Hypoelliptic estimates in radiative transfer

Christophe Gomez, [*,1], Olivier Pinaud [†,2], and Lenya Ryzhik [‡,3]

[1]Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France

[2]Department of Mathematics, Colorado State University, Fort Collins CO, 80523

[3]Department of Mathematics, Stanford University, Stanford CA 94305

August 13, 2014

Abstract

We derive the hypoelliptic estimates for a kinetic equation of the form

$$\partial_t f + k \cdot \nabla_x f = (-\Delta_d)^\beta h, \qquad \text{for} \quad (t, x, k) \in \mathbb{R} \times \mathbb{R}^{d+1} \times \mathbb{S}^d,$$

where $d \geq 1$, $\beta > 0$, $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$ and Δ_d is the Laplace-Beltrami operator on $\mathbb{S}^d$. Such equations arise in the modeling of high frequency waves in random media with long-range correlations. Assuming some (fractional) Sobolev regularity in the momentum variable $k \in \mathbb{S}^d$, we obtain estimates for the fractional derivatives of f in the (t, x) variables. Our proof follows the method of [9] based on the regularization of the momentum variable and on averaging lemmas on the sphere.

1 Introduction

This work is motivated by the radiative transfer equations of the form

$$\partial_t f + k \cdot \nabla_x f = \mathcal{L} f, \qquad \text{for} \qquad (t, x, k) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \times \mathbb{S}^d. \tag{1.1}$$

Here $d \geq 1$, $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$ and $\mathcal{L}$ is a linear operator acting on the momentum variable $k \in \mathbb{S}^d$, typically of the form

$$\mathcal{L} f(k) = \int_{\mathbb{S}^d} K(k, p)(f(p) - f(k)) d\sigma(p). \tag{1.2}$$

[*]christophe.gomez@univ-amu.fr
[†]pinaud@math.colostate.edu
[‡]ryzhik@math.stanford.edu

Here, and in the rest of the paper, $d\sigma(k)$ is the surface measure on $\mathbb{S}^d$. We are interested in this work in what happens when the collision kernel $K(k,p)$ is singular when $k=p$. In the context of gas dynamics and the Boltzmann equation, this happens in the so-called non cut-off case [1, 27], where the particle interactions are long-range. Our interest is rather in the energy transport of the solutions of the linear Schrödinger equation with a weak time-independent random potential, in particular, in the effect of the slow decay of the correlations of the random potential. The starting point is the Schrödinger equation

$$i\partial_t\psi^\varepsilon + \frac{1}{2}\Delta_x\psi^\varepsilon - \sqrt{\varepsilon}\,V\left(x\right)\psi^\varepsilon = 0.$$

Here, ε is the (small) variance of the random potential $V(x)$ which is mean-zero and statistically homogeneous in space. The random Schrödinger equation also arises as the parabolic (or paraxial) approximation to the propagation of a wave beam in a random medium [26]. The energy transport at an appropriate macroscopic scale is obtained via the asymptotics of the Wigner transform [15, 21, 28]: in a certain macroscopic limit we have

$$\mathbb{E}[|\psi^\varepsilon(t,x)|^2] \to \int W(t,x,k)dk,$$

where $\mathbb{E}$ denotes the expectation over realizations of the random medium. The limiting Wigner measure W is, formally, a solution to a transport equation similar to (1.1), with the collision operator of the form

$$\mathcal{L}W(k) = \int_{\mathbb{R}^{d+1}} \delta\left(\frac{|k|^2}{2} - \frac{|p|^2}{2}\right)\hat{R}(k-p)(W(p)-W(k))dp, \qquad (1.3)$$

and

$$K(k,p) = \hat{R}(k-p)\delta\left(\frac{|k|^2}{2} - \frac{|p|^2}{2}\right).$$

Here, δ is the Dirac measure and $\hat{R}$ is the power spectrum of V:

$$R(x) := \mathbb{E}\{V(x+y)V(y)\} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{d+1}} e^{ik\cdot x}\hat{R}(k)dk.$$

There is a large literature on the derivation of kinetic equations using Wigner transforms when the correlation function R is integrable – see [7, 14, 22, 25], and [16] for the long-range case with a slightly different scaling, as well as [8].

The Dirac measure in the scattering cross-section in (1.3) decouples the transport equations for different values of $|k|$, and we may set $|k|=1$, leading to a problem posed on the unit sphere in the k-variable. When the correlation function $R(x)$ decays only algebraically:

$$R(x) \sim \frac{1}{|x|^{2-2\alpha}}, \quad |x| \gg 1,$$

the power spectrum has a singularity at the origin:

$$\hat{R}(p) \sim \frac{1}{|p|^{2\alpha+d-1}}.$$

Hence, in that case the transport equation (1.1) will have a singlar scattering cross-section $K(k,p)$. This singularity is non-integrable when $\alpha \in (1/2, 1)$ (recall that the integration in (1.3) is carried over the d-dimensional sphere), and, essentially, we have (as $|k| = |p| = 1$)

$$\hat{R}(k-p) = \hat{R}(|k-p|) = \hat{R}(2\sqrt{1-\cos\theta}) \sim \theta^{-d-2\beta}, \text{ as } \theta \to 0,$$

where $\beta = 2\alpha - 1 \in (0,1)$ and $\theta = k \cdot p$. Compared to the classical integrable case, the physical consequence is that the mean free path, defined (up to multiplicative constants) as in inverse of

$$\int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^{d+1}} \delta\left(\frac{|k|^2}{2} - \frac{|p|^2}{2}\right) \hat{R}_0(k-p) dp dk = \infty,$$

is equal to zero. Hence, the wavefront is instantaneously washed off by the dynamics, which brings serious complications in applications, such as, for instance, the inverse problem of a source or a scatterer localization.

In the spirit of [2, 3, 9, 20], our interest is in the hypoelliptic type estimates that characterize the transfer of regularity from the momentum variable k to the spatial variable x for solutions of (1.1) with such singular kernels as above. The present paper is devoted to the derivation of these estimates, while the regularity theory and asymptotic analysis (such as the diffusion limit, and peaked-forward regime) will be addressed in the companion paper [17]. Similar issues have been considered in the Euclidean case where $k \in \mathbb{R}^{d+1}$ in [9, 20, 3, 2]. The main contribution of the present work is to extend these results to radiative transfer equations where the momentum is confined to the unit sphere.

In the case of the Boltzmann equation, the non-linearity in the collision operator makes the problem much harder than in our linear situation, see e.g. [4, 5, 18, 19], but the non-integrability of the collision kernel leads to comparable phenomena. As mentioned in [5], the operator $\mathcal{L}$ roughly acts as a fractional Laplacian in the momentum variables, and therefore in our case as a fractional Laplace-Beltrami operator on the sphere. A standard energy estimate then yields some Sobolev regularity in k, and our main question here is to figure how this regularity is propagated to the spatial variable x. Bootstrapping the estimates, we expect the solution to be $\mathcal{C}^\infty$ in all variables for any time $t > 0$, which is equivalent to saying that the operator

$$\partial_t + k \cdot \nabla_x - \mathcal{L}$$

is hypoelliptic. We note that compared to the Euclidean case $k \in \mathbb{R}^{d+1}$, the restriction $k \in \mathbb{S}^d$ brings additional technical difficulties. We will follow the approach of Bouchut [9] and build the estimates by regularization of the momentum variable and by using averaging lemmas on the sphere. These latter lemmas were established in [12, 10, 11, 13] in various configurations, and we will need to generalize them to the case where the right-hand side in the transport equations involves fractional derivatives on the sphere.

The paper is organized as follows: in Section 2, we introduce the setting, state our main theorems, and offer an outline of the proofs. Section 3 is devoted to the proof of the first theorem, while the proofs of auxiliary propositions and the second theorem are postponed to Section 4.

Acknowledgment. This work was supported in part by NSF grant DMS-1311903 and an AFOSR NSSEFF Fellowship.

2 Hypoelliptic estimates

2.1 Preliminaries

We consider the transport equation

$$\partial_t f + k \cdot \nabla_x f = (-\Delta_d)^\beta h, \tag{2.1}$$

with $\beta > 0$. This equations is posed in the space of distributions $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}_k^d)$, and Δ_d is the Laplace-Beltrami operator on $\mathbb{S}^d$, defined by

$$\begin{aligned} \Delta_d \varphi(z) &= \Delta \varphi\left(\frac{y}{\|y\|}\right)\Bigg|_{y=z}, \qquad z \in \mathbb{S}^d \\ &= \left(\Delta - \sum_{i=1}^{d+1}\sum_{j=1}^{d+1} z_i z_j \partial_{z_i}\partial_{z_j} - d\sum_{i=1}^{d+1} z_i \partial_{z_i}\right)\varphi(z). \end{aligned}$$

Here, Δ is the standard Laplacian on $\mathbb{R}^{d+1}$. The eigenfunctions of $-\Delta_d$ are the spherical harmonics $Y_{n,m}$, for $n \in \mathbb{N}$ and $m = 1, \cdots, M(d,n)$ where

$$M(d,n) = (2n + d - 1)\frac{\Gamma(n+d-1)}{\Gamma(d)\Gamma(n+1)},$$

and are associated with the eigenvalues $\lambda_n = n(n+d-1)$. Above, Γ is the gamma function. The Fourier representation of the fractional Laplace-Beltrami operator is then, for $\beta \in (0, \infty)$,

$$(-\Delta_d)^\beta \varphi(k) = \sum_{n=0}^{\infty}\sum_{m=1}^{M(d,n)} \lambda_n^\beta \, (\varphi, Y_{n,m}) Y_{n,m}(k),$$

where $(\cdot,\cdot)$ is the $L^2(\mathbb{S}^d)$ inner product and convergence is understood in $L^2(\mathbb{S}^d)$. The Sobolev space $H^\theta(\mathbb{S}^d)$, for $\theta > 0$, is defined by

$$H^\theta(\mathbb{S}^d) = \left\{\varphi \in L^2(\mathbb{S}^d), \quad (-\Delta_d)^{\frac{\theta}{2}}\varphi \in L^2(\mathbb{S}^d)\right\}.$$

We will use the following convention for the Fourier transform in $\mathbb{R}^{d+1}$:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^{d+1}} e^{-ix\cdot\xi} f(x)dx, \qquad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{d+1}} e^{ix\cdot\xi}\hat{f}(\xi)d\xi,$$

and introduce the fractional derivative as $\partial_{x_j}^\gamma f(x) = \mathcal{F}^{-1}[(i\xi_j)^\gamma \hat{f}(\xi)](x)$, with a similar definition for fractional derivatives involving the time variable.

2.2 Main results

The main result of the paper is the following:

Theorem 2.1 *Assume $h \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$, and let $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$ satisfy the transport equation (2.1). For some $\theta > 0$, suppose, in addition, that*

$$(-\Delta_d)^{\frac{\theta}{2}} f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d).$$

Then, for

$$\gamma = \frac{\theta}{2(1+2\beta)+\theta},$$

we have $\partial_{t,x}^{\gamma} f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$ with the estimate

$$\|\partial_{t,x}^{\gamma} f\|_{L^2} \leq C \left(\|(-\Delta_d)^{\frac{\theta}{2}} f\|_{L^2} + \|f\|_{L^2} + \|h\|_{L^2} \right).$$

Above, L^2 is a shorthand for $L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$, and this notation will be used in the rest of the paper. Note that there is a lesser gain in regularity compared to the usual Euclidean case (i.e. when $\mathbb{S}^d$ is replaced by $\mathbb{R}^{d+1}$), where the obtained γ equals $\theta/(1+2\beta+\theta)$ – see [9]. This loss can be attributed to some geometric factors, as will be made clear in the proof.

A relatively simple consequence of Theorem 2.1 and its method of proof, is a generalization of the averaging lemmas on the sphere of [12] to a r.h.s involving a fractional derivative with additional regularity for f in the k variable:

Theorem 2.2 *Let $\varphi \in \mathcal{C}^\infty(\mathbb{S}^d)$. Under the assumptions of Theorem 2.1, with now possibly $\theta = 0$, we have*

$$\left\| \int_{\mathbb{S}^d} f(\cdot,\cdot,k)\varphi(k) d\sigma(k) \right\|_{H_{t,x}^{\gamma'}} \leq C \left(\|\partial_{t,x}^{\gamma} f\|_{L^2} + \|h\|_{L^2} \right),$$

where $\gamma' = \frac{1+\gamma(1+4\beta)}{2(1+2\beta)}$ when $d \geq 2$, and $\gamma' = \frac{1+\gamma(1+4\beta)}{4(1+2\beta)}$ when $d = 1$.

2.3 Outline of the proof

The principle of proof follows the ideas of Bouchut [9]. The hypoelliptic estimates are essentially obtained in two steps: first by regularization in k so as to exploit the assumed regularity in the k variable, and, second, by using averaging lemmas on the sphere in order to gain regularity in the (t,x) variables. More precisely, the regularization is done by convolution on the sphere with a mollifier as follows: let ρ be a smooth function whose properties will be specified later on, and define for $\varepsilon \in (0,1]$,

$$\begin{aligned} f_0^\varepsilon(t,x,k) &= \int_{\mathbb{S}^d} \left(f(t,x,k) - f(t,x,p) \right) \rho_\varepsilon(k \cdot p) d\sigma(p) \qquad (2.2) \\ \rho_\varepsilon(s) &= \frac{1}{\varepsilon^{\frac{d}{2}}} \rho\left(\frac{1-s}{\varepsilon} \right), \qquad s \in [-1,1], \end{aligned}$$

where $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$ satisfies (2.1). The function f is then decomposed as

$$f = f_0^\varepsilon + f_1^\varepsilon + f_2^\varepsilon, \qquad (2.3)$$

where

$$
\begin{aligned}
f_1^\varepsilon(t,x,k) &= \int_{\mathbb{S}^d} f(t,x,p)\rho_\varepsilon(k\cdot p)d\sigma(p),\\
f_2^\varepsilon(t,x,k) &= f(t,x,k)\left(1-\int_{\mathbb{S}^d}\rho_\varepsilon(k\cdot p)d\sigma(p)\right).
\end{aligned}
$$

The term f_2^ε is not zero but small as $\varepsilon\to 0$ since ρ_ε will be chosen to integrate to one in the limit. The term f_0^ε will be treated using the regularity assumption

$$(-\Delta_d)^{\frac{\theta}{2}} f \in L^2(\mathbb{R}_t\times\mathbb{R}_x^{d+1}\times\mathbb{S}^d).$$

That is, under appropriate hypotheses on ρ, we will prove in Lemma 3.1 the following estimate, which holds a.e. in (ω,ξ):

$$\|\hat f_0^\varepsilon(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \le C\varepsilon^{\frac{\theta}{2}}\|(-\Delta_d)^{\frac{\theta}{2}}\hat f(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}. \tag{2.4}$$

Above, $\hat f(\omega,\xi,k)$ (resp. $\hat f_0^\varepsilon$) is the Fourier transform of f (resp. f_0^ε) in the (t,x) variables.

The term f_1^ε is treated using similar techniques of the averaging lemmas: $\hat f(\omega,\xi,k)$ satisfies

$$(\lambda+i(\omega+\xi\cdot k))\hat f(\omega,\xi,k) = (-\Delta_d)^\beta\hat h(\omega,\xi,k)+\lambda\hat f(\omega,\xi,k),$$

for any $\lambda>0$ (the specific choice of λ will become important later), so that f_1^ε has the form

$$
\begin{aligned}
\hat f_1^\varepsilon(\omega,\xi,k) &= \int_{\mathbb{S}^d}\frac{(-\Delta_d)^\beta\hat h(\omega,\xi,p)+\lambda\hat f(\omega,\xi,p)}{\lambda+i(\omega+\xi\cdot p)}\rho_\varepsilon(k\cdot p)d\sigma(p)\\
&:= (F_0+F_1)(\omega,\xi,k).
\end{aligned}\tag{2.5}
$$

The core of the proof then consists in estimating the L^2 norms of F_0 and F_1 in the k variable, the most difficult term, naturally, being F_0 because of the fractional Laplacian. For $|\omega|+|\xi|>0$, with an appropriate choice of λ, we will obtain estimates of the form, for any $\epsilon_0>0$,

$$\|f_1^\varepsilon(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \le \frac{C}{\varepsilon^{\gamma_1}(|\omega|+|\xi|)^{\gamma_2}}\|\hat h(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}+\epsilon_0\|\hat f(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}. \tag{2.6}$$

Here, γ_1 and γ_2 are positive coefficients that depend on β. The crucial and most difficult part of the proof is to obtain the best possible exponents γ_1, γ_2. Note that due to the spherical geometry and the presence of the fractional Laplacian, this task is considerably more technical than in the Euclidean case with integer derivatives. The coefficient γ_2 is somewhat direct to obtain, and follows in the same manner as in the averaging lemmas. Concerning γ_1, the parameter ε is chosen by optimizing the r.h.s of (2.4) and (2.6), so that $\varepsilon^{\frac{\theta}{2}+\gamma_1}=(|\omega|+|\xi|)^{-\gamma_2}$, and the lower the γ_1 the better the regularity gain in (t,x). What to expect for γ_1 when $d\ge 2$ is as follows (the case $d=1$ is actually simpler): let us focus on the term F_0 and define

$$g(\omega,\xi,p)=\frac{1}{\lambda+i(\omega+\xi\cdot p)}. \tag{2.7}$$

The first step consists in "integrating by parts" and applying the Laplacian to $\rho_\varepsilon g$. As is classical in averaging lemmas, one uses one dimension in the vector p (the coordinate in the direction ξ) in order to integrate g and gain some decrease in $|\omega| + |\xi|$, which leaves $(d-1)$ free dimensions to integrate in ρ_ε. Since ρ_ε is of order $\varepsilon^{-\frac{d}{2}}$, integrating over $(d-1)$ dimensions leaves a factor $\varepsilon^{-\frac{1}{2}}$. This is the correct result when $k \in \mathbb{R}^{d+1}$. However, when $k \in \mathbb{S}^d$, there is an additional geometric factor coming from the integration along ξ that introduces a loss of $\varepsilon^{-\frac{1}{2}}$. After some algebra, this leads to $\gamma_1 = 1$ when $\beta = 0$. When $\beta \neq 0$, the proof is much more technical but eventually we find $\gamma_1 = 1 + 2\beta$. Obtaining the latter factor is relatively direct for integer derivatives, but more difficult in the fractional case. The Fourier representation of the Laplacian is not very well adapted for this, and we will use an integral representation instead. Instead of using the integral formula for the fractional Laplacian, it is simpler to use an auxilary operator that has the same singularity as the Laplacian. Hence, we define the operator $\mathcal{R}_\beta$ via

$$\mathcal{R}_\beta \varphi(p) = \text{p.v.} \int_{\mathbb{S}^d} \frac{\varphi(p) - \varphi(q)}{|p-q|^{2\beta+d}} d\sigma(q), \quad p \in \mathbb{S}^d. \tag{2.8}$$

Note that in the Euclidean case, as well as when $d = 1$, $\mathcal{R}_\beta$ is nothing but the integral representation of fractional Laplacian (up to a constant). Then we write $\beta = \beta_0 + [\beta]$ where $[\cdot]$ is the integer part of β, and

$$(-\Delta_d)^\beta h = (-\Delta_d)^{[\beta]} (\mathcal{R}_{\beta_0} + I)(\mathcal{R}_{\beta_0} + I)^{-1} (-\Delta_d)^{\beta_0} h := (-\Delta_d)^{[\beta]} (\mathcal{R}_{\beta_0} + I) \mathcal{Q} h. \tag{2.9}$$

We will see that the operator $\mathcal{Q}$ can be extended to a bounded operator on $L^2(\mathbb{S}^d)$, so we will focus on the operator $(-\Delta_d)^{[\beta]}(\mathcal{R}_{\beta_0} + I)$, which is easier to handle than $(-\Delta_d)^\beta$ since it involves only integer powers of the Laplacian (that can be explicitly calculated) and the operator $\mathcal{R}_{\beta_0}$. The term F_0 in (2.5) then becomes

$$F_0(\omega, \xi, k) = \int_{\mathbb{S}^d} \mathcal{Q}\hat{h}(\omega, \xi, p)(\mathcal{R}_{\beta_0} + I)(-\Delta_d)^{[\beta]} \left(\frac{\rho_\varepsilon(k \cdot p)}{\lambda + i(\omega + \xi \cdot p)} \right) d\sigma(p) := F_{0,1} + F_{0,2}. \tag{2.10}$$

The term $F_{0,1}$ will give us the leading order.

