Wave Decoherence for the Random Schrödinger Equation with Long-Range Correlations

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Abstract

In this paper, we study the decoherence of a wave described by the solution to a Schrödinger equation with a time-dependent random potential. The random potential is assumed to have slowly decaying correlations. The main tool to analyze the decoherence behaviors is a properly rescaled Wigner transform of the solution of the random Schrödinger equation. We exhibit anomalous wave decoherence effects at different propagation scales.

Key words. Schrödinger equation, random media, long-range processes

AMS subject classification. 81S30, 82C70, 35Q40

Introduction.

The Schrödinger equation with a time-dependent random potential has attracted lots of attention because of its large domains of applications [10, 13, 14, 15, 22, 34]. It is widely used for instance in wave propagation under the paraxial or parabolic approximation [3, 4, 5, 6, 8]. This field of research was recently stimulated [8, 19, 28, 29, 33, 21] by data collections in wave propagation experiments showing that the medium of propagation presented some long-range effects [12, 32]. Most of the theoretical studies regarding wave propagation in long-range random media hold in one dimensional propagation media, which are very convenient for mathematical studies but not relevant in many applications.

We consider the random Schrödinger equation

$$
\begin{aligned}
i\partial_t\phi+\frac{1}{2}\Delta_{\mathbf{x}}\phi-\sqrt{\epsilon}V(t,\mathbf{x})\phi=0,\quad t\geq 0 \text{ and } \mathbf{x}\in\mathbb{R}^d,\\
\phi(0,\mathbf{x})=\phi_0(\mathbf{x}),
\end{aligned}
$$

with a random potential $V(t,\mathbf{x})$, which is a spatially and temporally homogeneous mean-zero random field. Here, $t\geq 0$ represents the temporal variable, $\mathbf{x}\in\mathbb{R}^d$ the spatial variable with $d\geq 1$, and $\epsilon\ll 1$ is a small parameter which represents the relative strength of the random fluctuations. A classical tool to study the decoherence of the field ϕ is the Wigner transform defined by

$$
W(t,\mathbf{x},\mathbf{k})=\frac{1}{(2\pi)^d}\int d\mathbf{y}e^{i\mathbf{k}\cdot\mathbf{y}}\phi\Big(t,\mathbf{x}-\frac{\mathbf{y}}{2}\Big)\overline{\phi\Big(t,\mathbf{x}+\frac{\mathbf{y}}{2}\Big)},
$$

which analyzes the correlations of the field ϕ around the point $\mathbf{x}$. Decaying correlations of ϕ with respect to time correspond to the wave decoherence phenomenon. If $V=0$, we have $W(t,\mathbf{x},\mathbf{k})=W_0(\mathbf{x}-t\mathbf{k},\mathbf{k})$. In this case, the momentum of W is preserved during the propagation, there is no variation of the momentum with respect to time, meaning there is no wave decoherence.

We refer to [20, 26] for the basic properties of the Wigner transform. In our problem the amplitude of the random perturbations are small, so significant effects are observed for long time and distance of propagation. Consequently, we consider the rescaled field

$$
\phi_\epsilon(t,\mathbf{x})=\phi\Big(\frac{t}{\epsilon^s},\frac{\mathbf{x}}{\epsilon^s}\Big)
$$

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which satisfies the scaled random Schrödinger equation

$$i\epsilon^s\partial_t\phi_\epsilon + \frac{\epsilon^{2s}}{2}\Delta_{\mathbf{x}}\phi_\epsilon - \sqrt{\epsilon}V\Big(\frac{t}{\epsilon^s}, \frac{\mathbf{x}}{\epsilon^s}\Big)\phi_\epsilon = 0, \quad t \geq 0 \text{ and } \mathbf{x} \in \mathbb{R}^d,$$
$$\phi_\epsilon(0, \mathbf{x}) = \phi_{0,\epsilon}(\mathbf{x}),$$

and where $s \in (0, 1]$ **is the propagation scale parameter**. If the random potential V has rapidly decaying correlations, it has been shown [8] that the field ϕ_ϵ does not show significant random perturbations before the propagation scale e^{-1} ($s = 1$). More precisely, the Fourier transform of the field ϕ_ϵ properly scaled converges point-wise to a stochastic complex Gaussian limit with a one-time statistic of an Ornstein-Uhlenbeck process. The wave decoherence has been studied in this context in many papers [4, 5, 13, 14, 15, 16, 17, 22, 27, 34] thanks to the Wigner transform for $s = 1$

$$\begin{aligned} W_\epsilon(t, \mathbf{x}, \mathbf{k}) &= \frac{1}{(2\pi)^d}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\epsilon\Big(t, \mathbf{x} - \epsilon\frac{\mathbf{y}}{2}\Big)\overline{\phi_\epsilon\Big(t, \mathbf{x} + \epsilon\frac{\mathbf{y}}{2}\Big)} \\ &= \frac{1}{(2\pi)^d}\int d\mathbf{y} e^{i\mathbf{k}\cdot\mathbf{y}}\phi\Big(\frac{t}{\epsilon}, \frac{\mathbf{x}}{\epsilon} - \frac{\mathbf{y}}{2}\Big)\overline{\phi\Big(\frac{t}{\epsilon}, \frac{\mathbf{x}}{\epsilon} + \frac{\mathbf{y}}{2}\Big)} \end{aligned} \tag{1}$$

which analyzes the decoherence of ϕ_ϵ on spatial correlation scales of order ϵ at the macroscopic propagation scale ϵ^{-1}, and the decoherence of ϕ on spatial correlation scales of order 1 at the microscopic propagation scale. The Wigner transform is also interpreted as the phase space density energy of ϕ_ϵ.

In several context involving rapidly decorrelating random potential, it has been shown that the expectation of the Wigner transform $\mathbb{E}[W_\epsilon(t, \mathbf{x}, \mathbf{k})]$ converges as ϵ goes to 0 to the solution W of the radiative transport equation

$$\partial_t W + \mathbf{k}\cdot\nabla_{\mathbf{x}}W = \int d\mathbf{p}\sigma(\mathbf{p}, \mathbf{k})(W(t, \mathbf{x}, \mathbf{p}) - W(t, \mathbf{x}, \mathbf{k})), \tag{2}$$

where the transfer coefficient $\sigma(\mathbf{p}, \mathbf{k})$ depends on the power spectrum of the two-point correlation function of the random potential V. Moreover, in some cases [3, 5, 6, 16], it has been shown that the limit W is often self-averaging, that is, W_ϵ converges in probability to the deterministic limit W for the weak topology on $L^2(\mathbb{R}^{2d})$. In other words, the wave decoherence described by the radiative transfer equation (2) does not depend on the particular realization of the random medium.

In this paper, we investigate the decoherence of a wave described by a Schrödinger equation involving a random potential V with slowly decaying correlations. In this context, the behavior of the field ϕ_ϵ evolves on different propagation scales e^{-s} [8, 21]. In [8, Theorem 1.2] the authors study ϕ_ϵ itself on the propagation scales e^{-s}, with $s = 1/(2\kappa)$ and $\kappa > 1/2$, and show that the Fourier transform properly scaled of ϕ_ϵ converge point-wise in distribution to a complex exponential function of a fractional Brownian motion with Hurst index κ, where κ depends on the statistic of the random potential. This result means that e^{-s}, with $s = 1/(2\kappa) < 1$, is the first propagation scale on which the random perturbations become significant, and induces a random phase modulation on the wave. In [21, Theorem 2.2] the author study the phase space density energy of ϕ_ϵ, which describes the wave decoherence, in the case $s = 1$, and shows that the Wigner transform (1) of ϕ_ϵ converges in probability for the weak topology on $L^2(\mathbb{R}^{2d})$ to the unique solution of a deterministic radiative transfer equation similar to (2) obtained under rapidly decaying correlations. In other words, the wave decoherence holds on the propagation scale e^{-1} ($s = 1$), and does not depend on the particular realization of the random potential. However, even if the radiative transfer equation has similar structure in both cases, for rapidly and slowly decaying correlations, the long-range correlations have a striking effect. In contrast with the rapidly decorrelating case [4], the scattering coefficient $\Sigma(\mathbf{k}) = \int d\mathbf{p}\sigma(\mathbf{k}, \mathbf{p}) = +\infty$ is not defined anymore. The radiative transfer equation is however still well defined because of the difference $W(t, \mathbf{x}, \mathbf{p}) - W(t, \mathbf{x}, \mathbf{k})$ which balances the singularity introduced by the long-range correlation assumption. Moreover, this long-range correlations imply an instantaneous regularizing effect of the radiative transfer equation [21, Theorem 3.1]. Consequently, these results show a qualitative and thorough difference between the rapidly and slowly deccorelating cases. In fact, in contrast with the rapidly decorrelating case, for which the phase and the phase space density evolve on the same propagation scale e^{-1} [8], the phase of ϕ_ϵ and its phase space energy now evolve on different propagation scales.

The main goal of this paper is to study the decoherence of a wave satisfying the Schrödinger equation involving a random potential V with slowly decaying correlations for the propagation scale parameters $s \in (1/(2\kappa), 1)$. In fact, for $s > 1/(2\kappa)$ the random phase modulation obtained for $s = 1/(2\kappa)$ produces very fast oscillations on the wave, so that for sufficiently large propagation scale parameters $s > 1/(2\kappa)$ the wave coherence is broken on a certain spatial scales. Let us note that for a given propagation scale parameter s, the wave decoherence can be too small on certain spatial scale. For instance, in [21], the author show using the Wigner transform (1) that there is no significant wave decoherence on ϕ_ϵ on spatial scales of order ϵ, before $s = 1$. As we will see in Section 3, for $s < 1$ wave decoherence takes place first on large spatial scales and then propagates to the smaller ones as the propagation scale parameter s increase. The larger the propagation scale parameter s is the smaller the spatial scale is to observe wave decoherence (see Figure 2). To exhibit wave decoherence for $s < 1$, we need a properly scaled Wigner transform of the field ϕ_ϵ. Depending on the propagation scale parameter s, we show that this scaled Wigner transform converges in probability, for the weak topology on $L^2(\mathbb{R}^{2d})$, to the unique solution of a fractional diffusion equation. This momentum diffusion equation describes the wave decoherence mechanism, and show that it does not depend on the particular realization of the random potential. The anomalous momentum diffusions obtained for $s \in (1/(2\kappa), 1]$ allow us to exhibit a damping coefficient, describing the decoherence rate, obeying a power law with exponent in $(0, 1)$.

The organization of this paper is as follows: In Section 1, we present the random Schrödinger equation that will be studied in this paper; then we present the construction of the random potential; finally, we introduce the long-range correlation assumption used throughout this paper. The results stated in Section 2 and Section 4 have been shown in [8] and [21] respectively, but we recall these results to provide a self-contained presentation of the wave decoherence phenomenon. In Section 2, we present the behavior of the field ϕ_ϵ on the scale $s = 1/(2\kappa)$. In Section 3, we state the main result of this paper. We present the asymptotic behavior in long-range random media of a properly scaled Wigner transform over the intermediate range of propagation scale parameter $s \in (1/(2\kappa), 1)$. In Section 4, we describe the asymptotic evolution in long-range random media of the phase space energy density of the solution of the random Schrödinger equation for $s = 1$. Finally, Section 5 is devoted to the proofs of Theorem 3.1 and Theorem 3.2.

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1 The Random Schrödinger equation

This section introduces first the random Schrödinger equation studied in this paper. Then, we present the construction of the random potential with long-range correlation properties. Finally, we introduce the Wigner transform which is the main tool in this paper to study the Schrödinger equation.

We consider the dimensionless form of the Schrödinger equation on $\mathbb{R}^d$ with a time-dependent random potential:

$$i\partial_t \phi + \frac{1}{2}\Delta_{\mathbf{x}}\phi - \epsilon^{\frac{1-\gamma}{2}} V\Big(\frac{t}{\epsilon^\gamma}, \mathbf{x}\Big)\phi = 0, \tag{3}$$

with $\gamma \in [0, 1)$. γ is a parameter which characterizes the correlation length in time. If $\gamma = 0$ the correlation length in space and time are of the same order, but if $\gamma \in (0, 1)$ the correlation length in time is small compared to the correlation length in space. In (3) the strength of the random perturbation are small, so we consider the rescaled field

$$\phi_\epsilon(t, \mathbf{x}) = \phi\Big(\frac{t}{\epsilon^s}, \frac{\mathbf{x}}{\epsilon^s}\Big), \quad \text{with} \quad s \in (0, 1],$$

to observe significant effects after a sufficiently large propagation distance and propagation time. The parameter $\boldsymbol{s \in (0, 1]}$ **represents the propagation scale parameter**. Therefore, the scaled field ϕ_ϵ satisfies the scaled Schrödinger equation

$$i\epsilon^s\partial_t\phi_\epsilon + \frac{\epsilon^{2s}}{2}\Delta_{\mathbf{x}}\phi_\epsilon - \epsilon^{\frac{1-\gamma}{2}} V\Big(\frac{t}{\epsilon^{s+\gamma}}, \frac{\mathbf{x}}{\epsilon^s}\Big)\phi_\epsilon = 0 \quad \text{with} \quad \phi_\epsilon(0, \mathbf{x}) = \phi_{0,\epsilon}(\mathbf{x}). \tag{4}$$

Here $\Delta_{\mathbf{x}}$ is the Laplacian on $\mathbb{R}^d$ given by $\Delta = \sum_{j=1}^d \partial^2_{x_j}$. $(V(t, \mathbf{x}), \mathbf{x} \in \mathbb{R}^d, t \geq 0)$ is the random potential, whose properties are described in the next section. The initial datum $\phi_{0,\epsilon}(\mathbf{x}) = \phi_{0,\epsilon}(\mathbf{x}, \zeta)$

is a random function with respect to a probability space $(S, \mu(d\zeta))$, and independent to the random potential V. This randomness on the initial data is called mixture of states. This terminology comes from the quantum mechanics, and the reason for introducing this additional randomness will be explained more precisely in Section 1.3.

1.1 Random potential

This section is devoted to the construction of the random potential V, and is also a short remainder about some properties of Gaussian random fields that we use in the proof of Theorem 3.1 and Theorem 3.2. All the properties of the random field V exposed in this section result from the standard properties of Gaussian random fields presented in [1, 2] for instance.

