frac{X_L}{N}\right)^2} \mathbf{1}_{\left(\frac{X_L}{N} \in \left\{\frac{1}{N}, \dots, 1\right\}\right)} \middle| \frac{X_0}{N} = \frac{n}{N}\right]$$
$$= \mathbb{E}\left[\sqrt{1 - \left(\sigma B_L + \frac{n}{N}\right)^2} \mathbf{1}_{\left(\sigma B_L + \frac{n}{N} \in [-1, 1]\right)}\right] + o(1),$$

and we have the following result.

LEMMA 1.

$$\lim_{N \to +\infty} \sum_{j=1}^N \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \cos\left(j\pi \frac{d_2 + d_1}{d}\right) \sin\left(j\pi \frac{d_2 - d_1}{d}\right) = 0.$$

Proof. It suffices to show that

$$\lim_{N \to +\infty} \sum_{j=1}^N \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \sin\left(2j\pi \frac{d_2}{d}\right) = 0.$$

Let $\eta \in (0, 1)$; we have

$$\sum_{j=1}^N \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \sin\left(2j\pi \frac{d_2}{d}\right) \leq \mathbb{P}\left(\frac{X_L}{N} \in \left\{\frac{1}{N}, \dots, \frac{[N\eta]}{N}\right\} \middle| \frac{X_0}{N} = \frac{n}{N}\right)$$
$$+ \frac{1}{[N\eta] + 1} \sum_{j=[N\eta]+1}^N \mathbb{P}(X_L = j | X_0 = n).$$

Therefore,

$$\overline{\lim_N} \sum_{j=1}^N \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \sin\left(2j\pi \frac{d_2}{d}\right) \leq \mathbb{P}(\sigma B_L \in [0, \eta]),$$

and we get the result by letting $\eta \searrow 0$. This completes the proof of the lemma. ∎

This lemma shows that the time-reversal coupling matrix does not play a role in the transverse profile of the mean refocused pulse. Consequently,

$$\lim_{\omega_0 \to +\infty} \left\langle H_{x_0}^{\alpha_M}(\omega_0, .) - \tilde{H}_{x_0}^{\alpha_M}(\omega_0, .), \phi_n \right\rangle_{L^2(0,d)} = 0,$$

where

$$\tilde{H}_{x_0}^{\alpha_M}(\omega_0, x) = \frac{d_2 - d_1}{2d} \sum_{l \geq 1} \mathbb{E}\left[\sqrt{1 - \left(\sigma B_L + \frac{l}{N}\right)^2} \mathbf{1}_{(\sigma B_L + \frac{l}{N} \in [-1,1])}\right] \phi_l(x_0)\phi_l(x).$$

LEMMA 2. *In the continuum limit, we have*

$$\frac{\lambda_0^{1-\alpha_M}}{2} \tilde{H}_{x_0}^{\alpha_M}(\omega_0, x_0 + \lambda_0 \tilde{x}) = \frac{\tilde{d}_2 - \tilde{d}_1}{d} e^{-2L\sigma^2\pi^2\tilde{x}^2} \int_0^1 \sqrt{1-u^2} \cos(2\pi\tilde{x}u)\, du. \tag{4.16}$$

Proof. The proof is an application of the Poisson formula,

$$\sum_{m \in \mathbb{Z}} \widehat{F}_u(m) e^{imv} = 2\pi \sum_{m \in \mathbb{Z}} F_u(v + 2m\pi),$$

with $\widehat{F}_u(m) = e^{-\frac{(m-Nu)^2}{2N^2\sigma^2 L}}$ and $F_u(t) = \frac{\sqrt{2\pi N^2 \sigma^2 L}}{2\pi} e^{-t^2 N^2 \frac{\sigma^2}{2} L + itNu}$. Thus, we obtain

$$\begin{aligned}
&\frac{d}{d_2 - d_1} \tilde{H}_{x_0}^{\alpha_M}(\omega_0, x_0 + \lambda_0 \tilde{x}) \\
&= \frac{N}{d} \sum_{l \in \mathbb{Z}} e^{-N^2 \frac{\sigma^2}{2} L \left(\frac{\pi}{d}\lambda_0 \tilde{x} + 2l\pi\right)^2} \int_0^1 \sqrt{1-u^2} \cos\left[\left(\frac{\pi}{d}\lambda_0\tilde{x} + 2l\pi\right) Nu\right] du \\
&- \frac{N}{d} \sum_{l \in \mathbb{Z}} e^{-N^2 \frac{\sigma^2}{2} L \left(\frac{\pi}{d}(\lambda_0 \tilde{x} + 2x_0) + 2l\pi\right)^2} \int_0^1 \sqrt{1-u^2} \cos\left[\left(\frac{\pi}{d}(\lambda_0\tilde{x} + 2x_0) + 2l\pi\right) Nu\right] du.
\end{aligned}$$

Finally, we take only the term $l = 0$ in the first sum on the right because the rest of the first sum and the second sum are of order $\mathcal{O}(e^{-CN^2})$ uniformly in $\tilde{x}$. Moreover, we have $\lim_{\omega_0} \lambda_0 N/(2d) = 1$. This completes the proof of the lemma and the proof of the proposition. ☐

In Figure 3, we illustrate the differences between the transverse profiles of the refocused wave in the homogeneous case and when a random section is inserted. To show that random inhomogeneities enhance refocusing of the time-reversed waves, we consider two configurations. (a) and (c) illustrate the case where $\sigma \ll 1$ (weak fluctuations). We can see that the focal spot in the case where we add a section can be larger than in the case where this section is missing. In contrast, (b) and (d) illustrate the case where σ is large enough to have side-lobe suppression and a focal spot which is narrower than in the case where the random section is missing. In Figure 4, we illustrate the improvement of resolution compared to σ by using the FWHM, that is the *full width at half maximum*, which is a useful tool for studying the width of peaks. In the case where the random perturbed section is missing, the FWHM of the transverse profile given in Proposition 3 is of order $\lambda_0/2$. However, when this section is inserted, the FWHM of the transverse profile given in Proposition 4 is narrower than in the previous case for σ large enough. Consequently, if σ is large enough, the resolution is $< \lambda_0/2$.

4.5. Statistical stability. Pulse stabilization is proved by a frequency decoherence argument; see [5] in the context of a one-dimensional medium and [9, Chapter 20] in the context of waveguides. In our case, to prove the self-averaging property, we study the second order moment of $\epsilon^{i\omega_0 \frac{t}{\epsilon^p}} p_{TR}^{\epsilon}(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, x, 0)$. As in section 5.1, we

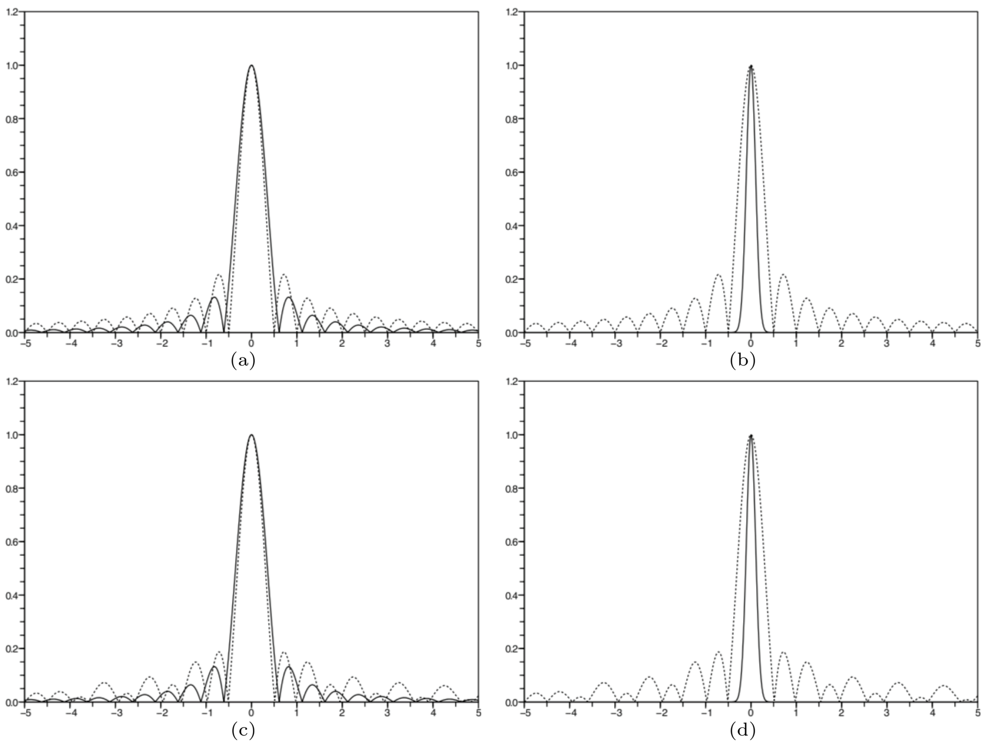


FIG. 3. *Renormalized modulus of the transverse profiles in a random waveguide. Here* $L = 1$, $d = 10$, $d_M = 6$, $\tilde{d}_2 = 3$, $\tilde{d}_1 = 2$, $\lambda_0 = 0.01$, *and* $x_0 = 6$. *In* (a) *and* (b), *we illustrate the case where* $\alpha_M \in [0, 1)$ *and in* (c) *and* (d) *the case where* $\alpha_M = 1$. *Dashed curves are transverse profiles in the case where the section is missing, and solid curves are transverse profiles in the case where we add a random a section, with* $\sigma = 0.01$ *in* (a) *and* (c) *and* $\sigma = 1.5$ *in* (b) *and* (d).