The proof is organized as follows: we prove first the estimate (2.4) on f_0^ε in Section 3.1 – it is elementary in the Euclidean geometry, but requires more work in the spherical case. Essentially, all of the rest deals with the "averaging lemma" term f_1^ε. We first prove in Section 3.2 some auxiliary lemmas that will be used throughout the proof and provide the full proof in the simple case with no fractional derivatives $\beta = 0$. Next, we treat the purely fractional (and the most technically involved) case $\beta \in (0, 1)$ in Section 3.3, which ends with the sketch of the (minor) modifications needed for $\beta \geq 1$. The proofs of the main auxiliary technical propositions and of Theorem 2.2 are contained in Section 4.

Throughout the proof, C denotes a generic constant independent of the variables of interest.

3 Proof of Theorem 2.1

We follow the program outlined in Section 2.3, and start with the convolution term. Note that after an appropriate cut-off and smoothing, it is enough to consider functions

f and h in $\mathcal{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$.

3.1 The convolution term f_0^ε

Recall that $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d)$, with

$$(-\Delta_d)^{\frac{\theta}{2}} f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^{d+1} \times \mathbb{S}^d).$$

We denote $N = [\frac{\theta}{2}] + 1$. We construct an explicit mollifier $\rho(s)$ as follows: for $d \geq 1$, we require that

$$2^{\frac{d-2}{2}} |S_{d-1}| \int_0^\infty \rho(s) s^{\frac{d-2}{2}} ds = 1, \qquad \int_0^\infty \rho(s) s^{\frac{d-2}{2}+l} ds = 0, \qquad l = 1, \cdots, N-1, \quad (3.1)$$

where $|S_{d-1}|$ is the surface area of $\mathbb{S}^{d-1}$. The second condition in (3.1) is only required for $N \geq 2$. More explicitly, we look for $\rho(s)$ in the form

$$\rho(s) = \sum_{i=0}^{N-1} a_i b_i^{\frac{2}{d}} e^{-s/b_i}, \qquad s \geq 0, \tag{3.2}$$

with the coefficients $b_i > 0$, and a_i to be determined. When $N \geq 2$, the conditions (3.1) yield a linear system of the form $Ax = y$ where

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ b_0 & b_1 & \cdots & b_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_0^{N-1} & b_1^{N-1} & \cdots & b_{N-1}^{N-1} \end{pmatrix}, \quad x = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{pmatrix}, \quad y = \begin{pmatrix} 2^{\frac{2-d}{2}} |S^{d-1}|^{-1} \Gamma(\frac{d}{2})^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix A is the Vandermonde matrix, and is invertible provided $b_i \neq b_j$, $i \neq j$. Taking then a vector $b_i > 0$ satisfying the latter condition, we find a_i, and therefore the function ρ. We will use the explicit form of the mollifier in the proofs of Propositions 3.4, 3.5 and 3.6, in order to simplify some computations. Recalling that

$$\rho_\varepsilon(t) = \rho((1-t)/\varepsilon)/\varepsilon^{\frac{d}{2}},$$

we have the following lemma:

Lemma 3.1 *Assume $\theta > 0$, $\varepsilon > 0$, and consider the function f_0^ε defined in* (2.2)*. With the mollifier ρ chosen as above, and $\hat{f}(\omega, \xi, k)$ (resp. $\hat{f}_0^\varepsilon$) the Fourier transform of f (resp. f_0^ε) in the (t, x) variables with $\hat{f}(\omega, \xi, \cdot) \in H^\theta(\mathbb{S}^d)$, we have the following estimate, (ω, ξ) a.e. in $\mathbb{R} \times \mathbb{R}^{d+1}$:*

$$\|\hat{f}_0^\varepsilon(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \leq C \varepsilon^{\frac{\theta}{2}} \|(-\Delta_d)^{\frac{\theta}{2}} \hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)}.$$

Proof. Since the variables (ω, ξ) are frozen here, we will omit them in the rest of proof for notational simplicity. The proof essentially relies on calculating the Fourier coefficients of $\hat{f}_0^\varepsilon$ and using appropriately (3.1) to recover the estimate of the lemma. We start by decomposing $\hat{f}$ as

$$\hat{f}(k) = \sum_{n=0}^{\infty} \sum_{m=1}^{M(d,n)} f_{n,m} Y_{n,m}(k), \qquad f_{n,m} = (\hat{f}, Y_{n,m}).$$

We used here the notation of the preliminaries. Using the Funk-Hecke formula ([6], Chapter 2, p. 36), we can write

$$\int_{\mathbb{S}^d} \hat{f}(p)\rho_\varepsilon(k \cdot p) d\sigma(p) = \sum_{n=0}^{\infty} \sum_{m=1}^{M(d,n)} \alpha_n f_{n,m} Y_{n,m}(k)$$
$$\alpha_n = |S_{d-1}| \int_{-1}^{1} \rho_\varepsilon(s) P_n(s)(1-s^2)^{\frac{d-2}{2}} ds.$$

Above, P_n is the Legendre polynomial of degree n in d dimensions. Using the latter decomposition, it follows that

$$\|\hat{f}_0^\varepsilon\|^2_{L^2(\mathbb{S}^d)} = \sum_{n=0}^{\infty} \sum_{m=1}^{M(d,n)} |f_{n,m}|^2 |\beta_n|^2$$
$$\beta_n = |S_{d-1}| \int_{-1}^{1} \rho_\varepsilon(s)(1 - P_n(s))(1-s^2)^{\frac{d-2}{2}} ds. \tag{3.3}$$

Since $P_n(1) = 1$ for all $n \in \mathbb{N}$, and $\rho_\varepsilon(s)$ is concentrated around $s = 1$, we use the following Taylor expansion for $n \geq 1$,

$$\left| P_n(s) - \sum_{l=0}^{N-1} \frac{(s-1)^l}{l!} P_n^{(l)}(1) \right| \leq \frac{|s-1|^N}{N!} \max_{\tau \in [-1,1]} |P_n^{(N)}(\tau)|.$$

We can now decompose β_n as

$$\beta_n = \beta_n^1 + \beta_n^2, \qquad \text{with} \qquad \beta_n^1 = -|S^{d-1}| \sum_{l=1}^{N-1} \frac{P_n^{(l)}(1)}{l!} c_l, \qquad n \geq 1,$$

and where β_n^2 verifies

$$|\beta_n^2| \leq \frac{|S^{d-1}|}{N!} \max_{\tau \in [-1,1]} |P_n^{(N)}(\tau)| \int_{-1}^{1} |\rho_\varepsilon(s)|s-1|^N (1-s^2)^{\frac{d-2}{2}} ds. \tag{3.4}$$

The coefficients c_l are given by

$$c_l = \int_{-1}^{1} \rho_\varepsilon(s)(s-1)^l (1-s^2)^{\frac{d-2}{2}} ds = (-1)^l \varepsilon^l \int_0^{\frac{2}{\varepsilon}} \rho(s) s^{l+\frac{d-2}{2}} (2-\varepsilon s)^{\frac{d-2}{2}} ds.$$

Splitting c_l as

$$c_l = \int_0^{\frac{1}{\varepsilon}} (\cdots) ds + \int_{\frac{1}{\varepsilon}}^{\frac{2}{\varepsilon}} (\cdots) ds := c_l^1 + c_l^2,$$

it follows from the definition of ρ that

$$|c_l^2| = \mathcal{O}(e^{-\frac{C}{\varepsilon}}). \tag{3.5}$$

Furthermore, expanding the term $(2-\varepsilon s)^{\frac{d-2}{2}}$ in c_l^1 up to the order $N-1$, and bounding the remainder using the estimate

$$(2-\varepsilon s)^{\frac{d-2}{2}-(N-1)} \leq C, \text{ for } s \in [0, 1/\varepsilon],$$

we find, with a coefficient $C(l,d,p)$ whose expression is not needed:

$$c_l^1 = \sum_{p=0}^{N-1} C(l,d,p)\varepsilon^{p+l} \int_0^{\frac{1}{\varepsilon}} \rho(s) s^{l+\frac{d-2}{2}+p} ds + \varepsilon^{N+l} R_\varepsilon,$$

where

$$|R_\varepsilon| \leq C \int_0^{\frac{1}{\varepsilon}} |\rho(s)| s^{l+\frac{d-2}{2}+N} ds \leq C.$$

We need some estimates now on the Legendre polynomials and their derivatives to conclude the proof. For this, the Markov inequality [23] yields first,

$$\max_{s\in[-1,1]} |P_n'(s)| \leq n^2 \max_{s\in[-1,1]} |P_n(s)|.$$

Iterating, and using the fact that $\max_{s\in[-1,1]} |P_n(s)| = 1$, we get

$$\max_{s\in[-1,1]} |P_n^{(l)}(t)| \leq n^{2l}. \tag{3.6}$$

Together with (3.4), this gives after the change of variables $1-s \to \varepsilon s$:

$$|\beta_n^2| \leq C\varepsilon^N n^{2N}. \tag{3.7}$$

Regarding β_n^1, we find using (3.5) and (3.6),

$$\left| \beta_n^1 + |S^{d-1}| \sum_{l=1}^{N-1} \sum_{p=0}^{N-1} C(l,d,p) \frac{P_n^{(l)}(1)}{l!} \varepsilon^{p+l} \int_0^{\frac{1}{\varepsilon}} \rho(s) s^{l+\frac{d-2}{2}+p} ds \right| \tag{3.8}$$
$$\leq C\varepsilon^{N+l} \max_{l\in\{1,\cdots,N-1\}} |P_n^{(l)}(1)| \leq C\varepsilon^{N+l} n^{2N-2}.$$

Using now (3.1) and (3.6), we find that the double sum in (3.8) verifies

$$\left| \sum_{l=1}^{N-1} \sum_{p=0}^{N-1} C(l,d,p) \frac{P_n^{(l)}(1)}{l!} \varepsilon^{p+l} \int_0^{\frac{1}{\varepsilon}} \rho(s) s^{l+\frac{d-2}{2}+p} ds \right|$$
$$= \left| \sum_{l+p\geq N} C(l,d,p) \frac{P_n^{(l)}(1)}{l!} \varepsilon^{p+l} \int_0^{\frac{1}{\varepsilon}} \rho(s) s^{l+\frac{d-2}{2}+p} ds \right| \leq C\varepsilon^N \max_{1\leq l\leq N-1} |P_n^{(l)}(1)| \leq C\varepsilon^N n^{2N-2}.$$

This yields, together with (3.7),

$$|\beta_n| \leq C\varepsilon^N n^{2N}. \tag{3.9}$$

Since $\max_{s\in[-1,1]}|P_n(t)| = 1$, we have as well the straightforward estimate:

$$|\beta_n| \leq C. \tag{3.10}$$

Recalling that $N = [\frac{\theta}{2}] + 1$, and interpoling between (3.9) and (3.10) finally yields,

$$|\beta_n| \leq C\varepsilon^{\frac{\theta}{2}} n^\theta.$$

We conclude the proof by going back to (3.3) and remembering that $\lambda_n \sim n^2$ as $n \to \infty$, as well as the fact that $\beta_0 = 0$ since $P_0(t) = 1$. □

3.2 Proof of Theorem 2.1 when $\beta = 0$

We first consider the case $\beta = 0$ which is significantly less involved than $\beta > 0$ that will be considered further.

3.2.1 Preliminary lemmas

We will need a couple of auxiliary results in the estimates for f_1^ε. The first lemma is a slight generalization of that in [12] and is at the computational core of the averaging lemmas on the sphere.

Lemma 3.2 *Let $\tau > 0$, $\gamma \in [0, 1)$ and $\alpha > 1/2$, and set*

$$I_\gamma(\omega, \tau) = \int_{-1}^{1} \frac{ds}{(1 + (\omega - \tau s)^2)^\alpha (1 - s^2)^\gamma}.$$

We have the estimate

$$I_\gamma(\omega, \tau) \leq \frac{C}{(|\omega| + \tau)^{1-\gamma}}.$$

Proof. We only treat the case $\omega \geq 0$, the converse situation follows by symmetry. Let

$$I_\gamma^\pm(\omega, \tau) = \pm \int_0^{\pm 1} \frac{ds}{(1 + (\omega - \tau s)^2)^\alpha (1 - s^2)^\gamma}.$$

Since $I_\gamma^- \leq I_\gamma^+$, we focus on I_γ^+. Suppose first that $\tau \leq \omega \leq 2\tau$, so that

$$\omega - \tau s \geq \tau(1 - s).$$

After the change of variable $\tau(1 - s) \to u$, we get

$$I_\gamma^+(\omega, \tau) \leq \frac{1}{\tau^{1-\gamma}} \int_0^\tau \frac{du}{(1 + u^2)^\alpha u^\gamma (2 - u/\tau)^\gamma} \leq \frac{C}{\tau^{1-\gamma}}.$$

Assume next that $0 \leq \omega \leq \tau$. The change of variable $\omega - \tau s \to u$ leads to

$$\begin{aligned} I_\gamma^+(\omega, \tau) &= \frac{1}{\tau^{1-2\gamma}} \int_{\omega-\tau}^{\omega} \frac{du}{(1 + u^2)^\alpha (\tau + \omega - u)^\gamma (\tau - \omega + u)^\gamma} \\ &\leq \frac{1}{\tau^{1-\gamma}} \int_0^\omega \frac{du}{(1 + u^2)^\alpha u^\gamma} + \frac{1}{\tau^{1-2\gamma}} \int_0^{\tau-\omega} \frac{du}{(1 + u^2)^\alpha (\tau + \omega + u)^\gamma (\tau - \omega - u)^\gamma} \\ &\leq \frac{C}{\tau^{1-\gamma}} + \frac{1}{\tau^{1-\gamma}} \int_0^{\tau-\omega} \frac{du}{(1 + (u - \tau + \omega)^2)^\alpha u^\gamma} \leq \frac{C}{\tau^{1-\gamma}}. \end{aligned}$$

It remains to treat the case $2\tau \leq \omega$, for which we have

$$\omega - \tau s \geq \omega(2-s)/2.$$

After the change of variables $\omega(2-s) \to u$, we find

$$\begin{aligned} I_\gamma^+(\omega,\tau) &\leq \frac{C}{\omega^{1-2\gamma}} \int_\omega^{2\omega} \frac{du}{(1+u^2)^\alpha (u-\omega)^\gamma (3\omega - u)^\gamma} \\ &\leq \frac{1}{\omega^{1-\gamma}} \int_0^\omega \frac{du}{(1+(u+\omega)^2)^\alpha u^\gamma} \leq \frac{C}{\omega^{1-\gamma}}. \end{aligned}$$

This ends the proof. □

Let us introduce the following function

$$\mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|) = \begin{cases} \dfrac{1}{\varepsilon^{\frac14}\lambda^{\frac12}(|\omega|+|\xi|)^{\frac12}}, & \text{when} \quad d \geq 3, \\ \dfrac{1}{\varepsilon^{\frac12}\lambda^{\frac12}(|\omega|+|\xi|)^{\frac12}}, & \text{when} \quad d = 2, \\ \dfrac{1}{\varepsilon^{\frac14}\lambda^{\frac34}(|\omega|+|\xi|)^{\frac14}}, & \text{when} \quad d = 1. \end{cases} \tag{3.11}$$

The next lemma builds on Lemma 3.2.

Lemma 3.3 *Let $G_\varepsilon(s) = G((1-s)/\varepsilon)/\varepsilon^{d/2}$, with $G \in \mathcal{S}(\mathbb{R})$ (the Schwartz class), and set*

$$\mathcal{I}_d(\omega,\xi) = \int_{\mathbb{S}^d} |g(\omega,\xi,p)|^2 G_\varepsilon(k\cdot p)\, d\sigma(p),$$

with g defined in (2.7). Then, for $d = 1, 2$ we have

$$|\mathcal{I}_d(\omega,\xi)| \leq C\big(\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|)\big)^2,$$

while for $d \geq 3$ we have

$$|\mathcal{I}_d(\omega,\xi)| \leq C\big(\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|)\big)^2/\sqrt{1-r^2},$$

where $r = (k\cdot\xi)/|\xi|$.