In this paper, the random potential $(V(t, \mathbf{x}), t \geq 0, x \in \mathbb{R}^d)$ is modeled using a stationary continuous random process in space and time. We construct our potential in the Fourier space as follows. Let $\widehat{R}_0$ be a nonnegative function with support included in a compact subset of $\mathbb{R}^d$ containing 0, such that $\widehat{R}_0 \in L^1(\mathbb{R}^d)$, $\widehat{R}_0(-\mathbf{p}) = \widehat{R}_0(\mathbf{p})$, and $\widehat{R}_0$ has a singularity in 0. Let us consider

$$\mathcal{H} = \left\{ \varphi \text{ such that } \int_{\mathbb{R}^d} d\mathbf{p} \widehat{R}_0(\mathbf{p}) |\varphi(\mathbf{p})|^2 < +\infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int d\mathbf{p} \ \widehat{R}_0(\mathbf{p}) \varphi(\mathbf{p}) \overline{\psi(\mathbf{p})} \quad \forall (\varphi, \psi) \in \mathcal{H}^2.$$

Let us consider $(\widehat{V}(t, \cdot))_{t \geq 0}$ be a stationary continuous zero-mean Gaussian field on $\mathcal{H}'$ with autocorrelation function given by

$$\mathbb{E}[\widehat{V}(t_1, d\mathbf{p}_1) \widehat{V}(t_2, d\mathbf{p}_2)] = (2\pi)^d R(t_1 - t_2, \mathbf{p}_1) \delta(\mathbf{p}_1 + \mathbf{p}_2),$$

and

$$\mathbb{E}[\widehat{V}(t_1, d\mathbf{p}_1) \overline{\widehat{V}(t_2, d\mathbf{p}_2)}] = (2\pi)^d R(t_1 - t_2, \mathbf{p}_1) \delta(\mathbf{p}_1 - \mathbf{p}_2),$$

and where $\mathcal{H}'$ is the dual space of $\mathcal{H}$. In other words, $\forall n \in \mathbb{N}^*$, $\forall (\varphi_1, \dots, \varphi_n) \in \mathcal{H}^n$ and $\forall (t_1 \dots, t_n) \in [0, +\infty)^n$,

$$\left(\langle \widehat{V}(t_1), \varphi_1 \rangle_{\mathcal{H}', \mathcal{H}}, \dots, \ \langle \widehat{V}(t_n), \varphi_n \rangle_{\mathcal{H}', \mathcal{H}} \right)$$

is a zero-mean Gaussian vector with covariance matrix given by: $\forall (j, l) \in \{1, \dots, n\}^2$

$$\mathbb{E}\left[\langle \widehat{V}(t_j), \varphi_j \rangle_{\mathcal{H}', \mathcal{H}} \langle \widehat{V}(t_l), \varphi_l \rangle_{\mathcal{H}', \mathcal{H}} \right] = \int_{\mathbb{R}} d\mathbf{p} \ \varphi_j(\mathbf{p}) \varphi_l(-\mathbf{p}) R(t_1 - t_2, \mathbf{p})$$

and

$$\mathbb{E}\left[\langle \widehat{V}(t_j), \varphi_j \rangle_{\mathcal{H}', \mathcal{H}} \overline{\langle \widehat{V}(t_l), \varphi_l \rangle_{\mathcal{H}', \mathcal{H}}} \right] = \int_{\mathbb{R}} d\mathbf{p} \ \varphi_j(\mathbf{p}) \overline{\varphi_l(\mathbf{p})} R(t_1 - t_2, \mathbf{p}).$$

Here, the spatial power spectrum is given by

$$R(t, \mathbf{p}) = e^{-\mathfrak{g}(\mathbf{p})|t|} \widehat{R}_0(\mathbf{p}), \tag{5}$$

where the nonnegative function $\mathfrak{g}$ is the spectral gap, and such that $\mathfrak{g}(\mathbf{p}) = \mathfrak{g}(-\mathbf{p})$. Particular assumptions involving the spectral gap $\mathfrak{g}$ will be introduced at the end of this section to ensure the long-range correlation property of the potential V in (4).

According to the shape of the autocorrelation function (5), we have the following proposition.

Proposition 1.1 *Let*

$$\mathcal{F}_t = \sigma(\widehat{V}(s, \cdot), s \leq t) \tag{6}$$

be the σ-algebra generated by $(\widehat{V}(s, \cdot), s \leq t)$. We have

$$\mathbb{E}\big[\widehat{V}(t+h, \cdot) | \mathcal{F}_t\big] = e^{-\mathfrak{g}(\mathbf{p})h} \widehat{V}(t, \cdot) \tag{7}$$

and $\forall (\varphi, \psi) \in \mathcal{H}^2$

$$\begin{aligned} \mathbb{E}\Big[\langle \widehat{V}(t+h), \varphi \rangle_{\mathcal{H}', \mathcal{H}} \langle \widehat{V}(t+h), \psi \rangle_{\mathcal{H}', \mathcal{H}} - \mathbb{E}\big[\langle \widehat{V}(t+h), \varphi \rangle_{\mathcal{H}', \mathcal{H}} | \mathcal{F}_t \big] \mathbb{E}\big[\langle \widehat{V}(t+h), \psi \rangle_{\mathcal{H}', \mathcal{H}} | \mathcal{F}_t \big] \Big| \mathcal{F}_t \Big] \\ = \int d\mathbf{p} \ \varphi(\mathbf{p}) \psi(-\mathbf{p}) \widehat{R}_0(\mathbf{p}) \left(1 - e^{-2\mathfrak{g}(\mathbf{p})h} \right). \end{aligned} \tag{8}$$

These two properties will be used in the proof of Theorem 3.1 and Theorem 3.2, which are based on the perturbed-test-function method.

Let us note that $\langle \widehat{V}, \varphi \rangle_{\mathcal{H}',\mathcal{H}}$ is a real-valued gaussian process once $\varphi \in \mathcal{H}$ satisfies $\overline{\varphi(\mathbf{p})} = \varphi(-\mathbf{p})$. According to this last remark, let us introduce the real random potential V defined by

$$V(t,\mathbf{x}) = \langle \widehat{V}(t,\cdot), e_{\mathbf{x}} \rangle_{\mathcal{H}',\mathcal{H}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{V}(t, d\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{9}$$

where $e_{\mathbf{x}} \in \mathcal{H}$ is defined by $e_{\mathbf{x}}(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{x}}/(2\pi)^d$. Consequently, the random potential V is a stationary real-valued zero-mean Gaussian field with a covariance function given by: $\forall (t_1,t_2) \in [0,+\infty)^2$ and $\forall (\mathbf{x}_1,\mathbf{x}_2) \in \mathbb{R}^{2d}$

$$\begin{aligned} R(t_1-t_2,\mathbf{x}_1-\mathbf{x}_2) = \mathbb{E}\big[V(t_1,\mathbf{x}_1)V(t_2,\mathbf{x}_2)\big] &= \frac{1}{(2\pi)^d}\int d\mathbf{p} R(t_1-t_2,\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \\ &= \frac{1}{(2\pi)^{d+1}}\int d\omega d\mathbf{p} \widehat{R}(\omega,\mathbf{p}) e^{i\omega(t_1-t_2)} e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)}, \end{aligned} \tag{10}$$

where

$$\widehat{R}(\omega,\mathbf{p}) = \frac{2\mathfrak{g}(\mathbf{p})\widehat{R}_0(\mathbf{p})}{\omega^2+\mathfrak{g}^2(\mathbf{p})}. \tag{11}$$

According to the previous construction and [2, Theorem 2.2.1] the random potential V is continuous and bounded with probability one on each compact subset K of $\mathbb{R}\times\mathbb{R}^d$. This fact comes from the continuity relation

$$\mathbb{E}\big[\big(V(t_1,x)-V(t_2,y)\big)^2\big]^{1/2} \leq C\left(\int d\mathbf{p}\widehat{R}_0(\mathbf{p})\right)\big(|t_1-t_2|+|\mathbf{x}-\mathbf{y}|\big),$$

$\forall (t_1,t_2,\mathbf{x},\mathbf{y}) \in [0,T]^2 \times K^2$. Moreover, we have the following proposition.

Proposition 1.2 *$\forall \mu > 0$, $\eta > 0$, and $\forall K$ compact subset of $\mathbb{R}^d$*

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\epsilon^\mu \sup_{\mathbf{x}\in K}\sup_{t\in[0,T]}\Big|V\Big(\frac{t}{\epsilon^{1+\gamma}},\frac{\mathbf{x}}{\epsilon}\Big)\Big| > \eta\Big) = 0. \tag{12}$$

According to [2, Theorem 2.1.1], this limit (12) holds exponentially fast as $\epsilon \to 0$.

1.2 Slowly decorrelating assumption

In this paper we are interested in the Schrödinger equation with a random potential with long-range correlations. Let us introduce some additional assumptions on the spectral gap $\mathfrak{g}$ of the spatial power spectrum (5) in order to give slowly decaying correlations to the random potential V defined by (9).

Let us note that $\forall t \geq 0$, the random field $V(t,\cdot)$ has spatial slowly decaying correlations. In fact, if we freeze the temporal variable, the autocorrelation function of the random potential $V(t,\cdot)$ is given by

$$R(t,\mathbf{x}) = \mathbb{E}[V(t,\mathbf{x}+\mathbf{y})V(t,\mathbf{y})] = \int d\mathbf{p}\widehat{R}_0(\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{p}}$$

where $\widehat{R}_0(\mathbf{p})$ is assumed to have a singularity in 0, so that $R(t,\cdot) \notin L^1(\mathbb{R}^d)$. As a result, $(V(t))_{t\geq 0}$ models a family of random fields on $\mathbb{R}^d$ with spatial long-range correlations which evolves with respect to time. However, since (3) is a time evolution problem, we have to take care of the evolution of the random perturbation V with respect to the temporal variable. In fact, if V has rapidly decaying correlation in time, $(V(t_1),V(t_2))$ has now rapidly decaying spatial correlations, and the evolution problem (3) behaves like in the mixing case addressed in [7]. As a result, even if at each fixed time the spatial correlations are slowly decaying, the resulting time evolution problem behaves as if the random potential has rapidly decaying correlations. Consequently, we have to introduce a long-range correlation assumption with respect to the temporal variable. Let us note that $\forall (s,\mathbf{x},\mathbf{y}) \in \mathbb{R}_+\times\mathbb{R}^{2d}$

$$\int_0^{+\infty} dt\big|\mathbb{E}\big[V(t+s,\mathbf{x}+\mathbf{y})V(s,\mathbf{y})\big]\big| = +\infty \Longleftrightarrow \int d\mathbf{p}\frac{\widehat{R}_0(\mathbf{p})}{\mathfrak{g}(\mathbf{p})} = +\infty. \tag{13}$$

Consequently, throughout this paper we say that the family $(V(t))_{t\geq 0}$ of random fields with spatial long-range correlations has slowly decaying correlations in time if

$$\int d\mathbf{p}\frac{\widehat{R}_0(\mathbf{p})}{\mathfrak{g}(\mathbf{p})} = +\infty, \tag{14}$$

and rapidly decaying correlations in time otherwise. For the sake of simplicity, we assume throughout this paper that,

$$\mathfrak{g}(\mathbf{p}) = \nu|\mathbf{p}|^{2\beta} \quad \text{and} \quad \widehat{R}_0(\mathbf{p}) = \frac{a(\mathbf{p})}{|\mathbf{p}|^{d+2(\alpha-1)}}. \tag{15}$$

where, a is a continuous function with a compact support such that $a(0) > 0$. This configuration has been considered in [8] to study the propagation of the field ϕ_ϵ in a random media with long-range correlations. To consider the same setting of [8] we have to assume that $\beta \in (0, 1/2]$, $\alpha \in (1/2, 1)$, and $\alpha + \beta > 1$. As a result,

$$\frac{\widehat{R}_0(\mathbf{p})}{\mathfrak{g}(\mathbf{p})} \sim \frac{a(0)}{|\mathbf{p}|^{d+\theta}} \quad \text{with} \quad \theta = 2(\alpha+\beta-1) \in (0,1). \tag{16}$$

These assumptions permit to model a random field $V(t, \mathbf{x})$ with spatial long-range correlations for each time $t \geq 0$ and with slowly decaying correlations in time.

1.3 Wigner transform

In this paper we study wave decoherence phenomena happening on different propagation scales. To exhibit these phenomena, we need to consider different spatial correlation scales (see Figure 2). In this paper we consider the Wigner transform of the field ϕ_ϵ, satisfying the Schrödinger equation (4) and averaged with respect to the randomness of the initial data, defined by:

$$\begin{aligned} W_\epsilon(t,\mathbf{x},\mathbf{k}) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d\times S} d\mathbf{y}\mu(d\zeta)e^{i\mathbf{k}\cdot\mathbf{y}}\phi\Big(\frac{t}{\epsilon^s},\frac{\mathbf{x}}{\epsilon^s}-\frac{\mathbf{y}}{2\epsilon^{s_c}},\zeta\Big)\overline{\phi\Big(\frac{t}{\epsilon^s},\frac{\mathbf{x}}{\epsilon^s}+\frac{\mathbf{y}}{2\epsilon^{s_c}},\zeta\Big)} \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d\times S} d\mathbf{y}\mu(d\zeta)e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\epsilon\Big(t,\mathbf{x}-\epsilon^{s-s_c}\frac{\mathbf{y}}{2},\zeta\Big)\overline{\phi_\epsilon\Big(t,\mathbf{x}+\epsilon^{s-s_c}\frac{\mathbf{y}}{2},\zeta\Big)}, \end{aligned} \tag{17}$$

where **$s_c \in [0, s]$ is the spatial correlation parameter**, and $(S, \mu(d\zeta))$ is a probability space. We discuss below the reason of introducing this probability space, and we refer to [20, 26] for the basic properties of the Wigner distribution.

The scaled Wigner transform (17) is well suited to study the evolution of the correlations of the field ϕ_ϵ around $\mathbf{x}$. It captures decorrelations of the field ϕ on the spatial correlation scale ϵ^{-s_c}, on the microscopic propagation scale. Or in the same way, it captures decorrelations of the field ϕ_ϵ on the spatial correlation scale ϵ^{s-s_c}, on the macroscopic propagation scale ϵ^{-s}. Let us note that the cases $s_c < s$ study the local decoherence of the wave, while the case $s = s_c$ study the nonlocal decoherence of the wave.

According to [21] no significant loss of coherence of the wave can be exhibited on the correlation scale $s_c = 0$ before $s = 1$. Then, the idea is to study the wave decoherence on larger spatial scales $s_c > 0$. Let us note that for a rapidly deccorelating potential V, no wave decoherence effects can be observed except for the radiative transfer scaling $s_c = 0$ and $s = 1$ [8]. The reason will be explain formally in Section 3.