prove a limit theorem for $(\overline{T^{\epsilon}_{.l}(\omega + \epsilon^p h, .)} T^{\epsilon}_{.n}(\omega, .))_\epsilon$ and show that, $\forall p \in (0, 1)$ and $\alpha \in \left(0, \frac{1}{4} \wedge \frac{1-p}{2}\right)$,

$$\mathbb{E}\left[\overline{T^{\epsilon}_{jl}(\omega + \epsilon^p h, L)} T^{\epsilon}_{mn}(\omega, L)\right] = \mathbb{E}\left[\overline{T^{\epsilon}_{jl}(\omega + \epsilon^p h, L)}\right] \mathbb{E}\left[T^{\epsilon}_{mn}(\omega, L)\right] + \mathcal{O}\left(\epsilon^{(1/2)\wedge(1-2\alpha-p)}\right)$$

$\forall K \geq 1$ and $\forall (j, l, m, n) \in \{1, \ldots, K\}^4$. Consequently, we can use the same argument as in [9, Chapter 20] in the broadband case and

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\left(e^{i\omega_0 \frac{t}{\epsilon^p}} p^{\epsilon}_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, x, 0\right)\right)^2\right] = \lim_{\epsilon \to 0} \left(\mathbb{E}\left[e^{i\omega_0 \frac{t}{\epsilon^p}} p^{\epsilon}_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, x, 0\right)\right]\right)^2$$

in $\mathcal{E}'$.

4.6. Quarter wavelength plate. In this section, we explain how the transmission coefficients through the interface $z = L/\epsilon^{1-\alpha}$ can be made of order one. We have seen that the previous transmission coefficients, defined by (4.2), are particularly small, of order $\epsilon^{\alpha/2}$. This poor transmission can be corrected by inserting a quarter wavelength plate. A description of this antireflective process can be found in [9, Chapter 3]. This method is often used in echographic imaging; it consists in adding

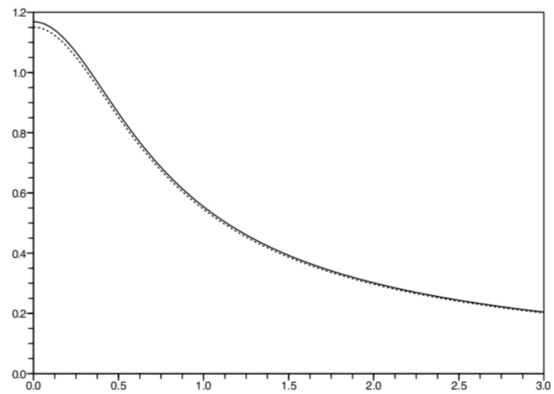


FIG. 4. *Ratio between the FWHM of the profile obtained when we add a section and that of the profile obtained when this section is missing, in terms of the standard deviation σ. Here $L = 1$, $d = 10$, $d_M = 6$, $\tilde{d}_2 = 3$, $\tilde{d}_1 = 2$, $\lambda_0 = 0.01$, and $x_0 = 6$. The solid curve represents the case where $\alpha_M \in [0, 1)$, and the dashed curve represents the case where $\alpha_M = 1$.*

a thin layer to enhance the transmission through an interface with the minimum loss of energy. In our situation, we will obtain a transmission of order one when it was of order $\epsilon^{\alpha/2}$ without this method. Here, we consider a source that emits a pulse of the form

$$f^\epsilon(t) = \frac{1}{2} f(\epsilon^p t) e^{-i\omega_0 t}.$$

Note that we no longer need the factor $1/\epsilon^\alpha$ as in (1.2) in order to get a refocused signal of order one. The medium parameters of this thin homogeneous layer located in the region $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ are given by

$$\rho^\epsilon(x, z) = \frac{\bar{\rho}}{\epsilon^{\alpha_\rho}} \text{ and } K^\epsilon(x, z) = \frac{\bar{K}}{\epsilon^{\alpha_K}} \quad \forall (x, z) \in \mathcal{D} \times (L/\epsilon^{1-\alpha}, L_c^\epsilon)$$
$$\text{with } L_c^\epsilon = \frac{L}{\epsilon^{1-\alpha}} + \epsilon^{\alpha/2} \frac{\lambda_0}{4}.$$

In the section $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$, the modal wavenumbers are

$$\tilde{\beta}_j^\epsilon(\omega) = \sqrt{\frac{k^2(\omega)}{\epsilon^\alpha} - j^2 \frac{\pi^2}{d^2}}, \quad j = 1, \dots, \left[\frac{k(\omega) d}{\epsilon^{\alpha/2} \pi} \right].$$

From the continuity of the pressure and velocity fields, the transmission coefficients of the layer become

$$\tau_j^{\epsilon,+}(\omega) = \frac{T_j^{0,\epsilon}(\omega) T_j^{1,\epsilon}(\omega) e^{i \tilde{\beta}_j^\epsilon(\omega)(L_c^\epsilon - L)}}{1 + R_j^{0,\epsilon}(\omega) R_j^{1,\epsilon}(\omega) e^{2i\tilde{\beta}_j^\epsilon(\omega)(L_c^\epsilon - L)}},$$

with

$$T_j^{0,\epsilon}(\omega) = \frac{2\sqrt{\beta_j^\epsilon(\omega)\tilde{\beta}_j^\epsilon(\omega)}}{\beta_j^\epsilon(\omega) + \tilde{\beta}_j^\epsilon(\omega)}, \qquad T_j^{1,\epsilon}(\omega) = \frac{2\sqrt{\beta_j(\omega)\tilde{\beta}_j^\epsilon(\omega)}}{\beta_j(\omega) + \tilde{\beta}_j^\epsilon(\omega)},$$
$$R_j^{0,\epsilon}(\omega) = \frac{\beta_j^\epsilon(\omega) - \tilde{\beta}_j^\epsilon(\omega)}{\beta_j^\epsilon(\omega) + \tilde{\beta}_j^\epsilon(\omega)}, \qquad R_j^{1,\epsilon}(\omega) = \frac{\tilde{\beta}_j^\epsilon(\omega) - \beta_j(\omega)}{\tilde{\beta}_j^\epsilon(\omega) + \beta_j(\omega)},$$

where $T_j^{0,\epsilon}$ and $R_j^{0,\epsilon}$ (resp., $T_j^{1,\epsilon}$ and $R_j^{1,\epsilon}$) are the transmission and reflection coefficients of the interface between the sections $(0, L/\epsilon^{1-\alpha})$ and $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ (resp., $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ and $(L_c^\epsilon, L_M/\epsilon^{1-\alpha})$). The refocused pulse is given by

(4.17)

$$\begin{aligned}
e^{-i\omega_0 \frac{t_1-t}{\epsilon}} p_{TR}^\epsilon\left(\frac{t}{\epsilon}, x, 0\right) = \frac{1}{16\pi} \int \sum_{j,m=1}^{N(\omega_0+\epsilon^p h)} \sum_{n=1}^{N_\epsilon(\omega_0+\epsilon^p h)} \sum_{l=1}^{\zeta} \sqrt{\frac{\beta_l^\epsilon(\omega_0+\epsilon^p h)\beta_m(\omega_0+\epsilon^p h)}{\beta_j(\omega_0+\epsilon^p h)\beta_n^\epsilon(\omega_0+\epsilon^p h)}} \\
\times \overline{T_{jl}^\epsilon(\omega+\epsilon^p h, L)} T_{mn}^\epsilon(\omega+\epsilon^p h, L) \overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)} \tau_m^{\epsilon,+}(\omega_0+\epsilon^p h) \\
\times M_{mj} \phi_l(x_0)\phi_n(x) \overline{\widehat{f}(h)} e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M}{\epsilon^{1-\alpha}} - L_c^\epsilon\right)} \\
\times e^{i(\beta_m^\epsilon(\omega_0+\epsilon h)-\beta_j^\epsilon(\omega_0+\epsilon h'))\frac{L}{\epsilon^{1-\alpha}}} e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)} dh' dh.
\end{aligned}$$

Note that the only difference between (4.9) and (4.17) is the expression for the product of transmission coefficients $\overline{\tau_j^{\epsilon,+}(\omega)}\tau_m^{\epsilon,+}(\omega)$. The limit as $\epsilon \to 0$ of this product is (4.11) in the absence of quarter wavelength plate. In the presence of the quarter wavelength plate, it is given by

$$\lim_{\epsilon\to 0} \overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)} \tau_m^{\epsilon,+}(\omega_0+\epsilon^p h) = 4\frac{\sqrt{\beta_j(\omega_0)\beta_m(\omega_0)}}{k(\omega_0)} \frac{1}{\left(\frac{\beta_j(\omega_0)}{k(\omega_0)}+1\right)\left(\frac{\beta_m(\omega_0)}{k(\omega_0)}+1\right)}.$$

From this result, we can analyze the mean refocused pulse and see that the statistical stability is not affected. The homogeneous spatial profile, with $\alpha_M = 1$, becomes

$$\frac{1}{2}\sum_{j=1}^{N(\omega_0)} \frac{\beta_j(\omega_0)}{k(\omega_0)} \frac{1}{\left(\frac{\beta_j(\omega_0)}{k(\omega_0)}+1\right)^2} M_{mj}\phi_j(x_0)\phi_j(x_0+\lambda_0\tilde{x}),$$

and in the case where $\alpha_M \in [0,1)$, we have in the continuum limit

$$\int_0^1 \frac{\sqrt{1-u^2}}{\left(1+\sqrt{1-u^2}\right)^2} \cos(2\pi\tilde{x}u)\, du.$$

In the random case, the expression of the mean refocused field (4.16) becomes

$$e^{-2L\sigma^2\pi^2\tilde{x}^2} \int_0^1 \frac{\sqrt{1-u^2}}{\left(1+\sqrt{1-u^2}\right)^2} \cos(2\pi\tilde{x}u)\, du$$

in the continuum limit.

