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Christophe GOMEZ

PROPAGATION ET RETOURNEMENT TEMPOREL DES ONDES DANS DES GUIDES D'ONDES ALÉATOIRES

Thèse dirigée par **Josselin GARNIER**

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M.	**Habib**	**AMMARI**	Ecole Polytechnique
M.	**Mark**	**ASCH**	Université de Picardie
M.	**Francis**	**COMETS**	Université Paris 7
M.	**Josselin**	**GARNIER**	Université Paris 7
M.	**Stefano**	**OLLA**	Université Paris 9 Dauphine
Mme.	**Marie**	**POSTEL**	Université Paris 6
M.	**Julien**	**DE ROSNY**	ESPCI

au vu des rapports de :

M.	**Mark**	**ASCH**	Université de Picardie
M.	**Lenya**	**RYZHIK**	Université de Stanford

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Chapter 1

Introduction

Le travail présenté dans cette thèse porte sur la propagation et le retournement temporel des ondes dans des guides d'ondes aléatoirement perturbés. Dans cette introduction, nous rappellerons dans un premier temps quelques aspects de la propagation d'ondes en milieu aléatoire, et dans un second temps nous présenterons un résumé des résultats exposés dans les trois principaux chapitres de ce manuscrit.

1.1 Propagation des ondes en milieux aléatoires

L'étude mathématique de la propagation d'ondes en milieu complexe est indispensable à la compréhension de certains phénomènes observés expérimentalement et au développement de nouvelles applications. La description exacte des perturbations présentes dans un milieu n'étant quasiment jamais possible, il devient très difficile de pouvoir résoudre certains problèmes de manière analytique ou numérique. En générale, on ne dispose que d'une description statistique des milieux complexes. Ainsi, l'approche consistant à considérer un milieu inhomogène comme aléatoirement perturbé apparaît beaucoup mieux adaptée, et permet une description statistique des effets produits sur des ondes se propageant dans un tel milieu.

Les perturbations d'un milieu peuvent avoir différentes origines : la présence d'impuretés, des différences de salinités, des imperfections géométriques, etc. Généralement, ces perturbations sont petites. Cependant, sur de longues distances de propagation, l'effet cumulé de ces imperfections peut devenir significatif.

A l'aide de considérations physiques, il est possible d'appréhender les échelles caractéristiques d'un problème et d'émettre des hypothèses sur leurs ordres de grandeurs. L'intérêt est de mettre en évidence les rapports d'échelles qui peuvent mener à des régimes asymptotiques remarquables. Cette technique de séparation d'échelles introduite par G. Papanicolaou *et al* [5], permet de développer une analyse asymptotique basée sur des théorèmes limites de solutions d'équations différentielles à coefficients aléatoires [48, 5, 25, 50, 33]. Le principal travail est d'arriver à caractériser les quantités physiques intéressantes à l'aide d'équations effectives, afin de pouvoir décrire l'allure des ondes après qu'elles se soient propagées dans le milieu aléatoire.

Les échelles de longueur que nous considérerons dans cette thèse, et qui sont aussi largement utilisées dans la littérature [5, 25, 9] sont : la longueur de propagation dans le milieu inhomogène, la longueur d'onde typique d'une onde, et les longueurs de corrélations spatiales des inhomogénéités. Nous considérons aussi l'amplitude des inhomogénéités, ainsi que la largeur du spectre de la source.

Les phénomènes de propagation d'ondes en milieux aléatoires, à travers la séparation d'échelles, ont été largement étudiés que se soit pour des milieux de dimension 1 (voir [5, 25] et leurs références), de dimension 3 stratifiés (voir [25] et ses références) ou en considérant l'approximation parabolique de l'équation des ondes [7, 8, 9, 13]. Pour des milieux de dimension 1, le phénomène de localisation des ondes, observé en premier par P.W. Anderson [4], a lieu même si les inhomogénéités sont faibles [32]. Pour les milieux stratifiés cela se passe essentiellement comme en dimension 1. En dimension 3 sous l'approximation parabolique, l'amplitude de l'onde cohérente décroît avec la distance de propagation et son énergie se transforme en fluctuations incohérentes. En revanche, l'énergie moyenne se propage de manière diffusive ou par transfert radiatif. Pour l'étude de ces phénomènes d'un point de vue physique, on peut se référer à [34].

Les modèles aléatoires de dimension 1 ne possèdent qu'un axe de propagation et aucune diversité spatiale, ce qui limite le réalisme de ces modèles. Les modèles ouverts de dimension 2 ou 3 possèdent une diversité spatiale mais n'ont pas de direction de propagation privilégiée, ce qui rend l'utilisation des outils de calcul stochastique difficile. Les modèles de guides d'ondes aléatoires sont donc à la fois physiquement pertinents et mathématiquement traitables.

En outre, l'étude de la propagation dans les guides d'ondes aléatoires est devenue indispensable face au grand nombre de situations pouvant se modéliser de cette manière : comme par exemple en télécommunication, en acoustique sous-marine ou en géophysique. Contrairement aux modèles de propagation de dimension un, ainsi qu'aux modèles ouverts de dimension supérieure, les guides d'ondes possèdent une diversité spatiale ainsi qu'un axe de propagation privilégié, ce qui permet aux ondes de ce propager sur de très longues distances (voir Figure 1.2). Il s'agit donc d'une situation intermédiaire qui permet de modéliser des phénomènes spatiaux échappant aux modèles ouverts.

Dans un guide d'onde idéal, la structure géométrique peut avoir une forme très générale. Les paramètres du milieu peuvent, eux aussi, avoir une forme générale mais restent constants le long de l'axe du guide d'onde. Il y a deux types de guide d'onde idéal : ceux qui entourent une région homogène avec des conditions aux bords entraînant le confinement des ondes, et ceux dont le confinement est assuré par les variations transverses de l'indice de réfraction. Dans les guides d'ondes, il y a deux types de dispersion. Tout d'abord, il y a la dispersion modale. Les modes propagatifs voyagent à travers le guide d'onde à des vitesses différentes, ce qui provoque un étalement de l'onde dans le temps et l'espace. Ensuite, les nombres d'ondes modaux ne sont pas linéaires par rapport à la fréquence, ce qui implique une dispersion supplémentaire liée au spectre des fréquences de l'onde [30, 25]. L'étude des phénomènes de propagation d'ondes dans des guides d'ondes aléatoires a fait l'objet de multiples études [38, 44, 54, 52, 39, 30, 25, 31, 29], dans lesquelles l'analyse du couplage des modes produit par les inhomogénéités du milieu est le point central.

L'amplitude de l'onde cohérente se propageant dans un guide d'onde décroît avec la distance de propagation, et se transforme alors en fluctuations incohérentes. Pour des ondes monochromatiques ou de spectre de fréquences étroit, l'énergie, par contre, se propage diffusivement [30, 25]. En revanche, dans le cas d'ondes à spectre de fréquences large, comme dans le cas des milieux de dimension 1, l'amplitude de l'onde cohérente décroît avec la distance de propagation. Cependant, dans ce cas, l'énergie est déterministe et décroît aussi avec la distance de propagation. L'énergie se transforme alors en fluctuations incohérentes de faible amplitude [25].

Le concept de retournement temporel des ondes a été introduit par M. Fink. Le principe de l'expérience de retournement temporel se compose de deux étapes. Dans un premier temps (voir Figure 1.1 (a)), une source émet un signal. Une onde se propage dans le milieu et est enregistrée par le miroir à retournement temporel. Un miroir à retournement temporel se

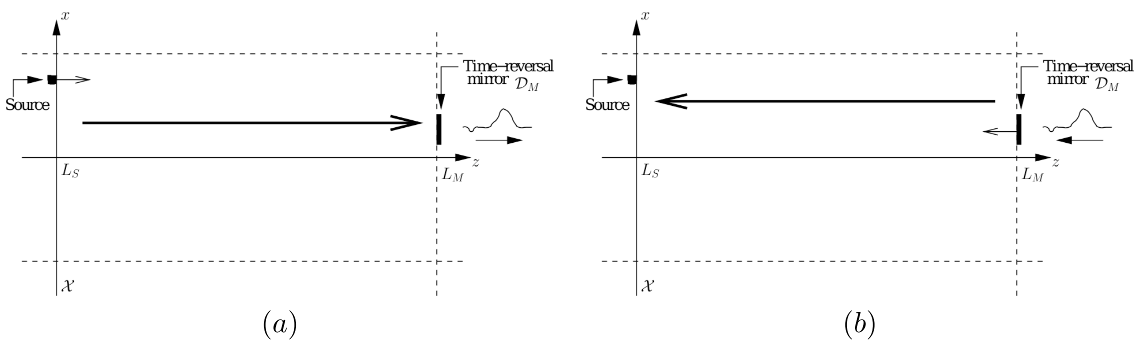


Figure 1.1: Illustration de l'expérience de retournement temporel dans un guide d'onde d'axe de propagation z et de section transverse $\mathcal{X}$. Une source est placée dans le plan $z = L_S$ et $\mathcal{D}_M \subset \mathcal{X}$ représente le miroir à retournement temporel placé dans le plan $z = L_M$. La figure (a) représente la première étape de l'expérience. Une source émet un signal, l'onde se propage dans le milieu et est enregistrée par le miroir à retournement temporel. Ce miroir retourne en temps le signal enregistré. La figure (b) représente la deuxième étape de l'expérience. Le miroir à retournement temporel réémet le signal retourné en temps dans le milieu en direction inverse. Ce qui a été enregistré en dernier repart en premier. L'onde émise par le miroir se propage en sens inverse et a la propriété de refocaliser au voisinage de la source d'origine.

compose de matrices de transducteurs piézoélectriques ayant la capacité de recevoir un signal, de l'enregistrer et de réémettre ce signal renversé en temps. Ce miroir retourne alors en temps le signal enregistré et le réémet dans le milieu en direction inverse. Ce qui a été enregistré en dernier repart en premier. Dans un second temps (voir Figure 1.1 (a)), l'onde émise par le miroir à retournement temporel se propage en sens inverse et a la propriété de refocaliser au voisinage de la source d'origine. De plus, des études théoriques et expérimentales [19, 22, 42] ont montré que les imperfections présentes dans le milieu améliorent la refocalisation de l'onde retournée en temps. Des recherches sur ce sujet ainsi que des applications sont présentées dans [21]. Des expériences de retournement temporel ont aussi été effectuées en mer à l'aide de réseaux de sonars par W. Kuperman et son équipe de San Diego [40, 57].

Les propriétés de refocalisation en milieu inhomogène du procédé de retournement temporel permettent de multiple applications, comme par exemple : la détection en contrôle non destructif, délivrer de l'énergie sur des petites cibles en lithotritie (destruction des calculs rénaux), la réduction des interférences en télécommunication sans fil.

Une étude mathématique est indispensable à la compréhension des phénomènes de refocalisation lors de l'expérience de retournement temporel et pour le développement de nouvelles applications. Les phénomènes de refocalisation ont été étudiés dans différents contextes : dans des milieux inhomogènes de dimension 1 [18, 25], de dimension 3 stratifiés aléatoirement [26], de dimension 3 sous l'approximation parabolique [15, 10, 49], ainsi que dans les guides d'ondes aléatoires [30, 25]. Dans les cas multidimensionnels, la taille de la tache focale principale, obtenue avec des milieux aléatoires est plus petite que la formule de Rayleigh $\lambda L/D$ (où λ est la longueur d'onde principale, L est la distance de propagation et D le diamètre du miroir), qui donne la taille de la tache focale principale obtenue dans un milieu homogène. La tache focale obtenue en milieu homogène a typiquement la forme d'un sinus cardinal. Les inhomogénéités du milieu permettent aussi la suppression des lobes latéraux. M. Fink et son groupe de l'ESPCI ont même proposé un dispositif avec un miroir à retournement temporel en champ lointain permettant de refocaliser sous la limite de diffraction $\lambda/2$ (où λ est la longueur d'onde principale). Ce dispositif consiste à ajouter un "peigne" de diffuseurs proche de la source.

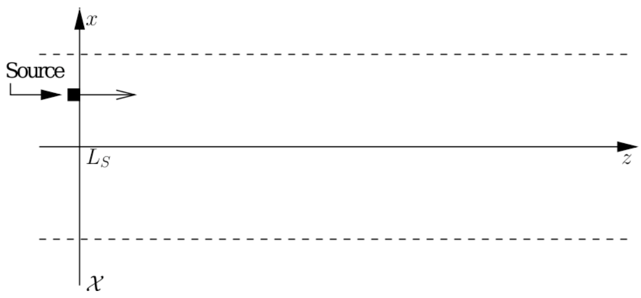


Figure 1.2: Illustration of a planar waveguide, with propagation axis in the z-direction, cross-section $\mathcal{X}$, and a source localized in the plane $z = L_S$.

La stabilité statistique est une autre propriété importante du retournement temporel des ondes. L'onde refocalisée ne dépend pas de la réalisation particulière du milieu. Dans le cas d'ondes à large bande de fréquences, la stabilité statistique a été étudiée dans différents contextes : dans des milieux de dimension 1 [18, 25], de dimension 3 aléatoirement stratifiés [24, 25], de dimension 3 sous l'approximation parabolique [15, 49], ainsi que dans les guides d'ondes [30, 25].

Les phénomènes intéressants, produits par le retournement temporel des ondes, sont intiment liés au fait de faire revivre, à l'onde enregistrée par le miroir à retournement temporel, sa "vie passée". Quelles sont alors les conséquences sur l'onde refocalisée, lorsqu'elle n'a pas exactement revécu sa "vie passée"? La question de l'influence d'un milieu changeant entre les deux étapes de l'expérience de retournement temporel a été étudiée dans plusieurs contextes : dans des milieux de dimension 1 [3], de dimension 3 sous l'approximation parabolique [12, 11]. Pour des milieux de dimension 1, l'influence du milieu entre les deux étapes de l'expérience entraîne une perte de stabilité statistique de l'onde refocalisée reliée au degré de corrélation des deux réalisations du milieu. Dans des milieux de dimension 3 sous l'approximation parabolique [12, 11], l'onde refocalisée reste statistiquement stable contrairement aux cas unidimensionnels.