Proof. Let us start with the case $d \geq 3$. We may assume without loss of generality that $k = e_d = (0,\ldots,0,1)$, and write

$$p = (\sqrt{1-s^2}u, s), \quad \xi = |\xi|(\sqrt{1-r^2}v, r),$$

with $r, s \in [-1,1]$, and $u, v \in \mathbb{S}^{d-1}$. We have

$$\mathcal{I}_d(\omega,\xi) = \int_{\mathbb{S}^{d-1}} \int_{-1}^1 \frac{G_\varepsilon(s)}{|\lambda + i(\omega + sr|\xi| + |\xi|\sqrt{1-s^2}\sqrt{1-r^2}(u\cdot v))|^2} (1-s^2)^{\frac{d-2}{2}} ds\, d\sigma(u).$$

Defining further $z = u \cdot v$, we get

$$\begin{aligned}
\mathcal{I}_d(\omega,\xi) &= |S_{d-2}| \int_{-1}^{1}\int_{-1}^{1} \frac{G_\varepsilon(s)(1-s^2)^{\frac{d-2}{2}}(1-z^2)^{\frac{d-3}{2}}}{|\lambda + i(\omega + sr|\xi| + |\xi|\sqrt{1-r^2}\sqrt{1-s^2}z)|^2} dzds \\
&\leq \frac{|S_{d-2}|}{\sqrt{1-r^2}} \int_{-1}^{1}\int_{-\sqrt{(1-s^2)(1-r^2)}}^{\sqrt{(1-s^2)(1-r^2)}} \frac{G_\varepsilon(s)(1-s^2)^{\frac{d-3}{2}}}{|\lambda + i(\omega + sr|\xi| + |\xi|z)|^2} dzds \\
&\leq \frac{C}{\sqrt{1-r^2}} \int_{-1}^{1}\int_{-1}^{1} \frac{G_\varepsilon(s)(1-s^2)^{\frac{d-3}{2}}}{|\lambda + i(\omega + sr|\xi| + |\xi|z)|^2} dzds.
\end{aligned}$$

We obtain, using Lemma 3.2 with $\gamma = 0$,

$$\begin{aligned}
|\mathcal{I}_d(\omega,\xi)| &\leq \frac{C}{\lambda\sqrt{1-r^2}} \int_{-1}^{1} \frac{|G_\varepsilon(s)|}{\lambda(|sr|\xi| + \omega| + |\xi|)} (1-s^2)^{\frac{d-3}{2}} ds \\
&\leq \frac{C}{\lambda\sqrt{1-r^2}(|\omega| + |\xi|)} \int_{-1}^{1} |G_\varepsilon(s)|(1-s^2)^{\frac{d-3}{2}} ds \\
&\leq \frac{C}{\varepsilon^{1/2}\sqrt{1-r^2}\lambda(|\omega| + |\xi|)}.
\end{aligned}$$

The second line above is straightforward when $|\xi| \geq |\omega|$, and follows when $|\xi| \leq |\omega|$ from the inequality

$$|sr|\xi| + \omega| + |\xi| \geq |\omega| - |sr||\xi| + |\xi| \geq |\omega|.$$

The last line is obtained after a change of variable $1 - s \to \varepsilon s$.

When $d = 1$ and $d = 2$, we bound G_ε directly by $C\varepsilon^{-d/2}$, which gives, using Lemma 3.2 with $\gamma = 0$ and $\gamma = 1/2$, respectively:

$$\begin{aligned}
|\mathcal{I}_d(\omega,\xi)| &\leq \frac{C}{\varepsilon^{d/2}} \int_{\mathbb{S}^d} |g(\omega,\xi,p)|^2 d\sigma(p) \leq \frac{C}{\varepsilon^{d/2}} \int_{-1}^{1} \frac{(1-s^2)^{\frac{d-2}{2}} ds}{|\lambda + i(\omega + s|\xi|)|^2} \\
&\leq \begin{cases} \dfrac{C}{\varepsilon\lambda(|\omega| + |\xi|)} & \text{when} \quad d = 2, \\ \dfrac{C}{\varepsilon^{\frac{1}{2}}\lambda^{\frac{3}{2}}(|\omega| + |\xi|)^{\frac{1}{2}}} & \text{when} \quad d = 1. \end{cases}
\end{aligned}$$

This concludes the proof of the lemma. □

3.2.2 The term f_1^ε

We focus mostly on the term F_0 in decomposition (2.5), since F_1 is very similar. We recall that F_0 and F_1 are

$$F_0(\omega,\xi,k) = \int_{\mathbb{S}^d} \frac{\hat{h}(\omega,\xi,p)\rho_\varepsilon(k\cdot p)}{\lambda + i(\omega + \xi\cdot p)} d\sigma(p), \quad F_1(\omega,\xi,k) = \lambda \int_{\mathbb{S}^d} \frac{\hat{f}(\omega,\xi,p)\rho_\varepsilon(k\cdot p)}{\lambda + i(\omega + \xi\cdot p)} d\sigma(p).$$

The Cauchy-Schwarz inequality gives, with $\hat{\xi} = \xi/|\xi|$,

$$\|F_0(\omega,\xi,\cdot)\|^2_{L^2(\mathbb{S}^d)} \leq \mathcal{J}(\hat{\xi}) \sup_{k\in\mathbb{S}^d} \left[(1-(k\cdot\hat{\xi})^2)^\alpha \mathcal{I}(\omega,\xi,k)\right],$$

where

$$
\begin{aligned}
\mathcal{I}(\omega,\xi,k) &= \int_{\mathbb{S}^d} |g(\omega,\xi,p)|^2 |\rho_\varepsilon(k\cdot p)| d\sigma(p) \\
\mathcal{J}(\hat\xi) &= \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \frac{|\hat h(p)|^2|\rho_\varepsilon(k\cdot p)|}{(1-(k\cdot\hat\xi)^2)^\alpha} d\sigma(k) d\sigma(p).
\end{aligned} \tag{3.12}
$$

We set above $\alpha = 1/2$ when $d \geq 3$, and $\alpha = 0$ when $d = 2$ or $d = 1$ and omit the dependence of $\hat h$ on (ω,ξ) for simplicity.

The term $\mathcal{I}$ is treated using Lemma 3.3 and yields

$$
\mathcal{I}(\omega,\xi,k) \leq C(1-(k\cdot\hat\xi)^2)^{-\alpha}(\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|))^2.
$$

Let us now consider the $\mathcal{J}$ term. When $d \geq 3$ (and thus $\alpha = 1/2$), there is a loss of a factor $\varepsilon^{-1/2}$ because of the geometric term $(1-(k\cdot\hat\xi)^2)^{-1/2}$. Indeed, we have

$$
\begin{aligned}
\mathcal{J}(\hat\xi) &\leq \|\hat h\|^2_{L^2(\mathbb{S}^d)} \sup_{p\in\mathbb{S}^d} \int_{(1-(k\cdot\hat\xi)^2)^{1/2}\leq\varepsilon^{1/2}} \frac{|\rho_\varepsilon(k\cdot p)| d\sigma(k)}{(1-(k\cdot\hat\xi)^2)^{1/2}} \\
&\quad + \|\hat h\|^2_{L^2(\mathbb{S}^d)} \sup_{p\in\mathbb{S}^d} \int_{(1-(k\cdot\xi)^2)^{1/2}>\varepsilon^{1/2}} \frac{|\rho_\varepsilon(k\cdot p)| d\sigma(k)}{(1-(k\cdot\hat\xi)^2)^{1/2}} \\
&\leq C\|\hat h\|^2_{L^2(\mathbb{S}^d)} \left(\int_{(1-s^2)^{1/2}\leq\varepsilon^{1/2}} \frac{(1-s^2)^{\frac{d-2}{2}} ds}{(1-s^2)}\right)^{\frac{1}{2}} \sup_{p\in\mathbb{S}^d} \|\rho_\varepsilon(p\,\cdot\,\cdot)\|_{L^2(\mathbb{S}^d)} \\
&\quad + C\varepsilon^{-1/2}\|\hat h\|^2_{L^2(\mathbb{S}^d)} \sup_{p\in\mathbb{S}^d} \|\rho_\varepsilon(p\,\cdot\,\cdot)\|_{L^1(\mathbb{S}^d)} \leq C\varepsilon^{-1/2}\|\hat h\|^2_{L^2(\mathbb{S}^d)}.
\end{aligned}
$$

When $d = 1$ and $d = 2$, there is no geometric factor and we find directly

$$
\mathcal{J}(\hat\xi) \leq \|\hat h\|^2_{L^2(\mathbb{S}^d)} \sup_{p\in\mathbb{S}^d} \|\rho_\varepsilon(p\,\cdot\,\cdot)\|_{L^1(\mathbb{S}^d)} \leq C\|\hat h\|^2_{L^2(\mathbb{S}^d)}.
$$

Combining our estimates on $\mathcal{I}$ and $\mathcal{J}$ leads to

$$
\|F_0(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \leq C\varepsilon^{-\frac{\alpha}{2}}\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|)\|\hat h(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}. \tag{3.13}
$$

In the same way, we find for F_1,

$$
\|F_1(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \leq C\lambda\varepsilon^{-\frac{\alpha}{2}}\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|)\|\hat f(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}. \tag{3.14}
$$

3.2.3 The term f_2^ε

Let us consider now the term f_2^ε:

$$
f_2^\varepsilon(t,x,k) = f(t,x,k)\left(1 - \int_{\mathbb{S}^d} \rho_\varepsilon(k\cdot p) d\sigma(p)\right).
$$

The analysis is straightforward:

$$
\begin{aligned}
&\int_{\mathbb{S}^d} \rho_\varepsilon(k\cdot p) d\sigma(p) = \frac{|S_{d-1}|}{\varepsilon^{d/2}} \int_{-1}^{1} \rho\left(\frac{1-s}{\varepsilon}\right)(1-s^2)^{(d-2)/2} ds \\
&= 2^{(d-2)/2}|S_{d-1}| \int_0^{\frac{2}{\varepsilon}} \rho(s) s^{(d-2)/2}(2-\varepsilon s)^{(d-2)/2} ds = \int_0^{\frac{1}{\varepsilon}} (\cdots) ds + \int_{\frac{1}{\varepsilon}}^{\frac{2}{\varepsilon}} (\cdots) ds.
\end{aligned}
$$

Due to the exponential decay of ρ, the second term above is an $\mathcal{O}(e^{-\frac{C}{\varepsilon}})$, while the first one, using the first condition in (3.1) and a Taylor expansion, is equal to $1 + \mathcal{O}(\varepsilon)$. Hence, there exists $\varepsilon_0 \in (0, 1)$, such that for all $\varepsilon \leq \varepsilon_0$,

$$|f_2^\varepsilon| \leq \frac{1}{2}|f|. \tag{3.15}$$

3.2.4 The end of the proof for $\beta = 0$

In order to conclude the proof for $\beta = 0$, we choose ε that depends on ξ and ω:

$$\varepsilon(|\omega|, |\xi|) = (|\omega| + |\xi|)^{-2/(2+\theta)}, \tag{3.16}$$

and write (recall that the expected regularity gain is $\gamma = \theta/(2 + \theta)$ for $\beta = 0$)

$$\int_{\mathbb{R}^{d+2}} (|\omega| + |\xi|)^{\frac{2\theta}{2+\theta}} \left\| \hat{f}(\omega, \xi, \cdot) \right\|^2_{L^2(\mathbb{S}^d)} d\omega d\xi = \int_{\varepsilon(|\omega|,|\xi|) \leq \varepsilon_0} (\cdots) d\omega d\xi + \int_{\varepsilon(|\omega|,|\xi|) > \varepsilon_0} (\cdots) d\omega d\xi.$$

The second integral above is directly bounded by $C\|\hat{f}\|_{L^2}$. For the first integral, we put together the result of Lemma 3.1, and estimates (3.13), (3.14), (3.15) and (2.3) to obtain, when $\varepsilon = \varepsilon(|\omega|, |\xi|) \leq \varepsilon_0$:

$$\begin{aligned} \left\| \hat{f}(\omega, \xi, \cdot) \right\|_{L^2(\mathbb{S}^d)} \leq\ & C\varepsilon^{\frac{\theta}{2}} \|(-\Delta_d)^{\frac{\theta}{2}} \hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \\ & + C\varepsilon^{-\frac{\alpha}{2}} \mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|) \left(\lambda \|\hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} + \|\hat{h}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \right). \end{aligned}$$

We treat separately the cases $d \geq 2$ and $d = 1$. We start with $d \geq 2$. Choosing

$$\lambda = \lambda(|\omega|, |\xi|) = \frac{\|\hat{h}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)}}{\|\hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)}} \tag{3.17}$$

yields

$$\left\| \hat{f}(\omega, \xi, \cdot) \right\|_{L^2(\mathbb{S}^d)} \leq C\varepsilon^{\frac{\theta}{2}} \|(-\Delta_d)^{\frac{\theta}{2}} \hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} + \frac{C\|\hat{f}(\omega, \xi, \cdot)\|^{\frac{1}{2}}_{L^2(\mathbb{S}^d)} \|\hat{h}(\omega, \xi, \cdot)\|^{\frac{1}{2}}_{L^2(\mathbb{S}^d)}}{\varepsilon^{\frac{1}{2}} (|\omega| + |\xi|)^{\frac{1}{2}}},$$

which leads to

$$\left\| \hat{f}(\omega, \xi, \cdot) \right\|_{L^2(\mathbb{S}^d)} \leq C\varepsilon^{\frac{\theta}{2}} \|(-\Delta_d)^{\frac{\theta}{2}} \hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} + \frac{C}{\varepsilon(|\omega| + |\xi|)} \|\hat{h}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)}. \tag{3.18}$$

The parameter $\varepsilon(|\omega|, |\xi|)$ was chosen so as to balance the two terms above, it follows that

$$\int_{\mathbb{R}^{d+2}} (|\omega| + |\xi|)^{\frac{2\theta}{2+\theta}} \left\| \hat{f}(\omega, \xi, \cdot) \right\|^2_{L^2(\mathbb{S}^d)} d\omega d\xi \leq C \left(\|f\|^2_{L^2} + \|(-\Delta_d)^{\frac{\theta}{2}} f\|^2_{L^2} + \|h\|^2_{L^2} \right).$$

This proves the result when $d \geq 2$. When $d = 1$, we have, with the same λ as above:

$$\begin{aligned} \left\| \hat{f}(\omega, \xi, \cdot) \right\|_{L^2(\mathbb{S}^1)} \leq\ & C\varepsilon^{\frac{\theta}{2}} \|(-\Delta_1)^{\frac{\theta}{2}} \hat{f}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^1)} \\ & + \frac{C}{\varepsilon^{\frac{1}{4}} (|\omega| + |\xi|)^{\frac{1}{4}}} \|\hat{f}(\omega, \xi, \cdot)\|^{\frac{3}{4}}_{L^2(\mathbb{S}^1)} \|\hat{h}(\omega, \xi, \cdot)\|^{\frac{1}{4}}_{L^2(\mathbb{S}^1)}, \end{aligned}$$

which leads to (3.18) as well, via Young's inequality, and concludes the proof in the case $\beta = 0$.

3.3 Proof of Theorem 2.1 when $\beta \in (0,1)$

We continue with $\beta \in (0,1)$, postponing the case $\beta \geq 1$ for later. The previously obtained estimates for f_0^ε and f_2^ε, as well as for F_1, still apply (though in the end we will choose different ε and λ), hence we focus on the term F_0.

Let us suppose momentarily that $\beta = 1$ – this is, strictly speaking, outside of the range $\beta \in (0,1)$ but will give us an idea what estimates to expect compared to $\beta = 0$. In this case, we have

$$F_0(\omega,\xi,k) = \int_{\mathbb{S}^d} \hat{h}(\omega,\xi,p)\Delta_d(g(\omega,\xi,p)\rho_\varepsilon(k\cdot p))d\sigma(p).$$

If we choose k as the North pole, and write $p = (\sqrt{1-s^2}u, s)$, where $u \in \mathbb{S}^{d-1}$ and $s \in [-1,1]$, the Laplacian is:

$$\Delta_d = (1-s^2)\frac{\partial^2}{\partial s^2} - ds\frac{\partial}{\partial s} + \frac{1}{1-s^2}\Delta_{d-1}.$$

In particular, the spherical Laplacian applied to $\rho_\varepsilon(k\cdot p)$ gives

$$\Delta_d\rho_\varepsilon = \left((1-s^2)\frac{\partial^2}{\partial s^2} - ds\frac{\partial}{\partial s}\right)\rho_\varepsilon. \tag{3.19}$$

The two derivatives in ρ_ε bring a factor ε^{-2}. As ρ_ε is localized around $s = 1$, the factor $1 - s^2$ in front of the second derivative cancels one of the ε, so that, roughly speaking, $\Delta_d\rho_\varepsilon$ is more singular than ρ_ε by a factor ε^{-1}. Things are different for $\Delta_d g$: the function g is not localized around $s = 1$, and $\Delta_d g$ is more singular than g by a factor of $(|\xi|/\lambda)^2$. Interpolating to fractional values of β, we can expect that F_0 will have two terms: one more singular by a factor $\varepsilon^{-\beta}$, and another by a factor $(|\xi|/\lambda)^{2\beta}$, compared to the case $\beta = 0$.