However, to observe decoherence effects of the field ϕ_ϵ on the spatial correlation scales ϵ^{s-s_c}, we need a proper initial condition $\phi_{0,\epsilon}$ in (4), which oscillates at the same scale (see Figure 1). Moreover, a natural way to introduce randomness on the initial condition is as follow. Let $S = \mathbb{R}^d$ and $\mu(\zeta)$ be a nonnegative rapidly decreasing function such that $\|\mu\|_{L^1(\mathbb{R}^d)} = 1$, and so that $(\mathbb{R}^d, \mu(d\zeta))$ is a probability space. Throughout this paper we assume that the initial condition $\phi_{0,\epsilon}$ in (4) is given by

$$\phi_{0,\epsilon}(\mathbf{x}) = \phi_0(\mathbf{x})\exp(i\zeta\cdot\mathbf{x}/\epsilon^{s-s_c}). \tag{18}$$

This initial condition represents a plane wave with initial propagation direction $\zeta \in \mathbb{R}^d$, oscillating on the scale ϵ^{s-s_c}, and with amplitude or envelope ϕ_0. The initial direction ζ of the wave is distributed according to $\mu(d\zeta)$, so that the Wigner transform (17) is average according to the distribution of

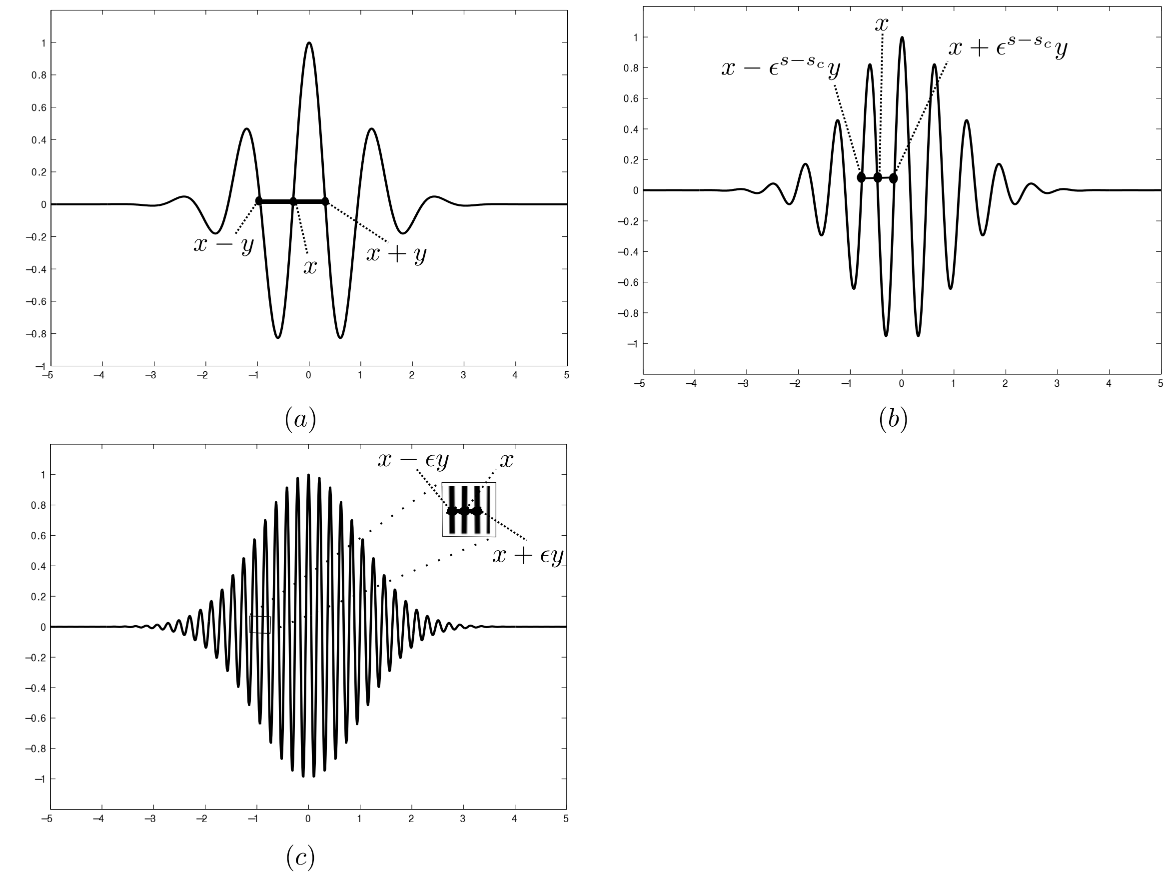

Figure 1: Illustration of the initial condition (18). (a) and (b) represent low spatial frequency initial conditions compared to the spatial frequency of the random medium ϵ^{-s} on the macroscopic scale ϵ^{-s}. (a) represents the case $s_c = s$ and has a spatial frequency of order 1 on the macroscopic scale ϵ^{-s} (and a spatial frequency of order ϵ^{s} on the microscopic scale). (b) represents the case $s_c < s$ and has a spatial frequency of order ϵ^{s-s_c} on the macroscopic scale ϵ^{-s} (and a spatial frequency of order ϵ^{s_c} on the microscopic scale). (c) represents the case $s_c = 0$ and $s = 1$, and has a spatial frequency of order ϵ on the macroscopic scale ϵ^{-1} (and a spatial frequency of order 1 on the microscopic scale).

the initial direction of the wave. Let us note that the spatial frequency of the initial condition ($\sim \epsilon^{s-s_c}$) is low compared to the one of the random medium ($\sim \epsilon^{s}$) on the macroscopic scale ϵ^{-s}. In rapidly deccorelating random media such low spatial frequency sources do not interact with the random medium, but as we will in Section 3, this kind of initial conditions interact strongly with slowly decorrelating random media. This result can be useful in passive imaging of a target in slowly deccorelating random media [18].

The main reason to introduce this additional randomness through the initial data $\phi_{0,\epsilon}$ is to make possible the weak convergence in $L^2(\mathbb{R}^{2d})$ of the initial Wigner transform, that is

$$\forall \lambda \in L^2(\mathbb{R}^{2d}), \quad \lim_{\epsilon} \big\langle W_\epsilon(0), \lambda \big\rangle_{L^2(\mathbb{R}^{2d})} = \big\langle W_0, \lambda \big\rangle_{L^2(\mathbb{R}^{2d})},$$

where

$$W_\epsilon(0,\mathbf{x},\mathbf{k}) = W_{0,\epsilon}(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} d\mathbf{y}\mu(d\zeta)e^{i(\mathbf{k}-\zeta)\cdot\mathbf{y}}\phi_0(\mathbf{x}-\epsilon^{s-s_c}\mathbf{y}/2)\overline{\phi_0(\mathbf{x}+\epsilon^{s-s_c}\mathbf{y}/2)}, \qquad (19)$$

with

$$W_0(\mathbf{x},\mathbf{k}) = |\phi_0(\mathbf{x})|^2\widehat{\mu}(\mathbf{k}),$$

if $s_c < c$, and

$$W_0(\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} d\mathbf{y}\mu(d\zeta)e^{i(\mathbf{k}-\zeta)\cdot\mathbf{y}}\phi_0(\mathbf{x}-\mathbf{y}/2)\overline{\phi_0(\mathbf{x}+\mathbf{y}/2)}$$

if $s = s_c$. Consequently, thanks to the Banach-Steinhaus Theorem, $W_{0,\epsilon}$ is uniformly bounded in $L^2(\mathbb{R}^{2d})$ with respect to ϵ. We need such a convergence on the initial Wigner transform $W_\epsilon(0)$ since we study W_ϵ in $L^2(\mathbb{R}^{2d})$ equipped with the weak topology. As it will be discussed in Section 3, it is not possible to expect a convergence result in $L^2(\mathbb{R}^{2d})$ equipped with the strong topology (except for the case $s = s_c$).

As described in Section 3, the spatial correlation parameter s_c depends on the propagation scale parameter s, it is proportional to $1 - s$. The larger the propagation scale parameter is the shorter the decoherence scale parameter is.

The Wigner distribution (17) satisfies the following evolution equation

$$\partial_t W_\epsilon(t, \mathbf{x}, \mathbf{k}) + \epsilon^{s_c}\mathbf{k} \cdot \nabla_{\mathbf{x}} W_\epsilon(t, \mathbf{x}, \mathbf{k}) = \epsilon^{(1-\gamma)/2-s} \int_{\mathbb{R}^d} \frac{\widehat{V}\Big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}\Big)}{(2\pi)^d i} e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon^s} \Big(W_\epsilon\Big(t, \mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\Big) - W_\epsilon\Big(t, \mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\Big)\Big), \tag{20}$$

with initial conditions $W_\epsilon(0, \mathbf{x}, \mathbf{k}) = W_{0,\epsilon}(\mathbf{x}, \mathbf{k})$, and where $W_{0,\epsilon}$ is defined by (19). Equation (20) can be recast in the weak sense as follows: $\forall \lambda \in \mathcal{S}(\mathbb{R}^{2d})$, where $\mathcal{S}(\mathbb{R}^{2d})$ stands for the space of rapidly decaying functions,

$$\big\langle W_\epsilon(t), \lambda \big\rangle_{L^2(\mathbb{R}^{2d})} - \big\langle W_\epsilon(0), \lambda \big\rangle_{L^2(\mathbb{R}^{2d})} = \int_0^t \big\langle W_\epsilon, \epsilon^{s_c}\mathbf{k} \cdot \nabla_{\mathbf{x}}\lambda + \epsilon^{(1-\gamma)/2-s}\mathcal{L}_\epsilon(s)\lambda \big\rangle_{L^2(\mathbb{R}^{2d})} ds,$$

where

$$\mathcal{L}_\epsilon \lambda(t, \mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d i} \int_{\mathbb{R}^d} \widehat{V}\Big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}\Big) e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon^s} \Big(\lambda\Big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\Big) - \lambda\Big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\Big)\Big). \tag{21}$$

Let us assume that $V = 0$, so that the Wigner transform is given by $W_\epsilon(t, \mathbf{x}, \mathbf{k}) = W_0(\mathbf{x} - \epsilon^{s_c} t\mathbf{k}, \mathbf{k})$. Therefore, the momentum of W_ϵ is conserve during the propagation, meaning there is no variation of the momentum with respect to time, that is there is no wave decoherence. The dispersion term $\mathbf{k} \cdot \nabla_{\mathbf{x}}$ is of order ϵ^{s_c} so that W_ϵ captures the wave dispersion only for $s_c = 0$. The transfer equation (20) describes the loss of coherence of the field ϕ_ϵ through the random operator $\mathcal{L}_\epsilon W_\epsilon(t)(t, \mathbf{x}, \mathbf{k})$. However, depending on the scale of observation the loss of coherence of the wave may not be significant. In fact, according to [21] no significant wave decoherence can be exhibited on the correlation scale $s_c = 0$ before $s = 1$.

Let us introduce some notations which are used in Section 3 and Section 4. Let

$$\mathcal{B}_r = \big\{\lambda \in L^2(\mathbb{R}^{2d}), \|\lambda\|_{L^2(\mathbb{R}^{2d})} \le r\big\}, \quad \text{with} \quad r = \sup_\epsilon \|W_{0,\epsilon}\|_{L^2(\mathbb{R}^{2d})} < +\infty$$

be the closed ball with radius r, and $\{g_n, n \ge 1\}$ be a dense subset of $\mathcal{B}_r$. We equip $\mathcal{B}_r$ with the distance $d_{\mathcal{B}_r}$ defined by

$$d_{\mathcal{B}_r}(\lambda, \mu) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \Big|\big\langle \lambda - \mu, g_n \big\rangle_{L^2(\mathbb{R}^{2d})}\Big|$$

$\forall (\lambda, \mu) \in (\mathcal{B}_r)^2$, so that $(\mathcal{B}_r, d_{\mathcal{B}_r})$ is a compact metric space. Therefore, $(W_\epsilon)_\epsilon$ is a family of process with values in $(\mathcal{B}_r, d_{\mathcal{B}_r})$, since $\|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2d})} = \|W_{0,\epsilon}\|_{L^2(\mathbb{R}^{2d})}$. The topology generated by the metric $d_{\mathcal{B}_r}$ is equivalent to the weak topology on $L^2(\mathbb{R}^{2d})$ restricted to $\mathcal{B}_r$.

The three following sections describe in a chronological order the effects produced by the random medium on the wave propagation.

2 Phase Modulation Scaling $s = 1/(2\kappa_\gamma)$

This section describes the first effects caused by the small random fluctuations of the medium on the wave propagation. The following theorem presents the asymptotic behavior of the phase of ϕ_ϵ solution of (4). Theorem 2.1 has been shown [8] in the case $\gamma = 0$ and $s_c = 0$, but nevertheless, its proof remains the same as the one of [8, Theorem 1.2]. We state this result in order to provide a complete and self-contained presentation of the wave propagation in long-range random media. Under the long-range correlation assumption in time (14) and medium parameters given by (15), we have the following result.