1.2 Presentation of the results

This section is an overview of the main results obtained in this thesis and presented in detail in the three following chapters. This presentation is in two parts. In the first part, we present the results obtained about wave propagation in random waveguides. In the second part, we present the results about time reversal of waves in random waveguides.

The present thesis is devoted to the study of the wave propagation and time reversal in randomly perturbed waveguides. However, throughout this manuscript, for the sake of simplicity, we consider planar waveguides. In this case a waveguide has a propagation axis with coordinate $z \in \mathbb{R}$, and a transverse section $\mathcal{X}$ which is an interval with coordinate $x \in \mathcal{X}$ (see Figure 1.2). Furthermore, the analysis developed in this manuscript can be extended to more general waveguides.

We consider acoustic wave propagation using the linearized equations of momentum and mass conservation for the pressure p and the velocity $\mathbf{u}$:

$$\rho(x,z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F},$$
$$\frac{1}{K(x,z)}\frac{\partial p}{\partial t} + \nabla . \mathbf{u} = 0,$$

where ρ is the density of the medium, and K is the bulk modulus. The source is modeled by the forcing term

$$\mathbf{F}(t,x,z) = \Psi(t,x)\delta(z - L_S)\mathbf{e}_z,$$

which is a source that emits a signal in the z-direction and localized in the plane $z = L_S$. $\Psi(t,x)$ represents the profile of the source and $\mathbf{e}_z$ is the unit vector pointing in the z-direction.

In this thesis, we are interested in phenomena which occur when the propagation distance L_0 is large compared to the typical wavelength of the source λ_0, and large compared to the length scales $l_{x,c}$ and $l_{z,c}$, which are the correlation lengths of the random perturbations in the transverse and longitudinal directions. Moreover, the typical amplitude σ of the random perturbations of the medium parameters is small, and in Chapters 2 and 3 we consider the case where the orders of $l_{x,c}$, $l_{z,c}$, and λ_0 are comparable. More precisely, we consider the regime in which

$$\frac{L_0}{\lambda_0} \gg 1, \quad \frac{l_{x,c}}{\lambda_0} \sim \frac{l_{z,c}}{\lambda_0} \sim 1, \quad \text{and} \quad \sigma \ll 1.$$

In the terminology of [25] this regime corresponds to the so-called weakly heterogeneous regime. Let $0 < \epsilon \ll 1$ be the ratio of λ_0 to the propagation distance. Then, we consider

$$L_0 = \frac{L}{\epsilon}, \quad \lambda_0 \sim l_{x,c} \sim l_{z,c} \sim 1 \quad \text{and} \quad \sigma = \sqrt{\epsilon}.$$

The scaling used in Chapter 4 is somewhat different, and it is described below.

1.2.1 Wave Propagation

Results of Chapter 2 First of all, we are interested in the wave propagation in a shallow-water acoustic waveguide model. The waveguide model that we consider can also be used for electromagnetic wave propagation in dielectric waveguides and optical fibers [43, 44, 52, 54, 62]. In shallow-water waveguides the transverse section $\mathcal{X}$ can be considered as being the semi-infinite interval $[0 + \infty)$. In this context, we assume that the medium parameters are given by

$$\frac{1}{K(x,z)} = \begin{cases} \frac{1}{\bar{K}}\left(n^2(x) + \sqrt{\epsilon}V(x,z)\right) & \text{if} \quad x \in [0,d], \quad z \in [0, L/\epsilon] \\ \frac{1}{\bar{K}}n^2(x) & \text{if} \quad \begin{cases} x \in [0,+\infty),\ z \in (-\infty,0) \cup (L/\epsilon,+\infty) \\ \text{or} \\ x \in (d,+\infty),\ z \in (-\infty,+\infty), \end{cases} \end{cases}$$

$$\rho(x,z) = \bar{\rho} \quad \text{if} \quad x \in [0,+\infty),\ z \in \mathbb{R},$$

and where the index of refraction $n(x)$ is given by

$$n(x) = \begin{cases} n_1 > 1 & \text{if} \quad x \in [0,d) \\ 1 & \text{if} \quad x \in [d,+\infty). \end{cases}$$

See Figure 1.3 for an illustration of this model. Here, $n(x)$ correspond to the Pekeris waveguide model with ocean depths d, and the random process $V(x,z)$ models the spatial inhomogeneities. Throughout this manuscript the process V is a continuous real-valued zero-mean Gaussian field with a covariance function given by

$$\mathbb{E}\left[V(x,t)V(y,s)\right] = \gamma_0(x,y)e^{-a|t-s|} \quad \forall(x,y) \in [0,d]^2 \text{ and } \forall(s,t) \in [0,+\infty)^2.$$

Here $a > 0$ and $\gamma_0 : [0,d] \times [0,d] \to \mathbb{R}$ is a function which is the kernel of a nonnegative operator. The properties of the random process V are described in Section 2.6.1.

In underwater acoustics the density of air is very small compared to the density of water, then it is natural to use a pressure-release boundary condition. The pressure is very weak

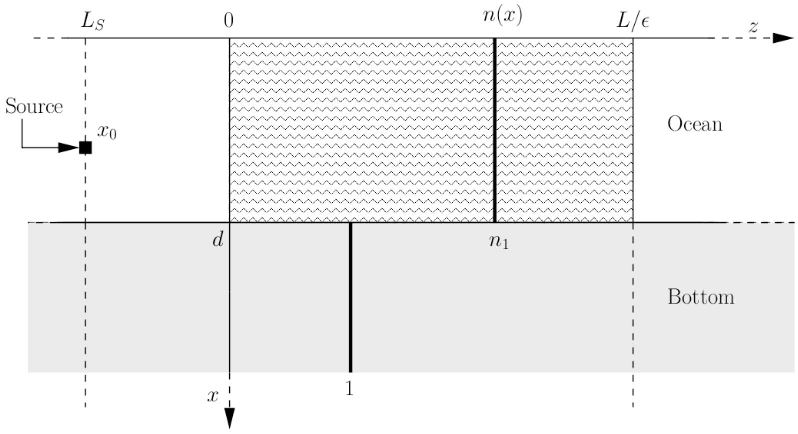


Figure 1.3: Illustration of the shallow-water random waveguide model with propagation axis in the z-direction, transverse section $[0, +\infty)$, and ocean depth d. The transverse index of refraction is the Pekeris profile $n(x)$, and a perturbed section is localized in the ocean section $[0, d]$ between the plane $z = 0$ and the plane $z = L/\epsilon$.

outside the waveguide, and by continuity, the pressure is zero at the free surface $x = 0$. This consideration leads us to the Dirichlet boundary conditions:

$$p(t, 0, z) = 0 \quad \forall (t, z) \in [0, +\infty) \times \mathbb{R}.$$

As we study linear models of propagation the pressure $p(t, x, z)$ can be expressed as the superposition of monochromatic waves by taking its Fourier transform:

$$\widehat{p}(\omega, x, z) = \int p(t, x, z) e^{i\omega t} dt.$$

With such a model, a wave field can be decomposed into three kinds of modes:

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega, z)\phi_j(\omega, x) + \int_{-\infty}^{k^2(\omega)} \widehat{p}_\gamma(\omega, z)\phi_\gamma(\omega, x) d\gamma,$$

where $(\phi_s(\omega, .))_{s \in \{1,\dots,N(\omega)\} \cup (-\infty, k^2(\omega))}$ is a basis of the Hilbert space $L^2(0, +\infty)$ defined in Section 2.2.1. We have $N(\omega)$ discrete propagating modes which propagate over long distances, a continuum $(-\infty, 0)$ of evanescent modes which decrease exponentially with the propagation distance, and a continuum $(0, k^2(\omega))$ of radiating modes representing modes which penetrate under the bottom of the water. Here, $k(\omega) = \omega/c$ is the wavenumber and $c = \sqrt{\bar{K}/\bar{\rho}}$ is the effective sound speed of the medium.

In this chapter, we essentially revisit in detail the paper of W. Kohler and G. Papanicolaou [39], but we take into account the three kinds of modes (see Sections 2.3).

According to the modal decomposition, we consider the profile $\Psi(t, x)$ given in the frequency domain by

$$\widehat{\Psi}(\omega, x) = \widehat{f}(\omega) \left[\sum_{j=1}^{N(\omega)} \phi_j(\omega, x_0)\phi_j(\omega, x) + \int_{(-S, -\xi) \cup (\xi, k^2(\omega))} \phi_\gamma(\omega, x_0)\phi_\gamma(\omega, x) d\gamma \right],$$

where $x_0 \in (0, d)$. The bound S in the spectral decomposition of the source profile is introduced to have $\widehat{\Psi}(\omega, .) \in L^2(0, +\infty)$, and $\xi > 0$ is introduced for technical reasons. Note that S can be arbitrarily large and ξ can be arbitrarily small. Therefore, the spatial profile of the source is an approximation of a Dirac distribution at x_0, which models a point source at x_0. We consider solutions of the form

$$\widehat{p}(\omega,x,z)=\sum_{j=1}^{N(\omega)}\widehat{p}_j(\omega,z)\phi_j(\omega,x)+\int_{(-\infty,-\xi)\cup(\xi,k^2(\omega))}\widehat{p}_\gamma(\omega,z)\phi_\gamma(\omega,x)d\gamma$$

for technical reasons and this assumption leads us to simplified algebra. In such a decomposition, the radiating and the evanescent modes are separated by the small band $(-\xi,\xi)$ with $\xi\ll 1$. The goal is to isolate the transition mode 0 between the radiating and the evanescent modes in the continuum of modes $(-\infty,k^2(\omega))$. Moreover, we assume that $\epsilon\ll\xi$ and therefore we have two distinct scales. Let us remark that in Chapters 2 and 3, we consider in a first step the asymptotic ϵ goes to 0 and in a second step the asymptotic ξ goes to 0.

Throughout this manuscript, we consider the forward scattering approximation discussed in Section 2.3.4, and which is widely used in underwater acoustics and in fiber optics. In this approximation the coupling between forward- and backward-propagating modes is assumed to be negligible compared to the coupling between the forward-propagating modes. After a long propagation distance the pressure field is essentially of the form

$$\begin{aligned}\widehat{p}\Big(\omega,x,\frac{L}{\epsilon}\Big)\underset{\epsilon\ll 1}{\simeq}&\sum_{j=1}^{N(\omega)}\frac{\mathbf{T}^{\xi,\epsilon}_j(\omega,L)(\widehat{a}_0(\omega))}{\sqrt{\beta_j(\omega)}}e^{i\beta_j(\omega)\frac{L}{\epsilon}}\phi_j(\omega,x)\\&+\int_\xi^{k^2(\omega)}\frac{\mathbf{T}^{\xi,\epsilon}_\gamma(\omega,L)(\widehat{a}_0(\omega))}{\gamma^{1/4}}e^{i\sqrt{\gamma}\frac{L}{\epsilon}}\phi_\gamma(\omega,x)d\gamma,\end{aligned}$$

where $\beta_j(\omega)$ are the modal wavenumbers. Here, $\mathbf{T}^{\xi,\epsilon}(\omega,L)$ is the transfer operator, from $\mathbb{C}^{N(\omega)}\times L^2(\xi,k^2(\omega))$ to itself, solution of a differential equation with random coefficients of the form

$$\frac{d}{dz}\mathbf{T}^{\xi,\epsilon}(\omega,z)=\frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^{\xi,\epsilon}(\omega,z)+\mathbf{G}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^{\xi,\epsilon}(\omega,z)$$

with $\mathbf{T}^{\xi,\epsilon}(\omega,0)=Id$. The transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega,L)$ describes the coupling between the three kinds of modes. $\mathbf{H}^{aa}$, defined by (2.30)-(2.33) page 45, describes the coupling between the propagating and radiating modes with themselves, while $\mathbf{G}^{aa}$, defined by (2.34)-(2.37) page 45, describes the coupling between the evanescent modes with the propagating and radiating modes. Moreover, the asymptotic behavior of $\mathbf{T}^{\xi,\epsilon}(\omega,L)$, as $\epsilon\to 0$ in first and $\xi\to 0$ in second, is described precisely in Section 2.4.1, and can be described in terms of a diffusion process with an infinitesimal generator which can be split into three parts and depends only on the $N(\omega)$-discrete propagating modes:

$$\mathcal{L}^\omega_1+\mathcal{L}^\omega_2+\mathcal{L}^\omega_3.$$

The first operator $\mathcal{L}^\omega_1$ describes the coupling between the $N(\omega)$-propagating modes. This part is of the form of the infinitesimal generator obtained in [25, 30], from which the total energy is conserved. The second operator $\mathcal{L}^\omega_2$ describes the coupling between the propagating modes with the radiating modes. This part implies a mode-dependent and frequency-dependant attenuation on the $N(\omega)$-propagating modes, and a mode-dependent and frequency-dependent phase modulation. The third operator $\mathcal{L}^\omega_3$ describes the coupling between the propagating and the evanescent modes, and implies a mode-dependent and frequency-dependent phase modulation. The frequency-dependent phase modulation does not remove energy from the propagating modes but gives an effective dispersion.