For a general $\beta \in (0,1)$, as we have mentioned in the outline of the proof, it is convenient to introduce the operator $\mathcal{R}_\beta$ via

$$\mathcal{R}_\beta\varphi(p) = \text{p.v.}\int_{\mathbb{S}^d} \frac{\varphi(p)-\varphi(q)}{|p-q|^{2\beta+d}}d\sigma(q), \quad p \in \mathbb{S}^d,$$

and make a decomposition (which defines the operator $\mathcal{Q}$)

$$(-\Delta_d)^\beta h = (\mathcal{R}_\beta + I)\mathcal{Q}h, \quad \mathcal{Q} = (\mathcal{R}_\beta + I)^{-1}(-\Delta_d)^\beta.$$

This leads to

$$F_0(\omega,\xi,k) = \int_{\mathbb{S}^d} \frac{(-\Delta_d)^\beta\hat{h}(\omega,\xi,p)}{\lambda + i(\omega+\xi\cdot p)}\rho_\varepsilon(k\cdot p)d\sigma(p) = F_{0,1}(\omega,\xi,k) + F_{0,2}(\omega,\xi,k),$$

with

$$F_{0,1}(\omega,\xi,k) = \int_{\mathbb{S}^d} \mathcal{Q}\hat{h}(\omega,\xi,p)\mathcal{R}_\beta\left(\frac{\rho_\varepsilon(k\cdot p)}{\lambda + i(\omega+\xi\cdot p)}\right)d\sigma(p),$$

and

$$F_{0,2}(\omega,\xi,k) = \int_{\mathbb{S}^d} \mathcal{Q}\hat{h}(\omega,\xi,p)\left(\frac{\rho_\varepsilon(k\cdot p)}{\lambda + i(\omega+\xi\cdot p)}\right)d\sigma(p).$$

The main contribution will come from $F_{0,1}$ and we look at it first. Let us split the operator $\mathcal{R}_\beta$ as

$$\mathcal{R}_\beta(\rho_\varepsilon g) = g(\mathcal{R}_\beta \rho_\varepsilon) + \rho_\varepsilon(\mathcal{R}_\beta g) + \mathcal{A}(g, \rho_\varepsilon), \tag{3.20}$$

where $\mathcal{A}$ is defined by

$$\mathcal{A}(g, \rho_\varepsilon)(\omega, \xi, k, p) = \text{p.v.} \int_{\mathbb{S}^d} \frac{(g(\omega, \xi, p) - g(\omega, \xi, q))(\rho_\varepsilon(k \cdot q) - \rho_\varepsilon(k \cdot p))}{|p - q|^{2\beta + d}} d\sigma(q).$$

Using (3.20), and setting $\underline{\hat{h}} = \mathcal{Q}\hat{h}$, we write

$$F_{0,1} = \int_{\mathbb{S}^d} \underline{\hat{h}} g(\mathcal{R}_\beta \rho_\varepsilon) d\sigma(p) + \int_{\mathbb{S}^d} \underline{\hat{h}} \rho_\varepsilon(\mathcal{R}_\beta g) d\sigma(p) + \int_{\mathbb{S}^d} \underline{\hat{h}} \mathcal{A}(g, \rho_\varepsilon) d\sigma(p) := \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_2.$$

The aforementioned heuristic argument for $\beta = 1$ indicates that we can expect that $\mathcal{F}_0$ will be more singular by a factor $\varepsilon^{-\beta}$ compared to the case $\beta = 0$, while $\mathcal{F}_1$ will be more singular by a factor $(|\xi|/\lambda)^{2\beta}$. This is confirmed in the following two propositions, proved in Section 4.

Proposition 3.4 *Suppose that $\varepsilon \leq 1/8$. Then, we have the estimate*

$$\|\mathcal{F}_0(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \leq C\varepsilon^{-\beta} \mathfrak{G}_d(\varepsilon, \lambda, |\omega|, |\xi|) \|\underline{\hat{h}}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)}, \tag{3.21}$$

where, for any $\delta \in (0, 1 - \beta]$,

$$\mathfrak{G}_d(\varepsilon, \lambda, |\omega|, |\xi|) = \begin{cases} \dfrac{1}{\varepsilon^{\frac{1}{4}}} \mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|) + \dfrac{e^{-\frac{C}{\varepsilon}}}{\lambda}, & \text{when} \quad d \geq 3 \\ \dfrac{1}{\varepsilon^{\frac{\beta}{2} + \frac{\delta}{2}}} \mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|), & \text{when} \quad d \leq 2. \end{cases} \tag{3.22}$$

We will see that $\mathcal{F}_1$ dominates both $\mathcal{F}_0$ and $\mathcal{F}_2$. Regarding $\mathcal{F}_1$, we have the estimate:

Proposition 3.5 *We have*

$$\|\mathcal{F}_1(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \leq \frac{C}{\varepsilon^{\alpha/2}} \left(1 + \left(\frac{|\xi|}{\lambda}\right)^{2\beta}\right) \mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|) \|\underline{\hat{h}}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)},$$

where $\alpha = 1/2$ when $d \geq 3$ and $\alpha = 0$ otherwise.

The term $\mathcal{F}_2$ can be seen as an interpolation term between $\mathcal{F}_0$ and $\mathcal{F}_1$ but we will simply control it in the same fashion as $\mathcal{F}_1$. The most direct result is the following:

Proposition 3.6 *Suppose that $(|\xi| + |\omega|)\varepsilon/\lambda \geq 1$, then*

$$\|\mathcal{F}_2(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)} \leq \frac{C}{\varepsilon^{\alpha/2}} \left(1 + \left(\frac{|\xi| + |\omega|}{\lambda}\right)^{2\beta}\right) \mathfrak{H}_d(\varepsilon, \lambda, |\omega|, |\xi|) \|\underline{\hat{h}}(\omega, \xi, \cdot)\|_{L^2(\mathbb{S}^d)},$$

where $\alpha = 1/2$ when $d \geq 3$ and $\alpha = 0$ otherwise.

We have everything now to conclude the proof. Let first $\varepsilon_0' = \varepsilon_0 \wedge \frac{1}{8}$, where ε_0 is chosen to guarantee (3.15) for $0 < \varepsilon < \varepsilon_0$. We will now choose ε and λ as

$$\lambda = \lambda(\omega,\xi) = (|\xi|+|\omega|)^{\frac{2\beta}{1+2\beta}} \frac{\left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{1}{1+2\beta}}}{\left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{1}{1+2\beta}}}, \qquad \varepsilon(\omega,\xi) = (|\omega|+|\xi|)^{-\frac{2}{2+4\beta+\theta}}. \quad (3.23)$$

Note that this choice agrees with (3.16) and (3.17) when $\beta = 0$. Recall that the expected regularity gain is

$$\gamma = \frac{\theta}{2(1+2\beta)+\theta},$$

and write

$$\int_{\mathbb{R}^{d+2}} (|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi = \sum_{i=0}^{2} \int_{\Omega_i} (|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi,$$

where

$$\begin{aligned}
\Omega_0 &= \left\{(\omega,\xi)\in\mathbb{R}^{d+2}, \quad \varepsilon(\omega,\xi)\le\varepsilon_0', \quad \varepsilon(\omega,\xi)(|\xi|+|\omega|)\ge\lambda(\omega,\xi)\right\}\\
\Omega_1 &= \left\{(\omega,\xi)\in\mathbb{R}^{d+2}, \quad \varepsilon(\omega,\xi)\le\varepsilon_0', \quad \varepsilon(\omega,\xi)(|\xi|+|\omega|)<\lambda(\omega,\xi)\right\}\\
\Omega_2 &= \left\{(\omega,\xi)\in\mathbb{R}^{d+2}, \quad \varepsilon(\omega,\xi)>\varepsilon_0'\right\}.
\end{aligned}$$

The domain Ω_1 is introduced to handle the case where the constraint $(|\xi|+|\omega|)\varepsilon \ge \lambda$ of Proposition 3.6 is not satisfied. The integral over Ω_2 is directly bounded by $\|\hat{f}\|_{L^2}$. Also, the condition

$$(|\xi|+|\omega|)\varepsilon(\omega,\xi) \le \lambda(\omega,\xi) \quad (3.24)$$

implies

$$(|\omega|+|\xi|)^{\frac{\theta}{2+\theta+4\beta}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)} \le \left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)},$$

so that the integral over Ω_1 is bounded by $\|\underline{\hat{h}}\|_{L^2}$.

It remains to bound

$$I_0 = \int_{\Omega_0} (|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi.$$

The terms f_0^ε and f_2^ε are treated as in the case $\beta = 0$. We write for f_1^ε:

$$\begin{aligned}
\left\|\hat{f}_1^\varepsilon(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)} &\le \|F_{0,1}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} + \|F_{0,2}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} + \|F_1(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}\\
&\le \sum_{i=0}^{2} \|\mathcal{F}_i(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} + \|F_{0,2}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} + \|F_1(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}.
\end{aligned}$$

With our current choice (3.23) of λ, we have, according to (3.11) and (3.14),

$$\|F_1(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \le \begin{cases} \dfrac{C}{\varepsilon^{\frac{1}{2}}(|\omega|+|\xi|)^{\frac{1}{2(1+2\beta)}}} \left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{1}{2(1+2\beta)}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{4\beta+1}{2(1+2\beta)}}, & d\ge 2,\\ \dfrac{C}{\varepsilon^{\frac{1}{4}}(|\omega|+|\xi|)^{\frac{1}{4(1+2\beta)}}} \left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{1}{4(1+2\beta)}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^{\frac{8\beta+3}{4(1+2\beta)}}, & d=1. \end{cases} \quad (3.25)$$

Using the Young inequality, we obtain, for all $d \geq 1$ and any $\epsilon_0 > 0$,

$$\|F_1(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} \leq \frac{C}{\varepsilon^{1+2\beta}(|\omega|+|\xi|)} \left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)} + \epsilon_0 \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}. \quad (3.26)$$

The definition (3.23) of λ, together with Propositions 3.5 and 3.6 implies that $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (3.25) and (3.26) as well, for $(\omega,\xi) \in \Omega_0$.

As the term $F_{0,2}$ does not involve $\mathcal{R}_\beta$, it is more regular than $F_{0,1}$ and can be shown to satisfy the same estimate as F_1 for large $|\xi|$ and $|\omega|$. Thus, it remains to treat $\mathcal{F}_0$. To this end, we use Proposition 3.4 to show that it is more regular than $\mathcal{F}_1 + \mathcal{F}_2$. Consider first $d \geq 3$: the contribution of the term $e^{-C/\varepsilon}$ in (3.21)-(3.22) to I_0 can be readily bounded by $C\|\underline{\hat{h}}\|_{L^2}^2 + C\|\hat{f}\|_{L^2}^2$. For the term proportional to $\mathfrak{H}_d$ in (3.21)-(3.22), we only need to show that there exists C independent of (ω,ξ) such that

$$\frac{C}{\varepsilon(\omega,\xi)^\beta} \leq \left(\frac{|\xi|+|\omega|}{\lambda(\omega,\xi)}\right)^{2\beta}.$$

As $0 < \varepsilon \leq \varepsilon_0$, this follows trivially from the condition

$$\varepsilon(|\xi|+|\omega|) \geq \lambda,$$

that holds on Ω_0. When $d \leq 2$, the condition for $\mathcal{F}_1 + \mathcal{F}_2$ to dominate $\mathcal{F}_0$ becomes, with $\delta \in (0, 1-\beta]$ as in (3.21)-(3.22)

$$\frac{C}{\varepsilon(\omega,\xi)^{\frac{3\beta+\delta}{2}}} \leq \left(\frac{|\xi|+|\omega|}{\lambda(\omega,\xi)}\right)^{2\beta},$$

which is also satisfied picking δ sufficiently small.

Collecting all our estimates together, we find for $d \geq 1$:

$$\begin{aligned}
&\int_{\mathbb{R}^{d+2}} (|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}} \left\|\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi \\
&\quad \leq C \int_{\mathbb{R}^{d+2}} \varepsilon^\theta(\omega,\xi)(|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}} \left\|(-\Delta_d)^{\frac{\theta}{2}}\hat{f}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi \\
&\qquad + \int_{\mathbb{R}^{d+2}} \varepsilon^{-2-4\beta}(\omega,\xi)(|\omega|+|\xi|)^{\frac{2\theta}{2+\theta+4\beta}-2} \left\|\underline{\hat{h}}(\omega,\xi,\cdot)\right\|_{L^2(\mathbb{S}^d)}^2 d\omega d\xi \\
&\qquad + C\|\underline{\hat{h}}\|_{L^2}^2 + C\|\hat{f}\|_{L^2}^2 \leq C\|(-\Delta_d)^{\frac{\theta}{2}}\hat{f}\|_{L^2}^2 + C\|\underline{\hat{h}}\|_{L^2}^2 + C\|\hat{f}\|_{L^2}^2.
\end{aligned}$$

It remains to show that

$$\|\underline{\hat{h}}\|_{L^2} = \|(\mathcal{R}_\beta + I)^{-1}(-\Delta_d)^\beta \hat{h}\|_{L^2} \leq C\|\hat{h}\|_{L^2}. \quad (3.27)$$

It is proved in [24] (with a slight adaptation of the constants), that the Fourier multipliers R_n associated with of $\mathcal{R}_\beta$ are

$$R_n = \frac{2^{2\beta}\pi^{\frac{d}{2}}\Gamma(\beta)}{\Gamma(\frac{d}{2}+\beta)} \left(\frac{\Gamma(n+\frac{d+2\beta}{2})}{\Gamma(n+\frac{d-2\beta}{2})} - \frac{\Gamma(\frac{d+2\beta}{2})}{\Gamma(\frac{d-2\beta}{2})}\right),$$

where Γ is the gamma function. When $d = 1$ and $\beta = 1/2$, we have, by convention, $\Gamma(\frac{d-2\beta}{2}) = \Gamma(0) = \infty$. The fact that $\Gamma(n+\alpha) \sim \Gamma(n)n^\alpha$ as $n \to \infty$ for $\alpha \in \mathbb{R}$, shows that R_n behaves like $n^{2\beta}$ for large n, which is the same asymptotics as the eigenvalues of $(-\Delta_d)^\beta$. This proves (3.27) and ends the proof of the theorem for $\beta \in (0,1)$.

Proof of Theorem 2.1 when $\beta \geq 1$

We only give a sketch explaining how the term F_0 in (2.5) is treated when $\beta \geq 1$, the main contribution to it coming from $F_{0,1}$ in (2.10). The fractional nature of the Laplacian is not an issue, since the fractional part is included in $\mathcal{R}_{\beta_0}$ in (2.10) (recall that $\beta_0 = \beta - [\beta]$ is the fractional part of β), and is treated in Propositions 3.4-3.6. Hence, the only extra technical part is to compute $(-\Delta_d)^{[\beta]}(\rho_\varepsilon g)$. A straightforward computation shows that $(-\Delta_d)^{[\beta]}(\rho_\varepsilon g)$ can be written as

$$
\begin{aligned}
&(-\Delta_d)^{[\beta]}(\rho_\varepsilon g) = \sum_{l=0}^{2[\beta]} \varepsilon^{-l} |\xi'|^{2[\beta]-l} C_l(\hat{\xi}, p, k) \rho_{\varepsilon,l} \, g_l, \qquad \hat{\xi} = \frac{\xi}{|\xi|}, \quad \xi' = \frac{\xi}{\lambda} \\
&\rho_{\varepsilon,l}(s) := \rho^{(l)}((1-s)/\varepsilon)/\varepsilon^{\frac{d}{2}}, \qquad g_l(s) := \lambda^{-1}(1 + i(\omega' + |\xi'|s))^{-(2[\beta]-l)-1}, \quad \omega' = \frac{\omega}{\lambda}
\end{aligned}
$$

where the C_l are smooth functions whose expressions are not needed. We can use Propositions 3.4-3.5-3.6 for each term in the sum above, with β replaced by $\beta_0 \in (0,1)$. As in the case $\beta \in (0,1)$, the powers of $|\xi'|$ coming from the derivation of g dominate the powers of ε^{-1} coming from the derivation of ρ_ε. The leading term in $(-\Delta_d)^{[\beta]}(\rho_\varepsilon g)$ is, therefore, $\rho_\varepsilon(-\Delta_d)^{[\beta]}(g)$, and the rest of the proof is similar to the case $\beta \in (0,1)$: we define λ and ε in the exact same way as in (3.23), and after a computation similar to $\beta \in (0,1)$, obtain the announced regularity coefficient $\gamma = \frac{\theta}{2+\theta+4\beta}$. This ends the proof of Theorem 2.1.

4 Proofs of the Propositions

4.1 Proof of Proposition 3.4

Let us define

$$
\begin{aligned}
\mathcal{I}(\omega, \xi, k) &= \int_{\mathbb{S}^d} |g(\omega, \xi, p)|^2 |\mathcal{R}_\beta \rho_\varepsilon(k, p)| d\sigma(p), \\
\mathcal{J}(\hat{\xi}) &= \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{|\hat{h}(p)|^2 |\mathcal{R}_\beta \rho_\varepsilon(k, p)|}{(1 - (k \cdot \hat{\xi})^2)^\alpha} d\sigma(k) d\sigma(p),
\end{aligned}
$$

where $\alpha = 1/2$ when $d \geq 3$, and $\alpha = 0$ when $d = 1, 2$. The Cauchy-Schwarz inequality yields

$$
\left\| \mathcal{F}_0(\omega, \hat{\xi}, \cdot) \right\|^2_{L^2(\mathbb{S}^d)} \leq \mathcal{J}(\hat{\xi}) \sup_{k \in \mathbb{S}^d} \left[(1 - (k \cdot \hat{\xi}))^\alpha \mathcal{I}(\omega, \xi, k) \right].
$$

We start by treating the somewhat most technical case $d \geq 3$ and postpone the simpler cases $d = 1, 2$ to the end of the section.