Theorem 2.1 *Let us assume that the autocorrelation function $R(t, \mathbf{x})$ of the random perturbations is given by* (5), (10), *and* (15), *and let*

$$\kappa_0 = \frac{\alpha + 2\beta - 1}{2\beta} \quad \textit{and} \quad \kappa_\gamma = \frac{\kappa_0}{1 - \gamma\left(\frac{\alpha+\beta-1}{\beta}\right)} \quad \textit{for } \gamma \in (0, 1].$$

Let us consider the process $\widehat{\zeta}_{\kappa_\gamma,\epsilon}(t, \mathbf{k})$ defined by

$$\widehat{\zeta}_{\kappa_\gamma,\epsilon}(t, \mathbf{k}) = \frac{1}{\epsilon^{d(s-s_c)}} \widehat{\phi}_\epsilon\Big(t, \frac{\mathbf{k}}{\epsilon^{s-s_c}}\Big) e^{i|\mathbf{k}|^2 t/(2\epsilon^{s-2s_c})}, \quad \textit{for} \quad s = 1/(2\kappa_\gamma), \textit{ and } s_c \leq s,$$

where ϕ_ϵ satisfies (4) *with initial data* (18). *For each $t \geq 0$, $\mathbf{k} \in \mathbb{R}^d$ fixed, $\widehat{\zeta}_{\kappa_\gamma,\epsilon}(t, \mathbf{k})$ converges in distribution to*

$$\widehat{\zeta}(t, \mathbf{k}) = \widehat{\zeta}_0(\mathbf{k}) \exp\Big(i\sqrt{D(\alpha, \beta, \mathbf{k})} B_{\kappa_\gamma}(t)\Big),$$

where $(B_{\kappa_\gamma}(t))_t$ is a standard fractional Brownian motion with Hurst index κ_γ. Here

$$\widehat{\zeta}_0(\mathbf{k}) = \widehat{\phi}_0(\zeta - \mathbf{k}) \quad \textit{if } s_c = s,$$

and

$$\widehat{\zeta}_0(\mathbf{k}) = \phi_0(0)\delta(\zeta - \mathbf{k}) \quad \textit{otherwise.}$$

Moreover,

$$D(\alpha, \beta, \mathbf{k}) = \frac{a(0)}{(2\pi)^d \kappa_\gamma(2\kappa_\gamma - 1)} \int_0^{+\infty} d\rho \frac{e^{-\nu\rho}}{\rho^{2\alpha-1}} \int_{\mathbb{S}^{d-1}} dS(u) e^{i|\mathbf{k}|\rho u\cdot \mathbf{e}_1} \quad \textit{if } \beta = \frac{1}{2}, \gamma = 0, \textit{ and } s_c = 0,$$

and

$$D(\alpha, \beta, \mathbf{k}) = D(\alpha, \beta) = \frac{a(0)\Omega_d}{(2\pi)^d \kappa_\gamma(2\kappa_\gamma - 1)} \int_0^{+\infty} d\rho \frac{e^{-\nu\rho^{2\beta}}}{\rho^{2\alpha-1}} \quad \textit{otherwise,}$$

where, Ω_d is the surface area of the unit sphere in $\mathbb{R}^d$, and $\mathbf{e}_1 \in \mathbb{S}^{d-1}$.

In Theorem 2.1, $\widehat{\zeta}_0$ represents the direction of the wave, and $\exp(i\sqrt{D(\alpha, \beta, \mathbf{k})} B_{\kappa_\gamma}(t))$ represents the random phase modulation induces by the slowly decorrelating perturbation of the medium. As a result, in long-range random media macroscopic effects may happen on the field ϕ_ϵ at a shorter scale $s = 1/(2\kappa_\gamma) < 1$ without induced loss of coherence of the field ϕ_ϵ. These effects are just a phase modulation given by a fractional Brownian motion with Hurst index κ_γ, which depends on the statistic of the random potential V. Consequently, the phase and the phase space energy of the field ϕ_ϵ do not evolve on the same scale. However, the scale $s = 1/(2\kappa_\gamma) < 1$ is "universal" in the sense that a random phase modulation appears on the wave whatever the order of the low frequency initial condition, that is $\forall s_c \in [0, 1/(2\kappa_\gamma)]$.

[8] is the first paper showing a qualitative difference between the random effects induced on a wave propagating in long range and in rapidly decorrelating random media in time (13), for propagation media of dimension strictly greater than 1. In fact, it has been shown in [8] that the field ϕ_ϵ propagating in a rapidly decorrelating medium does not evolve before the scale $s = 1$, more precisely the phase and the phase space energy evolves at the same propagation scale e^{-1} $(s = 1)$, so that no significant wave decoherence can be observed before this propagation scale. In rapidly decorrelating random media the scale $s = 1$ is "universal", in the sense that it does not depend on the statistic of the random potential V.

For propagation scale parameter $s > 1/(2\kappa_\gamma)$ fast random oscillations begin to appear on the wave up to break the wave coherence, first on the large spatial correlation scales, and then as s increase, the wave decoherence is transmitted to smaller spatial correlation scales (see Figure 2).

3 Wave Decoherence for $s \in (1/(2\kappa_\gamma), 1)$

This section presents the main result of this paper, it describes the loss of coherence of the field ϕ_ϵ occurring after the onset of the random phase modulation described in Section 2. On propagation scales e^{-s} with $s > 1/(2\kappa_\gamma)$ but strictly less than 1, the random phase modulation obtained in the Section 2 begins to oscillate very fast up to break the wave coherence, and produce momentum diffusion effect. However, according to [21] the wave decoherence does not take place for $s_c = 0$. To study this diffusion phenomenon we need the scaled Wigner transform (17) to capture the wave decoherence for spatial correlation parameters $s_c > 0$. Using the notation introduced in Section 1.3, we have the following results.

In Theorem 3.1, Theorem 3.2, and Theorem 4.1, we show that the good spatial correlation length is $s_c = (1-s)/\theta$, so that for $s_c \leq s$, we have $s \geq 1/(1+\theta) \geq 1/(2\kappa_\gamma)$, where θ is defined by (16) and κ_γ in Theorem 2.1. As a result, no decoherence effect can be observed before the propagation scaling $s = 1/(1+\theta)$ for any spatial correlation parameter $s_c \leq s$. Theorem 3.1 below deals with the case $s > 1/(1+\theta)$ for which one can describe the wave decoherence in term of a fractional diffusion, while Theorem 3.2 deals with the critical case $s_c = s = 1/(1+\theta)$ for which there is no wave decoherence, but a random phase modulation of the Wigner transform.

Theorem 3.1 *Let us assume that the autocorrelation function $R(t, \mathbf{x})$ of the random perturbations is given by* (5), (10), *and* (15). *For $s \in (1/(2\kappa_\gamma), 1)$, and*

$$s_c = \frac{1-s}{\theta} < s,$$

where $\theta \in (0,1)$ is defined by (16), *the family of scaled Wigner transform $(W_\epsilon)_{\epsilon \in (0,1)}$ defined by* (17) *and solution of the transport equation* (20), *converges in probability on $\mathcal{C}([0,+\infty), (\mathcal{B}_r, d_{\mathcal{B}_r}))$ as $\epsilon \to 0$ to a limit denoted by W. More precisely, $\forall T > 0$ and $\forall \eta > 0$,*

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,T]} d_{\mathcal{B}_r}(W_\epsilon(t), W(t)) > \eta \right) = 0.$$

W is the unique solution uniformly bounded in $L^2(\mathbb{R}^{2d})$ of the fractional diffusion equation

$$\partial_t W = -\sigma(\theta)(-\Delta_{\mathbf{k}})^{\theta/2} W, \tag{22}$$

with $W(0, \mathbf{x}, \mathbf{k}) = W_0(\mathbf{x}, \mathbf{k})$. Here, $(-\Delta_{\mathbf{k}})^{\theta/2}$ is the fractional Laplacian with Hurst index $\theta \in (0,1)$, and

$$\sigma(\theta) = \frac{2a(0)\theta\Gamma(1-\theta)}{(2\pi)^d} \int_{\mathbb{S}^{d-1}} dS(\mathbf{u}) |\mathbf{e}_1 \cdot \mathbf{u}|^\theta$$

with $\mathbf{e}_1 \in \mathbb{S}^{d-1}$ and $\Gamma(z) = \int_0^{+\infty} t^{1-z} e^{-t} dt$. Moreover, W is given by the following formula

$$W(t, \mathbf{x}, \mathbf{k}) = \frac{1}{(2\pi)^d} \int d\mathbf{q} e^{i\mathbf{k}\cdot\mathbf{q}} e^{-\sigma(\theta)|\mathbf{q}|^\theta} \widehat{W}_0^{\mathbf{k}}(\mathbf{x}, \mathbf{q}), \tag{23}$$

where $\widehat{W}_0^{\mathbf{k}}$ stands for the Fourier transform of W_0 with respect to the variable $\mathbf{k}$.

Let us note that we cannot expect a convergence on $L^2(\mathbb{R}^{2d})$ equipped with the strong topology. In fact, the following conservation relation $\|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2d})} = \|W_{0,\epsilon}\|_{L^2(\mathbb{R}^{2d})}$, is not true anymore for the limit W. Moreover, let us note that the Wigner distribution W_ϵ is self-averaging as ϵ goes to 0, that is the limit W is not random anymore. This self-averaging phenomenon of the Wigner distribution has already been observed in several studies [3, 5, 6, 21] for $s = 1$, and is very useful in applications. The proof of Theorem 3.1 is given in Section 5 and is based on an asymptotic analysis using perturbed-test-function and martingale techniques.

Equation (22) describes the loss of coherence of the field ϕ_ϵ for the particular spatial correlation parameter s_c through a momentum diffusion. This fractional diffusion exhibits a damping term obeying to a power law with exponent $\theta \in (0,1)$ describing the decoherence rate of the field ϕ_ϵ. An important point is that the wave decoherence mechanism is deterministic, it does not depend on the particular realization of the random medium. Finally, let us note that W does not evolves in $\mathbf{x}$. In fact, in (20) the dispersion term $\mathbf{k} \cdot \nabla_{\mathbf{x}}$ is of order ϵ^{s_c} since the wave dispersion evolves on a spatial scale $s_c = 0$, and W_ϵ captures only spatial variations happening on a spatial scale $s_c > 0$.

The following theorem investigates the special case $s_c = s = 1/(1+\theta)$, with either $\gamma > 0$ or $\beta < 1/2$. This special case study the decoherence of the wave envelop ϕ_0 (18) itself and not its local decoherence. The case $\gamma = 0$ and $\beta = 1/2$, and $s = s_c$ has been addressed in Theorem 3.1, since in this particular case $1/(1+\theta) = 1/(2\kappa_0)$. As a result, the Wigner transform does not evolves (no phase modulation as in Theorem 3.2 nor momentum diffusion) and then there is no wave decoherence.

Theorem 3.2 *Let us assume that the autocorrelation function $R(t,\mathbf{x})$ of the random perturbations is given by* (5), (10), *and* (15). *For either $\gamma > 0$ or $\beta < 1/2$, and*

$$s_c = s = \frac{1}{1+\theta},$$

where $\theta \in (0,1)$ is defined by (16), *the family of scaled Wigner transform $(W_\epsilon)_{\epsilon\in(0,1)}$ defined by* (17) *and solution of the transport equation* (20), *converges in distribution on $\mathcal{C}([0,+\infty), L^2(\mathbb{R}^{2d}))$ as $\epsilon \to 0$ to a limit W defined by*

$$W(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int d\mathbf{q}\widehat{W}_0^{\mathbf{k}}(\mathbf{x},\mathbf{q})\exp\Big(i\mathbf{k}\cdot\mathbf{q} + i\int \mathcal{B}_t(d\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}(e^{-i\mathbf{q}\cdot\mathbf{p}/2} - e^{i\mathbf{q}\cdot\mathbf{p}/2})\Big).$$

W is the unique solution of the stochastic differential equation

$$\begin{aligned} dW(t,\mathbf{x},\mathbf{k}) = & -\sigma(\theta)(-\Delta_{\mathbf{k}})^{\theta/2}W(t,\mathbf{x},\mathbf{k}) \\ & + \frac{2ia(0)}{(2\pi)^d}\int \mathcal{B}_t(d\mathbf{p})e^{i\mathbf{x}\cdot\mathbf{p}}\Big(W\big(t,\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\big) - W\big(t,\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\big)\Big), \end{aligned} \tag{24}$$

with $W(0,\mathbf{x},\mathbf{k}) = W_0(\mathbf{x},\mathbf{k})$. Here, $(\mathcal{B}_t)_t$ is a complex Brownian motion on $\mathcal{H}'_\theta$ the dual space of

$$\mathcal{H}_\theta = \Big\{\varphi \text{ such that } \int \frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}}|\varphi(\mathbf{p})|^2 < +\infty\Big\},$$

with covariance function

$$\mathbb{E}\big[\mathcal{B}_t(\varphi_1)\mathcal{B}_s(\varphi_2)\big] = s\wedge t\int \frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}}\varphi_1(\mathbf{p})\varphi_2(-\mathbf{p}) \quad \text{and} \quad \mathbb{E}\big[\mathcal{B}_t(\varphi_1)\overline{\mathcal{B}_s(\varphi_2)}\big] = s\wedge t\int \frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}}\varphi_1(\mathbf{p})\overline{\varphi_2(\mathbf{p})}.$$

Moreover, $(-\Delta_{\mathbf{k}})^{\theta/2}$ is the fractional Laplacian with Hurst index $\theta \in (0,1)$, and

$$\sigma(\theta) = \frac{2a(0)\theta\Gamma(1-\theta)}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} dS(\mathbf{u})|\mathbf{e}_1\cdot\mathbf{u}|^\theta$$

with $\mathbf{e}_1 \in \mathbb{S}^{d-1}$ and $\Gamma(z) = \int_0^{+\infty} t^{1-z}e^{-t}dt$.

This limiting Wigner transform is random because the wave does not propagate enough to be self-averaging. In fact, as shown in Theorem 3.1 $\forall s > 1/(1+\theta)$ the limiting Wigner transform is self-averaging and is equal to the expectation of the limiting Wigner transform obtain in the case $s_c = s = 1/(1+\theta)$.

Let us note that the convergence holds on $L^2(\mathbb{R}^{2d})$ equipped with the strong topology. In fact, the conservation relation $\|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2d})} = \|W_{0,\epsilon}\|_{L^2(\mathbb{R}^{2d})}$ is still true for the limiting process W. This conservation for the limiting process means that there is no loss of coherence of the wave for $s_c = s = 1/(1+\theta)$. However, in this scaling the limit W is a stochastic process, which is given in the Fourier domain by a random phase modulation. As illustrated in Figure 2, the random phase modulation of the Wigner transform is caused by the fast phase modulation of the field ϕ_ϵ itself obtained in Theorem 2.1. For $s \in (1/(2\kappa_\gamma), 1/(1+\theta))$ this random phase modulation oscillates very fast up to the scales $s = 1/(1+\theta)$, and produce a phase modulation (if $\gamma > 0$ or $\beta < 1/2$) on the Wigner transform. Afterwards, on the scales $s > 1/(1+\theta)$ this random phase modulation of the Wigner transform oscillates very fast and then breaks the wave coherence. The phase modulation of the Wigner transform average out and then describes the wave decoherence for $s \in (1/(1+\theta), 1)$.