To summarize, random inhomogeneities in the section $(0, L/\epsilon^{1-\alpha})$ ensure a conversion between low and high modes, and the quarter wavelength plate $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ ensures an efficient transmission from the perturbed section $(0, L/\epsilon^{1-\alpha})$ to the homogeneous medium $(L_c^\epsilon, L_M/\epsilon^{1-\alpha})$.

Conclusion. In this paper we have analyzed a time-reversal experiment in a homogeneous waveguide in which a heterogeneous section is inserted in the vicinity of the source. The role played by these inhomogeneities is quite different from the regime studied in [11], in which the random fluctuations are weak and distributed throughout the waveguide. In this case randomness enhances spatial refocusing up to the usual diffraction limit. But in our configuration, the random section permits us to refocus beyond this diffraction limit, and this effect is statistically stable in that it does not depend on the particular realization of the random section. The role of this random section is to ensure an strong conversion between low modes (that can propagate over large distances) and high modes (that carry the information about the small-scale features of the source). The insertion of a quarter wavelength plate completes the experimental set-up. It ensures an efficient transmission from the random section to the homogeneous one. It could be possible to build other experimental configurations (with a rough surface, for instance) in order to achieve superresolution. The important ingredient is that a time-reversible mechanism should convert high (evanescent) modes to low (propagating) modes in the vicinity of the source.

5. Appendix.

5.1. Proof of Theorem 1.

The proof of this theorem is based on a martingale approach using the perturbed-test-function method. We will first prove that $(U^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ converges in distribution in $\mathcal{C}([0,+\infty),\mathcal{H}_w)$, and we will conclude with an application of Ito's formula. To do this, we will prove the tightness of the family $(U^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ in $\mathcal{C}([0,+\infty),\mathcal{H}_w)$ using the criteria of Mitoma and Fouque [16, 8] and Theorem 4 in [14] which use the perturbed-test-function method. In a second part, we characterize all subsequence limits as solutions of a martingale problem in a Hilbert space. With the stochastic calculus in infinite-dimensional Hilbert spaces we will see that this martingale problem is well posed.

For any $\lambda\in\mathcal{H}$, we set $U^\epsilon_\lambda(\omega,z)=\langle\mathbf{U}^\epsilon(\omega,z),\lambda\rangle_{\mathcal{H}}$. According to the tightness criteria of Mitoma and Fouque [16, 8], the family $(U^\epsilon(\omega,.))_\epsilon$ is tight in $\mathcal{C}([0,+\infty),\mathcal{H}_w)$ if and only if the family $(U^\epsilon_\lambda(\omega,.))_\epsilon$ is tight in $\mathcal{C}([0,+\infty),\mathbb{C})$ $\forall\lambda\in\mathcal{H}$. Furthermore, $\|U^\epsilon(\omega,z)\|_{\mathcal{H}}=1$ $\forall z\geq 0\forall\epsilon\in(0,1)$, and $(U^\epsilon(\omega,.))_\epsilon$ is a family of continuous processes. Then, it is sufficient to prove that $(U^\epsilon_\lambda(\omega,.))_\epsilon$ is tight in $\mathcal{D}([0,+\infty),\mathbb{C})$ $\forall\lambda$ in a dense subset of $\mathcal{H}$. Let $\mathcal{E}_{\mathcal{H}}$ be the subspace of sequences with finite support equipped with the induced inner product. We have chosen $\mathcal{E}_{\mathcal{H}}$ for two reasons. First, $\mathcal{E}_{\mathcal{H}}$ is a dense subset of $\mathcal{H}$. Second, thanks to the band-limiting idealization, it allows one to avoid in (2.8) the unboundedness of $N_\epsilon(\omega)$ and the fact that $\epsilon\beta^\epsilon_j(\omega)$ goes to 0 for j of order $N_\epsilon(\omega)$ when $\epsilon\searrow 0$.

It will be convenient to consider the complex case for more convenient manipulations. Letting $\lambda\in\mathcal{E}_{\mathcal{H}}$, we consider the equation

$$\frac{d}{dt}U^\epsilon_\lambda(\omega,t)=\frac{1}{\sqrt{\epsilon}}F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t),C\left(\frac{t}{\epsilon}\right),\frac{t}{\epsilon}\right),$$

where

$$\begin{aligned}F^\epsilon_{jm}\left(\mathbf{U},C,s\right)=&\frac{-ik^2(\omega)}{2}\sum_{q=1}^{N_\epsilon(\omega)}\frac{C_{jq}}{\epsilon^\alpha\sqrt{\beta^\epsilon_j\beta^\epsilon_q}}e^{i\epsilon^\alpha\left(\beta^\epsilon_j-\beta^\epsilon_q\right)s}U_{qm}\\&+\frac{ik^2(\omega)}{2}\sum_{q=1}^{N_\epsilon(\omega)}\frac{C_{mq}}{\epsilon^\alpha\sqrt{\beta^\epsilon_m\beta^\epsilon_q}}e^{i\epsilon\left(\beta^\epsilon_q-\beta^\epsilon_m\right)s}U_{jq}.\end{aligned}$$

The proof of this theorem is based on the perturbed-test-function approach. Using the notion of a pseudogenerator, we prove tightness and characterize all subsequence limits.

5.1.1. Pseudogenerator. We recall the techniques developed by Kurtz and Kushner. Let $\mathcal{M}^\epsilon$ be the set of all $\mathcal{F}^\epsilon$-measurable functions $f(t)$ for which $\sup_{t\leq T} \mathbb{E}\left[|f(t)|\right] < +\infty$ and where $T > 0$ is fixed. The $p-\lim$ and the pseudogenerator are defined as follows. Let f and f^δ in $\mathcal{M}^\epsilon$ $\forall\delta > 0$. We say that $f = p-\lim_\delta f^\delta$ if

$$\sup_{t,\delta} \mathbb{E}[|f^\delta(t)|] < +\infty \quad \text{and} \quad \lim_{\delta\to 0} \mathbb{E}[|f^\delta(t) - f(t)|] = 0 \quad \forall t.$$

The domain of $\mathcal{A}^\epsilon$ is denoted by $\mathcal{D}(\mathcal{A}^\epsilon)$. We say that $f \in \mathcal{D}(\mathcal{A}^\epsilon)$ and $\mathcal{A}^\epsilon f = g$ if f and g are in $\mathcal{D}(\mathcal{A}^\epsilon)$ and

$$p - \lim_{\delta\searrow 0} \left[\frac{\mathbb{E}^\epsilon_t[f(t+\delta)] - f(t)}{\delta} - g(t)\right] = 0,$$

where $\mathbb{E}^\epsilon_t$ is the conditional expectation given $\mathcal{F}^\epsilon_t$ and $\mathcal{F}^\epsilon_t = \mathcal{F}_{\frac{t}{\epsilon}}$. A useful result about $\mathcal{A}^\epsilon$ is given by the following theorem.

THEOREM 3. *Let $f \in \mathcal{D}(\mathcal{A}^\epsilon)$. Then*

$$M^\epsilon_f(t) = f(t) - \int_0^t \mathcal{A}^\epsilon f(u)du \tag{5.1}$$

is an $(\mathcal{F}^\epsilon_t)$-martingale.

5.1.2. Tightness. We will consider the classical complex derivative with the following notation: If $v = \alpha + i\beta$, then $\partial_v = \frac{1}{2}(\partial_\alpha - i\partial_\beta)$ and $\partial_{\overline{v}} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$.

PROPOSITION 5. *$\forall\lambda \in \mathcal{E}_\mathcal{H}$, the family $(U^\epsilon_\lambda(\omega,.))_{\epsilon\in(0,1)}$ is tight in $\mathcal{D}([0,+\infty),\mathbb{C})$.*

Proof. According to Theorem 4 in Kushner [14], we need to show the following three lemmas. Let $\lambda \in \mathcal{E}_\mathcal{H}$, f be a smooth function, and $f^\epsilon_0(t) = f(U^\epsilon_\lambda(\omega,t))$. Thus,

$$\begin{aligned}\mathcal{A}^\epsilon f^\epsilon_0(t) = &\frac{1}{\sqrt{\epsilon}}\partial_v f(U^\epsilon_\lambda(\omega,t))\, F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t), C\left(\frac{t}{\epsilon}\right), \frac{t}{\epsilon}\right)\\ &+ \frac{1}{\sqrt{\epsilon}}\partial_{\overline{v}} f(U^\epsilon_\lambda(\omega,t))\, \overline{F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t), C\left(\frac{t}{\epsilon}\right), \frac{t}{\epsilon}\right)}.\end{aligned}$$

Let

$$\begin{aligned}f^\epsilon_1(t) = &\frac{1}{\sqrt{\epsilon}}\int_t^{+\infty} \mathbb{E}^\epsilon_t\left[F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t), C\left(\frac{u}{\epsilon}\right), \frac{u}{\epsilon}\right)\right]\partial_v f(U^\epsilon_\lambda(\omega,t))\, du\\ &+ \frac{1}{\sqrt{\epsilon}}\int_t^{+\infty} \mathbb{E}^\epsilon_t\left[\overline{F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t), C\left(\frac{u}{\epsilon}\right), \frac{u}{\epsilon}\right)}\right]\partial_{\overline{v}} f(U^\epsilon_\lambda(\omega,t))\, du.\end{aligned}$$

LEMMA 3. $\forall T > 0$, $\lim_\epsilon \sup_{0\leq t\leq T}|f^\epsilon_1(t)| = 0$ *almost surely, and* $\sup_{t\geq 0}\mathbb{E}\left[|f^\epsilon_1(t)|\right] = \mathcal{O}(\sqrt{\epsilon})$.