Then, in Section 2.5, we are interested in the study of the asymptotic mean mode powers of the propagating modes

$$\mathcal{T}^l_j(\omega,L)=\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[\big|\mathbf{T}^{\xi,\epsilon}_j(\omega,L)(y^l)\big|^2\Big],$$

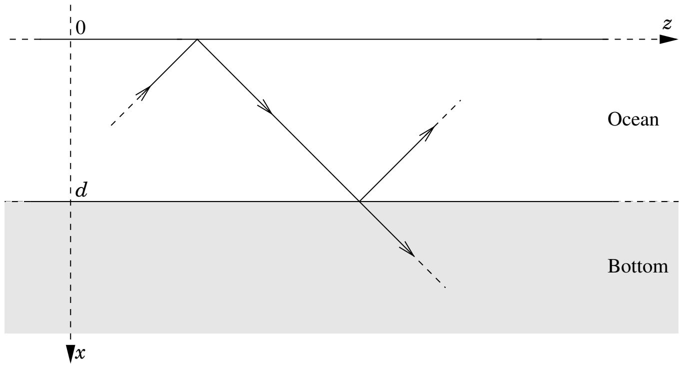


Figure 1.4: Illustration of the radiative loss in the shallow-water random waveguide model with propagation axis in the z-direction, transverse section $[0,+\infty)$, and ocean depth d.

where $y^l_j = \delta_{jl}$, and $y^l_\gamma = 0$ for $\gamma \in (0, k^2(\omega))$. The initial condition y^l means that an impulse equal to one charges only the lth propagating mode. $\mathcal{T}^l_j(\omega, L)$ is the expected power of the jth propagating mode at the propagation distance $z = L$, when at $z = 0$ the energy is concentrated on the lth propagating mode. The expected powers $\mathcal{T}^l_j(\omega, L)$ are solution of the following coupled power equations:

$$\frac{d}{dz}\mathcal{T}^l_j(\omega, z) = -\Lambda^c_j(\omega)\mathcal{T}^l_j(\omega, z) + \sum_{n=1}^{N(\omega)} \Gamma^c_{nj}(\omega)\left(\mathcal{T}^l_n(\omega, z) - \mathcal{T}^l_j(\omega, z)\right),$$

with initial conditions $\mathcal{T}^l_j(\omega, 0) = \delta_{jl}$, and where $\Gamma^c_{jl}(\omega)$ is defined in Theorem 2.1 page 51. These equations describe the transfer of energy between the propagating modes and $\Gamma^c(\omega)$ is the energy transport matrix. The initial condition means that an impulse equal to one charges only the lth propagating mode. In our context, we have the coefficients $\Lambda^c_j(\omega)$ given by the coupling between the propagating modes with the radiating modes. These coefficients, defined in Theorem 2.2 page 52, are responsible for the radiative loss of energy in the ocean bottom (see Figure 1.4). This loss of energy is described more precisely by the following result of Section 2.5.1.

Theorem *Let us assume that the energy transport matrix $\Gamma^c(\omega)$ is irreducible. Then, we have $\forall l \in \{1, \dots, N(\omega)\}$*

$$\lim_{L\to+\infty} \frac{1}{L} \ln\left[\sum_{j=1}^{N(\omega)} \mathcal{T}^l_j(\omega, L)\right] = -\Lambda_\infty(\omega)$$

with

$$\Lambda_\infty(\omega) = \inf_{X\in\mathcal{S}^{N(\omega)}_+} \left\langle\big(-\Gamma^c(\omega) + \Lambda^c_d(\omega)\big)X, X\right\rangle_{\mathbb{R}^{N(\omega)}},$$

which is positive as soon as one of the coefficients $\Lambda^c_j(\omega)$ is positive. Here,

$$\mathcal{S}^{N(\omega)}_+ = \left\{X \in \mathbb{R}^{N(\omega)},\ X_j \geq 0\ \forall j \in \{1, \dots, N(\omega)\} \text{ and } \|X\|^2_{2,\mathbb{R}^{N(\omega)}} = \langle X, X\rangle_{\mathbb{R}^{N(\omega)}} = 1\right\}$$

with

$$\Lambda^c_d(\omega) = diag\big(\Lambda^c_1(\omega), \dots, \Lambda^c_{N(\omega)}(\omega)\big),$$

and $\langle X, Y\rangle_{\mathbb{R}^{N(\omega)}} = \sum_{j=1}^{N(\omega)} X_j Y_j$ for $(X, Y) \in (\mathbb{R}^{N(\omega)})^2$.

This result means that the total energy carried by the expected powers of the propagating modes decay exponentially with the propagation distance, and the decay rate can be expressed in terms of a variational formula over a finite-dimensional space.

In Section 2.5.2, we show that under the assumption that nearest neighbor coupling is the main power transfer mechanism, the evolution of the mean mode powers of the propagating modes can be described, in the limit of a large number of propagating modes $N(\omega) \gg 1$, by a diffusion model. Let us note that the limit of a large number of propagating modes $N(\omega) \gg 1$ corresponds to the high-frequency regime $\omega \to +\infty$. This diffusive continuous model is equipped with boundary conditions which take into account the effect of the radiating modes at the bottom and the free surface of the waveguide (see Figure 1.4). Let, $\forall \varphi \in \mathcal{C}^0([0,1])$, $\forall u \in [0,1]$, and $z \geq 0$,

$$\mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi^{[N(\omega)u]}(z) = \sum_{j=1}^{N(\omega)} \varphi\Big(\frac{j}{N(\omega)}\Big)\mathcal{T}_j^{[N(\omega)u]}(\omega,z),$$

where $\varphi \mapsto \mathcal{T}_\varphi^{N(\omega)}(z,.)$ can be extended into an operator from $L^2(0,1)$ to itself.

Theorem *We have*

1. $\forall \varphi \in L^2(0,1)$ *and* $\forall z \geq 0$,

$$\lim_{\omega\to+\infty} \mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi(z,u) \quad \textit{in } L^2(0,1),$$

where $\mathcal{T}_\varphi(z,u)$ *satisfies the partial differential equation :* $\forall (z,u) \in (0,+\infty)\times(0,1)$,

$$\frac{\partial}{\partial z}\mathcal{T}_\varphi(z,u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_\varphi\right)(z,u),$$

with the boundary conditions

$$\frac{\partial}{\partial u}\mathcal{T}_\varphi(z,0) = 0, \quad \mathcal{T}_\varphi(z,1) = 0, \quad \textit{and} \quad \mathcal{T}_\varphi(0,u) = \varphi(u),$$

$\forall z > 0$.

2. $\forall u \in [0,1]$, $\forall z \geq 0$, *and* $\forall \varphi \in \mathcal{C}^0([0,1])$ *such that* $\varphi(1) = 0$, *we have*

$$\lim_{\omega\to+\infty} \mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi(z,u).$$

Here,

$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$

with $a_0 = \frac{\pi^2 S_0}{2 a n_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d\int_0^d \gamma_0(x_1,x_2)\cos\left(\frac{\pi}{d}x_1\right)\cos\left(\frac{\pi}{d}x_2\right)dx_1dx_2$. n_1 *is the index of refraction in the ocean section* $[0,d]$, $1/a = l_{z,x}$ *is the correlation length of the random inhomogeneities in the longitudinal direction, and* γ_0 *is the covariance function of the random inhomogeneities in the transverse direction.*

This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between the $N(\omega)$-discrete propagating modes, with a reflecting boundary condition at $u = 0$ (the top of the waveguide in Figure 1.3) and an absorbing boundary condition at $u = 1$ (the bottom of the waveguide in Figure 1.3) which represents the radiative loss (see Figure 1.4). In this high-frequency regime, we also observe in Section 2.5.2 that the energy carried by the continuum of propagating modes decays exponentially with the propagation distance.

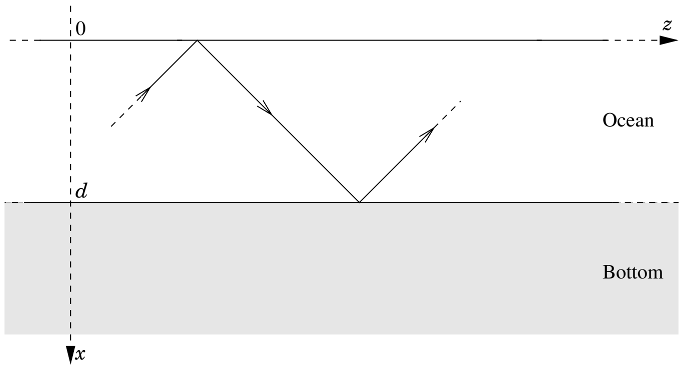


Figure 1.5: Illustration of negligible radiation losses in the shallow-water random waveguide model with propagation axis in the z-direction, transverse section $[0,+\infty)$, and ocean depth d.

Theorem *$\forall \varphi \in L^2(0,1) \setminus \{0\}$ such that $\varphi \geq 0$, and $\forall u \in [0,1)$,*

$$\lim_{L\to+\infty} \frac{1}{L} \ln \left[\mathcal{T}_\varphi(L,u)\right] = -\Lambda_\infty,$$

where

$$\Lambda_\infty = \inf_{\varphi\in\mathcal{D}} \int_0^1 a_\infty(v)\varphi'(v)^2 dv > 0$$

and

$$\mathcal{D} = \left\{ \varphi \in \mathcal{C}^\infty([0,1]), \quad \|\varphi\|_{L^2(0,1)} = 1, \quad \frac{\partial}{\partial v}\varphi(0) = 0, \quad \varphi(1) = 0 \right\}.$$

This result means that the energy carried by each propagating modes decays exponentially with the propagation distance, and the decay rate can be expressed in terms of a variational formula. Consequently, the spatial inhomogeneities of the medium and the geometry of the shallow-water waveguide lead us to an exponential decay phenomenon caused by the radiative loss into the ocean bottom.

In the case of negligible radiation losses, we also get in Section 2.5.3 a continuous diffusive model for the coupled power equations in the high-frequency regime or in the limit of a large number of propagating modes $N(\omega) \gg 1$. This diffusive continuous model is equipped with boundary conditions which take into account the negligible effect of the radiation losses at the bottom and the free surface of the waveguide (see Figure 1.5).

Theorem *We have*

1. *$\forall \varphi \in L^2(0,1)$ and $\forall z \geq 0$,*

$$\lim_{\omega\to+\infty} \mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi(z,u) \quad \textit{in } L^2(0,1),$$

where $\mathcal{T}_\varphi(z,u)$ satisfies the partial differential equation : $\forall (z,u) \in (0,+\infty)\times(0,1)$,

$$\frac{\partial}{\partial z}\mathcal{T}_\varphi(z,u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_\varphi\right)(z,u),$$

with the boundary conditions

$$\frac{\partial}{\partial u}\mathcal{T}_\varphi(z,0) = 0, \quad \frac{\partial}{\partial v}\mathcal{T}_\varphi(z,1) = 0, \quad \textit{and} \quad \mathcal{T}_\varphi(0,u) = \varphi(u),$$

$\forall z > 0.$

2. $\forall u \in [0,1)$, $\forall z \geq 0$, *and* $\forall \varphi \in \mathcal{C}^0([0,1])$ *such that* $\varphi(1) = 0$, *we have*

$$\lim_{\omega \to +\infty} \mathcal{T}_{\varphi}^{N(\omega)}(z,u) = \mathcal{T}_{\varphi}(z,u).$$

Here,

$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$

with $a_0 = \frac{\pi^2 S_0}{2an_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d \int_0^d \gamma_0(x_1, x_2) \cos\left(\frac{\pi}{d} x_1\right) \cos\left(\frac{\pi}{d} x_2\right) dx_1 dx_2$. n_1 *is the index of refraction in the ocean section* $[0, d]$, $1/a = l_{z,x}$ *is the correlation length of the random inhomogeneities in the longitudinal direction, and* γ_0 *is the covariance function of the random inhomogeneities in the transverse direction.*

This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between the $N(\omega)$-discrete propagating modes, with two reflecting boundary conditions at $u = 0$ (the top of the waveguide in Figure 1.3) and $u = 1$ (the bottom of the waveguide in Figure 1.3). Here, the two reflecting boundary conditions mean that there is no radiative loss anymore (see Figure 1.5). As a result, the energy is conserved and the modal energy distribution converges to a uniform distribution as $L \to +\infty$. This result was already obtained in [25, Chapter 20] and [30].

Theorem $\forall \varphi \in L^2(0,1)$ *and* $\forall u \in [0,1]$,

$$\lim_{L \to +\infty} \mathcal{T}_{\varphi}(L,u) = \int_0^1 \varphi(v) dv,$$

that is, the energy carried by the continuum of propagating modes converges exponentially fast to the uniform distribution over $[0,1]$ *as* $L \to +\infty$.

Results of Chapter 3 In Chapter 3, we extend the analysis of Chapter 2 to the propagation and the time-reversal of broadband pulses in the same waveguide model. In this chapter, the source profile $\Psi_q^\epsilon(t,x)$ is given, in the frequency domain, by

$$\begin{aligned}\widehat{\Psi}_q^\epsilon(\omega,x) = &\frac{1}{\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right)\\ &\times\left[\sum_{j=1}^{N(\omega)} \phi_j(\omega,x_0)\phi_j(\omega,x) + \int_{(-S,-\xi)\cup(\xi,k^2(\omega))} \phi_\gamma(\omega,x_0)\phi_\gamma(\omega,x)d\gamma\right],\end{aligned}$$

with $q > 0$, and where the family $(\phi_s(\omega,.))_{s\in\{1,\dots,N(\omega)\}\cup(-\infty,k^2(\omega))}$ is a basis of the Hilbert space $L^2(0,+\infty)$ defined in Section 2.2.1. The restriction $q > 0$ allows us to freeze the number of propagating and radiating modes, and gives simpler expressions of the transmitted wave. The term $\frac{1}{\epsilon^q}\widehat{f}(\frac{\omega-\omega_0}{\epsilon^q})$ is the Fourier transform of $f(\epsilon^q t)e^{-i\omega_0 t}$, which is a pulse with bandwidth of order ϵ^q and carrier frequency ω_0. In this chapter, we study the broadband case, that is $q \in (0,1)$. However, for the sake of simplicity we shall consider the case $q = 1/2$ but the analysis can be carried out for any $q \in (0,1)$.