Theorem 3.1 and Theorem 3.2 show a qualitative difference between the random effects induced on a wave propagating in long range and in rapidly decorrelating random media in time (13). As previously noted, it has been shown in [8] for rapidly decorrelating random media that the field ϕ_e does not evolve before the scale $s = 1$, so that there is no wave decoherence before $s = 1$. In this case the phase and the phase space energy of the wave, which describes the wave decoherence, evolve on the same scale. Let us give a probabilistic interpretation to illustrate the difference of the random effects caused by the long range and rapidly decorrelating random media. The wave decoherence is given by the random operator (21), which after homogenization gives rise to an operator of the form

$$\epsilon^{1-s}\int d\mathbf{p}\sigma(\mathbf{p})\Big(\lambda\big(\mathbf{k}+\frac{\mathbf{p}}{\epsilon^{s_c}}\big) - \lambda(\mathbf{k})\Big).$$

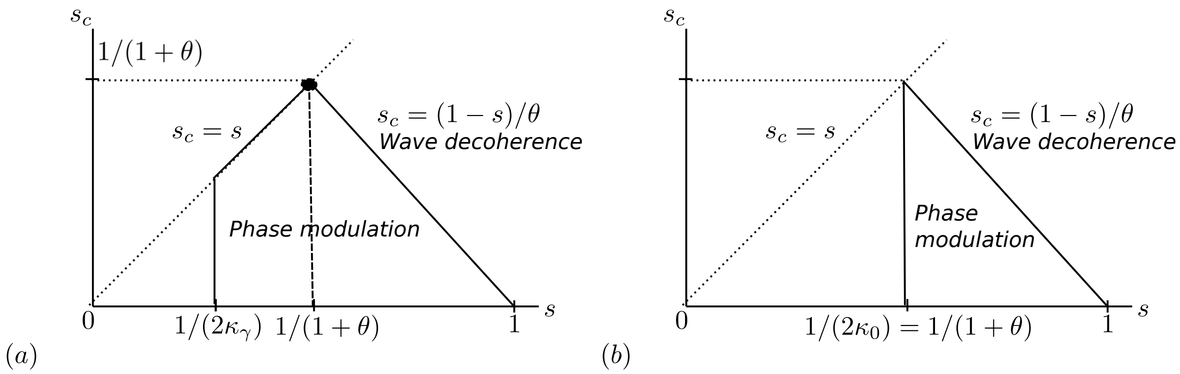

Figure 2: Schematic representation of the behavior of the wave. s is the propagation scale parameter and s_c is the spatial correlation parameter. The phase modulation effects appear for $s = 1/(2\kappa_\gamma)$. Afterward, wave decoherence appears on the low spatial correlation scales first, and then propagates to the higher one according to the formula $s_c = (1-s)/\theta$. In (a) we represent the behavior of the wave in the case where $\gamma > 0$ or $\beta < 1/2$. The dot represents the transition between the phase modulation effects and the wave decoherence effects (see Theorem 3.2). In (b) we represent the behavior of the wave in the case where $\gamma = 0$ and $\beta = 1/2$.

Formally, the momentum diffusion of the Wigner is given by the variations of a stochastic jump process [31] with infinitesimal generator given by the previous one, which characterizes its variations. Since $\sigma(\mathbf{p}) \in L^1(\mathbb{R})$, the variation of the jump process are therefore bounded by $\mathcal{O}(\epsilon^{1-s})$ for all s_c. That is why we cannot observe wave decoherence in rapidly decorrelating random media for $s_c > 0$. The variations of the jump process become significant only for the scaling $s = 1$ and $s_c = 0$, and that is why we can observe wave decoherence only at this scaling. However, for long-range random media the variations of the jump process are large and given by

$$\epsilon^{1-s} \int d\mathbf{p} \frac{a(p)}{|\mathbf{p}|^{d+\theta}} \Big(\lambda\big(\mathbf{k} + \frac{\mathbf{p}}{\epsilon^{s_c}}\big) - \lambda(\mathbf{k}) \Big) \underset{\epsilon \to 0}{\sim} \epsilon^{1-s-\theta s_c} a(0) \int \frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}} \Big(\lambda(\mathbf{k} + \mathbf{p}) - \lambda(\mathbf{k}) \Big).$$

As a result the large variations can balance the small term ϵ^{1-s} and give rise to significant momentum diffusion. In rapidly decorrelating random media the wave decoherence is only significant for $s_c = 0$ and for a sufficiently large propagation distance, while for the long-range random media the wave decoherence occurs first on the large correlation scale s_c and propagates to the smaller ones as the propagation scale increase, up to the scaling $s_c = 0$ and $s = 1$ (see Figure 2). We will see in the next section that the wave decoherence mechanism in the scaling $s = 1$ and $s_c = 0$ for long-range and rapidly decorrelating random media is exactly the same, but however, this decoherence mechanism has different regularity properties in both cases.

Let us remark that the spatial frequency of the initial condition (18) which is of order ϵ^{s-s_c} is low compared to the one of the random medium ($\sim \epsilon^s$) on the macroscopic scale ϵ^{-s}. In rapidly deccorelating random media such low frequency sources do not interact with the random medium. However, as we have seen in Theorem 3.1 low frequency initial conditions interact strongly with slowly decorrelating random media. This fact can be useful in passive imaging of a target in a slowly decorrelating random medium [18].

4 The Radiative Transport Scaling $s = 1$ and $s_c = 0$

This section describes the evolution of the phase space energy density, where the momentum diffusion and the dispersion are of order 1. The following results have been proved in [21], but we state these results in order to provide a complete and self-contained presentation of the wave propagation in long-range random media.

The radiative transport scaling is an important scaling since it is the only one for which wave decoherence happens for rapidly deccorelating random media. This scaling provides also wave decoherence for long-range random media, but the momentum diffusion is not exactly the fractional diffusion as previously obtained. Even if the decoherence mechanism are the same in long-range and rapidly decorrelating random media this result lightens an important qualitative difference between the two kind of random media.

In the radiative transfer scaling in addition to a momentum diffusion, now we also have a dispersion term $\mathbf{k}\cdot\nabla_{\mathbf{x}}$. Using the notation introduced in Section 1.3, we have the following result.

Theorem 4.1 *Let us assume that the autocorrelation function $R(t,\mathbf{x})$ of the random perturbations is given by (5), (10), and (15). For $\gamma>0$, the family $(W_\epsilon)_{\epsilon\in(0,1)}$ of Wigner transform, solution of the transport equation (20) with $s_c=0$, converges in probability on $\mathcal{C}([0,+\infty),(\mathcal{B}_r,d_{\mathcal{B}_r}))$ as $\epsilon\to 0$ to a limit denoted by W. More precisely, $\forall T>0$ and $\forall\eta>0$,*

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,T]} d_{\mathcal{B}_r}(W_\epsilon(t),W(t))>\eta\right)=0.$$

W is the unique classical solution uniformly bounded in $L^2(\mathbb{R}^{2d})$ of the radiative transfer equation

$$\partial_t W+\mathbf{k}\cdot\nabla_{\mathbf{x}}W=\mathcal{L}W, \tag{25}$$

with $W(0,\mathbf{x},\mathbf{k})=W_0(\mathbf{x},\mathbf{k})$. Here, $\mathcal{L}$ is defined by

$$\mathcal{L}\varphi(\mathbf{k})=\int d\mathbf{p}\sigma(\mathbf{p}-\mathbf{k})\big(\varphi(\mathbf{p})-\varphi(\mathbf{k})\big), \tag{26}$$

with $\varphi\in\mathcal{C}^\infty(\mathbb{R}^d)$ and where

$$\sigma(\mathbf{p})=\frac{2\widehat{R}_0(\mathbf{p})}{(2\pi)^d\mathfrak{g}(\mathbf{p})}=\frac{2a(\mathbf{p})}{(2\pi)^d|\mathbf{p}|^{d+\theta}}.$$

Moreover, W is given by

$$W(t,\mathbf{x},\mathbf{k})=\frac{1}{(2\pi)^{2d}}\int d\mathbf{y}d\mathbf{q}e^{i(\mathbf{x}\cdot\mathbf{y}+\mathbf{k}\cdot\mathbf{q})}e^{\int_0^t du\Psi(\mathbf{q}+u\mathbf{y})}\widehat{W}_0(\mathbf{y},\mathbf{q}+t\mathbf{y}),$$

where

$$\Psi(\mathbf{q})=\int d\mathbf{p}\sigma(\mathbf{p})(e^{i\mathbf{p}\cdot\mathbf{q}}-1),$$

so that $\forall t_0>0$ we have

$$W\in\mathcal{C}^0\Big((0,+\infty),\bigcap_{k\geq 0}H^k(\mathbb{R}^{2d})\Big)\cap L^\infty\Big([t_0,+\infty),\bigcap_{k\geq 0}H^k(\mathbb{R}^{2d})\Big).$$

In Theorem 4.1 $H^k(\mathbb{R}^{2d})$ stands for the kTh. Sobolev space on $\mathbb{R}^{2d}$, and $\widehat{W}_0$ stands for the Fourier transform in both variables $\mathbf{x}$ and $\mathbf{k}$. Let us note that the case $\gamma=0$, has not been addressed in [21], because it leads to much more difficult algebra than the cases $\gamma\in(0,1)$. More precisely, we show in this case the tightness of the family $(W_\epsilon)_\epsilon$ and show that all the subsequence limits are deterministic weak solutions of the same transport equation (25) with

$$\mathcal{L}\varphi(\mathbf{k})=\int d\mathbf{p}\sigma\Big(\mathbf{p}-\mathbf{k},\frac{|\mathbf{k}|^2-|\mathbf{p}|^2}{2}\Big)\big(\varphi(\mathbf{p})-\varphi(\mathbf{k})\big)\quad\text{with}\quad\sigma(\mathbf{p},\omega)=\frac{2\mathfrak{g}(\mathbf{p})\widehat{R}_0(\mathbf{p})}{(2\pi)^d(\mathfrak{g}(\mathbf{p})^2+\omega^2)}.$$

However, it is difficult to show the weak uniqueness of the limiting transfer equation in the slowly decorrelating case. First, the technique used in the proof of Theorem 4.1 to show the weak uniqueness leads to very difficult algebra. Second, it should be possible to use the techniques developed in [11]. This kind of techniques use a lower and an upper bound of σ in terms of $|\mathbf{k}-\mathbf{p}|^{-(d+\theta)}$. However, we just have an upper bound of this form. Nevertheless, we think that the transport equation obtained in the case $\gamma=0$ is still weakly well posed.

As in Section 3, we cannot expect a convergence on $L^2(\mathbb{R}^{2d})$ equipped with the strong topology since the conservation relation $\|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2d})}=\|W_{0,\epsilon}\|_{L^2(\mathbb{R}^{2d})}$ cannot be satisfy by the limit W. Moreover, the Wigner distribution W_ϵ is also self-averaging as ϵ goes to 0, that is the limit W is not random anymore. This self-averaging phenomenon of the Wigner distribution has already been observed in several studies [3, 5, 6, 21] and is very useful in applications.

Equation (25) describes the asymptotic evolution of the phase space energy distribution of the field ϕ_ϵ solution of the random Schrödinger equation (4). It describes the dispersion phenomenon through the transport term $\mathbf{k}\cdot\nabla_{\mathbf{x}}$, and the wave decoherence through the nonlocal transfer operator $\mathcal{L}$ defined by (26). Finally, the transfer coefficient $\sigma(\mathbf{p}-\mathbf{k})$ describes the energy transfer between the modes $\mathbf{k}$ and $\mathbf{p}$.

An interesting remark is that the result of Theorem 4.1 does not depend on whether $\int d\mathbf{p}\sigma(\mathbf{p})$ is finite or not. In other words, the radiative transfer equation (25) is valid in the two case, slowly and rapidly decaying correlations in time (13). However, as noted in Theorem 4.1, these equations, in the both cases, behave in different ways. As it has been discussed in Section 1.2, in the case of rapidly decaying correlations in time (13), that is $\int d\mathbf{p}\sigma(\mathbf{p}) < +\infty$, the radiative transfer equation (25) has the same properties as in the mixing case addressed in [7]. In the case of slowly decaying correlations in time (13), that is $\int d\mathbf{p}\sigma(\mathbf{p}) = +\infty$, we observe a regularizing effect of the solutions of (25) which cannot be observed in the case of rapidly decaying correlations in time. This regularizing property lightens a important qualitative difference between the cases rapidly and slowly decaying correlations.

Let us note that the momentum diffusion in (25) is not exactly the same diffusion mechanism as the one obtained in Section 3, but they are very closed. In fact the momentum diffusion given in Theorem 3.1 is described in terms of a fractional Laplacian, while in the radiative transfer regime the momentum diffusion is described in terms of a nonlocal operator which is not exactly a fractional Laplacian. However, this two diffusion mechanisms are anomalous diffusions since they lead to damping terms obeying to a power law with exponent $\theta \in (0,1)$. We have to wait for a long time of propagation in the radiative transfer regime to observe again the momentum diffusion given by a fractional Laplacian. This approximation in the radiative transfer scaling is proved in [21, Theorem 5.1]

Conclusion

In this paper we have studied the different behaviors happening on a wave propagating in a random media with long-range correlation properties. We have exhibited three different behaviors over a range of scale given by Theorem 2.1, Theorem 3.1, Theorem 3.2, and Theorem 4.1. These asymptotic behaviors differ strongly from those obtain with the random Schrödinger equation with rapidly decaying correlations [4, 7, 8], for which all the random effects appear on the wave on the same propagation scale e^{-1} ($s = 1$). In the context of long-range correlations, the effects of the randomness appear progressively according the scale of propagation. We have seen in Theorem 2.1 that the random perturbation induce a phase modulation in term of fractional Brownian motion on the wave itself. This phase modulation begins to oscillate very fast up to produce a phase modulation in the Wigner transform (Theorem 3.2) on the large spatial scale. Afterward, this phase modulation in the Wigner transform begins also to oscillate very fast up to average out and then breaks the wave coherence. The wave decoherence first happens on the large spatial scales and then propagates to smaller one as the propagation distance increase(Theorem 3.1, Theorem 4.1, and see Figure 2). The wave decoherence mechanism is described in term of an anomalous momentum diffusion (Theorem 3.1 and Theorem 4.1) since it obeys to a power law with exponent lying in $(0,1)$. Theorem 3.1 shows that low frequency initial conditions interact strongly with slowly decorrelating random media. This result can be useful in passive imaging of a target in a long-range random medium [18].

5 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the perturbed-test-function approach and follow the techniques of the proof of [21, Theorem 2.2]. Using the notion of a pseudogenerator, we prove tightness and characterize all subsequence limits.