4.1.1 The case $d \geq 3$

We split the operator $\mathcal{R}_\beta$ into several contributions:

$$
\begin{aligned}
\mathcal{R}_\beta \rho_\varepsilon(k,p) &= \lim_{\eta\to 0}\int_{|p-q|>\sqrt{2\eta}} \frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)}{|p-q|^{2\beta+d}} d\sigma(q) \\
&= \int_{|p-q|>\sqrt{2\varepsilon}} \frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)}{|p-q|^{2\beta+d}} d\sigma(q) \\
&+ \lim_{\eta\to 0}\int_{\Omega_{\varepsilon,\eta}(k,p)} \frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)-k\cdot(p-q)\rho_\varepsilon'(k\cdot p)}{|p-q|^{2\beta+d}} d\sigma(q) \\
&+ \lim_{\eta\to 0}\int_{\Omega^c_{\varepsilon,\eta}(k,p)} \frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)-k\cdot(p-q)\rho_\varepsilon'(k\cdot p)}{|p-q|^{2\beta+d}} d\sigma(q) \\
&+ \lim_{\eta\to 0}\int_{\sqrt{2\varepsilon}\geq|p-q|>\sqrt{2\eta}} \frac{k\cdot(p-q)\rho_\varepsilon'(k\cdot p)}{|p-q|^{2\beta+d}} d\sigma(q) := (D_1+D_2+D_3+D_4)(k,p),
\end{aligned}
$$

where $\rho_\varepsilon'(s) = -\rho'((1-s)/\varepsilon)/\varepsilon^{1+\frac{d}{2}}$ and

$$
\begin{aligned}
\Omega_{\varepsilon,\eta}(k,p) &= \left\{q\in\mathbb{S}^d,\ \sqrt{2\varepsilon}\geq|p-q|\geq\sqrt{2\eta},\ \max(k\cdot p,k\cdot q)\geq 0\right\} \\
\Omega^c_{\varepsilon,\eta}(k,p) &= \left\{q\in\mathbb{S}^d,\ \sqrt{2\varepsilon}\geq|p-q|\geq\sqrt{2\eta},\ \max(k\cdot p,k\cdot q)< 0\right\}.
\end{aligned}
$$

Let us comment on this decomposition. We introduce a cutoff at the scale $\sqrt{\varepsilon}$ in order to handle the localization of the function ρ_ε at the scale ε. Roughly speaking, when

$$k\cdot p\simeq k\cdot q\simeq 1-\varepsilon,$$

so that

$$\rho_\varepsilon(k\cdot p)\simeq\rho_\varepsilon(k\cdot q)\simeq\rho_\varepsilon(1),$$

then $|p-q|\simeq\sqrt{\varepsilon}$. Moreover, the domain $\sqrt{2\varepsilon}\geq|p-q|>\sqrt{2\eta}$ is split so as to justify the Taylor expansions in the term $\mathcal{I}_3$ below. The term D_4 removes the singularity in the principal value as usual and allows for $\beta>1/2$.

We then decompose $\mathcal{I}$ and $\mathcal{J}$ accordingly into

$$\mathcal{I} := \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,$$

and

$$\mathcal{J} := \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4.$$

The terms $\mathcal{I}$ and $\mathcal{J}$ are treated in a very similar fashion. We will provide the details in the estimates for $\mathcal{I}$, and only underline the differences for $\mathcal{J}$. We treat the most difficult terms $\mathcal{I}_1$, $\mathcal{I}_2$ (and $\mathcal{J}_1$, $\mathcal{J}_2$) first.

The terms $\mathcal{I}_1$ and $\mathcal{J}_1$

We proceed as in the proof of Lemma 3.3: choose the coordinate axes so that k is the North pole, and write

$$p=(\sqrt{1-t^2}u,t),\quad q=(\sqrt{1-s^2}u_1,s),$$

with $s,t\in[-1,1]$, and $u,u_1\in\mathbb{S}^{d-1}$. We also choose u as the North pole of $\mathbb{S}^{d-1}$ and write

$$u_1=(\sqrt{1-\tau^2}w,\tau),\quad w\in\mathbb{S}^{d-2}.$$

We also decompose

$$\hat{\xi}=(y_\xi\hat{\xi}_\perp,x_\xi),\quad \hat{\xi}=\xi/|\xi|,\quad \hat{\xi}_\perp\in\mathbb{S}^{d-1},\quad x_\xi=\hat{\xi}\cdot k,\quad y_\xi=\sqrt{1-x_\xi^2},\quad \hat{\xi}_\perp\cdot k=0.$$

Introducing, finally $z=u\cdot\hat{\xi}_\perp\in[-1,1]$, we write:

$$\begin{aligned}\mathcal{I}_1(\omega,\xi,k) \quad\leq\quad & |S_{d-2}|^2\int_{-1}^1\int_{-1}^1\frac{(1-t^2)^{\frac{d-2}{2}}(1-z^2)^{\frac{d-3}{2}}}{|\lambda+i(\omega+tk\cdot\xi+|\xi|y_\xi\sqrt{1-t^2}z)|^2} \\ & \times\left(\int_{\Omega_0^\varepsilon(t)}\frac{(1-s^2)^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}|\rho_\varepsilon(t)-\rho_\varepsilon(s)|}{|1-st-\sqrt{1-t^2}\sqrt{1-s^2}\tau|^{\beta+\frac{d}{2}}}dsd\tau\right)dtdz,\end{aligned}\tag{4.1}$$

where

$$\Omega_0^\varepsilon(t)=\{s\in[-1,1],\ \tau\in[-1,1],\ |1-st-\sqrt{1-t^2}\sqrt{1-s^2}\tau|\geq\varepsilon\}.$$

Using Lemma 3.2 with $\gamma=0$, we find, since $0\leq\sqrt{1-t^2}y_\xi\leq 1$,

$$\begin{aligned}\int_{-1}^1\frac{(1-z^2)^{\frac{d-3}{2}}dz}{|\lambda+i(\omega+tk\cdot\xi+|\xi|y_\xi\sqrt{1-t^2}z)|^2} &\leq\frac{1}{\sqrt{1-t^2}y_\xi}\int_{-1}^1\frac{dz}{|\lambda+i(\omega+tk\cdot\xi+|\xi|z)|^2} \\ &\leq\frac{C}{\lambda(|\omega+tk\cdot\xi|+|\xi|)\sqrt{1-t^2}y_\xi}\leq\frac{C}{\lambda(|\omega|+|\xi|)\sqrt{1-t^2}y_\xi}.\end{aligned}\tag{4.2}$$

The last inequality is straightforward when $|\xi|\geq|\omega|$, and follows when $|\xi|\leq|\omega|$ from the fact that $|\omega+tk\cdot\xi|+|\xi|\geq|\omega|-|\xi|+|\xi|$ since $|t|\leq 1$. Performing the changes of variables $t\to 1-\varepsilon t$ and $s\to 1-\varepsilon s$ leads to

$$\begin{aligned}\mathcal{I}_1(\omega,\xi,k)\quad\leq\quad & \frac{C}{\lambda\varepsilon^{\beta+\frac12}(|\omega|+|\xi|)y_\xi}\int_0^{\frac{2}{\varepsilon}}\int_{\Omega_1^\varepsilon(t)}t^{\frac{d-3}{2}}s^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}(2-\varepsilon t)^{\frac{d-3}{2}}(2-\varepsilon s)^{\frac{d-2}{2}} \\ & \times\frac{|\rho(t)-\rho(s)|}{d_0^\varepsilon(t,s,\tau)^{\beta+\frac{d}{2}}}dtdsd\tau,\end{aligned}$$

where we have introduced

$$\begin{aligned} & d_0^\varepsilon(t,s,\tau)=(\sqrt{t}-\sqrt{s})^2+2\sqrt{ts}\left(1-\tau\sqrt{1-\frac{\varepsilon t}{2}}\sqrt{1-\frac{\varepsilon s}{2}}-\frac{\varepsilon\sqrt{st}}{2}\right) \\ & \Omega_1^\varepsilon(t)=\left\{s\in[0,2/\varepsilon],\ \tau\in[-1,1],\ d_0^\varepsilon(t,s,\tau)\geq 1\right\}.\end{aligned}$$

Notice that in the domain $\Omega_1^\varepsilon(t)$, $d_0^\varepsilon(t,s,\tau)\geq(\sqrt{t}-\sqrt{s})^2+2\sqrt{ts}\,d_1^\varepsilon(t,s,\tau)$ where

$$\begin{aligned}d_1^\varepsilon(t,s) &= 1-(1-\frac{\varepsilon t}{2})^{\frac12}(1-\frac{\varepsilon s}{2})^{\frac12}-\frac{\varepsilon\sqrt{st}}{2} \\ &= \frac12\left((1-\frac{\varepsilon t}{2})^{\frac12}-(1-\frac{\varepsilon s}{2})^{\frac12}\right)^2+\frac{\varepsilon}{4}(\sqrt{t}-\sqrt{s})^2\geq 0.\end{aligned}$$

We will use d_1^ε later on in the estimation of $\mathcal{I}_2$. We then control $\mathcal{I}_1$ as

$$\mathcal{I}_1(\omega,\xi,k) \leq \frac{C}{\lambda\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)y_\xi}\int_0^\infty\int_{|\sqrt{t}-\sqrt{s}|\geq 1} t^{\frac{d-3}{2}}s^{\frac{d-2}{2}}\frac{|\rho(t)|+|\rho(s)|}{|\sqrt{t}-\sqrt{s}|^{2\beta+d}}dtds \tag{4.3}$$

$$+\frac{C}{\lambda\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)y_\xi}\int_0^\infty\int_{|\sqrt{t}-\sqrt{s}|\leq 1} t^{\frac{d-3}{2}}s^{\frac{d-2}{2}}(|\rho(t)|+|\rho(s)|)dtds \leq \frac{C}{\lambda\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)y_\xi}.$$

In the second line above, we used the fact that $d_0^\varepsilon \geq 1$ in the domain Ω_1^ε. This finishes the estimate for $\mathcal{I}_1$. As in the case $\beta=0$, the term y_ξ^{-1} will lead to a loss of a factor $1/\sqrt{\varepsilon}$.

Regarding $\mathcal{J}_1$, the variable k plays essentially the same role as the variable p in $\mathcal{I}_1$. Hence, we freeze p in D_1 as the North pole, and parametrize k as $k=(\sqrt{1-t^2}k_\perp, t)$, with $k_\perp\in\mathbb{S}^{d-1}$. Following the same lines, but also considering whether or not

$$1-(k\cdot\hat{\xi})^2\leq\varepsilon,$$

as we did for $\beta=0$, this gives

$$\begin{aligned}\mathcal{J}_1(\hat{\xi}) &\leq C\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\sup_{p\in\mathbb{S}^d}\int_{(1-(k\cdot\hat{\xi})^2)^\alpha\leq\varepsilon^\alpha}\frac{|D_1(k,p)|}{(1-(k\cdot\hat{\xi})^2)^\alpha}d\sigma(k)\\ &\quad +C\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\sup_{p\in\mathbb{S}^d}\int_{(1-(k\cdot\hat{\xi})^2)^\alpha>\varepsilon^\alpha}\frac{|D_1(k,p)|}{(1-(k\cdot\hat{\xi})^2)^\alpha}d\sigma(k)\\ &\leq C\varepsilon^{-\alpha}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\sup_{p\in\mathbb{S}^d}(\varepsilon^{\frac{d}{4}}\|D_1(\cdot,p)\|_{L^2(\mathbb{S}^d)}+\|D_1(\cdot,p)\|_{L^1(\mathbb{S}^d)}).\end{aligned}$$

Furthermore, we have

$$\|D_1(\cdot,p)\|_{L^1(\mathbb{S}^d)} \leq C\int_{-1}^1\int_{\Omega_0^\varepsilon(t)}\frac{(1-t^2)^{\frac{d-2}{2}}(1-s^2)^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}|\rho_\varepsilon(t)-\rho_\varepsilon(s)|}{|1-st-\sqrt{1-t^2}\sqrt{1-s^2}\tau|^{\beta+\frac{d}{2}}}dsd\tau dt \tag{4.4}$$

and

$$\|D_1(\cdot,p)\|^2_{L^2(\mathbb{S}^d)}\leq C\int_{-1}^1\left|\int_{\Omega_0^\varepsilon(t)}\frac{(1-s^2)^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}|\rho_\varepsilon(t)-\rho_\varepsilon(s)|}{|1-st-\sqrt{1-t^2}\sqrt{1-s^2}\tau|^{\beta+\frac{d}{2}}}dsd\tau\right|^2(1-t^2)^{\frac{d-2}{2}}dt.$$

The end is then very similar to the last steps for the term $\mathcal{I}_1$: we perform the changes of variables $t\to 1-\varepsilon t$ and $s\to 1-\varepsilon s$, and obtain for the L^2 norm of D_1:

$$\|D_1(\cdot,p)\|^2_{L^2(\mathbb{S}^d)}\leq\frac{C}{\varepsilon^{2\beta+\frac{d}{2}}}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\int_0^\infty\left|\int_{|\sqrt{t}-\sqrt{s}|\geq 1}s^{\frac{d-2}{2}}\frac{|\rho(t)|+|\rho(s)|}{|\sqrt{t}-\sqrt{s}|^{2\beta+d}}ds\right|^2 t^{\frac{d-2}{2}}dt$$

$$+\frac{C}{\varepsilon^{2\beta+\frac{d}{2}}}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\int_0^\infty\left|\int_{|\sqrt{t}-\sqrt{s}|\leq 1}s^{\frac{d-2}{2}}(|\rho(t)|+|\rho(s)|)ds\right|^2 t^{\frac{d-2}{2}}dt\leq\frac{C}{\varepsilon^{2\beta+\frac{d}{2}}}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)},$$

where we used a Cauchy-Schwarz inequality to bound the integrals. The L^1 norm of D_1 follows in a similar fashion, and it becomes

$$\mathcal{J}_1(\hat{\xi})\leq\frac{C}{\varepsilon^{\beta+\frac{1}{2}}}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}. \tag{4.5}$$

The terms $\mathcal{I}_2$ and $\mathcal{J}_2$

Regarding $\mathcal{I}_2$, we find, following the same lines as for $\mathcal{I}_1$:

$$\mathcal{I}_2(\omega,\xi,k) \le \frac{C}{\lambda\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)y_\xi}\int_{\Omega_2^\varepsilon} t^{\frac{d-3}{2}}s^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}(2-\varepsilon t)^{\frac{d-3}{2}}(2-\varepsilon s)^{\frac{d-2}{2}} \times \frac{|\rho(t)-\rho(s)-(t-s)\rho'(t)|}{d_0^\varepsilon(t,s,\tau)^{\beta+\frac{d}{2}}}dtdsd\tau \qquad (4.6)$$

$$\Omega_2^\varepsilon = \left\{(t,s)\in[0,2/\varepsilon]^2,\ \tau\in[-1,1],\ d_0^\varepsilon(t,s,\tau)\le 1,\ \min(t,s)\le 1/\varepsilon\right\}.$$

Notice that the condition $\max(k\cdot p, k\cdot q)\ge 0$ becomes $\min(t,s)\le 1/\varepsilon$ after the change of variables $t\to 1-\varepsilon t$ and $s\to 1-\varepsilon s$, with initially $t=k\cdot p$, $s=k\cdot q$. Let us denote by I the integral in the definition of $\mathcal{I}_2$. Making the change of variables below and using the Taylor formula

$$\rho(t)-\rho(s)-(t-s)\rho'(t) = \frac{(t-s)^2}{2}\rho''(\eta(t,s)), \qquad \eta(t,s)\in(s,t)$$

$$1-\tau\sqrt{1-\frac{\varepsilon t}{2}}\sqrt{1-\frac{\varepsilon s}{2}}-\frac{\varepsilon\sqrt{st}}{2}\to\tau,$$

yield the estimate

$$I\le C\int_{\Omega_3^\varepsilon}\frac{t^{\frac{d-3}{2}}s^{\frac{d-2}{2}}(2-\varepsilon t)^{\frac{d-5}{4}}(2-\varepsilon s)^{\frac{d-3}{4}}|t-s|^2}{|(\sqrt{t}-\sqrt{s})^2+2\sqrt{st}\tau|^{\beta+\frac{d}{2}}}|\rho''(\eta(t,s))|\left(d_1^\varepsilon(t,s)^{\frac{d-3}{2}}+\tau^{\frac{d-3}{2}}\right)dsd\tau,$$

where

$$\begin{aligned}\Omega_3^\varepsilon &= \left\{(t,s)\in[0,2/\varepsilon]^2,\tau\in\mathcal{T}_\varepsilon(s,t),\ (\sqrt{t}-\sqrt{s})^2+2\sqrt{ts}\tau\le 1,\ \min(t,s)\le 1/\varepsilon\right\},\\ \mathcal{T}_\varepsilon(s,t) &= \left[1-(1-\frac{\varepsilon t}{2})^{\frac12}(1-\frac{\varepsilon s}{2})^{\frac12}-\frac{\varepsilon\sqrt{st}}{2},1+(1-\frac{\varepsilon t}{2})^{\frac12}(1-\frac{\varepsilon s}{2})^{\frac12}-\frac{\varepsilon\sqrt{st}}{2}\right].\end{aligned}$$

We control now the term d_1^ε in order to make sense of the integral. This is where we use the division of the domain $\sqrt{2\varepsilon}\ge|p-q|>\sqrt{2\eta}$: noting that for $(s,t)\in\Omega_3^\varepsilon$, we have, since $\varepsilon\min(s,t)\le 1$,

$$\frac{\varepsilon|t-s|}{2-\varepsilon\min(s,t)}\le\varepsilon|\sqrt{t}-\sqrt{s}|(\sqrt{t}+\sqrt{s})\le\varepsilon(\sqrt{t}+\sqrt{s})\le 2\sqrt{2\varepsilon}<1 \quad \text{for} \quad \varepsilon<\frac{1}{8}.$$

This, after a direct Taylor expansion, leads to

$$\begin{aligned}\left|(1-\frac{\varepsilon t}{2})^{\frac12}-(1-\frac{\varepsilon s}{2})^{\frac12}\right| &= \left(1-\frac{\varepsilon\min(t,s)}{2}\right)^{\frac12}\left|\left(1+\varepsilon\frac{\min(t,s)-\max(t,s)}{2-\varepsilon\min(t,s)}\right)^{\frac12}-1\right|\\ &\le C\sqrt{\varepsilon}|\sqrt{t}-\sqrt{s}|\le C|\sqrt{t}-\sqrt{s}|.\end{aligned}$$

Note that we do not need the extra $\sqrt{\varepsilon}$ factor above. This gives

$$d_1^\varepsilon(s,t) = \frac12\left((1-\frac{\varepsilon t}{2})^{\frac12}-(1-\frac{\varepsilon s}{2})^{\frac12}\right)^2+\frac{\varepsilon}{4}(\sqrt{t}-\sqrt{s})^2\le C(\sqrt{t}-\sqrt{s})^2.$$

Inserting the last inequality into I, we obtain

$$I \leq C \int_{\Omega_3^\varepsilon} \frac{t^{\frac{d-3}{2}} s^{\frac{d-2}{2}} (2-\varepsilon t)^{\frac{d-5}{4}} (2-\varepsilon s)^{\frac{d-3}{4}} |t-s|^2}{|(\sqrt{t}-\sqrt{s})^2 + 2\sqrt{st}\tau|^{\beta+\frac{d}{2}}} |\rho''(\eta(t,s))| \big(|\sqrt{t}-\sqrt{s}|^{(d-3)} + \tau^{\frac{d-3}{2}}\big) ds d\tau.$$