In order to simplify the analysis of pulse propagation and time reversal, we assume that the source location $L_S < 0$ is sufficiently far away from 0 so that the evanescent modes generated by the source are negligible. However, according to Proposition 2.2 in Section 2.4, this assumption is not restrictive and all the results of this chapter are also valid for any $L_S < 0$. In fact, Proposition 2.2 means that, in the asymptotic $\epsilon \to 0$, the information about the evanescent part of the source profile is lost during the propagation in the random section $[0, L/\epsilon]$, and therefore it plays no role in the pulse propagation and in the time-reversal experiment. Moreover, in order to simplify the study of the pulse propagation and the time-reversal experiment, we assume in Chapter 3 that the coupling mechanism between the propagating and radiating modes with the evanescent modes is negligible. Furthermore, as it has been observed in Chapter 2 or in [25], this mechanism implies mode-dependent and frequency-dependent phase modulations, that is dispersion, but does not remove any energy from the propagating modes in the pulse propagation. Dispersion is compensated by the time-reversal mechanism and therefore plays no role in this experiment [25].

We observe the transmitted wave in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_0/ϵ, which is of the order the travel time for a distance of order $1/\epsilon$. The statistics of the transmitted wave is described in Section 3.3.2. The transmitted wave can be decomposed into two parts:

$$p_{tr}\left(\frac{t_0}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,\frac{L}{\epsilon}\right)e^{i\omega_0\left(\frac{t_0}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\right)}=p_{tr}^{1,\xi,\epsilon}(t_0,t,x,L)+p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L),$$

where $p_{tr}^{1,\xi,\epsilon}(t_0,t,x,L)$ and $p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L)$ are defined by (3.13) page 116. $p_{tr}^{1,\xi,\epsilon}(t_0,t,x,L)$ is the projection of the transmitted wave over the propagating modes, and $p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L)$ is the projection of the transmitted wave over the radiating modes.

First, we have $\mathbb{E}[p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L)]=\mathcal{O}(\sqrt{\epsilon})$ uniformly in t, and uniformly in x on each bounded subset of $[0,+\infty)$. Consequently, the amplitude of the radiating part of the transmitted wave is very small and it does not play any role in the pulse propagation. Second, in the broadband case the pulse width is of order $1/\sqrt{\epsilon}$, which is much smaller that the propagation distance, and therefore the propagating modes are separated in time by modal dispersion. As a result, we show in Section 3.3.2 that the transmitted wave can be decomposed into a sequence of modal waves with different arrival times and different modal speeds. Let $t_j=\beta_j'(\omega_0)L$, where $\beta_j'(\omega_0)$ is the derivative of the jth modal wavenumber with respect to the frequency, and let us consider

$$e^{-i\beta_j(\omega_0)\left(-L_S+\frac{L}{\epsilon}\right)}p_{tr}^{1,\xi,\epsilon}(t_j,t,x,L)=p_{tr,j}^{\xi,\epsilon}(t,x,L),$$

which is the transmitted wave observed in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_j/ϵ, which is of the order the travel time for a distance of order $1/\epsilon$.

Proposition *The jth-transmitted wave, observed around time t_j, $p_{tr,j}^{\xi,\epsilon}(t,x,L)$ converges in distribution as $\epsilon\to 0$ and as a continuous process in the three variables (t,x,L) to*

$$p_{tr,j}^{\xi}(t,x,L)=\frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)e^{iW_L^j}\tilde{K}_{j,L}^{\omega_0,\xi}*f(t),$$

where

$$\widehat{\tilde{K}_{j,L}^{\omega_0,\xi}}(\omega)=e^{\frac{1}{2}\left(\Gamma_{jj}^c(\omega_0)+i\Gamma_{jj}^s(\omega_0)-\Lambda_j^{c,\xi}(\omega_0)-i\Lambda_j^{s,\xi}(\omega_0)\right)L+i\beta_j''(\omega_0)\omega^2\frac{L}{2}},$$

and $(W^j)_j$ is a $N(\omega_0)$-dimensional Brownian motion with covariance matrix $\Gamma^1(\omega_0)$. Moreover, $p_{tr,j}^{\xi}(t,x,L)$ converges almost surely and uniformly in (t,x,L) as $\xi\to 0$ to

$$p_{tr,j}(t,x,L)=\frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)e^{iW_L^j}\tilde{K}_{j,L}^{\omega_0}*f(t),$$

where

$$\widehat{\tilde{K}_{j,L}^{\omega_0}}(\omega)=e^{\frac{1}{2}\left(\Gamma_{jj}^c(\omega_0)+i\Gamma_{jj}^s(\omega_0)-\Lambda_j^c(\omega_0)-i\Lambda_j^s(\omega_0)\right)L+i\beta_j''(\omega_0)\omega^2\frac{L}{2}}.$$

Here, $\Gamma_{jj}^c(\omega_0)$, $\Gamma_{jj}^s(\omega_0)$, $\Lambda^{c,\xi}(\omega_0)$, $\Lambda_j^{s,\xi}(\omega_0)$, $\Lambda^c(\omega_0)$, and $\Lambda_j^s(\omega_0)$ are defined in Section 2.4.1.

As in [25, Chpater 20], it is possible to observe coherent transmitted waves only around times t_j, $j \in \{1,\dots,N(\omega_0)\}$. The transmitted wave is composed of a sequence of transmitted waves which are separated from each other. Each pulse corresponds to a single mode. $\forall j \in \{1,\dots,N(\omega_0)\}$, the jth modal wave travels with the group velocity $1/\beta_j'(\omega_0)$. This result means that we have stabilization of the transmitted wave up to a random phase; that is one can observe deterministic intensity around the arrival times $t_0 = t_j\ \forall j \in \{1,\dots,N(\omega_0)\}$. The random phase is characterized in terms of a Brownian motion. The pulse intensities decrease exponentially with the propagation distance and the pulses spread dispersively through $\tilde{K}^{\omega_0}_{j,L}$. Moreover, there is no diffusion for the deterministic pulse profile.

In order to analyze the incoherent wave fluctuations at time $t_0 \neq t_j\ \forall j \in \{1,\dots,N(\omega_0)\}$, we study in Section 3.3.3 the statistics as $\epsilon \to 0$ and $\xi \to 0$ of the product of two transfer operators $\overline{\mathbf{T}^{\xi,\epsilon}(\omega+\epsilon s)} \otimes \mathbf{T}^{\xi,\epsilon}(\omega)$ at two nearby frequencies. This analysis was already carried out for waveguides with bounded cross-section in [30]. In our context, this leads to the system of transport equations which takes into account the radiation losses:

$$\frac{\partial}{\partial z}\mathcal{W}_j^l(\omega,r,z)+\beta_j'(\omega)\frac{\partial}{\partial r}\mathcal{W}_j^l(\omega,r,z) = -\Lambda_j^c(\omega)\mathcal{W}_j^l(\omega,r,z) + \sum_{n=1}^{N(\omega)}\Gamma^c_{nj}(\omega)\big(\mathcal{W}_n^l(\omega,r,z)-\mathcal{W}_j^l(\omega,r,z)\big),$$

with initial conditions $\mathcal{W}_j^l(\omega,.,0) = \delta_0(.)\delta_{jl}$. The system of transport equations describes the coupling between the $N(\omega)$-propagating modes. These equations are a generalization of the coupled power equations affected by the modal dispersion. In other words it is a space and time version of the coupled power equations with transport velocity equal to the group velocity $1/\beta_j'(\omega)$ for the jth mode.

Consequently, we can apply this result to the study of the incoherent wave fluctuations. For large propagation distance L/τ and small radiation losses $\tau\Lambda^c(\omega)$, with $\tau \ll 1$, we get in Section 3.3.3 that the limit mean transmitted intensity is given by

$$\lim_{\tau\to0}\lim_{\xi\to0}\lim_{\epsilon\to0}\frac{1}{\tau\sqrt{\epsilon}}\mathbb{E}\Big[\Big|p_{tr}^{\xi,\epsilon}\Big(\frac{t_0}{\tau},t,x,\frac{L}{\tau}\Big)\Big|^2\Big] = e^{-\overline{\Lambda}(\omega_0)L}H_{x_0}(\omega_0,x)\delta(t_0-\overline{\beta'(\omega_0)}L).$$

Here, the effective velocity of the incoherent wave fluctuations is the harmonic average of the modal group velocities $1/\overline{\beta'(\omega_0)}$, with

$$\overline{\beta'(\omega_0)} = \frac{1}{N(\omega_0)}\sum_{j=1}^{N(\omega_0)}\beta_j'(\omega_0),$$

and the effective radiative damping rate is the arithmetic average of the modal radiative damping rates

$$\overline{\Lambda}(\omega_0) = \frac{1}{N(\omega_0)}\sum_{j=1}^{N(\omega_0)}\Lambda_j^c(\omega_0).$$

As a result, the transmitted wave has also an incoherent part whose typical amplitude is of order $\epsilon^{1/4}$. Moreover, for the transverse profile $H_{x_0}(\omega_0,x)$, we have in the high-frequency regime or in the limit of large number of propagating modes $N(\omega_0) \gg 1$,

$$H_{x_0}(\omega_0,x) \underset{\omega_0\gg1}{\simeq} \frac{1}{4\lambda_{oc}d}\frac{\arcsin(\theta)}{\theta}\Big[\frac{\pi}{2}-\arccos(\theta)+\frac{1}{2}\sin(2\arccos(\theta))\Big],$$

$\forall x \in [0,d]$. Here, $\theta = \sqrt{1-1/n_1^2}$ and $\lambda_{oc} = \frac{2\pi c}{n_1\omega_0}$ is the carrier wavelength in the ocean section $[0,d]$ of the waveguide. Consequently, the mean intensity becomes uniform over the ocean

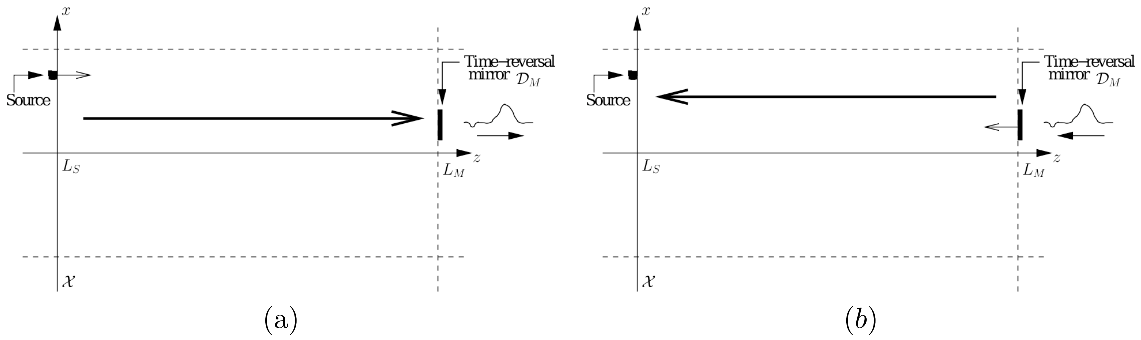


Figure 1.6: Illustration of the time-reversal experiment in a waveguide with propagation axis in the z-direction and cross-section $\mathcal{X}$. A source is localized in the plane $z = L_S$ and $\mathcal{D}_M \subset \mathcal{X}$ represents a time-reversal mirror located in the plane $z = L_M$. In (a) we illustrate the first step of the experiment. A source sends a pulse into a medium. The wave propagates and is recorded by the time-reversal mirror. The recorded signal is reversed in time by the mirror. In (b) we illustrate the second step of the experiment. The time-reversal mirror sends back the time-reversed wave. The part of the signal that is recorded first is sent back last. The back-propagating wave refocuses approximately at the source location.

cross-section $[0, d]$. The arrival time $\overline{\beta'(\omega_0)}L$ of the incoherent fluctuations takes a simple form in the high-frequency regime:

$$\lim_{\omega_0 \to +\infty} \overline{\beta'(\omega_0)}L = \frac{n_1}{c} \frac{\arcsin(\theta)}{\theta} L.$$

1.2.2 Time reversal

The time-reversal experiment is carried out in two steps. In a first step (see Figure 1.6 (a)), a source sends a broadband pulse into the medium. The wave propagates and is recorded by a device called a time-reversal mirror located in the plane $z = L_M/\epsilon$, and for a time interval $\left[\frac{t_0}{\epsilon}, \frac{t_1}{\epsilon}\right]$. We have chosen such a time window because it is of the order the travel time for a distance of order $1/\epsilon$. We assume that the time-reversal mirror occupies the transverse subdomain $\mathcal{D}_M \subset [0, d]$. A time-reversal mirror is a device that can receive a signal, record it, and resend it time-reversed into the medium. In other words, what is recorded in first is send in last. In a second step (see Figure 1.6 (b)), the wave emitted by the time-reversal mirror has the property of refocusing near the original source location, and it has been observed experimentally that random inhomogeneities enhance refocusing [19, 22, 42].

Results of Chapter 3 In this chapter we consider the shallow-water waveguide model introduced in Chapter 2. Here $L_M = L$, that is the time-reversal mirror is located at the end of the random section. However, the properties of the fluctuations of the medium may have changed between the two steps of the experiment. This situation is studied in detail in Section 3.4.5. We study the refocused wave in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_{obs}/ϵ, which is of the order the total travel time for a distance of order $1/\epsilon$. In the following proposition we observe the refocused wave at time $t_{obs} = t_1$, in which all propagating modes contribute to it. Let us note that we cannot observe the recompression of the radiating part of the recorded wave by time reversal, because it holds on a set with null Lebesgue measure.