5.1 Pseudogenerator

We recall the techniques developed by Kurtz and Kushner [25]. Let $\mathcal{M}^\epsilon$ be the set of all $\mathcal{F}^\epsilon$-measurable functions $f(t)$ for which $\sup_{t\leq T}\mathbb{E}\left[|f(t)|\right] < +\infty$ and where $T>0$ is fixed. Here, $\mathcal{F}^\epsilon_t = \mathcal{F}_{t/\epsilon}$ and $(\mathcal{F}_t)$ is defined by (6). The $p-\lim$ and the pseudogenerator are defined as follows. Let f and f^δ in $\mathcal{M}^\epsilon$

$\forall \delta > 0$. We say that $f = p - \lim_\delta f^\delta$ if

$$\sup_{t,\delta} \mathbb{E}[|f^\delta(t)|] < +\infty \quad \text{and} \quad \lim_{\delta\to 0} \mathbb{E}[|f^\delta(t) - f(t)|] = 0 \quad \forall t.$$

The domain of $\mathcal{A}^\epsilon$ is denoted by $\mathcal{D}(\mathcal{A}^\epsilon)$. We say that $f \in \mathcal{D}(\mathcal{A}^\epsilon)$ and $\mathcal{A}^\epsilon f = g$ if f and g are in $\mathcal{D}(\mathcal{A}^\epsilon)$ and

$$p - \lim_{\delta\to 0} \left[\frac{\mathbb{E}^\epsilon_t[f(t+\delta)] - f(t)}{\delta} - g(t) \right] = 0,$$

where $\mathbb{E}^\epsilon_t$ is the conditional expectation given $\mathcal{F}^\epsilon_t$. A useful result about pseudogenerator $\mathcal{A}^\epsilon$ is given by the following theorem.

Theorem 5.1 *Let $f \in \mathcal{D}(\mathcal{A}^\epsilon)$. Then*

$$M^\epsilon_f(t) = f(t) - f(0) - \int_0^t \mathcal{A}^\epsilon f(u) du$$

is an $(\mathcal{F}^\epsilon_t)$-martingale.

5.2 Tightness

According to the continuity of $(W_\epsilon)_{\epsilon\in(0,1)}$ and since $\sup_{t,\epsilon} \|W_\epsilon(t)\|_{L^2(\mathbb{R}^{2d})} < +\infty$, it suffices to prove the following proposition to show that the process $(W_\epsilon)_{\epsilon\in(0,1)}$ is tight in $\mathcal{C}([0,+\infty), L^2_w(\mathbb{R}^{2d}))$.

Proposition 5.1 $\forall \lambda \in \mathcal{C}^\infty_0(\mathbb{R}^{2d})$*, the family $(W_{\epsilon,\lambda})_{\epsilon\in(0,1)}$ is tight on $\mathcal{D}([0,+\infty),\mathbb{C})$.*

Proof (of Proposition 5.1) Throughout the proof Proposition 5.1, let $\lambda \in \mathcal{C}^\infty_0(\mathbb{R}^{2d})$, f be a bounded smooth function, and $f^\epsilon_0(t) = f(W_{\epsilon,\lambda}(t))$. According to the property of the Gaussian potential V [1, 2], we have $\forall T > 0$,

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|\mathcal{L}_\epsilon \lambda(t)\|^2_{L^2(\mathbb{R}^{2d})}\big] < +\infty,$$

so that the following pseudogenerator is well defined:

$$\mathcal{A}^\epsilon f^\epsilon_0(t) = f'(W_{\epsilon,\lambda}(t))\Big[\epsilon^{s_c} W_{\epsilon,\lambda_1}(t) + \epsilon^{(1-\gamma)/2-s} \big\langle W_\epsilon(t), \mathcal{L}_\epsilon(t)\lambda \big\rangle_{L^2(\mathbb{R}^{2d})}\Big],$$

where $\mathcal{L}_\epsilon$ is defined by (21), and

$$\lambda_1(\mathbf{x},\mathbf{k}) = \mathbf{k} \cdot \nabla_\mathbf{x} \lambda(\mathbf{x},\mathbf{k}).$$

Let

$$\begin{aligned} f^\epsilon_1(t) =& \epsilon^{(1-\gamma)/2-s} f'(W_{\epsilon,\lambda}(t)) \int d\mathbf{x} d\mathbf{k} W_\epsilon(t,\mathbf{x},\mathbf{k}) \\ & \times \int_t^{+\infty} \mathbb{E}^\epsilon_t\Big[\int \frac{\widehat{V}\big(\frac{u}{\epsilon^{s+\gamma}}, d\mathbf{p}\big)}{(2\pi)^d i} e^{i(u-t)\mathbf{p}\cdot\mathbf{k}/\epsilon^s} e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon^s} \times \Big(\lambda\big(\mathbf{x},\mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x},\mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big)\Big] du. \end{aligned}$$

Lemma 5.1 $\forall T > 0$*, and* $\eta > 0$

$$\lim_\epsilon \mathbb{P}\Big(\sup_{0\le t\le T} |f^\epsilon_1(t)| > \eta\Big) = 0, \quad \textit{and} \quad \lim_\epsilon \sup_{t\ge 0} \mathbb{E}\left[|f^\epsilon_1(t)|\right] = 0.$$

Proof (of Lemma 5.1) Using (7), we have

$$f^\epsilon_1(t) = \epsilon^{\frac{1+\gamma}{2}} f'(W_{\epsilon,\lambda}(t)) \big\langle W_\epsilon(t), \mathcal{L}_{1,\epsilon}\lambda(t)\big\rangle_{L^2(\mathbb{R}^{2d})}$$

with

$$\mathcal{L}_{1,\epsilon}\lambda(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d i} \int \frac{\widehat{V}\big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}\big)}{\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma \mathbf{k}\cdot\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon^s} \Big(\lambda\big(\mathbf{x},\mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x},\mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big).$$

Lemma 5.2

$$\lim_{\epsilon} \epsilon^{1+\gamma}\mathbb{E}\Big[\sup_{t\in[0,T]} \|\mathcal{L}_{1,\epsilon}\lambda(t)\|^2_{L^2(\mathbb{R}^{2d})}\Big] = 0.$$

Proof (of Lemma 5.2) First,

$$\|\mathcal{L}_{1,\epsilon}\lambda(t)\|^2_{L^2(\mathbb{R}^{2d})} \leq \frac{1}{(2\pi)^d}\int d\mathbf{x}d\mathbf{k}\Big|\int \frac{\widehat{V}\big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}\big)}{\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma \mathbf{k}\cdot\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon^s}\Big(\lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big)\Big|^2.$$

Let us fixe $\mathbf{x}$, $\mathbf{k}$, and u. Let

$$\phi_{\lambda,\mathbf{x},\mathbf{k},u}(\mathbf{p}) = \frac{e^{i\mathbf{p}\cdot\mathbf{x}/\epsilon}}{\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma \mathbf{k}\cdot\mathbf{p}}\Big(\lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big).$$

According to (16) $\phi_{\lambda,\mathbf{x},\mathbf{k},u} \in \mathcal{H}$. Consequently, $\tilde{V} = \big\langle \widehat{V}, \phi_{\lambda,\mathbf{x},\mathbf{k},u}\big\rangle_{\mathcal{H}',\mathcal{H}}$ is centered Gaussian process with a pseudo-metric m on $[0,T]$ given by

$$m(t_1,t_2) = \mathbb{E}\left[\Big(\tilde{V}\Big(\frac{t_1}{\epsilon^{s+\gamma}}\Big) - \tilde{V}\Big(\frac{t_2}{\epsilon^{s+\gamma}}\Big)\Big)^2\right]^{1/2}.$$

Then, $\forall (t_1,t_2)\in[0,T]^2$

$$m^2(t_1,t_2) \leq C\frac{|t_1-t_2|}{\epsilon^{s+\gamma+\theta s_c}}\Big(\sup_{\mathbf{x},\mathbf{k}}|\nabla_{\mathbf{k}}\lambda(\mathbf{x},\mathbf{k})|^2\int_{|\mathbf{p}|<1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta-1}} + \sup_{\mathbf{x},\mathbf{k}}|\lambda(\mathbf{x},\mathbf{k})|^2\int_{|\mathbf{p}|>1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta}}\Big)$$

and

$$\begin{aligned} diam_m^2([0,T]) \leq \frac{C}{\epsilon^{(\theta+2\beta)s_c}}\Big(&\int_{|\mathbf{p}|<1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta+2\beta-1}}\int_{-1/2}^{1/2} du|\nabla_{\mathbf{x}}\lambda(\mathbf{x},\mathbf{k}+u\mathbf{p})|^2 \\ &+ \int_{|\mathbf{p}|>1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta+2\beta}}\Big(\lambda\big(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big)^2\Big). \end{aligned}$$

Here, $diam_m([0,T])$ stands for the diameter of $[0,T]$ under the pseudo-metric m_2. According to [2, Theorem 2.1.3], we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\Big|\tilde{V}\Big(\frac{t}{\epsilon^{s+\gamma}}\Big)\Big|^2\right] \leq C_1\left(\int_0^{b(\mathbf{x},\mathbf{k})/\epsilon^{(\theta+2\beta)s_c}}\sqrt{\ln\Big(C_2\frac{T}{r^2\epsilon^{s+\gamma+\theta s_c}}\Big)}dr\right)^2,$$

where

$$\begin{aligned} b^2(\mathbf{x},\mathbf{k}) = C\Big(&\int_{|\mathbf{p}|<1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta+2\beta-2}}\int_{-1/2}^{1/2} du|\nabla_{\mathbf{x}}\lambda(\mathbf{x},\mathbf{k}+u\mathbf{p})|^2 \\ &+ \int_{|\mathbf{p}|>1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta+2\beta}}\Big(\lambda\big(\mathbf{x},\mathbf{k}-\frac{\mathbf{p}}{2}\big) - \lambda\big(\mathbf{x},\mathbf{k}+\frac{\mathbf{p}}{2}\big)\Big)^2\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} \epsilon^{1+\gamma}\mathbb{E}\big[\sup_{t\in[0,T]}\|\mathcal{L}_{1,\epsilon}\lambda(t)\|^2_{L^2(\mathbb{R}^{2d})}\big] &\leq C\int d\mathbf{x}d\mathbf{k}\Big(\int_0^{\epsilon^{s+\gamma-2\beta s_c}b(\mathbf{x},\mathbf{k})}\sqrt{\ln\Big(C_2\frac{T}{r^2}\Big)}dr\Big)^2 \\ &\leq C\epsilon^{s+\gamma-2\beta s_c}\int d\mathbf{x}d\mathbf{k}b(\mathbf{x},\mathbf{k})\int_0^1 \ln\Big(C_2\frac{T}{r^2}\Big)dr < +\infty, \end{aligned}$$

which concludes the proof of Lemma 5.2, since $s_c = (1-s)/\theta$ and $s_c < s < (s+\gamma)/(2\beta)$. □

Then, the proof of Lemma 5.1 is a direct consequence of Lemma 5.2.□

The following lemma insures the tightness of the process $(W_\epsilon)_\epsilon$.

Lemma 5.3 *$\forall T>0$, $\big\{\mathcal{A}^\epsilon\big(f_0^\epsilon + f_1^\epsilon\big)(t), \epsilon\in(0,1), 0\leq t\leq T\big\}$ is uniformly integrable.*

Proof (of Lemma 5.3) First, Lemma 5.4 insures that the pseudogenerator at $\mathcal{A}^\epsilon\big(f_0^\epsilon + f_1^\epsilon\big)(t)$ is well defined and it is also needed to show the uniform integrability.

Lemma 5.4

$$\sup_{\epsilon\in(0,1)} \epsilon^{1-s}\mathbb{E}\left[\|\mathcal{L}_\epsilon\big(\mathcal{L}_{1,\epsilon}\lambda(t)\big)(t)\|^2_{L^2(\mathbb{R}^{2d})}\right] < +\infty.$$

Proof (of Lemma 5.4) A short computation gives

$$\begin{aligned}
&\mathcal{L}_\epsilon\big(\mathcal{L}_{1,\epsilon}\lambda\big)(t,\mathbf{x},\mathbf{k}) = \\
&\quad\iint \widehat{V}\big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_1\big)\widehat{V}\big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_2\big)e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot\mathbf{x}/\epsilon^s}\\
&\quad\times\Big(\frac{1}{\mathfrak{g}(\mathbf{p}_2) - i\epsilon^\gamma\big(\mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big)\cdot\mathbf{p}_2}\big(\lambda\big(\mathbf{x},\mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x},\mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big)\big)\\
&\quad - \frac{1}{\mathfrak{g}(\mathbf{p}_2) - i\epsilon^\gamma\big(\mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big)\cdot\mathbf{p}_2}\big(\lambda\big(\mathbf{x},\mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x},\mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big)\big)\Big).
\end{aligned}$$

Moreover, the forth order moment of our Gaussian field $\widehat{V}$ is given by

$$\begin{aligned}
\mathbb{E}\big[\widehat{V}(t_1,d\mathbf{p}_1)\widehat{V}(t_2,d\mathbf{p}_2)\widehat{V}^*(t_3,d\mathbf{p}_3)\widehat{V}^*(t_4,d\mathbf{p}_4)\big] =&\\
&(2\pi)^{2d}\tilde{R}(t_1-t_2,\mathbf{p}_1)\tilde{R}(t_3-t_4,\mathbf{p}_3)\delta(\mathbf{p}_1+\mathbf{p}_2)\delta(\mathbf{p}_3+\mathbf{p}_4)\\
&+(2\pi)^{2d}\tilde{R}(t_1-t_3,\mathbf{p}_1)\tilde{R}(t_2-t_4,\mathbf{p}_3)\delta(\mathbf{p}_1-\mathbf{p}_3)\delta(\mathbf{p}_2-\mathbf{p}_4)\\
&+(2\pi)^{2d}\tilde{R}(t_1-t_4,\mathbf{p}_1)\tilde{R}(t_2-t_3,\mathbf{p}_3)\delta(\mathbf{p}_1-\mathbf{p}_4)\delta(\mathbf{p}_2-\mathbf{p}_3),
\end{aligned}$$

so that, using the smoothness of λ, (16) and the change of variable $\mathbf{p}' = \mathbf{p}/\epsilon^{s_c}$, we obtain

$$\begin{aligned}
&\epsilon^{1-s}\mathbb{E}\big[\|\mathcal{L}_\epsilon\big(\mathcal{L}_{1,\epsilon}\lambda\big)(t)\|^2_{L^2(\mathbb{R}^{2d})}\big] \leq \epsilon^{1-s-\theta s_c}\\
&\quad\times C\Big[\Big(\int_{|\mathbf{p}|<1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta-1}}\Big)^2 + \Big(\int_{|\mathbf{p}|>1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta}}\Big)^2\Big]\Big(\|\nabla_\mathbf{x}\lambda\|^2_{L^2(\mathbb{R}^{2d})} + \|\lambda\|^2_{L^2(\mathbb{R}^{2d})}\Big) < +\infty,
\end{aligned}$$

That concludes the proof of Lemma 5.4 since $s_c = (1-s)/\theta$. □.