Proof of Lemma 3. By the Markov property of the Gaussian field, we get

$$
\begin{aligned}
f_1^\epsilon(t) = \sqrt{\epsilon}\partial_v f\left(U_\lambda^\epsilon(\omega,t)\right)\Bigg[\sum_{j,m} &-\frac{ik^2}{2}\sum_{|q-j|\le 1}\frac{C_{jq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_q^\epsilon}}e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)\frac{t}{\epsilon}} \\
&\times U_{qm}^\epsilon(\omega,t)\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2}\overline{\lambda_{jm}} \\
&+\frac{ik^2}{2}\sum_{|q-m|\le 1}\frac{C_{mq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_m^\epsilon\beta_q^\epsilon}}e^{i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)\frac{t}{\epsilon}}U_{jq}^\epsilon(\omega,t)\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2}\overline{\lambda_{jm}}\Bigg] \\
&+\sqrt{\epsilon}\partial_{\overline{v}} f\left(U_\lambda^\epsilon(\omega,t)\right)\Bigg[\sum_{j,m}\frac{ik^2}{2}\sum_{|q-j|\le 1}\frac{C_{jq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_q^\epsilon}}e^{i\epsilon(\beta_q^\epsilon-\beta_j^\epsilon)\frac{t}{\epsilon}} \\
&\times \overline{U_{qm}^\epsilon(\omega,t)}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_j^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_j^\epsilon)^2}\lambda_{jm} \\
&-\frac{ik^2}{2}\sum_{|q-m|\le 1}\frac{C_{mq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_m^\epsilon\beta_q^\epsilon}}e^{i\epsilon^\alpha(\beta_m^\epsilon-\beta_q^\epsilon)\frac{t}{\epsilon}}\overline{U_{jq}^\epsilon(\omega,t)}\frac{a+i\epsilon^\alpha(\beta_m^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_m^\epsilon-\beta_q^\epsilon)^2}\lambda_{jm}\Bigg].
\end{aligned}
$$

Using (1.4), we obtain

$$\mathbb{E}\left[|f_1^\epsilon(t)|\right] \le \sqrt{\epsilon}K(f,\lambda).$$

For the first part, we get

$$|f_1^\epsilon(t)| \le K(\lambda,f,T)\sqrt{\epsilon}\sup_{0\le t\le T}\sup_{x\in[0,d]}\left|V\left(x,\frac{t}{\epsilon}\right)\right|,$$

and we conclude with (1.3). ◻

LEMMA 4. $\{\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon\right)(t), \epsilon\in(0,1), 0\le t\le T\}$ *is uniformly integrable.*

Proof of Lemma 4. After a computation, we get

$$\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon\right)(t) = \tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t), \left(C_{jl}\left(\frac{t}{\epsilon}\right)C_{mn}\left(\frac{t}{\epsilon}\right)\right)_{j,l,m,n}, \frac{t}{\epsilon}\right),$$

where

$$
\begin{aligned}
\tilde{F}_\lambda^\epsilon\left(\mathbf{U},C,s\right) &= \partial_v f(U)\tilde{F}_\lambda^{1,\epsilon}(U,C,s) + \partial_{\overline{v}} f(U)\overline{\tilde{F}_\lambda^{1,\epsilon}(U,C,s)} \\
&+ \partial_v^2 f(U)\tilde{F}_\lambda^{2,\epsilon}(U,C,s) + \partial_{\overline{v}}^2 f(U)\overline{\tilde{F}_\lambda^{2,\epsilon}(U,C,s)} \\
&+ \partial_{\overline{v}}\partial_v f(U)\tilde{F}_\lambda^{3,\epsilon}(U,C,s) + \partial_v\partial_{\overline{v}} f(U)\overline{\tilde{F}_\lambda^{3,\epsilon}(U,C,s)},
\end{aligned}
$$

with

$$\begin{aligned}
&\tilde{F}^{1,\epsilon}_\lambda(U,C,s)\\
&= \frac{k^4}{4}\sum_{j,m}\left[\sum_{q,q'=1}^{N_\epsilon} -\frac{C_{jqqq'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_q\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_{q'})s} U_{q'm}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\right.\\
&+\frac{C_{jqmq'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_m\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q+\beta^\epsilon_{q'}-\beta^\epsilon_m)s} U_{qq'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\\
&+\frac{C_{jq'mq}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_{q'}\beta^\epsilon_m\beta^\epsilon_q}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_{q'}+\beta^\epsilon_q-\beta^\epsilon_m)s} U_{q'q}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\\
&\left.-\frac{C_{mqqq'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_m\beta^\epsilon_q\beta^\epsilon_q\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_{q'}-\beta^\epsilon_m)s} U_{jq'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\right]\overline{\lambda_{jm}},
\end{aligned}$$

$$\begin{aligned}
&\tilde{F}^{2,\epsilon}_\lambda(U,C,s)\\
&= \frac{k^4}{4}\sum_{\substack{j,m\\j',m'}}\left[\sum_{q,q'=1}^{N_\epsilon} -\frac{C_{jqj'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_{j'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q+\beta^\epsilon_{j'}-\beta^\epsilon_{q'})s} U_{qm}U_{q'm'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\right.\\
&+\frac{C_{jqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_{m'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q+\beta^\epsilon_{q'}-\beta^\epsilon_{m'})s} U_{qm}U_{j'q'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\\
&+\frac{C_{j'q'mq}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_{j'}\beta^\epsilon_{q'}\beta^\epsilon_m\beta^\epsilon_q}} e^{i\epsilon^\alpha(\beta^\epsilon_{j'}-\beta^\epsilon_{q'}+\beta^\epsilon_q-\beta^\epsilon_m)s} U_{jq}U_{q'm'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\\
&\left.-\frac{C_{mqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_m\beta^\epsilon_q\beta^\epsilon_{m'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m+\beta^\epsilon_{q'}-\beta^\epsilon_{m'})s} U_{jq}U_{j'q'}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\right]\overline{\lambda_{jm}\lambda_{j'm'}},
\end{aligned}$$

$$\begin{aligned}
&\tilde{F}^{3,\epsilon}_\lambda(U,C,s)\\
&= \frac{k^4}{4}\sum_{\substack{j,m\\j',m'}}\left[\sum_{q,q'=1}^{N_\epsilon} \frac{C_{jqj'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_{j'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q-\beta^\epsilon_{j'}+\beta^\epsilon_{q'})s} U_{qm}\overline{U_{q'm'}}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\right.\\
&-\frac{C_{jqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_j\beta^\epsilon_q\beta^\epsilon_{m'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q-\beta^\epsilon_{q'}+\beta^\epsilon_{m'})s} U_{qm}\overline{U_{j'q'}}\frac{a+i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_j-\beta^\epsilon_q)^2}\\
&-\frac{C_{j'q'mq}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_{j'}\beta^\epsilon_{q'}\beta^\epsilon_m\beta^\epsilon_q}} e^{i\epsilon^\alpha(-\beta^\epsilon_{j'}+\beta^\epsilon_{q'}+\beta^\epsilon_q-\beta^\epsilon_m)s} U_{jq}\overline{U_{q'm'}}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\\
&\left.+\frac{C_{mqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta^\epsilon_m\beta^\epsilon_q\beta^\epsilon_{m'}\beta^\epsilon_{q'}}} e^{i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m-\beta^\epsilon_{q'}+\beta^\epsilon_{m'})s} U_{jq}\overline{U_{j'q'}}\frac{a+i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)}{a^2+\epsilon^{2\alpha}(\beta^\epsilon_q-\beta^\epsilon_m)^2}\right]\overline{\lambda_{jm}}\lambda_{j'm'}.
\end{aligned}$$

From this expression, using (1.4), we can check that $\sup_{\epsilon,t}\mathbb{E}[|\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon\right)(t)|^2] < +\infty$. $\square$

LEMMA 5.

$$\lim_{K\to+\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{0\le t\le T}|U^{\epsilon}_{\lambda}(\omega,t)|\ge K\right)=0.$$

Proof of Lemma 5. We have

$$|U^{\epsilon}_{\lambda}(\omega,t)|=\left|\sum_{j,m\ge 1}U^{\epsilon}_{jm}(\omega,t)\overline{\lambda_{jm}}\right|\le\|\lambda\|_{\mathcal{H}}.$$

5.1.3. Martingale problem. In this section, using a well-posed martingale problem, we characterize all subsequence limits. Let a converging subsequence of $(\mathbf{U}^{\epsilon}(\omega,.))_{\epsilon\in(0,1)}$ to $\mathbf{U}(\omega,.)$ in $\mathcal{C}([0,+\infty),\mathcal{H}_w)$ that we also denote by $(\mathbf{U}^{\epsilon}(\omega,.))_{\epsilon\in(0,1)}$.