In order to conclude the estimate for $\mathcal{I}_2$, we simply use basic interpolation to obtain

$$\begin{aligned} \frac{t^{\frac{d-3}{2}} s^{\frac{d-2}{2}} |t-s|^2 (|\sqrt{t}-\sqrt{s}|^{(d-3)} + \tau^{\frac{d-3}{2}})}{|(\sqrt{t}-\sqrt{s})^2 + 2\sqrt{st}\tau|^{\beta+\frac{d}{2}}} &\leq C \frac{t^{\frac{d-4+\delta}{2}} s^{\frac{d-3+\delta}{2}} (\sqrt{t}+\sqrt{s})^2}{|\sqrt{t}-\sqrt{s}|^{1-2\delta} \tau^{1-\delta}} \\ + C \frac{t^{\frac{d-5+2\delta}{4}} s^{\frac{d-3+2\delta}{4}} (\sqrt{t}+\sqrt{s})^2}{|\sqrt{t}-\sqrt{s}|^{1-2\delta} \tau^{1-\delta}} &\leq C \frac{(\sqrt{t}+\sqrt{s})^2}{|\sqrt{t}-\sqrt{s}|^{1-2\delta} \tau^{1-\delta}} \phi(t,s), \end{aligned}$$

where $\beta = 1 - 2\delta$, for some $\delta \in (0, \frac{1}{2})$, and

$$\phi(t,s) = t^{\frac{d-4+\delta}{2}} s^{\frac{d-3+\delta}{2}} + t^{\frac{d-5+2\delta}{4}} s^{\frac{d-3+2\delta}{4}}.$$

We are now ready to conclude: using the exponential decay of $|\rho''(\min(t,s))|$, the facts that $d \geq 3$ and $\delta \in (0, \frac{1}{2})$, as well as the property that ϕ is locally integrable, we have

$$I \leq C \int_{\Omega_3^\varepsilon} \frac{(2-\varepsilon t)^{\frac{d-5}{4}} (\sqrt{t}+\sqrt{s})^2}{|\sqrt{t}-\sqrt{s}|^{1-2\delta}} \phi(t,s) |\rho''(\min(t,s))| dt ds \leq C,$$

which yields

$$\mathcal{I}_2(\omega, \xi, k) \leq \frac{C}{\lambda \varepsilon^{\beta+\frac{1}{2}} (|\omega| + |\xi|) y_\xi}. \tag{4.7}$$

The term $\mathcal{J}_2$ is treated combining the methods of $\mathcal{J}_1$ in order to handle the factor y_ξ^{-1}, and of I above in order to remove the singularity in the principal value. We then obtain the estimate

$$\mathcal{J}_2(\hat{\xi}) \leq \frac{C}{\varepsilon^{\beta+\frac{1}{2}}} \|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}. \tag{4.8}$$

The terms $\mathcal{I}_3$, $\mathcal{I}_4$, $\mathcal{J}_3$ and $\mathcal{J}_4$

We consider now $\mathcal{I}_3$ and $\mathcal{I}_4$ that are treated in a very similar fashion. Calculations are somewhat more direct than the other terms since we simply bound $|g|$ by λ^{-2}, and then only use the exponential decay of ρ_ε. We find, for some $\eta \in (k \cdot p, k \cdot q)$:

$$|\mathcal{I}_3(\omega, \xi, k)| \leq \frac{C}{\lambda^2} \int_{\mathbb{S}^d} \int_{\Omega^c_{\varepsilon,0}(k,p)} \frac{|k \cdot (p-q)|^2}{|p-q|^{2\beta+d}} |\rho''_\varepsilon(\eta)| d\sigma(p) d\sigma(q)$$

We then decompose the integral according to the (p,q) such that

$$\max(k \cdot p, k \cdot q) = k \cdot q,$$

and those with

$$\max(k \cdot p, k \cdot q) = k \cdot p.$$

For the first part, we choose a parametrization of the sphere with q as the North pole, and write p as $p = (\sqrt{1-t^2}\hat{u}, t)$, with $\hat{u} \in S^{d-1}$. This gives

$$k \cdot (p-q) = (t-1)k \cdot q + \sqrt{1-t^2}k \cdot \hat{u}.$$

Let also $s = k \cdot q$, and perform the change of variables $s \to 1 - \varepsilon s$. Defining

$$\tilde{\eta}_\varepsilon = (1-\eta)/\varepsilon \in (s, (1-(1-\varepsilon s)t - \sqrt{1-t^2}k \cdot \hat{u})/\varepsilon),$$

we observe by the choice of the integration domain that

$$\varepsilon^{-1} \leq s \leq \tilde{\eta}_\varepsilon.$$

According to the definition (3.2) of ρ, $|\rho''(x)|$ can be bounded by a linear combination of decreasing exponentials that we denote by $\varphi(x)$. We have $|\rho''(\eta)| \leq \varphi(s)$, and the first part of $\mathcal{I}_3$ can therefore be controlled by

$$\begin{aligned}
&\frac{C}{\lambda^2 \varepsilon^{2+\frac{d}{2}}} \int_{|1-t|\leq 1} \int_{\frac{1}{\varepsilon}}^{\frac{2}{\varepsilon}} \int_{S^{d-1}} \frac{|(1-\varepsilon s)(t-1) + \sqrt{1-t^2}k \cdot \hat{u}|^2}{|1-t|^{\beta+\frac{d}{2}}} \varphi(s)(1-t^2)^{\frac{d-2}{2}} s^{\frac{d-2}{2}} dt ds d\sigma(\hat{u}) \\
&\leq \frac{C}{\lambda^2 \varepsilon^{2+\frac{d}{2}}} \int_{|1-t|\leq 1} \int_{\frac{1}{\varepsilon}}^{\frac{2}{\varepsilon}} \frac{\varphi(s)}{|1-t|^\beta} dt ds \leq \frac{Ce^{-\frac{C}{\varepsilon}}}{\lambda^2}.
\end{aligned}$$

The second part of $\mathcal{I}_3$ yields the same estimate as above and is treated in a similar fashion except the vector p is now taken as the North pole, leading to

$$\mathcal{I}_3(\omega, \xi, k) \leq \frac{Ce^{-\frac{C}{\varepsilon}}}{\lambda^2}. \tag{4.9}$$

Regarding the term $\mathcal{J}_3$, we have, for some $\eta \in (k \cdot p, k \cdot q)$ and $\alpha = 0$:

$$\mathcal{J}_3(\hat{\xi}) \leq C \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \int_{\Omega^c_{\varepsilon,0}(k,p)} \frac{|k \cdot (p-q)|^2}{|p-q|^{2\beta+d}} |\rho''_\varepsilon(\eta)| |\underline{\hat{h}}(p)|^2 d\sigma(k) d\sigma(p) d\sigma(q)$$

Again, the roles of p and k are exchanged here: when $\max(k \cdot p, k \cdot q) = k \cdot q$, we set q as the North pole, and follow the same lines as in $\mathcal{I}_3$. The integral over this set can then be controlled by

$$\mathcal{J}_3(\hat{\xi}) \leq \frac{C}{\varepsilon^{2+\frac{d}{2}}} \int_{\mathbb{S}^d} \int_{|1-t|\leq 1} \int_{\frac{1}{\varepsilon}}^{\frac{2}{\varepsilon}} |\underline{\hat{h}}(p)|^2 |1-t|^{-\beta} \varphi(s) s^{\frac{d-2}{2}} dt ds d\sigma(p) \leq Ce^{-\frac{C}{\varepsilon}} \|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}.$$

When $\max(k \cdot p, k \cdot q) = k \cdot p$, we take p as the North pole, and obtain the same estimate as above:

$$\mathcal{J}_3(\hat{\xi}) \leq Ce^{-\frac{C}{\varepsilon}} \|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}. \tag{4.10}$$

The term D_4 is direct and treated using p as the North pole, leading to

$$\begin{aligned}
D_4(k,p) &= \rho'_\varepsilon(k \cdot p) \lim_{\eta \to 0} \int_{\varepsilon \geq |1-s| \geq \eta} \int_{\mathbb{S}^{d-1}} \frac{k \cdot p(1-s) - \sqrt{1-s^2}k \cdot v}{|1-s|^{\beta+\frac{d}{2}}} (1-s^2)^{\frac{d-2}{2}} ds d\sigma(v) \\
&= \rho'_\varepsilon(k \cdot p)(k \cdot p)|S_{d-1}| \int_{\varepsilon \geq |1-s|} \frac{(1-s)}{|1-s|^{\beta+\frac{d}{2}}} (1-s^2)^{\frac{d-2}{2}} ds,
\end{aligned}$$

so that

$$|D_4(k,p)| \le C|\rho_\varepsilon'(k\cdot p)|\varepsilon^{1-\beta}.$$

We finally find

$$\mathcal{I}_4 \le \frac{C}{\varepsilon^\beta}\int_{\mathbb{S}^d}|g(\omega,\xi,p)|^2|\varepsilon\rho_\varepsilon'(k\cdot p)|d\sigma(p), \quad \mathcal{J}_4 \le \frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}\sup_{p\in\mathbb{S}^d}\int_{\mathbb{S}^d}\frac{|\varepsilon\rho_\varepsilon'(k\cdot p)|}{(1-(k\cdot\hat{\xi})^2)^\alpha}d\sigma(k),$$

which gives, using Lemma 3.3 with $d\ge 3$ for $\mathcal{I}_4$, and proceeding for $\mathcal{J}_4$ as in the term $\mathcal{J}$ in the case $\beta=0$,

$$\mathcal{I}_4 \le \frac{C}{\lambda\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)y_\xi}, \qquad \mathcal{J}_4 \le \frac{C}{\varepsilon^{\beta+\frac{1}{2}}}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}. \tag{4.11}$$

Collecting (4.3), (4.5), (4.7), (4.8)-(4.11) finally gives (3.21). This concludes the case $d\ge 3$.

4.1.2 The cases $d=1$ and $d=2$

We take care now of the simpler cases $d=1$ and $d=2$, which are more direct since the estimates do not need be as sharp as in the case $d\ge 3$. We split the operator $\mathcal{R}_\beta$ into only three terms:

$$\begin{aligned}\mathcal{R}_\beta\rho_\varepsilon(k,p) &= \int_{|p-q|>\sqrt{2\varepsilon}}\frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)}{|p-q|^{2\beta+d}}d\sigma(q)\\ &+\lim_{\eta\to 0}\int_{\sqrt{2\varepsilon}\ge|p-q|>\sqrt{2\eta}}\frac{\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q)-k\cdot(p-q)\rho_\varepsilon'(k\cdot p)}{|p-q|^{2\beta+d}}d\sigma(q)\\ &+\lim_{\eta\to 0}\int_{\sqrt{2\varepsilon}\ge|p-q|>\sqrt{2\eta}}\frac{k\cdot(p-q)\rho_\varepsilon'(k\cdot p)}{|p-q|^{2\beta+d}}d\sigma(q) := (D_1+D_2+D_3)(k,p).\end{aligned}$$

Let us start with $d=1$ and the term $\mathcal{J}$. Since $\alpha=0$ and there is no geometric factor, estimate (4.5) is straightforwardly replaced by

$$\begin{aligned}\mathcal{J}_1(\hat{\xi}) &\le \frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(S^1)}\int_0^{\frac{2}{\varepsilon}}\int_{|\sqrt{t}-\sqrt{s}|\ge 1, s\le 2/\varepsilon}\frac{|\rho(t)|+|\rho(s)|}{|\sqrt{t}-\sqrt{s}|^{2\beta+1}}\frac{dtds}{\sqrt{ts}\sqrt{2-\varepsilon t}\sqrt{2-\varepsilon s}} \\ &+\frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(S^1)}\int_0^{\frac{2}{\varepsilon}}\int_{|\sqrt{t}-\sqrt{s}|\le 1, s\le 2/\varepsilon}(|\rho(t)|+|\rho(s)|)\frac{dtds}{\sqrt{ts}\sqrt{2-\varepsilon t}\sqrt{2-\varepsilon s}} \le \frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^1)}.\end{aligned} \tag{4.12}$$

Above, we treated the square roots $\sqrt{2-\varepsilon t}\sqrt{2-\varepsilon s}$ in the denominator by splitting $[0,2/\varepsilon]$ into $[0,1/\varepsilon]$ and $[1/\varepsilon,2/\varepsilon]$, and by integrating by parts and using the exponential decay of ρ. For the term $\mathcal{J}_2$, we set $k\cdot p=\cos(\theta)=t$. When $\theta\in(0,\pi)$, we have

$$k = tp+\sqrt{1-t^2}p_\perp,$$

while when $\theta\in(\pi,2\pi)$, we have

$$k = tp-\sqrt{1-t^2}p_\perp.$$

Depending on θ, the vector p can thus be written as $p = tk \pm \sqrt{1-t^2}k_\perp$. With the representation $q = sk \pm \sqrt{1-s^2}k_\perp$, we find

$$\frac{1}{2}|p-q|^2 = 1 - p \cdot q = 1 - ts \pm \sqrt{1-s^2}\sqrt{1-t^2}.$$

Using the latter leads to

$$\mathcal{J}_2(\hat{\xi}) \le \frac{C}{\varepsilon^\beta} \int_{\mathbb{S}^1} |\underline{\hat{h}}(p)|^2 d\sigma(p) \int_{\Omega_4^\varepsilon} \frac{|\rho(t) - \rho(s) - (t-s)\rho'(t)|}{(\sqrt{t} - \sqrt{s})^{2\beta+1}} \frac{dtds}{\sqrt{ts}\sqrt{2-\varepsilon t}\sqrt{2-\varepsilon s}},$$

where

$$\Omega_4^\varepsilon = \left\{(t,s) \in [0, 2/\varepsilon]^2,\ |\sqrt{t} - \sqrt{s}| \le 1\right\}$$

The final steps of the estimate are then similar to these for $\mathcal{I}_2$ and $\mathcal{J}_2$, we leave the details to the reader. Again, since $\alpha = 0$, the case $d = 2$ follows directly by estimating the L^1 norm of D_1, using $d = 2$ into (4.4). The first conclusion is that for $d = 1, 2$,

$$\mathcal{J}_1(\hat{\xi}) + \mathcal{J}_2(\hat{\xi}) \le \frac{C}{\varepsilon^\beta} \|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}. \tag{4.13}$$

Let us focus now on the terms $\mathcal{I}_1$ and $\mathcal{I}_2$. We bound for this ρ_ε directly by $C/\varepsilon^{\frac{d}{2}}$. This yields, after familiar changes of variables:

$$\begin{aligned}
\mathcal{I}_1(\omega, \xi, k) &\le \frac{C}{\varepsilon^{\frac{d}{2}}} \int_{-1}^{1} \frac{(1-t^2)^{\frac{d-2}{2}} dt}{|(\lambda + i(\omega + t|\xi|))|^2} \int_{|1-s| \ge \varepsilon} \frac{1}{|1-s|^{\beta + \frac{1}{2}}} \frac{1}{(1-s)^{\frac{1}{2}}} ds \\
&\le \begin{cases} \dfrac{C}{\lambda \varepsilon^{1+\beta}(|\omega| + |\xi|)}, & \text{when} \quad d = 2, \\[2ex] \dfrac{C}{\lambda^{\frac{3}{2}} \varepsilon^{\frac{1}{2}+\beta}(|\omega| + |\xi|)^{\frac{1}{2}}}, & \text{when} \quad d = 1, \end{cases}
\end{aligned}$$

and for any $\delta \in (0, 1-\beta]$,

$$\begin{aligned}
\mathcal{I}_2(\omega, \xi, k) &\le \frac{C}{\varepsilon^{2\beta + 2\delta + \frac{d}{2}}} \int_{S^1} \frac{d\sigma(p)}{|(\lambda + i(\omega + \xi \cdot p))|^2} \int_{|p-q| \le \sqrt{2\varepsilon}} \frac{|p-q|^{2\beta+2\delta}}{|p-q|^{2\beta+d}} d\sigma(q) \\
&\le \frac{C}{\varepsilon^{2\beta + 2\delta + \frac{d}{2}}} \int_{-1}^{1} \frac{(1-t^2)^{\frac{d-2}{2}} dt}{|(\lambda + i(\omega + t|\xi|))|^2} \int_{|1-s| \le \varepsilon} \frac{1}{|1-s|^{\frac{1}{2} - \delta}} \frac{1}{(1-s)^{\frac{1}{2}}} ds \\
&\le \begin{cases} \dfrac{C}{\lambda \varepsilon^{1+2\beta+\delta}(|\omega| + |\xi|)}, & \text{when} \quad d = 2, \\[2ex] \dfrac{C}{\lambda^{\frac{3}{2}} \varepsilon^{\frac{1}{2}+2\beta+\delta}(|\omega| + |\xi|)^{\frac{1}{2}}}, & \text{when} \quad d = 1. \end{cases}
\end{aligned}$$

Above, we used Lemma 3.2, both with $\gamma = \frac{1}{2}$ and $\gamma = 0$, and controlled

$$\rho_\varepsilon(k \cdot p) - \rho_\varepsilon(k \cdot q) - k \cdot (p-q)\rho'_\varepsilon(k \cdot p)$$

by

$$\frac{C}{\varepsilon^{2\beta+2\delta}} |p-q|^{2\beta+2\delta}.$$

The term D_3 is straightforward:

$$D_3 = \varepsilon^{1-\beta}\rho'_\varepsilon(k\cdot p)(k\cdot p)\int_{|t|\leq 1}\frac{(2-\varepsilon t)^{\frac{d-2}{2}}dt}{t^\beta},$$

and using Lemma 3.3 for $d=1$, we have

$$\mathcal{I}_3 \leq \frac{C}{\lambda^{\frac{3}{2}}\varepsilon^{\beta+\frac{1}{2}}(|\omega|+|\xi|)^{\frac{1}{2}}}, \qquad \mathcal{J}_3 \leq \frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^d)}, \tag{4.14}$$

and now with $d=2$:

$$\mathcal{I}_3 \leq \frac{C}{\lambda^{\frac{1}{2}}\varepsilon^{\beta+1}(|\omega|+|\xi|)}, \qquad \mathcal{J}_3 \leq \frac{C}{\varepsilon^\beta}\|\underline{\hat{h}}\|^2_{L^2(\mathbb{S}^2)}. \tag{4.15}$$

Gathering the previous estimates on $\mathcal{I}_i$, $\mathcal{J}_i$, $i=1,2,3$ ends the proof of the proposition.