In the same way as Lemma 5.4 we have

$$\sup_{\epsilon\in(0,1)} \mathbb{E}\big[\|\mathcal{L}_\epsilon\lambda(t)\|^2_{L^2(\mathbb{R}^{2d})}\times\|\mathcal{L}_{1,\epsilon}\lambda(t)\|^2_{L^2(\mathbb{R}^{2d})}\big] < +\infty.$$

As a result, we obtain

$$\begin{aligned}
\mathcal{A}^\epsilon(f_0^\epsilon + f_1^\epsilon)(t) =&\epsilon^{1-s}f'(W_{\epsilon,\lambda}(t))\Big[W_{\epsilon,\lambda_1}(t) + \big\langle W_\epsilon(t), \mathcal{L}_\epsilon\big(\mathcal{L}_{1,\epsilon}\lambda(t)\big)(t)\big\rangle_{L^2(\mathbb{R}^{2d})}\Big]\\
&+\epsilon^{1-s}f''(W_{\epsilon,\lambda}(t))\big\langle W_\epsilon(t), \mathcal{L}_{1,\epsilon}\lambda(t)\big\rangle_{L^2(\mathbb{R}^{2d})}\big\langle W_\epsilon(t), \mathcal{L}_\epsilon\lambda(t)\big\rangle_{L^2(\mathbb{R}^{2d})}\\
&+\mathcal{O}(\epsilon^{(1+\gamma)/2+s_c}),
\end{aligned}$$

and $\sup_{\epsilon,t}\mathbb{E}[|\mathcal{A}(f_0^\epsilon + f_1^\epsilon)(t)|^2] < +\infty$. That conclude the proof of Lemma 5.3 and then the proof of Proposition 5.1. □ ■

5.3 Identification of all subsequence limits

In this section, we identify all the subsequence limits of the process $(W_\epsilon)_\epsilon$ as solutions of a deterministic diffusion equation. Let us note that in this case all the limit processes are therefore deterministic. This fact implies that the convergence of the process $(W_\epsilon)_\epsilon$ also holds in probability. We will see that this fractional diffusion equation is well posed. In particular, this will imply the convergence of the process $(W_\epsilon)_\epsilon$ itself to the unique solution of this diffusion equation.

Proposition 5.2 *Let W be an accumulation point of $(W_\epsilon)_\epsilon$. Then, W is the unique strong solution of the diffusion equation*

$$\partial_t W = -\sigma(\theta)(-\Delta_\mathbf{k})^{\theta/2}W, \qquad (27)$$

with $W(t) = W_0$, and

$$\sigma(\theta) = \frac{2a(0)\theta\Gamma(1-\theta)}{(2\pi)^d}\int_{\mathbb{S}^{d-1}} dS(\mathbf{u})|\mathbf{e}_1\cdot\mathbf{u}|^\theta.$$

Proof (of Proposition 5.2) In this proof we use the following notation

$$\varphi \otimes \psi(\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) = \varphi(\mathbf{x}_1, \mathbf{k}_1)\psi(\mathbf{x}_2, \mathbf{k}_2).$$

Let

$$\begin{aligned} f_2^\epsilon(t) =& \epsilon^{1-s} f'(W_{\epsilon,\lambda}(t)) \big\langle W_\epsilon(t), H_{1,\epsilon}(t) \big\rangle_{L^2(\mathbb{R}^{2d})} \\ &+ \epsilon^{1-s} f''(W_{\epsilon,\lambda}(t)) \big\langle W_\epsilon(t) \otimes W_\epsilon(t), H_{2,\epsilon}(t) \big\rangle_{L^2(\mathbb{R}^{4d})}, \end{aligned}$$

where

$$\begin{aligned} &H_{1,\epsilon}(t,\mathbf{x},\mathbf{k}) = \\ &\quad \frac{1}{(2\pi)^{2d} i^2} \int_t^{+\infty} du \iint \Big(\mathbb{E}_t^\epsilon \big[\widehat{V}(u/\epsilon^{s+\gamma}, d\mathbf{p}_1) \widehat{V}(u/\epsilon^{s+\gamma}, d\mathbf{p}_2) \big] - \mathbb{E}\big[\widehat{V}(0, d\mathbf{p}_1) \widehat{V}(0, d\mathbf{p}_2) \big] \Big) \\ &\quad \times e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot \mathbf{x}/\epsilon^s} e^{i(u-t)(\mathbf{p}_1+\mathbf{p}_2)\cdot \mathbf{k}/\epsilon^s} \\ &\quad \times \Big[\frac{1}{\mathfrak{g}(\mathbf{p_2}) - i\epsilon^\gamma \big(\mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}_2} \Big(\lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) \Big) \\ &\qquad - \frac{1}{\mathfrak{g}(\mathbf{p_2}) - i\epsilon^\gamma \big(\mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}_2} \Big(\lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) \Big) \Big], \end{aligned}$$

and

$$\begin{aligned} &H_{2,\epsilon}(t,\mathbf{x}_1, \mathbf{k}_1, \mathbf{x}_2, \mathbf{k}_2) = \\ &\quad \frac{1}{(2\pi)^{2d} i^2} \int_t^{+\infty} du \iint \Big(\mathbb{E}_t^\epsilon \big[\widehat{V}(u/\epsilon^{s+\gamma}, d\mathbf{p}_1) \widehat{V}(u/\epsilon^{s+\gamma}, d\mathbf{p}_2) \big] - \mathbb{E}\big[\widehat{V}(0, d\mathbf{p}_1) \widehat{V}(0, d\mathbf{p}_2) \big] \Big) \\ &\quad \times e^{i\mathbf{p}_1\cdot \mathbf{x}_1/\epsilon^s} e^{i\mathbf{p}_2\cdot \mathbf{x}_2/\epsilon^s} e^{i(u-t)(\mathbf{p}_1\cdot\mathbf{k}_1+\mathbf{p}_2\cdot \mathbf{k}_2)/\epsilon^s} \frac{1}{\mathfrak{g}(\mathbf{p_1}) - i\epsilon^\gamma \mathbf{k}_1 \cdot \mathbf{p}_1} \\ &\quad \times \Big(\lambda\big(\mathbf{x}_1, \mathbf{k}_1 - \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_1, \mathbf{k}_1 + \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) \Big) \\ &\quad \times \Big(\lambda\big(\mathbf{x}_2, \mathbf{k}_2 - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_2, \mathbf{k}_2 + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) \Big). \end{aligned}$$

However, according to (8)

$$\begin{aligned} &\mathbb{E}_t^\epsilon \Big[\widehat{V}\Big(u + \frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_1\Big) \widehat{V}\Big(u + \frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_2\Big) \Big] - \mathbb{E}\Big[\widehat{V}(0, d\mathbf{p}_1) \widehat{V}(0, d\mathbf{p}_2) \Big] \\ &\qquad = e^{-(\mathfrak{g}(\mathbf{p}_1)+\mathfrak{g}(\mathbf{p}_2))u} \widehat{V}\Big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_1\Big) \widehat{V}\Big(\frac{t}{\epsilon^{s+\gamma}}, d\mathbf{p}_2\Big) - (2\pi)^d e^{-2\mathfrak{g}(\mathbf{p}_1)u} \widehat{R}_0(\mathbf{p}_1) \delta(\mathbf{p}_1 + \mathbf{p}_2). \end{aligned}$$

Consequently,

$$f_2^\epsilon(t) = \epsilon^{1+\gamma} \Big[f'(W_{\epsilon,\lambda}(t)) \big\langle W_\epsilon(t), \tilde{H}_{1,\epsilon}(t) \big\rangle_{L^2(\mathbb{R}^{2d})} + f''(W_{\epsilon,\lambda}(t)) \big\langle W_\epsilon(t) \otimes W_\epsilon(t), \tilde{H}_{2,\epsilon}(t) \big\rangle_{L^2(\mathbb{R}^{4d})} \Big],$$

where

$$\begin{aligned} \tilde{H}_{1,\epsilon}(t, \mathbf{x},\mathbf{k}) =& \frac{1}{(2\pi)^{2d} i^2} \iint \widehat{V}(t/\epsilon^{s+\gamma}, d\mathbf{p}_1) \widehat{V}(t/\epsilon^{s+\gamma}, d\mathbf{p}_2) \frac{e^{i(\mathbf{p}_1+\mathbf{p}_2)\cdot \mathbf{x}/\epsilon^s}}{(\mathfrak{g}(\mathbf{p}_1) + \mathfrak{g}(\mathbf{p}_2) - i\epsilon^\gamma \mathbf{k}\cdot(\mathbf{p}_1+\mathbf{p}_2))} \\ &\times \Big[\frac{1}{\mathfrak{g}(\mathbf{p_2}) - i\epsilon^\gamma \big(\mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}_2} \Big(\lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) \Big) \\ &\qquad - \frac{1}{\mathfrak{g}(\mathbf{p_2}) - i\epsilon^\gamma \big(\mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}_2} \Big(\lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}_1}{2\epsilon^{s_c}} + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) \Big) \Big] \\ &- \frac{1}{(2\pi)^d} \int d\mathbf{p} \frac{\widehat{R}_0(\mathbf{p})}{2\mathfrak{g}(\mathbf{p})} \Big[\frac{1}{\mathfrak{g}(\mathbf{p}) + i\epsilon^\gamma \big(\mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}} \Big(\lambda\big(\mathbf{x}, \mathbf{k} - \frac{\mathbf{p}}{\epsilon^{s_c}}\big) - \lambda(\mathbf{x}, \mathbf{k}) \Big) \\ &\qquad - \frac{1}{\mathfrak{g}(\mathbf{p}) + i\epsilon^\gamma \big(\mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) \cdot \mathbf{p}} \Big(\lambda(\mathbf{x}, \mathbf{k}) - \lambda\big(\mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{\epsilon^{s_c}}\big) \Big) \Big], \end{aligned}$$

and

$$\begin{aligned}
\tilde{H}_{2,\epsilon}(t,\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) =& \frac{1}{(2\pi)^{2d} i^2} \iint \widehat{V}(t/\epsilon^{s+\gamma}, d\mathbf{p}_1)\widehat{V}(t/\epsilon^{s+\gamma}, d\mathbf{p}_2) \\
&\times \frac{e^{i\mathbf{p}_1\cdot\mathbf{x}_1/\epsilon^s} e^{i\mathbf{p}_2\cdot\mathbf{x}_2/\epsilon^s}}{(\mathfrak{g}(\mathbf{p_1}) - i\epsilon^\gamma \mathbf{k}_1\cdot\mathbf{p}_1)(\mathfrak{g}(\mathbf{p_1}) + \mathfrak{g}(\mathbf{p_2}) - i\epsilon^\gamma(\mathbf{k}_1\cdot\mathbf{p}_1 + \mathbf{k}_2\cdot\mathbf{p}_2))} \\
&\times \Big(\lambda\big(\mathbf{x}_1, \mathbf{k}_1 - \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_1, \mathbf{k}_1 + \frac{\mathbf{p}_1}{2\epsilon^{s_c}}\big)\Big) \\
&\times \Big(\lambda\big(\mathbf{x}_2, \mathbf{k}_2 - \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_2, \mathbf{k}_2 + \frac{\mathbf{p}_2}{2\epsilon^{s_c}}\big)\Big) \\
&- \frac{1}{(2\pi)^d} \int d\mathbf{p} \frac{\widehat{R}_0(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)/\epsilon^s}}{(\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma \mathbf{k}_1\cdot\mathbf{p})(2\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{p})} \\
&\times \Big(\lambda\big(\mathbf{x}_1, \mathbf{k}_1 - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_1, \mathbf{k}_1 + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big) \\
&\times \Big(\lambda\big(\mathbf{x}_2, \mathbf{k}_2 - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_2, \mathbf{k}_2 + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big).
\end{aligned}$$

Lemma 5.5

$$\lim_\epsilon \sup_{t\geq 0} \mathbb{E}[|f_2^\epsilon(t)|] = 0.$$

Proof (of Lemma 5.5) In the same way as in Lemma 5.4, we have

$$\begin{aligned}
\sup_t \mathbb{E}[\|\tilde{H}_{1,\epsilon}(t)\|^2_{L^{\mathbb{R}^{2d}}} + \|\tilde{H}_{2,\epsilon}(t)\|^2_{L^{\mathbb{R}^{2d}}}] \leq& \epsilon^{-(\theta+2\beta)s_c} \\
&\times C\Big[\Big(\int_{|\mathbf{p}|<1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta-1}}\Big)^2 + \Big(\int_{|\mathbf{p}|>1} d\mathbf{p}\frac{1}{|\mathbf{p}|^{d+\theta}}\Big)^2\Big] \\
&\times \Big(\|D^2_{\mathbf{x}}\lambda\|^2_{L^2(\mathbb{R}^{2d})} + \|\nabla_{\mathbf{x}}\lambda\|^2_{L^2(\mathbb{R}^{2d})} + \|\lambda\|^2_{L^2(\mathbb{R}^{2d})}\Big),
\end{aligned} \tag{28}$$

and

$$1+\gamma-(\theta+2\beta)s_c > 0 \iff s > (1-\gamma\theta/(2\beta))/(1+\theta/(2\beta)) = 1/(2\kappa_\gamma),$$

that concludes the proof of Lemma 5.5 □

Consequently, after a long but straightforward computation, we get

$$\begin{aligned}
\mathcal{A}^\epsilon(f_0^\epsilon + f_1^\epsilon + f_2^\epsilon)(t) =& \epsilon^{1-s} f'(W_{\epsilon,\lambda}(t))\big\langle W_\epsilon(t), G_{1,\epsilon}\lambda\big\rangle_{L^2(\mathbb{R}^{2d})} \\
&+ \epsilon^{1-s} f''(W_{\epsilon,\lambda}(t))\big\langle W_\epsilon(t)\otimes W_\epsilon(t), G_{2,\epsilon}\lambda\big\rangle_{L^2(\mathbb{R}^{4d})} \\
&+ o(1),
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
G_{1,\epsilon}\lambda(\mathbf{x},\mathbf{k}) = -\frac{1}{(2\pi)^d} \int d\mathbf{p}\widehat{R}_0(\mathbf{p}) \Big[& \frac{1}{\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma\big(\mathbf{k} - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\cdot\mathbf{p}} \Big(\lambda(\mathbf{x},\mathbf{k}) - \lambda\big(\mathbf{x},\mathbf{k} - \frac{\mathbf{p}}{\epsilon^{s_c}}\big)\Big) \\
&- \frac{1}{\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma\big(\mathbf{k} + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\cdot\mathbf{p}} \Big(\lambda\big(\mathbf{x},\mathbf{k} + \frac{\mathbf{p}}{\epsilon^{s_c}}\big) - \lambda(\mathbf{x},\mathbf{k})\Big)\Big],
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
G_{2,\epsilon}\lambda(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) =& -\frac{1}{(2\pi)^d} \int d\mathbf{p} \frac{\mathfrak{g}(\mathbf{p})\widehat{R}_0(\mathbf{p}) e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)/\epsilon^s}}{(\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma\mathbf{k}_1\cdot\mathbf{p})(2\mathfrak{g}(\mathbf{p}) - i\epsilon^\gamma(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{p})} \\
&\times \Big(\lambda\big(\mathbf{x}_1, \mathbf{k}_1 - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_1, \mathbf{k}_1 + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big) \\
&\times \Big(\lambda\big(\mathbf{x}_2, \mathbf{k}_2 - \frac{\mathbf{p}}{2\epsilon^{s_c}}\big) - \lambda\big(\mathbf{x}_2, \mathbf{k}_2 + \frac{\mathbf{p}}{2\epsilon^{s_c}}\big)\Big).
\end{aligned} \tag{31}$$