Proposition *The refocused wave* $p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}$ *converges in distribution as* $\epsilon\to 0$ *as a continuous process in* $(t,x)\in\mathbb{R}\times[0,+\infty)$ *to*

$$p^{\xi}_{TR}(t_1,t,x,L_S)=f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}M_{jj}(\omega_0)X^{\xi,l}_j(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x).$$

Moreover, $p^{\xi}_{TR}(t_1,t,x,L_S)$ *converges in distribution as* $\xi\to 0$ *as a continuous process in* (t,x) *to*

$$p_{TR}(t_1,t,x,L_S)=f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}M_{jj}(\omega_0)X^{l}_j(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x),$$

where $\big(X^l(\omega_0,.)\big)_{j\in\{1,\dots,N(\omega_0)\}}$ *is the unique solution of the system of coupled Stratonovich stochastic differential equations*

$$dX^l_j(\omega_0,z)=\mathcal{L}^{\mu}\big(X^l(\omega_0,z)\big)(j)dz+i\sqrt{2(1-\mu)}\,X^l_j(\omega_0,z)\circ dZ_j(\omega_0,z),$$

with $X^l_j(\omega_0,0)=\delta_{jl}$*, and*

$$\mathcal{L}^{\mu}\phi(j)=-\Lambda^c_j(\omega_0)\phi(j)+(1-\mu)\Gamma^c_{jj}(\omega_0)\phi(j)+\mu\sum_{n=1}^{N(\omega_0)}\Gamma^c_{nj}(\omega_0)\big(\phi(n)-\phi(j)\big).$$

Here $(\phi_s(\omega_0,.))_{s\in\{1,\dots,N(\omega)\}\cup(-\infty,k^2(\omega))}$ is a basis of the Hilbert space $L^2(0,+\infty)$ defined in Section 2.2.1. Consequently, the spatial profile of the refocused wave at the source location is the superposition of the $N(\omega_0)$-discrete propagating modes with random weights which depend on:

1. the time-reversal mirror through the coefficients $M_{jj}(\omega_0)=\int_{\mathcal{D}_M}\phi^2_j(\omega_0,x)dx$,
2. the solution of a stochastic differential equation driven by the family of Brownian motions $Z(\omega_0,.)$ with covariance matrix $\Gamma^1(\omega_0)$.

Here, $\mu\in[0,1]$ is a parameter which describes the degree of correlation between the two realizations of the random medium (see Section 3.4 and (3.23) page 137). Then, we can observe that the quality and the loss of statistical stability in the time-reversal experiment is related to the degree of correlation between the two realizations of the random medium.

In the case $\mu=1$, which corresponds to the case of two realizations of the random medium that are fully correlated, we observe the stabilization phenomenon of time-reversal refocusing. This means that the profile of the refocused wave is deterministic. The case $\mu=1$ is also the case in which the quality of the refocusing is maximal. In the other cases ($\mu\in[0,1)$), even if the radiation losses are negligible, we show in Section 3.4.5 that the amplitude of the refocused wave decays exponentially with the propagation distance. In fact the mean refocused wave is given by

$$\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)\Big]e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}=f(-t)H^{\alpha_M}_{x_0}(\omega_0,x,L).$$

Here,

$$H^{\alpha_M}_{x_0}(\omega_0,x,L)=\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}M_{jj}(\omega_0)\tilde{\mathcal{T}}^l_j(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x),$$

where

$$\tilde{\mathcal{T}}^l_j(\omega_0,L)=\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[\overline{\mathbf{T}^{1,\xi,\epsilon}_j(\omega_0,L)(y^l)}\mathbf{T}^{2,\xi,\epsilon}_j(\omega_0,L)(y^l)\Big],$$

with $y^l_j = \delta_{jl}$, and $y^l_\gamma = 0$ for $\gamma \in (0, k^2(\omega))$. Here, $\mathbf{T}^{1,\xi,\epsilon}$ is the transfer operator for the first step of the time-reversal experiment, and $\mathbf{T}^{2,\xi,\epsilon}$ is the transfer operator for the second step of the experiment. $\tilde{\mathcal{T}}^l_j(\omega_0, L)$ is the asymptotic covariance for the jth propagating mode of the transfer operators at distance $z = L$, with respect to the two steps of the time-reversal experiment.The initial condition y^l means that an impulse equal to one charges the lth propagating mode at $z = 0$. $\tilde{\mathcal{T}}^l_j(\omega_0, z)$ are the solutions of the coupled power equations:

$$\begin{aligned}\frac{d}{dz}\tilde{\mathcal{T}}^l_j(\omega_0, z) = &- \big[\Lambda^c_j(\omega_0) + (1-\mu)\big(\Gamma^1_{jj}(\omega_0) - \Gamma^c_{jj}(\omega_0)\big)\big]\tilde{\mathcal{T}}^l_j(\omega_0, z)\\ &+ \mu \sum_{n=1}^{N(\omega_0)} \Gamma^c_{nj}(\omega_0)\big(\tilde{\mathcal{T}}^l_n(\omega_0, z) - \tilde{\mathcal{T}}^l_j(\omega_0, z)\big)\end{aligned}$$

and $\tilde{\mathcal{T}}^l_j(\omega_0, 0) = \delta_{jl}$. These equations permit us to study the influence of the degree of correlation, between the two realizations of the random medium, on the amplitude of the refocused wave. We have the following result on the asymptotic covariances $\tilde{\mathcal{T}}^l_j(\omega_0, L)$.

Theorem *Let us assume that the energy transport matrix $\Gamma^c(\omega_0)$ is irreducible. Then, we have*

$$\lim_{L\to+\infty} \frac{1}{L} \ln \left[\sum_{j=1}^{N(\omega)} \tilde{\mathcal{T}}^l_j(\omega_0, L) \right] = -\tilde{\Lambda}_\infty(\omega_0)$$

with

$$\tilde{\Lambda}_\infty(\omega_0) = \inf_{X \in \mathcal{S}^{N(\omega_0)}_+} \big\langle \big(-\mu\Gamma^c(\omega_0) + D_d(\omega_0)\big)X, X\big\rangle_{\mathbb{R}^{N(\omega_0)}} > 0,$$

and where

$$D_d(\omega_0) = diag\big(D_1(\omega_0), \dots, D_{N(\omega)}(\omega_0)\big),$$

with

$$D_j(\omega_0) = \Lambda^c_j(\omega_0) + (1-\mu)\big[\Gamma^1_{jj}(\omega_0) - \Gamma^c_{jj}(\omega_0)\big].$$

This result means that if the two realizations of the random medium are not fully correlated ($\mu \in [0, 1)$), the amplitude of the refocused wave decays exponentially with the propagation distance even if the radiation losses are negligible.

In the case $\mu = 1$, we study in Section 3.4.7 the transverse profile of the deterministic refocused wave field using the continuous diffusive model introduced in Chapter 2. We consider a time-reversal mirror $\mathcal{D}_M = [d_1, d_2]$ with a size of order $\lambda_{oc}^{\alpha_M}$, where $\alpha_M \in [0, 1]$ and $\lambda_{oc} = \frac{2\pi c}{n_1\omega_0}$ is the carrier wavelength in the ocean section $[0, d]$ of the waveguide. In the case of a homogeneous waveguide, we get in Section 3.4.3 that the width of the focal spot is diffraction limited.

Proposition *For $\alpha_M \in [0, 1)$, the transverse profile of the refocused wave in the high-frequency regime $\omega_0 \to +\infty$ is given by*

$$H^{\alpha_M}_{x_0}(\omega_0, x, L) \underset{\omega_0 \gg 1}{\simeq} \frac{\theta}{\lambda_{oc}} \frac{d_2 - d_1}{d} \text{sinc}\Big(2\pi\frac{x - x_0}{\lambda_{oc}}\theta\Big).$$

The width of the focal spot is given by $\lambda_{oc}/(2\theta)$, where λ_{oc} is the carrier wavelength in the ocean section $[0, d]$.

In the case of a random waveguide with radiation losses, we have in Section 3.4.7 the following result.

Proposition *The transverse profile of the refocused wave in the high-frequency regime $\omega_0 \to +\infty$ is given by*

$$H_{x_0}^{\alpha_M}\Big(\omega_0, x, L\Big) \underset{\omega_0 \gg 1}{\simeq} \frac{\theta}{\lambda_{oc}} \frac{d_2 - d_1}{d} H\Big(\frac{x - x_0}{\lambda_{oc}}\theta, L\Big),$$

where

$$H(\tilde{x}, L) = \int_0^1 \mathcal{T}_1(L, u) \cos(2\pi u \tilde{x}) du,$$

and $\mathcal{T}_1(L, u)$ is the solution of

$$\frac{\partial}{\partial z}\mathcal{T}_1(z, u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_1\right)(z, u),$$

with the boundary conditions:

$$\frac{\partial}{\partial u}\mathcal{T}_1(z, 0) = 0, \quad \mathcal{T}_1(z, 1) = 0 \quad \text{and} \quad \mathcal{T}_1(0, u) = 1,$$

$\forall z > 0$. *Here,*

$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$

with $a_0 = \frac{\pi^2 S_0}{2a n_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d \int_0^d \gamma_0(x_1, x_2) \cos\left(\frac{\pi}{d}x_1\right) \cos\left(\frac{\pi}{d}x_2\right) dx_1 dx_2$. n_1 is the index of refraction in the ocean section $[0, d]$, $1/a = l_{z,x}$ is the correlation length of the random inhomogeneities in the longitudinal direction, and γ_0 is the covariance function of the random inhomogeneities in the transverse direction.

The transverse profile of the refocused wave can be expressed in terms of the diffusive continuous model introduced in Section 2.5.2, with a reflecting boundary condition at $u = 0$ (the top of the waveguide) and an absorbing boundary condition at $u = 1$ (the bottom of the waveguide) which represents the radiative loss (see Figure 1.4). As it is illustrated in Figures 1.7 and 1.8 the radiation losses degrade the quality of the refocusing: the amplitude of the refocused wave decays exponentially with the propagation distance (see Section 2.5.2), and the width of the focal spot increases and converges to an asymptotic value that is significantly larger than the diffraction limit $\lambda_{oc}/(2\theta)$, where λ_{oc} is the carrier wavelength in the ocean section $[0, d]$.

In the case of a random waveguide, if we assume that the radiation losses are negligible, we have in Section 3.4.7 the following result.

Proposition *For $\alpha_M \in [0, 1]$, with negligible radiation losses, the transverse profile of the refocused wave in the high-frequency regime $\omega_0 \to +\infty$ is given by*

$$H_{x_0}^{0,\alpha_M}\Big(\omega_0, x, L\Big) \underset{\omega_0 \gg 1}{\simeq} \frac{\theta}{\lambda_{oc}} \frac{d_2 - d_1}{d} \operatorname{sinc}\Big(2\pi \frac{x - x_0}{\lambda_{oc}}\theta\Big).$$

In the case of negligible radiation losses (see Figure 1.5) the energy is conserved (see Section 2.5.3). The sinc profile obtained in Proposition 3.13 page 147 is the best transverse profile that we can obtained.

Let us remark that, in the case of a random waveguide, the order of magnitude α_M of the time-reversal mirror plays no role in the transverse profile compared to the homogeneous case (see Section 3.4.7).

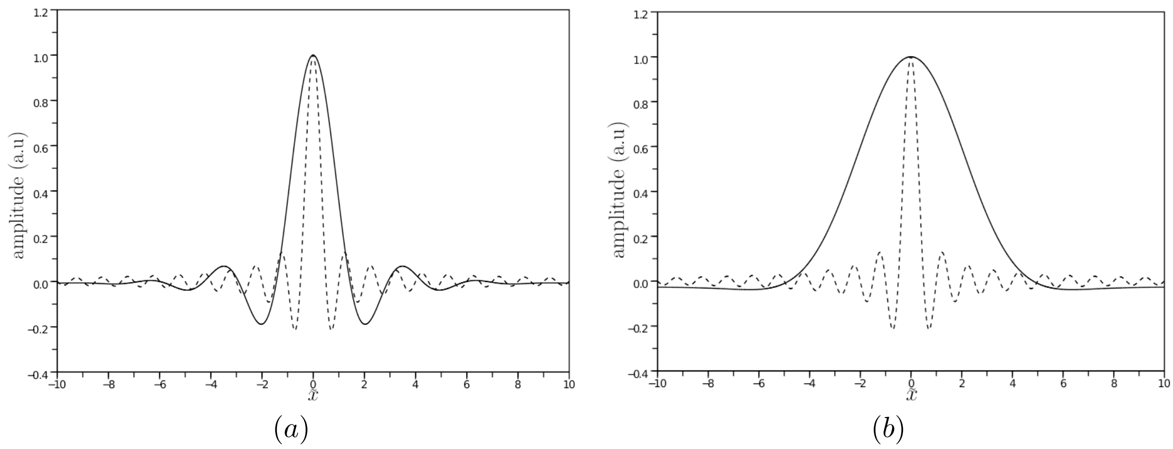


Figure 1.7: Normalized transverse profile. In (a) and (b) the dashed curves are the transverse profiles in the case where the radiation losses are negligible $\text{sinc}(2\pi\tilde{x})$, and the solid curves represent the transverse profile $H(\tilde{x}, L)$ in the presence of radiative loss. In (a) we represent $H(\tilde{x}, L)$ with $L = 75$, and in (b) we represent $H(\tilde{x}, L)$ with $L = 250$.

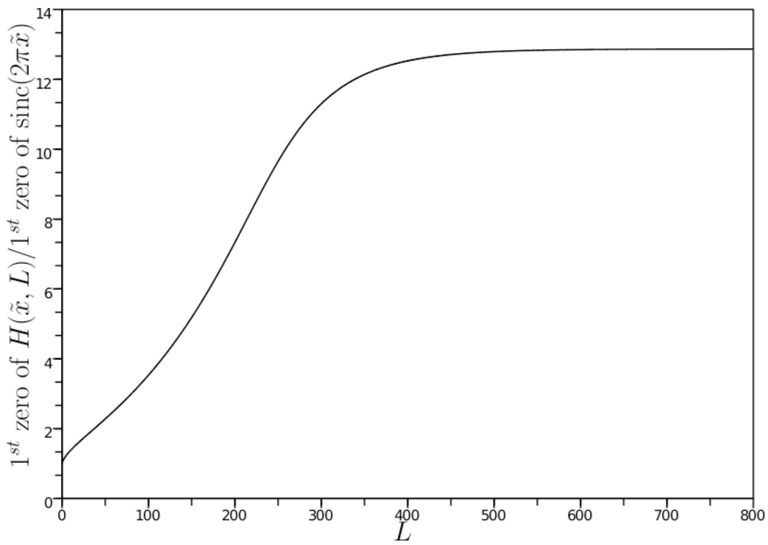


Figure 1.8: Representation of the evolution of the resolution with respect to the propagation distance L. Here $a_0 = 1$, $a = 1$, $n_1 = 2$, and $d = 20$.