4.2 Proof of Proposition 3.5

We start by rescaling (ω,ξ) by λ: let $(\omega',\xi')=(\omega,\xi)/\lambda$ so that

$$g(\omega,\xi,p)=\lambda^{-1}g(\omega',\xi',p).$$

Again, we consider first the case $d\geq 3$. Setting p as the North pole, and noticing that

$$g(\omega,\xi,p)-g(\omega,\xi,q)=i\xi\cdot(q-p)g(\omega,\xi,p)g(\omega,\xi,q), \tag{4.16}$$

we split $\mathcal{R}_\beta g$ into two contributions:

$$\begin{aligned}\mathcal{R}_\beta g &= \text{p.v.}\int_{\mathbb{S}^d}\left(\frac{g(\omega',\xi',p)-g(\omega',\xi',q)}{\lambda|p-q|^{2\beta+d}}\right)d\sigma(q)\\ &= ig(\omega,\xi,p)\ \text{p.v.}\int_{\mathbb{S}^d}\frac{(\xi'\cdot(q-p))}{(1+i(\omega'+\xi'\cdot q))|p-q|^{2\beta+d}}d\sigma(q) := A_1+A_2,\end{aligned}$$

where

$$\begin{aligned}A_1 &= -ig(\omega,\xi,p)\int_{\mathbb{S}^{d-1}}\int_{-1}^{1}\frac{(\xi'\cdot p)(1-t^2)^{\frac{d-2}{2}}dtd\sigma(\hat{u})}{2^{\beta+\frac{d}{2}}(1+i(\omega'+\xi'\cdot(tp+\sqrt{1-t^2}\hat{u}))(1-t)^{\beta+\frac{d}{2}-1}}\\ A_2 &= ig(\omega,\xi,p)\lim_{\eta\to 0}\int_{\mathbb{S}^{d-1}}\int_{|1-t|>\eta}\frac{(\xi'\cdot\hat{u})(1-t^2)^{\frac{d-1}{2}}dtd\sigma(\hat{u})}{(1+i(\omega'+\xi'\cdot(tp+\sqrt{1-t^2}\hat{u}))(2(1-t))^{\beta+\frac{d}{2}}}.\end{aligned}$$

Let us start by treating A_1. Note that we removed the principal value since the integral is well-defined. It is easy to see that A_1 can be immediately controlled by $C|\xi'|$, which is fine when $\beta\geq 1/2$ since then $|\xi'|\leq|\xi'|^{2\beta}$ for large $|\xi'|$, but is not optimal when $\beta<1/2$. In the latter case, write

$$\hat{\xi}=x_\xi p+y_\xi\hat{\xi}_\perp,\ \text{ with } \hat{\xi}_\perp\in\mathbb{S}^{d-1},\ \hat{\xi}_\perp\cdot p=0,\ x_\xi\in[-1,1] \text{ and } y_\xi=(1-x_\xi^2)^{1/2}.$$

Setting $\tau=\hat{u}\cdot\hat{\xi}_\perp\in[-1,1]$, A_1 becomes

$$A_1=-\frac{i(\xi'\cdot p)g(\omega,\xi,p)|S_{d-2}|}{2^{\beta+\frac{d}{2}}}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-t^2)^{\frac{d-2}{2}}(1-\tau^2)^{\frac{d-3}{2}}dtd\tau}{(1+i(\omega'+x_\xi|\xi'|t+y_\xi\sqrt{1-t^2}|\xi'|\tau))(1-t)^{\beta+\frac{d}{2}-1}}.$$

We would like to integrate in τ, in order to obtain some decay in $|\xi'|$. However, this will introduce a factor y_ξ^{-1}, which will eventually lead to undesired powers of ε^{-1} when $y_\xi \sim 0$. We need, therefore, to distinguish the region where y_ξ is small. There, we integrate in t instead of τ, in order to gain decay in $|\xi'|$. Hence, let us define

$$\Omega(\hat{\xi} \cdot p) = \left\{ t \in [-1,1], \frac{y_\xi |t|}{|x_\xi| \sqrt{1-t^2}} \leq \frac{1}{2} \right\}, \tag{4.17}$$

and denote by $\Omega^c(\hat{\xi} \cdot p)$ its complement in $[-1,1]$. We accordingly split A_1 into

$$A_1 = A_{1,1} + A_{1,2}.$$

For the term $A_{1,1}$, we use the Hölder inequality to obtain, for any $\delta > 0$,

$$\begin{aligned} |A_{1,1}| &\leq C|x_\xi||\xi'||g(\omega,\xi,p)| \int_{-1}^{1} (I(\omega,\xi',p,\tau))^{\frac{1}{1+\delta}} d\tau \left(\int_{-1}^{1} \frac{dt}{|1-t|^\beta} \right)^{\frac{\delta}{1+\delta}} \\ I(\omega,\xi',p,\tau) &= \int_{\Omega(\hat{\xi}\cdot p)} \frac{dt}{|1+i(\omega' + x_\xi|\xi'|t + y_\xi \sqrt{1-t^2}|\xi'|\tau))|^{1+\delta}(1-t)^\beta}. \end{aligned} \tag{4.18}$$

Now, split the integral I as (with $a = |\xi'|^{-1}$),

$$I = \int_{\Omega(\hat{\xi}\cdot p)\cap\{|1-t|\leq a\}} (\cdots)dt + \int_{\Omega(\hat{\xi}\cdot p)\cap\{|1-t|>a\}} (\cdots)dt := I_1 + I_2.$$

We have, by a direct computation

$$\begin{aligned} I_1 &\leq \int_{|1-t|\leq a} \frac{dt}{|1-t|^\beta} \leq C|\xi'|^{\beta-1} \\ I_2 &\leq |\xi'|^\beta \int_{\Omega(\hat{\xi}\cdot p)} \frac{dt}{|1+i(\omega' + x_\xi|\xi'|t + y_\xi\sqrt{1-t^2}|\xi'|\tau)|^{1+\delta}}. \end{aligned} \tag{4.19}$$

We control the above integral using the change of variables

$$z = j(t) = x_\xi t + y_\xi \sqrt{1-t^2}\tau,$$

which yields, since $t \in \Omega(\hat{\xi} \cdot p)$ and $\tau \in [-1,1]$,

$$|j'(t)| = |x_\xi| \left| 1 - \frac{y_\xi t \tau}{x_\xi \sqrt{1-t^2}} \right| \geq \frac{|x_\xi|}{2}.$$

Using Lemma 3.2 with $\gamma = 0$, we deduce that I_2 satisfies

$$I_2 \leq \frac{C|\xi'|^\beta}{|x_\xi|} \int_{-1}^{1} \frac{dz}{|1+i(\omega' + |\xi'|z)|^{1+\delta}} \leq \frac{C|\xi'|^\beta}{|x_\xi|(|\xi'| + |\omega'|)} \leq \frac{C|\xi'|^{\beta-1}}{|x_\xi|}.$$

Combining this with (4.18) and (4.19) implies that, for any $\delta > 0$,

$$|A_{1,1}| \leq C|\xi'|^{\frac{\beta+\delta}{1+\delta}} |g(\omega,\xi,p)|. \tag{4.20}$$

For the term $A_{1,2}$, we find for any $\delta' > 0$, using once more the Hölder inequality and Lemma 3.2 with $\gamma = 0$,

$$\begin{aligned}
&\left| \int_{-1}^{1} \frac{(1-\tau^2)^{\frac{d-3}{2}} d\tau}{(1+i(\omega' + x_\xi|\xi'|t + y_\xi|\xi'|\sqrt{1-t^2}\tau))} \right| \\
&\qquad \leq C \left(\int_{-1}^{1} \frac{d\tau}{|1+i(\omega' + x_\xi|\xi'|t + y_\xi|\xi'|\sqrt{1-t^2}\tau)|^{1+\delta'}} \right)^{\frac{1}{1+\delta'}} \\
&\qquad \leq \frac{C}{(y_\xi\sqrt{1-t^2}(|\xi'| + |\omega'|))^{\frac{1}{1+\delta'}}}. \qquad (4.21)
\end{aligned}$$

Owing to the latter estimate, and accounting for the fact that $t \in \Omega^c(\hat{\xi} \cdot p)$ in order to bound y_ξ from below, we see that for any δ' such that $\beta + \frac{1}{1+\delta'} < 1$:

$$\begin{aligned}
|A_{1,2}| &\leq C|x_\xi||\xi'|^{\frac{\delta'}{1+\delta'}}|g(\omega,\xi,p)| \int_{\Omega^c(\hat{\xi}\cdot p)} \frac{dt}{(y_\xi\sqrt{1-t^2})^{\frac{1}{1+\delta'}}(1-t)^\beta} \\
&\leq C|\xi'|^{\frac{\delta'}{1+\delta'}}|g(\omega,\xi,p)| \int_{-1}^{1} \frac{dt}{(1+t)^{\frac{1}{1+\delta'}}(1-t)^{\beta+\frac{1}{1+\delta'}}}.
\end{aligned}$$

Setting, for instance,

$$\frac{\delta'}{1+\delta'} = \frac{3\beta}{2},$$

we find

$$|A_{1,2}| \leq C|\xi'|^{\frac{3\beta}{2}}|g(\omega,\xi,p)| \int_{-1}^{1} \frac{dt}{(1+t)^{1-\frac{3\beta}{2}}(1-t)^{1-\frac{\beta}{2}}} \leq C|\xi'|^{\frac{3\beta}{2}}|g(\omega,\xi,p)|.$$

Hence, together with (4.20) with $\delta = \beta/(3-2\beta)$, we finally obtain an estimate for A_1 when $\beta \leq 1/2$:

$$|A_1| \leq C|\xi'|^{\frac{3\beta}{2}}|g(\omega,\xi,p)| \leq C(1+|\xi'|^{2\beta})|g(\omega,\xi,p)|, \qquad (4.22)$$

which is better than the naive estimate in $|\xi'|$. Recall that we already know that (4.22) holds when $\beta > 1/2$.

Let us consider now the term A_2. We split it as

$$A_2 = \int_{|p-q|>\sqrt{2}a} (\cdots)d\sigma(q) + \lim_{\eta\to 0} \int_{\sqrt{2}a \geq |p-q| > \sqrt{2\eta}} (\cdots)d\sigma(q) := A_{2,1} + A_{2,2},$$

where $a = |\xi'|^{-1}$. When $\beta > 1/2$, we have, after the change of variables $1-t \to a^2 t$:

$$|A_{2,1}| \leq C|\xi'||g(\omega,\xi,p)| \int_{|1-t|>a^2} \frac{dt}{(1-t)^{\beta+\frac{1}{2}}} \leq C|\xi'|^{2\beta}|g(\omega,\xi,p)|.$$

Note that the technical difficulty posed by the term y_ξ^{-1} in A_1 does not arise anymore in A_2 due to a cancellation, as the term

$$\hat{\xi} \cdot \hat{u} = y_\xi \hat{\xi}_\perp \cdot \hat{u} = y_\xi \tau$$

is in the numerator in the definition of A_2. Hence, we use (4.21) for some $\delta' > 0$ such that

$$\beta + \frac{1}{2} + \frac{1}{2(1+\delta')} > 1,$$

and obtain

$$|A_{2,1}| \leq C|\xi'|^{\frac{\delta'}{1+\delta'}}|g(\omega,\xi,p)| \int_{|1-t|>a^2} \frac{dt}{(1-t)^{\beta+\frac{1}{2}+\frac{1}{2(1+\delta')}}} \leq C|\xi'|^{2\beta}|g(\omega,\xi,p)|. \qquad (4.23)$$

Regarding $A_{2,2}$, we need to remove the singularity at $t=1$. We write

$$\begin{aligned} A_{2,2} &= ig(\omega,\xi,p) \lim_{\eta\to 0} \int_{\mathbb{S}^{d-1}} \int_{a^2\geq|1-t|>\eta} (\xi'\cdot\hat{u})^2 \frac{(1+t)^{\frac{d}{2}}}{2^{\beta+\frac{d}{2}}(1-t)^{\beta}} \\ &\qquad \times \frac{dt d\sigma(\hat{u})}{(1+i(\omega'+\xi'\cdot(tp+\sqrt{1-t^2}\hat{u})))(1+i(\omega'+\xi'\cdot tp))}. \end{aligned}$$

After the change of variables $1-t \to a^2 t$, it follows easily that

$$|A_{2,2}| \leq C|\xi'|^{2\beta}|g(\omega,\xi,p)|. \qquad (4.24)$$

Combining (4.24) with (4.22)-(4.23), we find

$$|\mathcal{R}_\beta g| \leq C(1+|\xi'|^{2\beta})|g|.$$

In order to conclude, let us introduce, as usual,

$$\begin{aligned} \mathcal{I}(\omega,\xi,k) &= \int_{\mathbb{S}^d} |\rho_\varepsilon(k\cdot p)||\mathcal{R}_\beta g(\omega,\xi,p)|^2 d\sigma(p) \\ \mathcal{J}(\hat{\xi}) &= \int_{\mathbb{S}^d}\int_{\mathbb{S}^d} \frac{|\hat{\underline{h}}(p)|^2|\rho_\varepsilon(k\cdot p)|}{(1-(k\cdot\hat{\xi})^2)^\alpha} d\sigma(k)d\sigma(p), \end{aligned}$$

where $\alpha = 1/2$ when $d \geq 3$, and $\alpha = 0$ when $d = 1,2$. The Cauchy-Schwarz inequality then yields

$$\left\|\mathcal{F}_1(\omega,\hat{\xi},\cdot)\right\|^2_{L^2(\mathbb{S}^d)} \leq \mathcal{J}(\hat{\xi}) \sup_{k\in\mathbb{S}^d} \left[(1-(k\cdot\hat{\xi}))^\alpha \mathcal{I}(\omega,\xi,k)\right].$$

Using (3.13) for the term $\mathcal{J}$, we finally get, using Lemma 3.3,

$$\begin{aligned} \left\|\mathcal{F}_1(\omega,\hat{\xi},\cdot)\right\|^2_{L^2(\mathbb{S}^d)} &\leq \frac{C(1+|\xi'|^{2\beta})^2\|\hat{\underline{h}}\|^2_{L^2}}{\varepsilon^{\frac{1}{2}}} \sup_{k\in\mathbb{S}^d}\left[(1-(k\cdot\hat{\xi}))^\alpha \int_{\mathbb{S}^d} |\rho_\varepsilon(k\cdot p)||g(\omega,\xi,p)|^2 d\sigma(p)\right] \\ &\leq \frac{C(1+|\xi'|^{2\beta})^2}{\varepsilon^{\frac{1}{2}}} (\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|))^2, \end{aligned}$$

which concludes the computation in the case $d \geq 3$.

The case $d \leq 2$ is more straightforward. We first write

$$\mathcal{R}_\beta g = \int_{|p-q|>\sqrt{2}a} (\cdots) d\sigma(q) + \lim_{\eta\to 0}\int_{\sqrt{2}a\geq|p-q|>\sqrt{2\eta}} (\cdots) d\sigma(q) := R_1 + R_2,$$

and split $\mathcal{I}$ into $\mathcal{I}=\mathcal{I}_1+\mathcal{I}_2$ accordingly. The Cauchy-Schwarz inequality, together with bounding $|\rho_\varepsilon|$ directly by $C/\varepsilon^{\frac{d}{2}}$, leads to

$$\begin{aligned}
\mathcal{I}_1 &\leq \int_{\mathbb{S}^d} |\rho_\varepsilon(k\cdot p)| \left(\int_{|p-\hat{w}|>\sqrt{2}a} \frac{|g(\omega,\xi,p)|^2+|g(\omega,\xi,\hat{w})|^2}{|p-\hat{w}|^{2\beta+d}} d\sigma(\hat{w})\right) \\
&\quad \times \left(\int_{|p-q|>\sqrt{2}a} \frac{d\sigma(q)}{|p-q|^{2\beta+d}}\right) d\sigma(p) \\
&\leq \frac{C|\xi'|^{2\beta}}{\varepsilon^{\frac{d}{2}}} \int_{\mathbb{S}^d}\int_{|p-\hat{w}|>\sqrt{2}a} \frac{|g(\omega,\xi,p)|^2+|g(\omega,\xi,\hat{w})|^2}{|p-\hat{w}|^{2\beta+d}} d\sigma(\hat{w})\sigma(p) \\
&\leq \frac{C|\xi'|^{4\beta}}{\varepsilon^{\frac{d}{2}}} \int_{\mathbb{S}^d} |g(\omega,\xi,p)|^2\sigma(p) \leq C|\xi'|^{4\beta}(\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|))^2.
\end{aligned}$$

Above, we used Lemma 3.2 for the last line. Regarding $\mathcal{I}_2$, and, therefore, R_2, we have when $d=2$,

$$R_2 = ig(\omega,\xi,p)\lim_{\eta\to 0}\int_{\mathbb{S}^1}\int_{\sqrt{2}a>|1-t|>\eta} \frac{[(t-1)\xi'\cdot p+\sqrt{1-t^2}\xi'\cdot u]dtd\sigma(u)}{(1+i(\omega'+\xi'\cdot(tp+\sqrt{1-t^2}u))(2(1-t))^{\beta+1}},$$

which yields, following similar lines as for the term $A_{2,2}$, the same estimate as (4.24). When $d=1$, the decomposition is slightly different, and we set $q\cdot p=\cos(\theta)=t$. When $\theta\in(0,\pi)$, we use

$$q = tp+\sqrt{1-t^2}p_\perp, \text{ with } p_\perp\in\mathbb{S}^1 \text{ and } p_\perp\cdot p=0,$$

while when $\theta\in(\pi,2\pi)$, we write

$$q = tp-\sqrt{1-t^2}p_\perp.$$

Depending on θ, the vector q can thus be written as $q=tp\pm\sqrt{1-t^2}p_\perp$. This gives for the term R_2:

$$R_2 = ig(\omega,\xi,p)\sum_{\pm}\lim_{\eta\to 0}\int_{\sqrt{2}a>|1-t|>\eta} \frac{[(t-1)\xi'\cdot p\pm\sqrt{1-t^2}\xi'\cdot p_\perp]}{(1+i(\omega'+\xi'\cdot(tp\pm\sqrt{1-t^2}p_\perp))} \frac{dt}{(1-t)^{\beta+\frac{1}{2}}(1+t)^{\frac{1}{2}}},$$

which also yields an estimate like (4.24) after the change of variables $1-t\to a^2t$. We are, therefore, done with the term $\mathcal{I}$. It remains to estimate $\mathcal{J}$ which follows from (3.13) with $\alpha=0$. This ends the proof of the proposition.