PROPOSITION 6 (convergence result). *$\forall\lambda\in\mathcal{E}_{\mathcal{H}}$ and $\forall f$ smooth test functions,*

$$\begin{aligned}
f\left(\boldsymbol{U}_{\lambda}(\omega,z)\right)-\int_0^z &\partial_v f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\langle J(\boldsymbol{U}(\omega,s)),\lambda\rangle_{\mathcal{H}}+\partial_{\overline{v}}f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\overline{\langle J(\boldsymbol{U}(\omega,s)),\lambda\rangle_{\mathcal{H}}}\\
&+\partial_v^2 f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\langle K\left(\boldsymbol{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\\
&+\partial_{\overline{v}}^2 f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\overline{\langle K\left(\boldsymbol{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}\\
&+\partial_{\overline{v}}\partial_v f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\langle L\left(\boldsymbol{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\\
&+\partial_v\partial_{\overline{v}} f\left(\boldsymbol{U}_{\lambda}(\omega,s)\right)\overline{\langle L\left(\boldsymbol{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}ds
\end{aligned}$$

is a martingale, where

$$\begin{aligned}
J(x)_{jm}&=\Lambda\left[(x_{j+1j+1}\delta_{jm}-x_{jm})+(x_{j-1j-1}\delta_{jm}-x_{jm})\right],\\
K(x)(\lambda)_{jm}&=\frac{\Lambda}{2}\left[x_{j-1m}\left(\langle x_{j-1},\lambda_j\rangle_2-\langle x_j,\lambda_{j-1}\rangle_1\right)\right.\\
&\qquad\left.+x_{j+1m}\left(\langle x_{j+1},\lambda_j\rangle_2-\langle x_j,\lambda_{j+1}\rangle_1\right)\right]\\
&\quad+\frac{\Lambda}{2}\left[x_{jm-1}\left(\langle x_{m-1},\lambda_m\rangle_1-\langle x_m,\lambda_{m-1}\rangle_2\right)\right.\\
&\qquad\left.+x_{jm+1}\left(\langle x_{m+1},\lambda_m\rangle_1-\langle x_m,\lambda_{m+1}\rangle_2\right)\right],\\
L(x)(\lambda)_{jm}&=\frac{\Lambda}{2}\left[x_{j-1m}\left(\overline{\langle x_{j-1},\lambda_j\rangle_1}-\overline{\langle x_j,\lambda_{j-1}\rangle_2}\right)\right.\\
&\qquad\left.+x_{j+1m}\left(\overline{\langle x_{j+1},\lambda_j\rangle_1}-\overline{\langle x_j,\lambda_{j+1}\rangle_2}\right)\right]\\
&\quad+\frac{\Lambda}{2}\left[x_{jm-1}\left(\overline{\langle x_{m-1},\lambda_m\rangle_2}-\overline{\langle x_m,\lambda_{m-1}\rangle_1}\right)\right.\\
&\qquad\left.+x_{jm+1}\left(\overline{\langle x_{m+1},\lambda_m\rangle_2}-\overline{\langle x_m,\lambda_{m+1}\rangle_1}\right)\right],
\end{aligned}$$

with

$$\langle\lambda_j,\mu_j\rangle_1=\sum_{m\ge 1}\lambda_{jm}\overline{\mu_{jm}},\quad\langle\lambda_m,\mu_m\rangle_2=\sum_{j\ge 1}\lambda_{jm}\overline{\mu_{jm}}$$

$\forall j,m\ge 1$, and for $(x,\lambda,\mu)\in\mathcal{H}\times\mathcal{E}_{\mathcal{H}}\times\mathcal{E}_{\mathcal{H}}$.

Proof of Proposition 6. Let

$$\begin{aligned}
f_2^{\epsilon}(t)=\int_t^{+\infty}&\mathbb{E}_t^{\epsilon}\left[\tilde{F}^{\epsilon}_{\lambda}\left(\mathbf{U}^{\epsilon}(\omega,t),\left(C_{jl}\left(\frac{u}{\epsilon}\right)C_{mn}\left(\frac{u}{\epsilon}\right)\right)_{j,l,m,n},\frac{u}{\epsilon}\right)\right]\\
&-\tilde{F}^{\epsilon}_{\lambda}\left(\mathbf{U}^{\epsilon}(\omega,t),\left(\mathbb{E}[C_{jl}(0)C_{mn}(0)]\right)_{j,l,m,n},\frac{u}{\epsilon}\right)du.
\end{aligned}$$

LEMMA 6.

$$\sup_{t\geq 0}\mathbb{E}\left[|f_2^\epsilon(t)|\right]=\mathcal{O}\left(\epsilon\right)$$

and

$$\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon+f_2^\epsilon\right)(t)=\tilde{F}_\lambda^\epsilon\left(\boldsymbol{U}^\epsilon(\omega,t),\left(S(j-l,m-n)\right)_{j,l,m,n},\frac{t}{\epsilon}\right)+A(\epsilon,t),$$

where $\sup_{t\geq 0}\mathbb{E}\left[|A(\epsilon,t)|\right]=\mathcal{O}(\sqrt{\epsilon})$.

Proof of Lemma 6. A change of variable gives

$$\begin{aligned}
f_2^\epsilon(t)=\epsilon\int_0^{+\infty}&\mathbb{E}_t^\epsilon\left[\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(C_{jl}\left(u+\frac{t}{\epsilon}\right)C_{mn}\left(u+\frac{t}{\epsilon}\right)\right)_{j,l,m,n},u+\frac{t}{\epsilon}\right)\right]\\
&-\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(\mathbb{E}[C_{jl}(0)C_{mn}(0)]\right)_{j,l,m,n},u+\frac{t}{\epsilon}\right)du\\
&=\epsilon B(\epsilon,t).
\end{aligned}$$

By a computation, we can check that $\sup_{\epsilon,t\geq 0}\mathbb{E}\left[|B(\epsilon,t)|\right]<+\infty$. The second part of this lemma follows a long but straightforward computation. ◻

We note that $\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\frac{t}{\epsilon}\right)=\tilde{F}_\lambda^\epsilon(\mathbf{U}^\epsilon(\omega,t),(S(j-l,m-n))_{j,l,m,n},\frac{t}{\epsilon})$ and let

$$f_3^\epsilon(t)=-\int_0^t\left[\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\frac{u}{\epsilon}\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds\right]du.$$

LEMMA 7. *We have*

$$\sup_{t\geq 0}\mathbb{E}\left[|f_3^\epsilon(t)|\right]=\mathcal{O}(\epsilon^{1-2\alpha}).$$

Then, we need to have $\alpha\in(0,1/2)$.

Proof of Lemma 7. After a change of variable, we get

$$f_3^\epsilon(t)=-\epsilon\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),u\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds\right]du,$$

and

$$\sup_{t,\epsilon}\mathbb{E}\left[\left|\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),u\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds\right]du\right|\right]\leq\frac{K}{\epsilon^{2\alpha}}.\qquad ◻$$

Let $f^\epsilon(t)=f_0^\epsilon(t)+f_1^\epsilon(t)+f_2^\epsilon(t)+f_3^\epsilon(t)$. With the boundness condition (1.4), a computation gives

$$\mathcal{A}^\epsilon f^\epsilon(t)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds+C(\epsilon,t).$$

We assume that the following nondegeneracy condition holds. $\forall\epsilon\in(0,1)$, the wavenumbers $\beta_j^\epsilon(\omega)=\beta_j(\omega/\epsilon^\alpha)$ are distinct along with their sums and differences. Consequently, we get

$$
\begin{aligned}
\mathcal{A}^\epsilon f^\epsilon(t) = {} & \partial_v f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\langle J(\mathbf{U}^\epsilon(\omega,t)),\lambda\rangle_{\mathcal{H}} + \partial_{\overline{v}} f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\overline{\langle J(\mathbf{U}^\epsilon(\omega,t)),\lambda\rangle_{\mathcal{H}}} \\
& + \partial^2_v f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\langle K\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}} \\
& + \partial^2_{\overline{v}} f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\overline{\langle K\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}} \\
& + \partial_{\overline{v}}\partial_v f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\langle L\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}} \\
& + \partial_v\partial_{\overline{v}} f\left(\mathbf{U}^\epsilon_\lambda(\omega,t)\right)\overline{\langle L\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}} \\
& + C(\epsilon,t),
\end{aligned} \tag{5.2}
$$

where $\sup_{t\geq 0}\mathbb{E}\left[|C(\epsilon,t)|\right] = \mathcal{O}(\epsilon^{\frac{1}{2}-2\alpha})$. Then, we need to have $\alpha\in(0,1/4)$. By Theorem 3, $(M^\epsilon_{f^\epsilon}(t))_{t\geq 0}$ is an $(\mathcal{F}^\epsilon_t)$-martingale; this implies that for every bounded continuous function h and every sequence $0 < s_1 < \cdots < s_n \leq s < t$ we have

$$
\mathbb{E}\left[h\left(\mathbf{U}^\epsilon_\lambda(\omega,s_j), 1\leq j\leq n\right)\left(f^\epsilon(t) - f^\epsilon(s) - \int_s^t \mathcal{A}^\epsilon f^\epsilon(u)du\right)\right] = 0.
$$

Finally, using (5.2) and (2.7) with Lemmas 3, 6, and 7, we get the announced result. $\square$