First, making the change of variable $\mathbf{p}' = \mathbf{p}/\epsilon^{s_c}$, and using the Riemann-Lebesgue Lemma and the dominated convergence Theorem with (28), we have

$$\lim_\epsilon \|G_{2,\epsilon}\lambda\|^2_{L^2(\mathbb{R}^{4d})} = 0 \tag{32}$$

since $s_c < s$, and also

$$\lim_{\epsilon} \|(G_{1,\epsilon} - G_1)\lambda\|^2_{L^2(\mathbb{R}^{2d})} = 0, \tag{33}$$

where

$$\begin{aligned} G_1\lambda(\mathbf{x},\mathbf{k}) &= \frac{a(0)}{(2\pi)^d} \int d\mathbf{p} \frac{1}{|\mathbf{p}|^{d+\theta}} \Big(\lambda(\mathbf{x},\mathbf{k}+\mathbf{p}) + \lambda(\mathbf{x},\mathbf{k}-\mathbf{p}) - 2\lambda(\mathbf{x},\mathbf{k})\Big) \\ &= \frac{2a(0)}{(2\pi)^d} \int d\mathbf{p} \frac{1}{|\mathbf{p}|^{d+\theta}} \Big(\lambda(\mathbf{x},\mathbf{k}+\mathbf{p}) - \lambda(\mathbf{x},\mathbf{k})\Big) \\ &= -\sigma(\theta)(-\Delta)^{\theta/2}\lambda. \end{aligned} \tag{34}$$

Let $f^\epsilon(t) = f_0^\epsilon(t) + f_1^\epsilon(t) + f_2^\epsilon(t)$. According to Theorem 5.1, $\big(M^\epsilon_{f^\epsilon}(t)\big)_{t\geq 0}$ is an $(\mathcal{F}^\epsilon_t)$-martingale. That is, for every bounded continuous function Φ, every sequence $0 < s_1 < \cdots < s_n \leq s < t$, and every family $(\mu_j)_{j\in\{1,\dots,n\}} \in L^2(\mathbb{R}^d)^n$, we have

$$\mathbb{E}\Big[\Phi\big(W_{\epsilon,\mu_j}(s_j), 1\leq j\leq n\big)\Big(f^\epsilon(t) - f^\epsilon(s) - \int_s^t \mathcal{A}^\epsilon f^\epsilon(u)du\Big)\Big] = 0.$$

Using (29), Lemma 5.1, Lemma 5.5, (32), and (33) we obtain that

$$M_{f,\lambda}(t) = f(W_\lambda(t)) - f(W_\lambda(0)) - \int_0^t \partial_v f(W_\lambda(u))\big\langle W(u), G_1\lambda\big\rangle_{L^2(\mathbb{R}^{2d})}du$$

is a martingale. More particularly, let us consider f be a smooth function so that $f(v) = v$, $\forall v$ such that $|v| \leq r\|\lambda\|_{L^2(\mathbb{R}^{2d})}$, then

$$M_\lambda(t) = W_\lambda(t) - W_\lambda(0) - \int_0^t \big\langle W(u), G_1\lambda\big\rangle_{L^2(\mathbb{R}^{2d})}du$$

is a martingale with a quadratic variation equal to 0. Consequently, $M_\lambda = 0$, that is W is a deterministic weak solution of the diffusion equation (27). To show the weak uniqueness of this equation, let us assume that $W_0 = 0$. Moreover, it is clear that this diffusion equation with initial condition $\lambda_0 \in L^2(\mathbb{R}^d)$ admits a unique strong solution that we denote by λ^θ of the form (23). As result, $\forall T > 0$

$$\big\langle W(T), \lambda_0\big\rangle_{L^2(\mathbb{R}^{2d})} = \big\langle W(T), \tilde\lambda^\theta(T)\big\rangle_{L^2(\mathbb{R}^{2d})} = \int_0^T \big\langle W(t), \partial_t\tilde\lambda^\theta(t) + \sigma(\theta)(-\Delta)^{\theta/2}\tilde\lambda^\theta(t)\big\rangle_{L^2(\mathbb{R}^{2d})}dt = 0,$$

for $\tilde\lambda^\theta(t) = \lambda^\theta(T-t)$. As a result, all the accumulation points are also strong solutions of the diffusion equation (27). That concludes the proof a Proposition 5.2. ■

6 Proof of Theorem 3.2

The proof of this theorem is quite similar to the previous one. The proof of the tightness is exactly the same. However, in this theorem we have assumed $s_c = s$, but also either $\beta < 1/2$ or $\gamma > 0$. To characterize the accumulation points we use exactly the same perturbed test functions as in the proof of Theorem 3.1, to obtain

$$\begin{aligned} \mathcal{A}^\epsilon(f_0^\epsilon + f_1^\epsilon + f_2^\epsilon)(t) =& \epsilon^{1-s} f'(W_{\epsilon,\lambda}(t))\big\langle W_\epsilon(t), G_{1,\epsilon}\lambda\big\rangle_{L^2(\mathbb{R}^{2d})} \\ &+ \epsilon^{1-s} f''(W_{\epsilon,\lambda}(t))\big\langle W_\epsilon(t)\otimes W_\epsilon(t), G_{2,\epsilon}\lambda\big\rangle_{L^2(\mathbb{R}^{4d})} \\ &+ o(1), \end{aligned} \tag{35}$$

where $G_{1,\epsilon}\lambda$ is defined by (30), and $G_{2,\epsilon}\lambda$ by (31) with $s = s_c$. However, we still have (33) for the drift term f', but with the change of variable $\mathbf{p}' = \mathbf{p}/\epsilon^s$ there are no fast phases anymore, and we get that

$$\begin{aligned} M_{f,\lambda}(t) = f(W_\lambda(t)) - f(W_\lambda(0)) - \int_0^t & f'(W_\lambda(u))\big\langle W(u), G_1\lambda\big\rangle_{L^2(\mathbb{R}^{2d})} \\ &+ f''(W_\lambda(u))\big\langle W(u)\otimes W(u), G_2(\lambda,\lambda)\big\rangle_{L^2(\mathbb{R}^{4d})}du \end{aligned}$$

is a martingale where G_1 is defined by (34), and

$$G_2(\lambda_1,\lambda_2)(\mathbf{x}_1,\mathbf{k}_1,\mathbf{x}_2,\mathbf{k}_2) = -\frac{1}{(2\pi)^d}\int \frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}} e^{i\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \times\Big(\lambda_1\big(\mathbf{x}_1,\mathbf{k}_1-\frac{\mathbf{p}}{2}\big)-\lambda_1\big(\mathbf{x}_1,\mathbf{k}_1+\frac{\mathbf{p}}{2}\big)\Big) \times\Big(\lambda_2\big(\mathbf{x}_2,\mathbf{k}_2-\frac{\mathbf{p}}{2}\big)-\lambda_2\big(\mathbf{x}_2,\mathbf{k}_2+\frac{\mathbf{p}}{2}\big)\Big). \tag{36}$$

In this theorem the second order term f'' has not been killed by the Riemann-Lebesgue Lemma, so that the limiting point W is not deterministic anymore. As a result we need to study the finite dimensional distributions

$$\lim_\epsilon \mathbb{E}\Big[\prod_{j=1}^N \langle W_\epsilon(t_j),\lambda_j\rangle^{n_j}_{L^2(\mathbb{R}^{2d})}\Big]$$

to characterize all the accumulation points of $(W_\epsilon)_\epsilon$. To do that we follow the technique used in [30] and let us consider the tensorial process $W^M_\epsilon(t)=\bigotimes_{j=1}^M W_\epsilon(t)$ on $L^2(\mathbb{R}^{2dM})$. In the same way as the case $M=1$, we can show that $W^M_\epsilon(t)$ is a tight process in $L^2(\mathbb{R}^{2dM})$ equipped with the weak topology and for all its accumulation points W^M

$$M_\lambda(t) = W^M_\lambda(t) - W^M_\lambda(0) - \int_0^t \langle W^M(u), G^M_1\lambda+\tilde G^M_2\lambda\rangle_{L^2(\mathbb{R}^{2dM})}du$$

is a martingale $\forall\lambda\in L^2(\mathbb{R}^{2dM})$. Here, G^M_1 and $\tilde G^M_2$ are defined by

$$G^M_1\lambda = -\sum_{j=1}^M \sigma(\theta)(-\Delta_{\mathbf{k_j}})^{\theta/2}\lambda$$

and

$$\begin{aligned}&\tilde G^M_2\lambda(\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_M,\mathbf{k}_M)\\ &\quad= -\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^M \frac{1}{(2\pi)^d}\int\frac{d\mathbf{p}}{|\mathbf{p}|^{d+\theta}}e^{i\mathbf{p}\cdot(\mathbf{x}_{j_1}-\mathbf{x}_{j_2})}\\ &\quad\times\Big(\lambda\big(\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_{j_1},\mathbf{k}_{j_1}-\frac{\mathbf{p}}{2},\cdots,\mathbf{x}_M,\mathbf{k}_M\big)-\lambda\big(\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_{j_1},\mathbf{k}_{j_1}+\frac{\mathbf{p}}{2},\cdots,\mathbf{x}_M,\mathbf{k}_M\big)\Big)\\ &\quad\times\Big(\lambda\big(\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_{j_2},\mathbf{k}_{j_2}-\frac{\mathbf{p}}{2},\cdots,\mathbf{x}_M,\mathbf{k}_M\big)-\lambda\big(\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_{j_2},\mathbf{k}_{j_2}+\frac{\mathbf{p}}{2},\cdots,\mathbf{x}_M,\mathbf{k}_M\big)\Big).\end{aligned}$$

As a result, $\mathbb{E}[W^N]$ is a weak solution of the differential equation

$$\partial_t\lambda^M = (G^M_1+G^M_2)\lambda^M. \tag{37}$$

Let $\lambda_0\in L^2(\mathbb{R}^{2dM})$ such that its Fourier transform with respect to $(\mathbf{k}_1,\cdots,\mathbf{k}_M)$, $\widehat\lambda^{\mathbf{k}}_0$, belongs to $\mathcal{C}^\infty_0(\mathbb{R}^{2dM})$. Solving (37) in the Fourier domain, we show the existence and uniqueness of a smooth function λ^M in the strong sense of (37) with initial condition λ_0. As result, if $\mathbb{E}[W^M(0)]=0$, $\forall T>0$

$$\begin{aligned}\big\langle\mathbb{E}[W^M(T)],\lambda_0\big\rangle_{L^2(\mathbb{R}^{2dM})} &= \big\langle\mathbb{E}[W^M(T)],\tilde\lambda^M(T)\big\rangle_{L^2(\mathbb{R}^{2dM})}\\ &= \int_0^T \big\langle\mathbb{E}[W^M(t)],\partial_t\tilde\lambda^M(t)+\sigma(\theta)(-\Delta)^{\theta/2}\tilde\lambda^M(t)\big\rangle_{L^2(\mathbb{R}^{2dM})}dt = 0,\end{aligned}$$

for $\tilde\lambda^M(t)=\lambda^M(T-t)$. Consequently, by a density argument, we obtain the weak uniqueness of (37), and the moments

$$\mathbb{E}\Big[W^M(t,\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_M,\mathbf{k}_M)\Big] = \mathbb{E}\Big[\prod_{j=1}^M W(t,\mathbf{x}_j,\mathbf{k}_j)\Big]$$

are therefore uniquely determined for all accumulation point W. Let

$$\tilde W(t,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int d\mathbf{q}\widehat{W}^{\mathbf{k}}_0(\mathbf{x},\mathbf{q})\exp\Big(i\mathbf{k}\cdot\mathbf{q}+i\int\mathcal{B}_t(d\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}(e^{-i\mathbf{q}\cdot\mathbf{p}/2}-e^{i\mathbf{q}\cdot\mathbf{p}/2})\Big).$$

Using the îto's formula and the weak uniqueness of (37), we obtain for all accumulation point W^M of $(W_\epsilon)_\epsilon$

$$\mathbb{E}\Big[W^M(t,\mathbf{x}_1,\mathbf{k}_1,\cdots,\mathbf{x}_M,\mathbf{k}_M)\Big]=\mathbb{E}\Big[\prod_{j=1}^M\tilde{W}(t,\mathbf{x}_j,\mathbf{k}_j)\Big].$$

Consequently, we have identified the one-dimensional finite distributions for all accumulation point W

$$\lim_\epsilon\mathbb{E}\Big[\prod_{j=1}^N\big\langle W_\epsilon(t),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big]=\mathbb{E}\Big[\prod_{j=1}^N\big\langle W(t_j),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big]=\mathbb{E}\Big[\prod_{j=1}^N\big\langle \tilde{W}(t),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big].$$

To conclude the proof of Theorem 3.2, following a classical argument regarding the proof of uniqueness of martingale problems [23, Proposition 4.27]: If the one-dimensional distributions of two solutions are the same, then their finite dimensional distributions are also the same. Consequently,

$$\lim_\epsilon\mathbb{E}\Big[\prod_{j=1}^N\big\langle W_\epsilon(t_j),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big]=\mathbb{E}\Big[\prod_{j=1}^N\big\langle W(t_j),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big]=\mathbb{E}\Big[\prod_{j=1}^N\big\langle \tilde{W}(t_j),\lambda_j\big\rangle_{L^2(\mathbb{R}^{2d})}^{n_j}\Big].$$

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