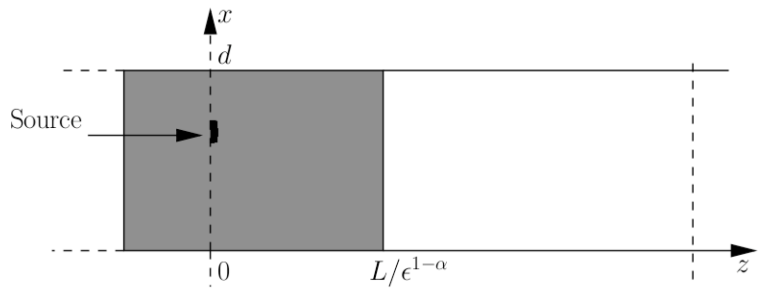


Figure 1.9: Representation of the waveguide model with propagation axis in the z-direction, bounded cross-section $[0, d]$, and two sections $(-\infty, L/\epsilon^{1-\alpha})$ and $(L/\epsilon^{1-\alpha}, +\infty)$.

Results of Chapter 4 For the sake of simplicity, we do not consider in this chapter the same waveguide model as in Chapters 2 and 3. The waveguide considered in this chapter is the same as in [25, Chapter 20] and [30], that is, the transverse section $\mathcal{X}$ is a bounded interval $[0, d]$. Consequently, in this chapter we do not consider the influence of the radiative loss on the time-reversal experiment.

In [25, Chapter 20] and [30] the authors show that the size of the focal spot in the time-reversal experiment is limited by the diffraction limit $\lambda_0/2$ (where λ_0 is the carrier wavelength). We show in Chapter 4 that the main focal spot can be smaller than the diffraction limit by inserting a random section in the vicinity of the source.

In this chapter the medium parameters are given by

$$\frac{1}{K^\epsilon(x,z)} = \begin{cases} \epsilon^{2\alpha_K} \frac{1}{K}\left(1+\sqrt{\epsilon}V\left(x, \frac{z}{\epsilon^\alpha}\right)\right) & \text{if} \quad x \in (0,d), \quad z \in [0, L/\epsilon^{1-\alpha}] \\ \epsilon^{2\alpha_K}\frac{1}{K} & \text{if} \quad x \in (0,d), \quad z \in (-\infty, 0) \\ \frac{1}{K} & \text{if} \quad x \in (0,d), \quad z \in (L/\epsilon^{1-\alpha}, +\infty), \end{cases}$$

$$\rho^\epsilon(x,z) = \begin{cases} \epsilon^{-2\alpha_\rho}\bar{\rho} & \text{if} \quad x \in (0,d), \quad z \in (-\infty, L/\epsilon^{1-\alpha}] \\ \bar{\rho} & \text{if} \quad x \in (0,d), \quad z \in (L/\epsilon^{1-\alpha}, +\infty), \end{cases}$$

where α_ρ and α_K are such that $\alpha_\rho - \alpha_K = \alpha \in (0, 1]$ (see Figure 1.9). The random process V, described more precisely in Section 2.6.1, models the spatial inhomogeneities. We consider a broadband source localized in the plane $z = 0$:

$$\mathbf{F}^\epsilon(t,x,z) = f^\epsilon(t)\Psi(x)\delta(z)\mathbf{e}_z, \text{ where } f^\epsilon(t) = \frac{1}{2\epsilon^\alpha} f(\epsilon^p t)e^{-i\omega_0 t} \text{ with } p \in (0,1),$$

and $\Psi(x)$ is the transverse profile of the source. The source amplitude is large, of order $1/\epsilon^\alpha$, because transmission coefficients at the interface $z = L/\epsilon^{1-\alpha}$ are small, of order $\epsilon^{\alpha/2}$. However, in Section 4.4.6, we show that the transmission coefficients can be made of order one by inserting a quarter wavelength plate. In this Chapter, the two realizations of the random medium during the time-reversal experiment are the same, and as in Chapter 3 the condition $p \in (0, 1)$ (broadband case) ensures the statistical stability property (see Section 4.4.5).

The important parameter is α, because it determines the order of magnitude of the sound speed c_1 of the first section $(-\infty, L/\epsilon^{1-\alpha})$. This configuration means that the order of magnitude of the sound speed $c_1 \sim \epsilon^\alpha$ is small compared to that $c_0 \sim 1$ of the section $(L/\epsilon^{1-\alpha}, +\infty)$. The first section can represent a solid with random inhomogeneities, and the second can represent a homogeneous gas or liquid. The particular case $\alpha = 0$ is equivalent to that studied in [30] and [25, Chapter 20], in which no superresolution effect can be detected. The parameter α represents a possible configuration of the waveguide model, but in order to apply an asymptotic analysis we take $\alpha \in (0, 1/4)$. The regime of the first section

$(-\infty, L/\epsilon^{1-\alpha})$ is given by

$$\frac{L_0}{\lambda_0} = \frac{K_1}{\epsilon} \gg 1, \quad \frac{l_{z,c}}{\lambda_0} = K_2 \frac{\epsilon^\alpha}{\epsilon^\alpha} \sim 1, \quad \frac{l_{x,c}}{\lambda_0} = \frac{K_3}{\epsilon^\alpha} \gg 1, \quad \text{and} \quad \sigma = \sqrt{\epsilon}.$$

This regime is somewhat different from the weakly heterogeneous regime dicussed at the begining of this presentation. However, in the z-direction, which is the main propagation axis, this regime corresponds to the weakly heterogeneous regime.

As we study linear models of propagation the pressure $p(t,x,z)$ can be expressed as the superposition of monochromatic waves by taking its Fourier transform:

$$\widehat{p}(\omega,x,z) = \int p(t,x,z)e^{i\omega t}dt.$$

Moreover, a wave field can be decomposed as follows:

$$\widehat{p}(\omega,x,z) = \sum_{j\geq 1} \widehat{p}_j(\omega,z)\phi_j(x),$$

where $(\phi_j(.))_{j\geq 1}$ is the basis of the Hilbert space $L^2(0,d)$ defined by

$$\phi_j(x) = \sqrt{\frac{2}{d}}\sin\left(\frac{j\pi}{d}x\right) \text{ with } \lambda_j = \frac{j^2\pi^2}{d^2} \text{ for } j \geq 1,$$

and corresponds to the eigenvectors and eigenvalues of the unperturbed waveguide.

- In the second section $(L/\epsilon^{1-\alpha}, +\infty)$: for $j \leq N(\omega) = [\frac{\omega d}{\pi c_0}]$, the modes $\phi_j(x)$ are the propagating modes for the waveguide with homogeneous parameters $K(x,z) = \bar{K}$ and $\rho(x,z) = \bar{\rho}$, and we call these modes low modes; for $j > N(\omega)$, these modes are the evanescent modes for the waveguide, and we call these modes high modes.
- In the first section $(-\infty, L/\epsilon^{1-\alpha})$: for $j \leq N_\epsilon(\omega) = [\frac{\omega d}{\pi c_0 \epsilon^\alpha}]$, the modes $\phi_j(x)$ are the propagating modes for the waveguide with homogeneous parameters $K(x,z) = \bar{K}/\epsilon^{2\alpha_K}$ and $\rho(x,z) = \bar{\rho}/\epsilon^{2\alpha_\rho}$; for $j > N_\epsilon(\omega)$, these modes are the evanescent modes for the same waveguide.

We know that for the waveguide with homogeneous parameters $K(x,z) = \bar{K}$ and $\rho(x,z) = \bar{\rho}$ the information on the small-scale features (position and shape) of the source, which are carried by the high modes, is lost [30]. Let us remark that in Chapter 2, with a randomly perturbed waveguide, we show in Proposition 2.2 page 50, that the information on the small-scale features of the source is lost because of a low coupling mechanism between the high modes and the low modes. Let us remark that the number of propagating modes of the first section $(-\infty, L/\epsilon^{1-\alpha})$ goes to $+\infty$ as $\epsilon \to 0$. This implies that high modes are propagating modes of the first section. By adding random inhomogeneities in the first section, we get an efficient coupling between high modes and low modes.

In this chapter the source profile is given by

$$\Psi(x) = \sum_{j=1}^{\zeta} \phi_j(x_0)\phi_j(x) \quad \forall x \in [0,d],$$

with $\zeta \gg N(\omega_0)$ to have a large number of high modes. This profile is an approximation of a Dirac distribution which models a point source at x_0.

In order to study the refocused wave around the original source location (see Sections 4.4.3 and 4.4.4), and under the assumption that nearest neighbor coupling is the main power transfer mechanism, we analyze the asymptotic behavior of the product of two transfer matrices $\mathbf{U}^\epsilon_{jm}(\omega,z) = \overline{\mathbf{T}^\epsilon_{jl}(\omega,z)}\mathbf{T}^\epsilon_{mn}(\omega,z)$ at the same frequency. Here, $\mathbf{T}^\epsilon(\omega,z)$ is the solution of the $N_\epsilon(\omega)\times N_\epsilon(\omega)$ system of differential equations with random coefficients of the form:

$$\frac{d}{dz}\mathbf{T}^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{a,\epsilon}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^\epsilon(\omega,z) \quad \text{with} \quad \mathbf{T}^\epsilon(\omega,0)=\mathbf{I}.$$

Here $\mathbf{H}^{a,\epsilon}$ is defined by (4.10) page 158 and represents the coupling of the $N_\epsilon(\omega)$-propagating modes of the first section $(-\infty, L/\epsilon)$ due to the random heterogeneities. The asymptotic behavior as $\epsilon\to 0$ of the statistical properties of the matrix $\mathbf{U}^\epsilon$ is described in Section 4.3 in terms of the diffusion model given by the infinite-dimensional stochastic differential equation:

$$d\mathbf{U}(\omega,z) = J^\omega(\mathbf{U}(\omega,z))dz + \psi^\omega_1(\mathbf{U}(\omega,z))(dB^1_z) + \psi^\omega_2(\mathbf{U}(\omega,z))(dB^2_z),$$

with $\mathbf{U}_{jm}(\omega,0) = \delta_{jl}\delta_{mn}$, where $(B^\eta_{jm})_{\substack{\eta=1,2\\ j,m\geq 1}}$ is a family of independent one-dimensional standard Brownian motions, and J^ω, ψ^ω_1 and ψ^ω_2 are defined in Theorem 4.1 page 160. As a result, the transverse profile of the refocused wave in the asymptotic $\epsilon\to 0$ is essentially given by

$$H^{\alpha_M}_{x_0}(\omega_0,x) = \frac{1}{2}\sum_{l\geq 1}\sum_{j=1}^{N(\omega_0)}\frac{\beta_j(\omega_0)}{k(\omega_0)}\mathcal{T}^l_j(\omega_0,L)M_{jj}\phi_l(x)\phi_l(x_0),$$

up to an error which decays exponentially to 0 in the high-frequency regime (see Section 4.4.4). Here, $M_{jj} = \int_{\mathcal{D}_M}\phi^2_j(x)dx$, $k(\omega_0)=\omega_0/c_0$ is the carrier wavenumber of the second section $(L/\epsilon^{1-\alpha},+\infty)$ of the waveguide, and $\beta_j(\omega_0)=\sqrt{k^2(\omega_0)-\lambda_j}$ is the jth modal wavenumber of the second section of the waveguide.

Consequently, we are interested in the study of the asymptotic mean mode powers of the propagating modes

$$\mathcal{T}^l_j(\omega,L) = \lim_{\epsilon\to 0}\mathbb{E}\Big[\big|\mathbf{T}^\epsilon_{jl}(\omega,L)\big|^2\Big].$$

$\mathcal{T}^l_j(\omega,L)$ is the expected power of the jth propagating mode at the propagation distance $z=L$, when at $z=0$ the energy is concentrated on the lth propagating mode. The expected powers $\mathcal{T}^l_j(\omega,L)$ are solution of the following coupled power equations:

$$\begin{aligned}
\frac{d}{dz}\mathcal{T}^l_j(\omega,z) &= \Lambda(\omega)\left[\mathcal{T}^l_{j+1}(\omega,z)+\mathcal{T}^l_{j-1}(\omega,z)-2\mathcal{T}^l_j(\omega,z)\right], \quad j\geq 1,\\
\frac{d}{dz}\mathcal{T}^l_1(\omega,z) &= \Lambda(\omega)\left[\mathcal{T}^l_2(\omega,z)-\mathcal{T}^l_1(\omega,z)\right],
\end{aligned}$$

with $\mathcal{T}^l_j(\omega,0)=\delta_{jl}$. These equations describe the transfer of energy between the propagating modes and $\Lambda(\omega)$ is the energy transport coefficient. The initial condition means that an impulse equal to one charges only the lth propagating mode. As in Chapter 2, the evolution of the mean mode powers is described in the high-frequency regime by a continuous diffusive model (see Section 4.3). Moreover, as in Chapter 3, this continuous diffusive model can be used to study the refocused transverse profile (see Section 4.4.4). In the high-frequency regime, we can consider $(\mathcal{T}^l(\omega,L))_{l\geq 1}$ as a family of probability measures on $\mathbb{R}_+$. Let $\forall\varphi\in\mathcal{C}^0_b([0,+\infty))$, $\forall u\in[0,+\infty)$, and $z\geq 0$,

$$\mathcal{T}^{N(\omega)}_\varphi(z,u) = \mathcal{T}^{[N(\omega)u]}_\varphi(\omega,z) = \sum_{j\geq 1}\varphi\Big(\frac{j}{N(\omega)}\Big)\mathcal{T}^{[N(\omega)u]}_j(z).$$

Theorem *$\forall u\geq 0$, $\forall z\geq 0$, and $\forall\varphi\in\mathcal{C}^0_b([0,+\infty))$, we have*

$$\lim_{\omega\to+\infty}\mathcal{T}^{N(\omega)}_\varphi(z,u) = \mathcal{T}_\varphi(z,u) = \int_{\mathbb{R}_+}\varphi(v)\mathcal{W}(z,u,v)dv,$$