Uniqueness. To show uniqueness, we will decompose $\mathbf{U}(\omega,.)$ into real and imaginary parts and consider the new process

$$
\mathbf{Y}(\omega,t) = \begin{bmatrix} Y^1(\omega,t) \\ Y^2(\omega,t)\end{bmatrix}, \text{ where } Y^1(\omega,t) = \mathrm{Re}\left(\mathbf{U}(\omega,t)\right) \text{ and } Y^2(\omega,t) = \mathrm{Im}\left(\mathbf{U}(\omega,t)\right).
$$

Let $\mathcal{G} = l^2\left(E,\mathbb{R}\right)$. $\mathcal{G}\times\mathcal{G}$ is endowed with the inner product defined by

$$
\langle x,y\rangle_{\mathcal{G}\times\mathcal{G}} = \sum_{j,m\geq 1} x^1_{jm}y^1_{jm} + x^2_{jm}y^2_{jm}
$$

$\forall(x,y)\in\mathcal{G}\times\mathcal{G}$. We also use the notation $\mathbf{Y}_\lambda(\omega,t) = \langle\mathbf{Y}(\omega,t),\lambda\rangle$ with $\lambda\in\mathcal{G}\times\mathcal{G}$. We introduce the operator φ on $\mathcal{G}\times\mathcal{G}$ given by

$$
\begin{aligned}
\varphi : \mathcal{G}\times\mathcal{G} & \longrightarrow \mathcal{G}\times\mathcal{G}, \\
\begin{bmatrix} x^1 \\ x^2\end{bmatrix} & \longmapsto \begin{bmatrix} x^2 \\ -x^1\end{bmatrix}.
\end{aligned}
$$

Let f be a smooth function on $\mathbb{R}$. By Proposition 6, we get the following.

PROPOSITION 7. $\forall\lambda\in\mathcal{E}_{\mathcal{G}\times\mathcal{G}}$,

$$
\begin{aligned}
f\left(\boldsymbol{Y}_\lambda(\omega,t)\right) - \int_0^t & \langle J(\boldsymbol{Y}(\omega,s)),\lambda\rangle_{\mathcal{G}\times\mathcal{G}} f'\left(\boldsymbol{Y}_\lambda(\omega,s)\right) \\
& + \frac{1}{2}\langle A\left(\boldsymbol{Y}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{G}\times\mathcal{G}} f''\left(\boldsymbol{Y}_\lambda(\omega,s)\right)ds
\end{aligned}
$$

is a martingale, where

$$\begin{aligned}
A(x)(\lambda)_{jm} = \frac{\Lambda}{2} \Big[& x_{j+1m} \left[\langle x_{j+1}, \lambda_j\rangle_1 - \langle x_j, \lambda_{j+1}\rangle_2 + \langle x_{j+1}, \lambda_j\rangle_2 - \langle x_j, \lambda_{j+1}\rangle_1\right] \\
& + x_{j-1m} \left[\langle x_{j-1}, \lambda_j\rangle_1 - \langle x_j, \lambda_{j-1}\rangle_2 + \langle x_{j-1}, \lambda_j\rangle_2 - \langle x_j, \lambda_{j-1}\rangle_1\right] \\
& + x_{jm+1} \left[\langle x_{m+1}, \lambda_m\rangle_1 - \langle x_m, \lambda_{m+1}\rangle_2 + \langle x_{m+1}, \lambda_m\rangle_2 - \langle x_m, \lambda_{m+1}\rangle_1\right] \\
& + x_{jm-1} \left[\langle x_{m-1}, \lambda_m\rangle_1 - \langle x_m, \lambda_{m-1}\rangle_2 + \langle x_{m-1}, \lambda_m\rangle_2 - \langle x_m, \lambda_{m-1}\rangle_1\right] \\
& + \varphi(x)_{j+1m} \left[\langle \varphi(x)_{j+1}, \lambda_j\rangle_1 - \langle \varphi(x)_j, \lambda_{j+1}\rangle_2 \right.\\
& \qquad \left. - \langle \varphi(x)_{j+1}, \lambda_j\rangle_2 + \langle \varphi(x)_j, \lambda_{j+1}\rangle_1\right] \\
& + \varphi(x)_{j-1m} \left[\langle \varphi(x)_{j-1}, \lambda_j\rangle_1 - \langle \varphi(x)_j, \lambda_{j-1}\rangle_2 \right.\\
& \qquad \left. - \langle \varphi(x)_{j-1}, \lambda_j\rangle_2 + \langle \varphi(x)_j, \lambda_{j-1}\rangle_1\right] \\
& + \varphi(x)_{jm+1} \left[\langle \varphi(x)_{m+1}, \lambda_m\rangle_2 - \langle \varphi(x)_m, \lambda_{m+1}\rangle_1 \right.\\
& \qquad \left. - \langle \varphi(x)_{m+1}, \lambda_m\rangle_1 + \langle \varphi(x)_m, \lambda_{m+1}\rangle_2\right] \\
& + \varphi(x)_{jm-1} \left[\langle \varphi(x)_{m-1}, \lambda_m\rangle_2 - \langle \varphi(x)_m, \lambda_{m-1}\rangle_1 \right.\\
& \qquad \left. - \langle \varphi(x)_{m-1}, \lambda_m\rangle_1 - \langle \varphi(x)_m, \lambda_{m-1}\rangle_2\right]\Big]
\end{aligned}$$

for $(x,\lambda) \in (\mathcal{G}\times\mathcal{G})^2$.

Proof of Proposition 7. By Proposition 6,

$$\begin{aligned}
f\left(\mathbf{Y}_\lambda(\omega,t)\right) - \int_0^t & \langle J(\mathbf{Y}(\omega,s)), \lambda\rangle_{\mathcal{G}\times\mathcal{G}}\, f'\left(\mathbf{Y}_\lambda(\omega,s)\right) \\
& + \frac{1}{2}\mathrm{Re}\left(\langle (L+K)\left(\mathbf{U}(\omega,s)\right)(\lambda), \lambda\rangle_{\mathcal{H}}\right) f''\left(\mathbf{Y}_\lambda(\omega,s)\right) ds
\end{aligned}$$

is a martingale, where we have also denoted by λ the sequence $\lambda^1 + i\lambda^2$. In addition,

$$\mathrm{Re}\left(\langle \mathbf{U}(\omega,t)_j, \lambda_j\rangle\right) = \langle \mathbf{Y}(\omega,t)_j, \lambda_j\rangle \text{ and } \mathrm{Im}\left(\langle \mathbf{U}(\omega,t)_j, \lambda_j\rangle\right) = \langle \varphi\left(\mathbf{Y}(\omega,t)\right)_j, \lambda_j\rangle,$$

and we get $\mathrm{Re}\left(\langle (L+K)\left(\mathbf{U}(\omega,s)\right)(\lambda), \lambda\rangle_{\mathcal{H}}\right) = \langle A\left(\mathbf{Y}(\omega,s)\right)(\lambda), \lambda\rangle_{\mathcal{G}\times\mathcal{G}}$. ∎

From this last proposition, for $f(x) = x$ and $f(x) = x^2$, we get that

$$M_\lambda(t) = \left\langle \mathbf{Y}(\omega,t) - \int_0^t J(\mathbf{Y}(\omega,s))ds, \lambda \right\rangle_{\mathcal{G}\times\mathcal{G}}$$

is a martingale with quadratic variation given by

$$\langle M_\lambda\rangle(t) = \int_0^t \langle A(\mathbf{Y}(\omega,s))(\lambda), \lambda\rangle_{\mathcal{G}\times\mathcal{G}}\, ds.$$

PROPOSITION 8. $\forall f \in \mathcal{C}_b^2(\mathcal{G}\times\mathcal{G})$,

$$f(\boldsymbol{Y}(\omega,t)) - \int_0^t Lf(\boldsymbol{Y}(\omega,s))ds \tag{5.3}$$

is a martingale, where $\forall x \in \mathcal{G}\times\mathcal{G}$

$$Lf(x) = \frac{1}{2}\mathrm{trace}\left(A(x)D^2 f(x)\right) + \langle J(x), Df(x)\rangle_{\mathcal{G}\times\mathcal{G}}\,.$$

Moreover, the associated martingale problem is well posed.

Proof of Proposition 8. We begin with the following lemma.