4.3 Proof of Proposition 3.6

The proof builds on the proof of Proposition 3.5. We will underline the main differences, and will not go into detail of similar computations. Since $\mathcal{A}$ is an interpolation term between $g\mathcal{R}_\beta\rho_\varepsilon$ and $\rho_\varepsilon\mathcal{R}_\beta g$, we decided, in order to minimize the calculations, to put all the weight on $\rho_\varepsilon\mathcal{R}_\beta g$. We will thus obtain a similar estimate as in Proposition 3.5. There is a technical condition for this, which is that $\rho_\varepsilon\mathcal{R}_\beta g$ is more "singular" than $g\mathcal{R}_\beta\rho_\varepsilon$, which translates into the inequality $\varepsilon(|\xi'|+|\omega'|)\geq 1$.

Let $a = (|\xi'| + |\omega'|)^{-1}$. We separate in the integral in the definition $\mathcal{A}$ the q such that $|p-q| > \sqrt{2}a$ from the ones such that $|p-q| < \sqrt{2}a$. We then split $\mathcal{A}$ accordingly into $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$:

$$\begin{aligned}\mathcal{A}(g, \rho_\varepsilon) &= \int_{|p-q|>\sqrt{2}a} (\cdots) d\sigma(q) + \lim_{\eta\to 0} \int_{\sqrt{2}a \geq |p-q| > \sqrt{2}\eta} (\cdots) d\sigma(q) \\ &:= \mathcal{A}_1 + \mathcal{A}_2,\end{aligned}$$

and write for $\mathcal{A}_1$,

$$\begin{aligned}\mathcal{A}_1 &= \rho_\varepsilon(k\cdot p) \int_{|p-q|>\sqrt{2}a} \frac{(g(\omega,\xi,p) - g(\omega,\xi,q))}{|p-q|^{2\beta+d}} d\sigma(q) \\ &\quad - \int_{|p-q|>\sqrt{2}a} \rho_\varepsilon(k\cdot q) \frac{(g(\omega,\xi,p) - g(\omega,\xi,q))}{|p-q|^{2\beta+d}} d\sigma(q) \\ &:= \mathcal{A}_{1,1} + \mathcal{A}_{1,2}.\end{aligned}$$

This leads to the following decomposition of $\mathcal{F}_2(\omega, \xi, k)$:

$$\mathcal{F}_2 = \mathcal{F}_{2,1,1} + \mathcal{F}_{2,1,2} + \mathcal{F}_{2,2}.$$

The term $\mathcal{A}_{1,1}$ is exactly $\rho_\varepsilon \mathcal{R}_\beta g$ and is treated in Proposition 3.5. We focus therefore first on $\mathcal{F}_{2,1,2}$, and secondly on $\mathcal{F}_{2,2}$, for both of which we use (4.16). Assume that $d \geq 3$. As usual, we obtain after from the Cauchy-Schwarz inequality

$$\|\mathcal{F}_{2,1,1}(\omega,\xi,\cdot)\|^2_{L^2(\mathbb{S}^d)} \leq \mathcal{J}_{2,1,2}(\hat{\xi}) \sup_{k\in\mathbb{S}^d} \left[(1-(k\cdot\hat{\xi}))^\alpha \mathcal{I}_{2,1,2}(\omega,\xi,k) \right],$$

where $\alpha = 1/2$ when $d \geq 3$ and $\alpha = 0$ otherwise, and where we have introduced

$$\mathcal{I}_{2,1,2}(\omega,\xi,k) = \int_{\mathbb{S}^d} \int_{|p-q|>\sqrt{2}a} \frac{|\rho_\varepsilon(k\cdot q)||g(\omega,\xi,q)|^{1+\delta}|g(\omega,\xi,p)|^2|\xi\cdot(p-q)|}{|p-q|^{2\beta+d}} d\sigma(p)d\sigma(q)$$

$$\mathcal{J}_{2,1,2}(\hat{\xi}) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \int_{|p-q|>\sqrt{2}a} \frac{|\underline{\hat{h}}(p)|^2|\rho_\varepsilon(k\cdot q)||g(\omega,\xi,q)|^{1-\delta}|\xi\cdot(p-q)|}{(1-(k\cdot\hat{\xi})^2)^\alpha|p-q|^{2\beta+d}} d\sigma(p)d\sigma(q)d\sigma(k).$$

The term $\mathcal{J}_{2,1,2}$ is controlled by

$$\mathcal{J}_{2,1,2}(\hat{\xi}) \leq \|\underline{\hat{h}}\|^2_{L^2} \sup_{q\in\mathbb{S}^d} \int_{\mathbb{S}^d} \frac{|\rho_\varepsilon(k\cdot q)|}{(1-(k\cdot\hat{\xi})^2)^\alpha} d\sigma(k) \sup_{p\in\mathbb{S}^d} \int_{|p-q|>\sqrt{2}a} \frac{|g(\omega,\xi,q)|^{1-\delta}|\xi\cdot(p-q)|}{|p-q|^{2\beta+d}} \sigma(q).$$

The term involving ρ_ε is treated as in (3.13) and is bounded by $C\varepsilon^{-\alpha}$. The one with g is treated in a similar way as the terms A_1 and $A_{2,1}$ in Proposition 3.5, only the definition of a and the power $1-\delta$ differ. We can therefore control this term by

$$C(1 + (|\xi'| + |\omega'|)^{2\beta}),$$

which eventually yields for $\mathcal{J}_{2,1,2}$:

$$\mathcal{J}_{2,1,2}(\hat{\xi}) \leq C\varepsilon^{-\alpha}(1 + (|\xi'| + |\omega'|)^{2\beta})\|\underline{\hat{h}}\|^2_{L^2}. \tag{4.25}$$

Concerning the term $\mathcal{I}_{2,1,2}$, we have

$$\mathcal{I}_{2,1,2}(\hat{\xi}) \leq \int_{\mathbb{S}^d} |\rho_\varepsilon(k\cdot q)||g(\omega,\xi,q)|^{1+\delta} d\sigma(q) \sup_{q\in\mathbb{S}^d} \int_{|p-q|>\sqrt{2}a} \frac{|g(\omega,\xi,p)|^2|\xi\cdot(p-q)|}{|p-q|^{2\beta+d}}\sigma(p).$$

The second term above is again very similar to A_1 and $A_{2,1}$ in Proposition 3.5, and can therefore be controlled by

$$C(1+(|\xi'|+|\omega'|)^{2\beta}).$$

The first term can be estimated with Lemma 3.3, which gives

$$\mathcal{I}_{2,1,2}(\hat{\xi}) \leq C(1+(|\xi'|+|\omega'|)^{2\beta})(\mathfrak{H}_d(\varepsilon,\lambda,|\omega|,|\xi|))^2. \tag{4.26}$$

When $d\leq 2$, we treat $\mathcal{A}_{2,1}$ following the lines of the term R_1 in proposition 3.5 and find that (4.25)-(4.26) hold with $\alpha=0$. The conclusion is that $\mathcal{F}_{2,1,1}+\mathcal{F}_{2,1,2}$ satisfies the estimate claimed in the proposition. Regarding $\mathcal{A}_2$, we have using (4.16), for any $d\geq 1$,

$$\mathcal{A}_2 = ig(\omega,\xi,p)\lim_{\eta\to 0}\int_{\sqrt{2}a\geq|p-q|>\sqrt{2\eta}} \frac{(\rho_\varepsilon(k\cdot p)-\rho_\varepsilon(k\cdot q))(\xi'\cdot(q-p))}{(1+i(\omega'+\xi'\cdot q))|p-q|^{2\beta+d}} d\sigma(q).$$

We use now the expression (3.2) of the mollifier ρ and write, with $t=k\cdot p$ and $s=k\cdot q$,

$$\begin{aligned}
|\rho_\varepsilon(t)-\rho_\varepsilon(s)| &= \varepsilon^{-\frac{d}{2}}\left|\sum_{i=0}^{N-1} a_i b_i^{\frac{2}{d}} e^{-\frac{1-t}{\varepsilon b_i}}(1-e^{\frac{s-t}{\varepsilon b_i}})\right| \leq \varepsilon^{-\frac{d}{2}-1}|t-s|\sum_{i=0}^{N-1} e^{\frac{\sqrt{2}}{b_i}}|a_i| b_i^{\frac{2}{d}-1} e^{-\frac{1-t}{\varepsilon b_i}} \\
&:= \varepsilon^{-1}|t-s|\varphi_\varepsilon(t).
\end{aligned}$$

We have used the fact that for

$$|p-q|\leq\sqrt{2}a=\sqrt{2}(|\xi'|+|\omega'|)^{-1} \text{ and } (|\xi'|+|\omega'|)\varepsilon\geq 1,$$

we have

$$\varepsilon^{-1}|k\cdot q-k\cdot p|\leq\sqrt{2}\varepsilon^{-1}(|\xi'|+|\omega'|)^{-1}\leq\sqrt{2},$$

and therefore

$$|1-e^{\frac{k\cdot q-k\cdot p}{\varepsilon b_i}}|\leq e^{\frac{\sqrt{2}}{b_i}}|k\cdot q-k\cdot p|/(\varepsilon b_i).$$

Using this in $\mathcal{A}_2$, and once more that $\varepsilon(|\xi'|+|\omega'|)\geq 1$, leads to

$$\begin{aligned}
|\mathcal{A}_2| &\leq C\varepsilon^{-1}(|\xi'|+|\omega'|)|g(\omega,\xi,p)|\varphi_\varepsilon(k\cdot p)\int_{\sqrt{2}a\geq|p-q|}|p-q|^{-(2\beta+d-2)}d\sigma(q) \\
&\leq C(|\xi'|+|\omega'|)^{2\beta}|g(\omega,\xi,p)|\varphi_\varepsilon(k\cdot p)|/(\varepsilon(|\xi'|+|\omega'|)) \\
&\leq C(|\xi'|+|\omega'|)^{2\beta}|g(\omega,\xi,p)|\varphi_\varepsilon(k\cdot p).
\end{aligned}$$

The conclusion then follows by proceeding as in the case $\beta=0$ and by using Lemma 3.3. This ends the proof.

4.4 Proof of Theorem 2.2

As in the proof of Theorem 2.1, it is enough to consider regular functions f and h. Let $u = \partial_x^\gamma f$, so that

$$\partial_t u + k \cdot \nabla u = (-\Delta_d)^\beta \partial_x^\gamma h,$$

and set

$$u_\varphi(t,x) = \int_{\mathbb{S}^d} u(t,x,k)\varphi(k)d\sigma(k).$$

It can be written, for any $\lambda > 0$, as

$$\hat{u}_\varphi(\omega,\xi) \;=\; \int_{\mathbb{S}^d} \frac{(-\Delta_d)^\beta (i\xi)^\gamma \hat{h}(\omega,\xi,k) + \lambda \hat{u}(\omega,\xi,k)}{\lambda + i(\omega + \xi \cdot p)} \varphi(k)d\sigma(k).$$

The proof of the theorem is direct due to what was previously done for Theorem 2.1: we introduce the operator $\mathcal{R}_{\beta_0}$, and observe that the leading term in $\hat{u}_\varphi(\omega,\xi)$ involves $\mathcal{R}_{\beta_0}\big((-\Delta_d)^{[\beta]}g\big)$. An easy variant of Proposition 3.5 shows that when $d \geq 2$,

$$\left| \int_{\mathbb{S}^d} \mathcal{Q}\hat{h}(\omega,\xi,k)\varphi(k)\mathcal{R}_{\beta_0}\big((-\Delta_d)^{[\beta]}g(\omega,\xi,p)\big)d\sigma(k) \right| \leq \frac{C|\xi|^{2\beta}\|\hat{h}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}}{\lambda^{2\beta+\frac{1}{2}}(|\omega|+|\xi|)^{\frac{1}{2}}}.$$

This leads to, for large $|\xi|$ and $d \geq 2$,

$$|\hat{u}_\varphi(\omega,\xi)| \leq C\frac{|\xi|^{2\beta+\gamma-\frac{1}{2}}}{\lambda^{\frac{1}{2}+2\beta}}\|\hat{h}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)} + C\frac{\lambda^{\frac{1}{2}}}{|\xi|^{\frac{1}{2}}}\|\hat{u}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^d)}.$$

Setting $\lambda = |\xi|^{\frac{2\beta+\gamma}{1+2\beta}}$ yields the desired result. When $d = 1$, we rather find

$$|\hat{u}_\varphi(\omega,\xi)| \leq C\frac{|\xi|^{2\beta+\gamma-\frac{1}{4}}}{\lambda^{\frac{3}{4}+2\beta}}\|\hat{h}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^1)} + C\frac{\lambda^{\frac{1}{4}}}{|\xi|^{\frac{1}{4}}}\|\hat{u}(\omega,\xi,\cdot)\|_{L^2(\mathbb{S}^1)},$$

and conclude by setting the same λ as when $d \geq 2$. The estimate for the time derivative follows in a similar manner.

References

[1] R. Alexandre, *A review of Boltzmann equation with singular kernels*, Kinet. Relat. Models, 2 (2009), pp. 551–646.

[2] R. Alexandre, *Fractional order kinetic equations and hypoellipticity*, Anal. Appl. (Singap.), 10 (2012), pp. 237–247.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, *Uncertainty principle and kinetic equations*, J. Funct. Anal., 255 (2008), pp. 2013–2066.

[4] ——, *Global existence and full regularity of the Boltzmann equation without angular cutoff*, Comm. Math. Phys., 304 (2011), pp. 513–581.

[5] R. Alexandre and C. Villani, *On the Boltzmann equation for long-range interactions*, Comm. Pure Appl. Math., 55 (2002), pp. 30–70.

[6] K. Atkinson and W. Han, *Spherical harmonics and approximations on the unit sphere: an introduction*, vol. 2044 of Lecture Notes in Mathematics, Springer, Heidelberg, 2012.

[7] G. Bal, T. Komorowski, and L. Ryzhik, *Kinetic limits for waves in a random medium*, Kinet. Relat. Models, 3 (2010), pp. 529–644.

[8] G. Bal, T. Komorowski, and L. Ryzhik, *Asymptotics of the phase of the solutions of the random schrödinger equation*, ARMA, (2011), pp. 13–64.

[9] F. Bouchut, *Hypoelliptic regularity in kinetic equations*, J. Math. Pures Appl. (9), 81 (2002), pp. 1135–1159.

[10] N. Bournaveas and S. Gutiérrez, *On the regularity of averages over spheres for kinetic transport equations in hyperbolic Sobolev spaces*, Rev. Mat. Iberoam., 23 (2007), pp. 481–512.

[11] ——, *Averages over spheres for kinetic transport equations with velocity derivatives in the right-hand side*, SIAM J. Math. Anal., 40 (2008), pp. 653–674.

[12] N. Bournaveas and B. Perthame, *Averages over spheres for kinetic transport equations; hyperbolic Sobolev spaces and Strichartz inequalities*, J. Math. Pures Appl. (9), 80 (2001), pp. 517–534.

[13] N. Bournaveas and H. Wang, *Velocity averaging lemmas in hyperbolic Sobolev spaces for the kinetic transport equation with velocity field on the sphere*, NoDEA Nonlinear Differential Equations Appl., 16 (2009), pp. 131–142.

[14] L. Erdös and H. T. Yau, *Linear Boltzmann equation as scaling limit of quantum Lorentz gas*, Advances in Differential Equations and Mathematical Physics. Contemporary Mathematics, 217 (1998), pp. 137–155.

[15] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math., 50 (1997), pp. 323–380.

[16] C. Gomez, *Radiative transport limit for the random Schrödinger equation with long-range correlations*, J. Math. Pures Appl. (9), 98 (2012), pp. 295–327.

[17] C. Gomez, O. Pinaud, and L. Ryzhik, *Radiative transfer with long-range interactions: regularity and asymptotics*, in preparation.

[18] P. T. Gressman and R. M. Strain, *Global classical solutions of the Boltzmann equation with long-range interactions*, Proc. Natl. Acad. Sci. USA, 107 (2010), pp. 5744–5749.

[19] P. T. Gressman and R. M. Strain, *Global classical solutions of the Boltzmann equation without angular cut-off*, J. Amer. Math. Soc., 24 (2011), pp. 771–847.

[20] N. Lerner, Y. Morimoto, and K. Pravda-Starov, *Hypoelliptic estimates for a linear model of the Boltzmann equation without angular cutoff*, Comm. Partial Differential Equations, 37 (2012), pp. 234–284.

[21] P.-L. Lions and T. Paul, *Sur les mesures de Wigner*, Rev. Mat. Iberoamericana, 9 (1993), pp. 553–618.

[22] J. Lukkarinen and H. Spohn, *Kinetic limit for wave propagation in a random medium*, Arch. Ration. Mech. Anal., 183 (2007), pp. 93–162.

[23] A. A. Markov, *On a problem of D. I. Mendeleev*, Zap. Imp. Akad. Nauk (Russian), St. Petersburg, 62 (1889), pp. 1–24.

[24] S. Samko, *On inversion of fractional spherical potentials by spherical hypersingular operators*, in Singular integral operators, factorization and applications, vol. 142 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2003, pp. 357–368.

[25] H. Spohn, *Derivation of the transport equation for electrons moving through random impurities*, J. Stat. Phys., 17 (1977), pp. 385–412.

[26] F. Tappert, *The parabolic approximation method, Lecture notes in physics, vol. 70, Wave Propagation and Underwater Acoustics*, Ed. J.B. Keller and J.S. Papadakis, Springer-Verlag, pp. 224-287, 1977.

[27] C. Villani, *A review of mathematical topics in collisional kinetic theory*, in Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 2002, pp. 71–305.

[28] E. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev. 40, (1932), pp. 742–749.