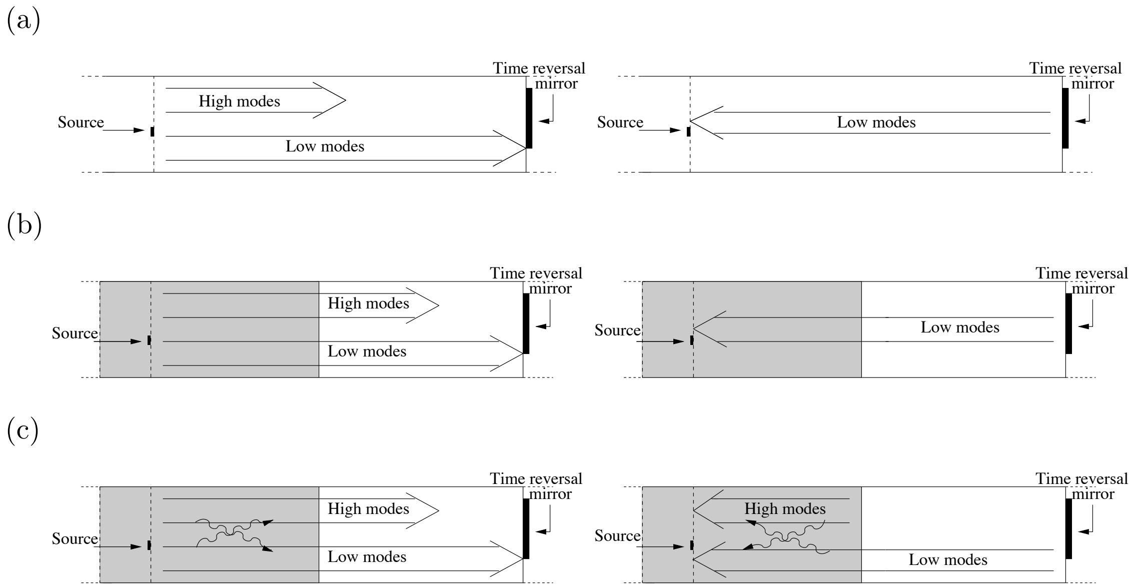


Figure 1.10: Representation of mode propagation in the time reversal experiment. The pictures on the left-hand side illustrate the first step of the experiment. A source sends a pulse into a medium. The wave propagates and is recorded by the time-reversal mirror. The recorded signal is reversed in time by the mirror. The pictures on the right-hand side illustrate the second step of the experiment. The time-reversal mirror sends back the time-reversed wave. The part of the signal that is recorded first is sent back last. The back-propagating wave refocuses approximately at the source location. In (a) we represent a homogeneous waveguide, in (b) we add a homogeneous section with low speed propagation, and in (c) we add a randomly heterogeneous section with low background propagation speed.

where $\forall z > 0$ *and* $(u, v) \in [0, +\infty)^2$,

$$\frac{\partial}{\partial z}\mathcal{W}(z, u, v) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\mathcal{W}(z, u, v),$$

with

$$\frac{\partial}{\partial u}\mathcal{W}(z, 0, v) = 0 \text{ and } \mathcal{W}(0, u, v) = \delta(u - v).$$

Here, $\sigma^2 = \frac{\pi^2}{d^2 a}S(1, 1)$ *and* $S(1, 1) = \frac{4}{d^2}\int_0^d\int_0^d \gamma_0(x_1, x_2)\cos\left(\frac{\pi}{d}x_1\right)\cos\left(\frac{\pi}{d}x_2\right)dx_1dx_2$. *Moreover,* $1/a = l_{z,x}$ *is the correlation length of the random inhomogeneities in the longitudinal direction, and* γ_0 *is the covariance function of the random inhomogeneities in the transverse direction.*

Let us note that $\mathcal{W}(z, u, v)$ can be computed. We have, $\forall z > 0$ and $\forall(u, v) \in [0, +\infty)^2$,

$$\mathcal{W}(z, u, v) = \frac{1}{\sqrt{2\pi\sigma^2 z}}\left(e^{-\frac{(v-u)^2}{2\sigma^2 z}} + e^{-\frac{(v+u)^2}{2\sigma^2 z}}\right).$$

As in Chapter 3, this result is used to study the refocused transverse profile. Now, let us describe the important mechanisms which lead us to the *superresolution* effect.

First, the case of a waveguide with homogeneous sound speed c_0 (see Figure 1.10 (a)) is well known; see for instance [25], where the authors obtain the classical diffraction limit $\lambda_0/2$. In this case, the small-scale features (position and shape) of the source are carried by the high modes that decay exponentially fast with the propagation distance. Consequently, these modes do not reach the time-reversal mirror, which is located in the far field. Only low modes are recorded by the time-reversal mirror. In the second step of the time-reversal experiment, the mirror sends back the recorded low modes that carry only the large-scale features of the original source. This loss of information is responsible for the diffraction-limited transverse profile, and is described by the following result of Section 4.4.3. We consider a time-reversal mirror $\mathcal{D}_M = [d_1, d_2]$ with a size of order $\lambda_0^{\alpha_M}$, where $\alpha_M \in [0,1]$ and $\lambda_0 = \frac{2\pi c}{\omega_0}$ is the carrier wavelength in the second section $(L/\epsilon^{1-\alpha}, +\infty)$ of the waveguide.

Proposition *For $\alpha_M \in [0,1)$, the spatial profile in the high-frequency regime is given by*

$$H^{\alpha_M}_{x_0,\text{no section}}(\omega_0, x) \underset{\omega_0 \gg 1}{\simeq} \frac{d_2 - d_1}{\lambda_0 d} \text{sinc}\Big(2\pi \frac{x - x_0}{\lambda_0}\Big).$$

In this chapter, we are interested in comparing the two following cases with the previous one. First, we assume that a homogeneous section with low sound speed $c_1 \ll 1$ is inserted in the vicinity of the source, as illustrated in Figure 1.10 (b), such that some high modes of the previous case are propagating modes in this first section. However, we assume that the major part of the waveguide has sound speed c_0 so the high modes and the small-scale features of the source do not reach the time-reversal mirror. Therefore, as in the homogeneous case, only low modes are recorded by the time-reversal mirror and the small-scale features of the source are lost. Then, we get in Section 4.4.3 the following result.

Proposition *For $\alpha_M \in [0,1)$, the transverse profile of the refocused wave in the high-frequency regime is given by*

$$H^{\alpha_M}_{x_0}(\omega_0, x) \underset{\omega_0 \gg 1}{\simeq} \frac{d_2 - d_1}{\lambda_0 d} H^{(1)}\Big(\frac{x - x_0}{\lambda_0}\Big),$$

where

$$H^{(1)}(\tilde{x}) = \int_0^1 \sqrt{1-u^2} \cos(2\pi \tilde{x} u)\, du.$$

Second, if the additional section has low sound speed and is randomly perturbed, then coupling mechanisms, between all the propagating modes of the first section, allow small-scale features of the source, which are carried by the high modes, to be transferred to low modes. Even if the high modes do not propagate over large distances in the second part of the waveguide and are not recorded by the time-reversal mirror, a part of the small-scale features of the source reaches the time-reversal mirror since they are carried by the low modes which are recorded by the time-reversal mirror. This fact is illustrated in Figure 1.10 (c). These low modes, time-reversed, come back to the randomly perturbed section in the second step of the time-reversal experiment, and by coupling mechanisms they regenerate high modes with the small-scale features of the source. This regeneration of small-scale features of the source is responsible for the superresolution effect and we get in Section 4.4.4 the following result.

Proposition *For $\alpha_M \in [0,1]$, in the high-frequency regime, we have*

$$\tilde{H}^{\alpha_M}_{x_0}(\omega_0, x) = \frac{d_2 - d_1}{\lambda_0 d} H^{(2)}\Big(\frac{x - x_0}{\lambda_0}, L\Big),$$

where

$$H^{(2)}(\tilde{x}, L) = e^{-\tilde{x}^2/r_c^2} H^{(1)}(\tilde{x}) = e^{-\tilde{x}^2/r_c^2} \int_0^1 \sqrt{1-u^2} \cos(2\pi \tilde{x} u)\, du,$$

with

$$r_c = \frac{1}{\pi\sigma\sqrt{2L}} = \frac{d}{\pi^2}\sqrt{\frac{a}{2LS(1,1)}},$$

and $S(1,1) = \frac{4}{d^2}\int_0^d\int_0^d \gamma_0(x_1,x_2)\cos\big(\frac{\pi}{d}x_1\big)\cos\big(\frac{\pi}{d}x_2\big)dx_1dx_2$. Moreover, $1/a = l_{z,x}$ is the correlation length of the random inhomogeneities in the longitudinal direction, and γ_0 is the covariance function of the random inhomogeneities in the transverse direction.

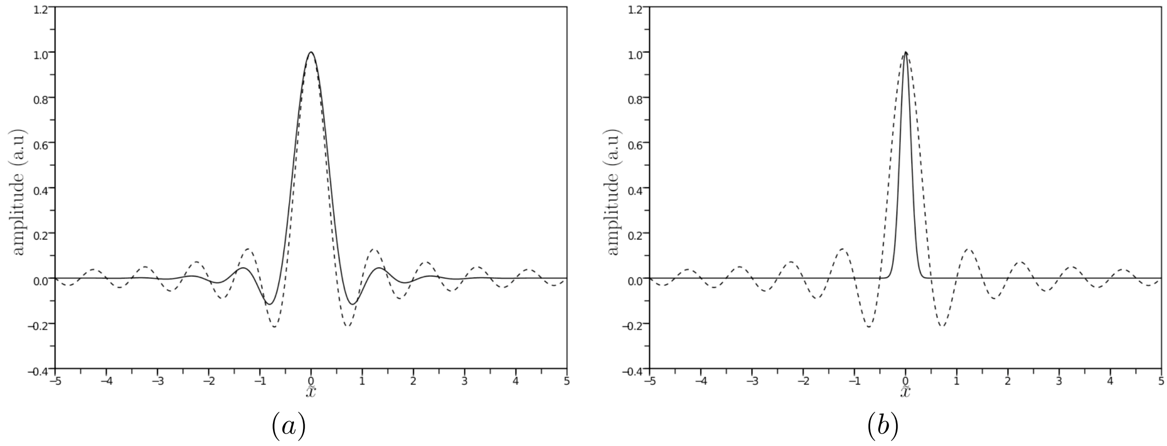


Figure 1.11: Normalized transverse profiles in a random waveguide. Here $L = 1$. In (a) and (b) we illustrate the case where $\alpha_M \in [0, 1)$. The dashed curves are the transverse profiles in the case where the section is missing, and the solid curves are the transverse profiles $H^{(2)}(\tilde{x}, L)$ in the case where we add a random section, with $\sigma = 0.5$ in (a), and $\sigma = 7$ in (b).

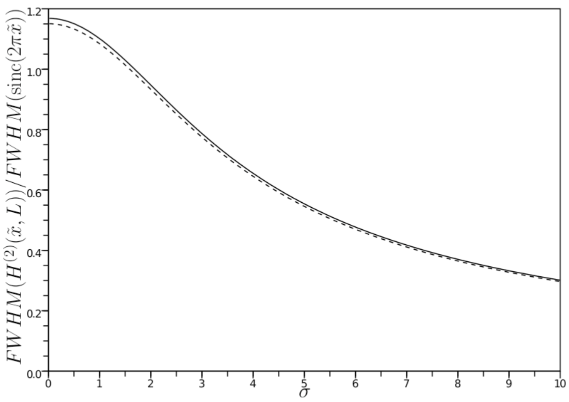


Figure 1.12: Ratio between the FWHM (Full Width at Half Maximum) of the profile $H^{(2)}(\tilde{x}, L)$ obtained when we add a random section and that of the profile obtained when this section is missing $\text{sinc}(2\pi\tilde{x})$, in terms of the standard deviation σ of the random fluctuations. Here $L = 1$. The solid curve represents the case where $\alpha_M \in [0, 1)$, and the dashed curve represents the case where $\alpha_M = 1$.

Moreover, let us remark that, in the case where the additional section is randomly perturbed, the order of magnitude α_M of the time-reversal mirror plays no role in the transverse profile compared to the homogeneous cases (see Section 4.4.4).

Chapter 2

Wave Propagation in Shallow-Water Acoustic Random Waveguides

Introduction

Acoustic wave propagation in shallow-water waveguides has been studied for a long time because of its numerous domains of applications. One of the most important applications is submarine detection with active or passive sonars, but it can also be used in underwater communication, mines or archaeological artifacts detection, and to study the ocean's structure or ocean biology. Shallow-waters are complicated media because they have indices of refraction with spatial and time dependences. However, the sound speed in water, which is about 1500 m/s, is sufficiently large with respect to the motions of water masses that we can consider this medium as being time independent. Moreover, the presence of spatial inhomogeneities in the water produces a mode coupling and can induce significant effects over large propagation distances.

In shallow-water waveguides the transverse section can be represented as a semi-infinite interval (see Figure 2.1) and then a wave field can be decomposed over three kinds of modes: the propagating modes which propagate over long distances, the evanescent modes which decrease exponentially with the propagation distance, and the radiating modes representing modes which penetrate under the bottom of the water. The main purpose of this chapter is to analyze how the propagating mode powers are affected by the radiating and evanescent modes. This analysis is carried out using an asymptotic analysis based on a separation of scale technique, where the wavelength and the correlation lengths of the inhomogeneities, which are of the same order, are small compared to the propagation distance, and the fluctuations of the medium are small compared to the wavelength. In the terminology of [25] this is the so-called weakly heterogeneous regime.