LEMMA 8.

$$A : \mathcal{G}\times\mathcal{G} \longrightarrow L_1^+(\mathcal{G}\times\mathcal{G}),$$
$$J : \mathcal{G}\times\mathcal{G} \longrightarrow \mathcal{G}\times\mathcal{G},$$

where $L_1^+(\mathcal{G}\times\mathcal{G})$ is a set of nonnegative operators with finite trace. We have, $\forall x \in \mathcal{G}\times\mathcal{G}$, $A(x) = \sigma^*(x)\circ\sigma(x)$ with

$$\sigma : \mathcal{G}\times\mathcal{G} \longrightarrow L_2(\mathcal{G}\times\mathcal{G}),$$

where $L_2(\mathcal{G}\times\mathcal{G})$ is the set of Hilbert–Schmidt operators on $\mathcal{G}\times\mathcal{G}$, σ^* is the adjoint operator of σ, and

$$\begin{aligned}
\sigma^1(x)(\lambda)_{jm} &= \sqrt{\frac{\Lambda}{2}}\left(\langle x_{j+1},\lambda_j\rangle_1 + \langle x_{j+1},\lambda_j\rangle_2 - \langle x_j,\lambda_{j+1}\rangle_1 - \langle x_j,\lambda_{j+1}\rangle_2\right)\delta_{j+1m},\\
\sigma^2(x)(\lambda)_{jm} &= \sqrt{\frac{\Lambda}{2}}\Big(\langle\varphi(x)_{j+1},\lambda_j\rangle_1 - \langle\varphi(x)_{j+1},\lambda_j\rangle_2 + \langle\varphi(x)_j,\lambda_{j+1}\rangle_1\\
&\qquad - \langle\varphi(x)_j,\lambda_{j+1}\rangle_2\Big)\delta_{j+1m}.
\end{aligned}$$

Proof. $\forall(x,\lambda,\mu)\in(\mathcal{G}\times\mathcal{G})^3$, we have

$$\begin{aligned}
\langle A(x)(\lambda),\mu\rangle_{\mathcal{G}\times\mathcal{G}} = \frac{\Lambda}{2}\sum_{j\geq 1} &\left(\langle x_{j+1},\lambda_j\rangle_1 + \langle x_{j+1},\lambda_j\rangle_2 - \langle x_j,\lambda_{j+1}\rangle_1 - \langle x_j,\lambda_{j+1}\rangle_2\right)\\
&\times\left(\langle x_{j+1},\mu_j\rangle_1 + \langle x_{j+1},\mu_j\rangle_2 - \langle x_j,\mu_{j+1}\rangle_1 - \langle x_j,\mu_{j+1}\rangle_2\right)\\
&+\left(\langle\varphi(x)_{j+1},\lambda_j\rangle_1 - \langle\varphi(x)_{j+1},\lambda_j\rangle_2 + \langle\varphi(x)_j,\lambda_{j+1}\rangle_1 - \langle\varphi(x)_j,\lambda_{j+1}\rangle_2\right)\\
&\times\left(\langle\varphi(x)_{j+1},\mu_j\rangle_1 - \langle\varphi(x)_{j+1},\mu_j\rangle_2 + \langle\varphi(x)_j,\mu_{j+1}\rangle_1 - \langle\varphi(x)_j,\mu_{j+1}\rangle_2\right)\\
&= \langle\sigma(x)(\lambda),\sigma(x)(\mu)\rangle_{\mathcal{G}\times\mathcal{G}}.
\end{aligned}$$

Let $(e^\eta_{jl})_{\substack{\eta=1,2\\ j,l\geq 1}}$ be the family of elements in $\mathcal{G}\times\mathcal{G}$ defined by

$$e^1_{jl} = \begin{bmatrix}\delta_{jl}\\ \mathbf{0}\end{bmatrix} \text{ and } e^2_{jl} = \begin{bmatrix}\mathbf{0}\\ \delta_{jl}\end{bmatrix}.$$

This family defines a basis of $\mathcal{G}\times\mathcal{G}$ and $\forall x\in\mathcal{G}\times\mathcal{G}$,

$$\operatorname{trace}(A(x)) = \sum_{\substack{\eta=1,2\\ j,l\geq 1}}\left\langle A(x)(e^\eta_{jl}),e^\eta_{jl}\right\rangle_{\mathcal{G}\times\mathcal{G}} = \sum_{\substack{\eta=1,2\\ j,l\geq 1}}\|\sigma(x)(e^\eta_{jl})\|^2_{\mathcal{G}\times\mathcal{G}} \leq 16\|x\|^2_{\mathcal{G}\times\mathcal{G}}. \qquad \square$$

From this lemma and Theorem 4.1.4 in [20], (5.3) is a martingale. By Theorems 3.2.2 and 4.4.1 in [20], the martingale problem is well posed since σ is linear in x and $\forall x\in\mathcal{G}\times\mathcal{G}$

$$\|\sigma(x)\| = \|\sigma^*(x)\| \leq 4\|x\|_{\mathcal{G}\times\mathcal{G}}. \qquad \square$$

Using the representation theorem, Theorem 4.3.5, in [20], there exists a cylindrical Brownian motion $(B_t)_{t\geq 0}$ defined on $\mathcal{G}\times\mathcal{G}$ such that

$$\mathbf{Y}(\omega,t) = \mathbf{Y}(\omega,0) + \int_0^t J(\mathbf{Y}(\omega,s))ds + \int_0^t \sigma^*(\mathbf{Y}(\omega,s))dB_s.$$

By the definition of the last integral, we have

$$\left\langle e^{\eta}_{jl}, \int_0^t \sigma^*(\mathbf{Y}(\omega,s))dB_s \right\rangle_{\mathcal{G}\times\mathcal{G}} = \int_0^t \left\langle \sigma(\mathbf{Y}(\omega,s))(e^{\eta}_{jl}), dB_s \right\rangle_{\mathcal{G}\times\mathcal{G}}$$
$$= \sum_{\substack{\theta=1,2\\ r,s\geq 1}} \int_0^t \left\langle \sigma(\mathbf{Y}(\omega,s))(e^{\eta}_{jl}), e^{\theta}_{rs} \right\rangle_{\mathcal{G}\times\mathcal{G}} dB_s(e^{\theta}_{rs}).$$

By Theorem 3.2.2 in [12], $(B_t)_{t\geq 0}$ can be decomposed as follows:

$$B_t(h) = \sum_{\substack{\eta=1,2\\ j,l\geq 1}} \left\langle e^{\eta}_{jl}, h \right\rangle_{\mathcal{G}\times\mathcal{G}} B^{\eta}_{jl}(t) \quad \forall h \in \mathcal{G}\times\mathcal{G},$$

with $(B^{\eta}_{jl})_{\substack{\eta=1,2\\ j,l\geq 1}}$ a family of independent one-dimensional Brownian motions. Finally, a computation gives

$$\begin{aligned} d\mathbf{U}(\omega,t) &= d\mathbf{Y}^1(\omega,t) + i\, d\mathbf{Y}^2(\omega,t) \\ &= J(\mathbf{U}(\omega,t))dt + \psi_1(\mathbf{U}(\omega,t))(dB^1_t) + \psi_2(\mathbf{U}(\omega,t))(dB^2_t). \end{aligned}$$

Using the Ito formula given by Theorem 3.1.3 in [20], we have

$$\|\mathbf{U}(\omega,t)\|_{\mathcal{H}} = 1 \quad \forall t \geq 0.$$

This result shows that the process belongs to $\mathcal{C}([0,+\infty),\mathcal{H})$, and consequently the convergence also holds in $\mathcal{C}([0,+\infty),\mathcal{H})$.

5.2. Proof of Theorem 2. The proof of this theorem follows ideas developed in [19, Chapter 11]. In a first step, we introduce a new process; it is an adapted version of the first which has a symmetric state space about 0 and which is more convenient for manipulations. In a second step we will show the tightness using Theorem 3 in [14]. Moreover, the size of the jumps are equal to $1/N$. Then, all accumulated points are supported by the set of continuous functions. Consequently, the last step consists of adapting Lemmas 11.1.1 and 11.1.3 in [19] to the Skorokhod topology.

We begin by introducing a new process. Let $(Y_t)_{t\geq 0}$ be a jump Markov process on $\mathbb{Z}$ with generator $\tilde{\mathcal{L}}$ given by

$$\begin{aligned} \tilde{\mathcal{L}}\phi(j) &= \Lambda(\omega)(\phi(j+1) + \phi(j-1) - 2\phi(j)), \quad j \neq 0, \\ \tilde{\mathcal{L}}\phi(0) &= \frac{\Lambda(\omega)}{2}(\phi(1) + \phi(-1) - 2\phi(0)), \quad j = 0. \end{aligned}$$

One can check that, starting from the same point and $\forall t \geq 0$, X_t and $1 + |Y_t|$ have the same law. In what follows, we will denote by $\mathbb{Q}^N_{d(N)}$ the law of the renormalized process $(Y_t/N)_{t\geq 0}$ starting from $d(N) = (l(N)-1)/N$. According to Theorem 3 in [14], we will not directly prove the tightness of the renormalized process, but of truncations of this process, and we will be able to conclude thanks to an adapted version of Lemma 11.1.1 in [19] to the Skorokhod topology on $\mathcal{D}([0,+\infty),\mathbb{R})$. We also introduce some notation. Let $\mathcal{M} = \sigma(x(u), u \geq 0)$, $\mathcal{M}_t = \sigma(x(u), u \leq t)$, and

$$M^N_f(t) = f(x(u)) - f(x(0)) - \int_0^t \tilde{\mathcal{L}}^N f(x(s))ds,$$

which is an $(\mathcal{M}_t)$-martingale under $\mathbb{Q}^N_{d(N)}$ and where

$$\tilde{\mathcal{L}}^N\phi(j) = \Lambda(\omega)\left[\phi\left(\frac{j+1}{N}\right) + \phi\left(\frac{j-1}{N}\right) - 2\phi\left(\frac{j}{N}\right)\right], \quad j \neq 0,$$
$$\tilde{\mathcal{L}}^N\phi(0) = \frac{\Lambda(\omega)}{2}\left[\phi\left(\frac{1}{N}\right) + \phi\left(\frac{-1}{N}\right) - 2\phi(0)\right], \quad j = 0.$$

5.2.1. Tightness of $(\mathbb{Q}^{N,M}_{d(N)})_N$. Let $M \geq 1$, large enough to have $\sup_N d(N) \leq M$, and $\tau_M = \inf(u \geq 0, |x(u)| \geq M)$. We denote by $\mathbb{Q}^{N,M}_{d(N)}$ the law of $(Y_{t\wedge\tau_M}/N)_{t\geq 0}$ starting from $d(N)$. We remark that $\mathbb{Q}^{N,M}_{d(N)} = \mathbb{Q}^N_{d(N)}$ on $\mathcal{M}_{\tau_M}$. It becomes easy to see that

$$\lim_{K\to+\infty} \mathbb{Q}^{N,M}_{d(N)}\left(\sup_{u\geq 0}|x(u)| \geq K\right) = 0.$$

Moreover, $(M^N_f(t\wedge\tau_M))_{t\geq 0}$ is an $(\mathcal{M}_t)$-martingale under $\mathbb{Q}^N_{d(N)}$. Consequently, $\forall\ 0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}\left[(x(t)-x(s))^2\right] &= \mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}\left[(M^N_{Id}(t) - M^N_{Id}(s))^2\right] \\
&= \mathbb{E}_s^{\mathbb{Q}^{N}_{d(N)}}\left[\langle M^N_{Id}\rangle_{t\wedge\tau_M} - \langle M^N_{Id}\rangle_{s\wedge\tau_M}\right] \\
&\leq 2\frac{\Lambda}{N^2}(t-s),
\end{aligned}$$

where $\mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}$ is the conditional expectation under $\mathbb{Q}^{N,M}_{d(N)}$ given $\mathcal{M}_s$. Thus, by Theorem 3 in [14], $(\mathbb{Q}^{N,M}_{d(N)})_N$ is tight in $\mathcal{D}([0,+\infty),\mathbb{R})$.

5.2.2. Convergence. We consider f a smooth function and $(\mathbb{Q}^{N',M}_{d(N')})_{N'}$ a converging subsequence to $\mathbb{Q}^M_y$. Let $0 \leq s \leq t$ and Φ be a bounded continuous $\mathcal{M}_s$-measurable function. We have

$$\mathbb{E}^{\mathbb{Q}^{N',M}_{d(N')}}\left[M^{N'}_f(t\wedge\tau_M)\Phi\right] = \mathbb{E}^{\mathbb{Q}^{N',M}_{d(N')}}\left[M^{N'}_f(s\wedge\tau_M)\Phi\right]. \tag{5.4}$$

However, $\Lambda(\omega) = \frac{k^2(\omega)S(1,1)}{2a} \underset{\omega\gg 1}{\sim} N^2\frac{\sigma^2}{2}$,

$$\lim_{N\to+\infty} \sup_{x\in\left[-M,-\frac{1}{N}\right]\cup\left[\frac{1}{N},M\right]} \left|\tilde{\mathcal{L}}^N f\left(\frac{[Nx]}{N}\right) - \frac{\sigma^2}{2}f''(x)\right| = 0,$$
$$\text{and } \lim_{N\to+\infty}\tilde{\mathcal{L}}^N f(0) = \frac{1}{2}\frac{\sigma^2}{2}f''(0).$$

To correct the problem in 0, we have the following lemma.

Lemma 9.

$$\mathbb{E}^{\mathbb{Q}^N_{d(N)}}\left[\int_0^t \mathbf{1}_{(x(u)=0)}du\right] = \mathcal{O}\left(\frac{1}{N^2}\right).$$

Proof. $\mathbb{E}^{\mathbb{Q}^N_{d(N)}}[\int_0^t \mathbf{1}_{(x(u)=0)}du]$ is the mean time spent by $(\frac{Y_t}{N})_{t\geq 0}$ in the state 0. We denote by $(\tilde{X}_t)_{t\geq 0}$ the traffic of the $M/M/1$ queue with traffic rate $\rho = 1$. In addition to the Markov property,

$$\mathbb{E}^{\mathbb{Q}^N_{d(N)}}\left[\int_0^t \mathbf{1}_{(x(u)=0)}du\right] = \mathbb{E}_{d(N)}\left[\int_0^t \mathbf{1}_{(\tilde{X}_u=0)}du\right] \leq \mathbb{E}_0\left[\int_0^t \mathbf{1}_{(\tilde{X}_u=0)}du\right]$$
$$\leq \int_0^t \mathbb{P}_0(\tilde{X}_u=0)du.$$

However, explicit expressions of the transition probabilities for this queue can be found in [2, Theorem 8.5]. In our case, $\mathbb{P}_0(\tilde{X}_t=0) = e^{-2\Lambda t}\left(I_0(2\Lambda t) + I_1(2\Lambda t)\right)$, where I_n is the modified Bessel function of order n, given by $I_n(t) = \sum_{k\geq 0}\frac{t^{n+2k}}{2^{n+2k}k!(n+k)!}$. Then, we get

$$\int_0^t \mathbb{P}_0(\tilde{X}_u=0)du \leq \frac{1}{2\Lambda}e^{-2\Lambda t}\sum_{k\geq 0}\frac{(2\Lambda)^{2k+1}}{(2k+1)!} + \frac{(2\Lambda)^{2k+2}}{(2k+2)!} \leq \frac{1}{2\Lambda}. \qquad \square$$

Consequently, letting $N'\to+\infty$ in (5.4), we obtain that under $\mathbb{Q}^M_y$,

$$f(x(t\wedge\tau_M)) - f(x(0)) - \frac{\sigma^2}{2}\int_0^{t\wedge\tau_M} f^{''}(x(u))du$$

is an $(\mathcal{M}_t)$-martingale. If we denote by $\mathbb{W}^\sigma_y$ the law of the process $(\sigma B_t + y)_{t\geq 0}$, we have $\mathbb{Q}^M_y = \mathbb{W}^\sigma_y$ on $\mathcal{M}_{\tau_M}$. Finally, we conclude that $\mathbb{Q}^N_{d(N)}$ converges to $\mathbb{W}^\sigma_y$ thanks to an adapted version of Lemma 11.1.1 in [19] to the Skorokhod topology on $\mathcal{D}([0,+\infty),\mathbb{R})$. $\square$

Acknowledgment. I wish to thank my Ph.D. supervisor Josselin Garnier for his suggestions, help, and support.

REFERENCES

[1] R. Adler, *The Geometry of Random Fields*, Wiley, London, 1981.
[2] S. Asmussen, *Applied Probability and Queues*, Springer-Verlag, New York, 2003.
[3] G. Bal, G. Papanicolaou, and L. Ryzhik, *Self-averaging in time reversal for the parabolic wave equation*, Stoch. Dyn., 2 (2002), pp. 507–531.
[4] P. Blomgren, G. Papanicolaou, and H. Zhao, *Super-resolution in time-reversal acoustics*, J. Acoust Soc. Amer, 111 (2002), pp. 230–248.
[5] J.-F. Clouet and J.-P. Fouque, *A time-reversal method for an acoustical pulse propagating in randomly layered media*, Wave Motion, 25 (1997), pp. 361–368.
[6] A. Derode, P. Roux, and M. Fink, *Robust acoustic time reversal with high-order multiple scattering*, Phys. Rev. Lett., 75 (1995), pp. 4206–4209.
[7] M. Fink, D. Casserau, A. Derode, C. Prada, P. Roux, M. Tanter, J.-L. Thomas, and F. Wu, *Time-reversed acoustic*, Rep. Progr. Phys., 63 (2000), pp. 1933–1995.
[8] J.-P. Fouque, *La convergence en loi pour les processus à valeur dans un espace nucléaire*, Ann. Inst. H. Poincaré Probab. Statist. 20 (1984), pp. 225–245.
[9] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer-Verlag, New York, 2007.
[10] J.-P. Fouque, J. Garnier, and K. Sølna, *Time reversal super resolution in randomly layered media*, Wave Motion, 43 (2006), pp. 646–666.
[11] J. Garnier and G. Papanicolaou, *Pulse propagation and time reversal in random waveguides*, SIAM J. Appl. Math, 67 (2007), pp. 1718–1739.
[12] G. Kallianpur and J. Xiong, *Stochastic Differential Equations in Infinite-Dimensional Spaces*, Institute of Mathematical Statistics, Lecture Notes—Monograph Series 26, Institute of Mathematical Statistics, Hayward, CA, 1995.
[13] W. Kohler and G. Papanicolaou, *Wave propagation in randomly inhomogeneous ocean*, in Wave Propagation and Underwater Acoustics, Lecture Notes in Phys. 70, J. B. Keller and J. S. Papadakis, eds., Springer-Verlag, Berlin, 1977, pp. 153–223.

[14] H.-J. Kushner, *Approximation and Weak Convergence Methods for Random Processes*, MIT Press, Cambridge, MA, 1984.

[15] G. Lerosey, J. De Rosny, A. Tourin, and M. Fink, *Focusing beyond the diffraction limit with far-field time reversal*, Science, 315 (2007), pp. 1120–1122.

[16] I. Mitoma, *On the sample continuity of $\mathcal{S}'$-processes*, J. Math. Soc. Japan, 35 (1983), pp. 629–636.

[17] G. Papanicolaou and W. Kohler, *Asymptotic theory of mixing stochastic ordinary differential equations*, Comm. Pure Appl. Math., 27 (1974), pp. 641–668.

[18] G. Papanicolaou, L. Ryzhik, and K. Sølna, *Statistical stability in time reversal*, SIAM J. Appl. Math., 64 (2004), pp. 1133–1155.

[19] D.W. Stroock and S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, 1979.

[20] M. Yor, *Existence et unicité de diffusion à valeurs dans un espace de Hilbert*, Ann. Inst. H. Poincaré Sect. B, 10 (1974), pp. 55–88.