Wave propagation in random waveguides with a bounded cross-section and Dirichlet boundary conditions (see Figure 2.1) has been studied in [25, Chapter 20] or [30] for instance. In this case we have only two kinds of modes, the propagating and the evanescent modes. In such a model an asymptotic analysis of the mode powers show total energy conservation and a uniform distribution of the energy carried by the propagating modes. In [30] coupled power equations are derived under the assumption that evanescent modes are negligible. In [29] the role of evanescent modes is studied in absence of radiating modes. In this chapter we take into account the influence of the radiating and the evanescent modes on the coupled power equations. In this case we show a mode-dependent and frequency-dependent attenuation on the propagating modes in Theorem 2.3, that is, the total energy carried by the propagating modes decreases exponentially with the size of the random section and we give an expression

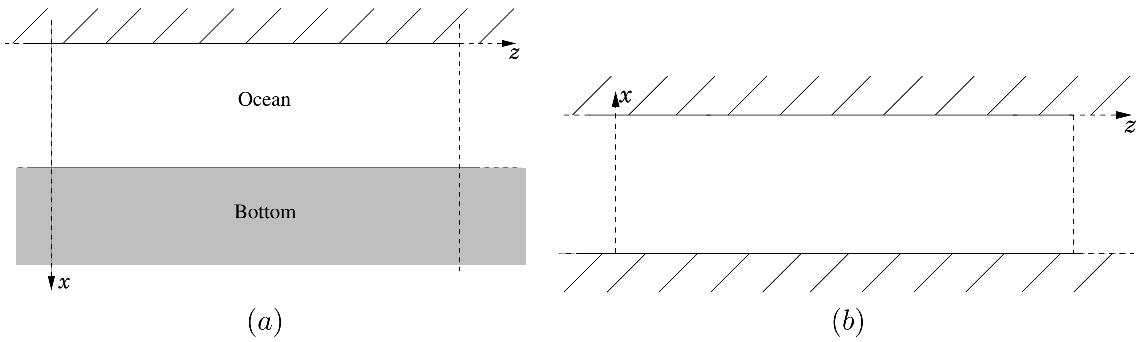


Figure 2.1: Illustration of two kinds of waveguides. In (a) we represent a shallow-water waveguide model with an unbounded cross-section. In (b) we represent a waveguide with a bounded cross-section.

of the decay rate. Moreover, in the high-frequency regime, we show in Theorems 2.4 and 2.6 that the propagating mode powers converge to the solution of a diffusion equation. All the results of this chapter are also valid for electromagnetic wave propagation in dielectric waveguides and optical fibers [43, 44, 52, 54, 62].

The organization of this chapter is as follows. In Section 2.1 we present the waveguide model that we consider in Chapter 2 and Chapter 3, and in Section 2.2 we present the mode decomposition associated to that model and studied in detail in [61]. In Section 2.3 we study the mode coupling when there are the three kinds of modes. In the same spirit as in [25, chapter 20], we derive the coupled mode equation, we study the energy flux for the propagating and the radiating modes, and the influence of the evanescent modes on the two other kinds of modes. In Section 2.4, under the forward scattering approximation, we study the asymptotic form of the joint distribution of the propagating and radiating mode amplitudes. We apply this result in Section 2.5 to derive the coupled power equations for the propagating modes, which was already obtained in [39] or [44] for instance. In this section, we study the influence of the radiating and evanescent modes on the mean propagating mode powers. We show that the total energy carried by the propagating modes decreases exponentially with the size of the random section and we give an expression of the decay rate. In other words, the radiating modes induce a mode-dependent attenuation on the propagating modes, that is why these modes are sometimes called dissipative modes. Moreover, under the assumption that nearest neighbor coupling is the main power transfer mechanism, we show, in the high-frequency regime or in the limit of large number of propagating modes, that the mean propagating mode powers converge to the solution of a diffusion equation. We can refer to [39, 44] for further references and discussions about diffusion models. In that regime, we can also observe the exponential decay behavior caused by the radiative loss.

2.1 Waveguide Model

We consider a two-dimensional linear acoustic wave model. The conservation equations of mass and linear momentum are given by

$$\begin{aligned} \rho(x,z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{F}, \\ \frac{1}{K(x,z)}\frac{\partial p}{\partial t} + \nabla . \mathbf{u} &= 0, \end{aligned} \tag{2.1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ is the density of the medium, K is the bulk modulus, and the source is modeled by the forcing term $\mathbf{F}(t,x,z)$. The third

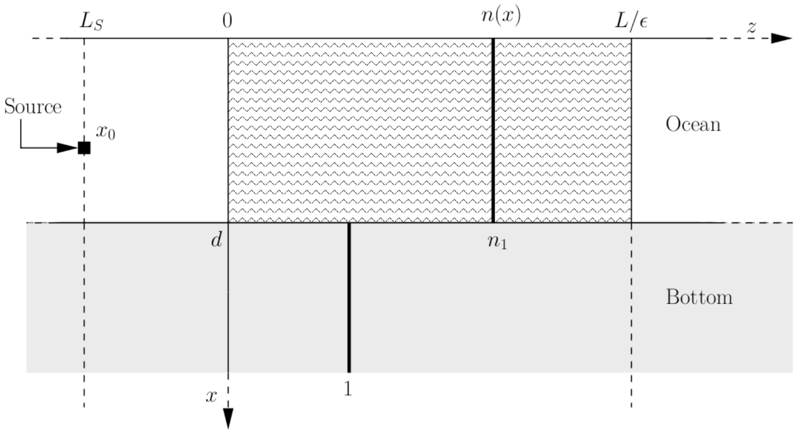


Figure 2.2: Illustration of the shallow-water waveguide model.

coordinate z represents the propagation axis along the waveguide. The transverse section of the waveguide is the semi-infinite interval $[0,+\infty)$, and $x \in [0,+\infty)$ represents the transverse coordinate. Let $d > 0$, we assume that the medium parameters are given by

$$\frac{1}{K(x,z)} = \begin{cases} \frac{1}{\bar{K}}\left(n^2(x) + \sqrt{\epsilon}V(x,z)\right) & \text{if} \quad x \in [0,d], \quad z \in [0,L/\epsilon] \\ \frac{1}{\bar{K}} n^2(x) & \text{if} \quad \begin{cases} x \in [0,+\infty),\ z \in (-\infty,0) \cup (L/\epsilon,+\infty) \\ \text{or} \\ x \in (d,+\infty),\ z \in (-\infty,+\infty). \end{cases} \end{cases}$$

$$\rho(x,z) = \bar{\rho} \quad \text{if} \quad x \in [0,+\infty),\ z \in \mathbb{R}.$$

In Chapters 2 and 3, we consider the Pekeris waveguide model. This kind of model has been studied for half a century [51] and in this model the index of refraction $n(x)$ is given by

$$n(x) = \begin{cases} n_1 > 1 & \text{if} \quad x \in [0,d) \\ 1 & \text{if} \quad x \in [d,+\infty). \end{cases}$$

This profile can model an ocean with a constant sound speed. Such conditions can be found during the winter in Earth's mid latitudes and in water shallower than about 30 meters. The Pekeris profile leads us to simplified algebra but it underestimates the complexity of the medium. However, the analysis that we present in Chapters 2 and 3 can be extended to more general profiles $n(x)$ with general boundary conditions. In the Pekeris model that we consider n_1 represents the index of refraction of the ocean section $[0,d]$, where d is the depth of the ocean, and we consider that the index of refraction of the bottom of the ocean is equal to 1. This model can also be used to study the propagation of electromagnetic waves in a dielectric slab with randomly perturbed index of refraction and optical fiber [43, 44, 54, 62].

We consider a source that emits a signal in the z-direction, which is localized in the plane $z = L_S$.

$$\mathbf{F}(t,x,z) = \Psi(t,x)\delta(z - L_S)\mathbf{e}_z. \tag{2.2}$$

$\Psi(t,x)$ represents the profile of the source and $\mathbf{e}_z$ is the unit vector pointing in the z-direction. $L_S < 0$ is the location of the source on the propagating axis.

The random process $(V(x,z), x \in [0,d], z \geq 0)$ that we consider, and which represents the spatial inhomogeneities is presented in Section 2.6.1. However, one can remark that the process V is unbounded. This fact implies that the bulk modulus can take negative values. In order to avoid this situation, we can work on the event

$$\Big(\forall (x,z) \in [0,d] \times [0,L/\epsilon],\ n_1 + \sqrt{\epsilon}V(x,z) > 0\Big).$$

In fact, the property (2.55) implies

$$\lim_{\epsilon\to 0}\mathbb{P}\Big(\exists (x,z)\in[0,d]\times[0,L/\epsilon]:\ n_1+\sqrt{\epsilon}V(x,z)\le 0\Big)$$
$$\le \lim_{\epsilon\to 0}\mathbb{P}\Big(\sqrt{\epsilon}\sup_{z\in[0,L]}\sup_{x\in[0,d]}\left|V\left(x,\frac{z}{\epsilon}\right)\right|\ge n_1\Big)=0.$$

2.2 Wave Propagation in a Homogeneous Waveguide

In this section, we assume that the medium parameters are given by

$$\rho(x,z)=\bar\rho \text{ and } K(x,z)=\frac{\bar K}{n^2(x)},\quad \forall (x,z)\in[0,+\infty)\times\mathbb{R}.$$

From the conservation equations (2.1), we can derive the wave equation for the pressure field,

$$\Delta p-\frac{1}{c(x)^2}\frac{\partial^2 p}{\partial t^2}=\nabla.\mathbf{F}, \tag{2.3}$$

where $c(x)=c/n(x)$ with $c=\sqrt{\frac{\bar K}{\bar\rho}}$, and $\Delta=\partial_x^2+\partial_z^2$.

In underwater acoustics the density of air is very small compared to the density of water, then it is natural to use a pressure-release condition. The pressure is very weak outside the waveguide, and by continuity, the pressure is zero at the free surface $x=0$. This consideration leads us to consider the Dirichlet boundary conditions

$$p(t,0,z)=0\quad \forall (t,z)\in[0,+\infty)\times\mathbb{R}.$$

Throughout this manuscript, we consider linear models of propagation. Therefore, the pressure $p(t,x,z)$ can be expressed as the superposition of monochromatic waves by taking its Fourier transform. Here, the Fourier transform and the inverse Fourier transform, with respect to time, are defined by

$$\widehat f(\omega)=\int f(t)e^{i\omega t}dt,\quad f(t)=\frac{1}{2\pi}\int \widehat f(\omega)e^{-i\omega t}d\omega.$$

In the half-space $z>L_S$ (resp., $z<L_S$), taking the Fourier transform in (2.3), we get that $\widehat p(\omega,x,z)$ satisfies the time harmonic wave equation without source term

$$\partial_z^2\widehat p(\omega,x,z)+\partial_x^2\widehat p(\omega,x,z)+k^2(\omega)n^2(x)\widehat p(\omega,x,z)=0, \tag{2.4}$$

where $k(\omega)=\frac{\omega}{c}$ is the wavenumber, and with Dirichlet boundary conditions $\widehat p(\omega,0,z)=0$ $\forall z\in\mathbb{R}$. The source term implies the following jump conditions for the pressure field across the plane $z=L_S$

$$\begin{aligned}\widehat p(\omega,x,L_S^+)-\widehat p(\omega,x,L_S^-)&=\widehat\Psi(\omega,x),\\ \partial_z\widehat p(\omega,x,L_S^+)-\partial_z\widehat p(\omega,x,L_S^-)&=0.\end{aligned} \tag{2.5}$$

2.2.1 Spectral Decomposition in Unperturbed Waveguides

This section is devoted to the presentation of the spectral decomposition of the Pekeris operator $\partial_x^2+k^2(\omega)n^2(x)$. The spectral analysis of this operator is carried out in [61]. Throughout Chapters 2 and 3, we shall be interested in solutions of (2.4) such that

$$\widehat p(\omega,.,.)\mathbf{1}_{(L_S,+\infty)}(z)\in\mathcal{C}^0\Big((L_S,+\infty),H_0^1(0,+\infty)\cap H^2(0,+\infty)\Big)\cap\mathcal{C}^2\Big((L_S,+\infty),H\Big),$$
$$\widehat p(\omega,.,.)\mathbf{1}_{(-\infty,L_S)}(z)\in\mathcal{C}^0\Big((-\infty,L_S),H_0^1(0,+\infty)\cap H^2(0,+\infty)\Big)\cap\mathcal{C}^2\Big((-\infty,L_S),H\Big),$$

where $H = L^2(0,+\infty)$. H is equipped with the inner product defined by

$$\forall (h_1,h_2) \in H \times H, \quad \langle h_1, h_2 \rangle_H = \int_0^{+\infty} h_1(x)\overline{h_2(x)}dx.$$

Consequently, in the half-space $z > L_S$ (resp., $z < L_S$), we can consider (2.4) as the operational differential equation

$$\frac{d^2}{dz^2}\widehat{p}(\omega,.,z) + R(\omega)\big(\widehat{p}(\omega,.,z)\big) = 0 \tag{2.6}$$

in H, where $R(\omega)$ is an unbounded operator on H with domain

$$\mathcal{D}(R(\omega)) = H_0^1(0,+\infty) \cap H^2(0,+\infty),$$

and defined by

$$R(\omega)(y) = \frac{d^2}{dx^2}y + k^2(\omega)n^2(x)y \quad \forall y \in \mathcal{D}(R(\omega)).$$

According to [61], $R(\omega)$ is a self-adjoint operator on the Hilbert space H, and its spectrum is given by

$$Sp(R(\omega)) = \Big(-\infty, k^2(\omega)\Big] \cup \{\beta^2_{N(\omega)}(\omega),\dots,\beta^2_1(\omega)\}. \tag{2.7}$$

More precisely, we have $\beta_j(\omega) > 0$ $\forall j \in \{1,\dots,N(\omega)\}$, and

$$k^2(\omega) < \beta^2_{N(\omega)}(\omega) < \dots < \beta^2_1(\omega) < n_1^2 k^2(\omega).$$

Moreover, there exists a resolution of the identity Π_ω of $R(\omega)$ such that $\forall y \in H$, $\forall r \in \mathbb{R}$,

$$\begin{aligned}\Pi_\omega(r,+\infty)(y)(x) = &\sum_{j=1}^{N(\omega)} \langle y, \phi_j(\omega,.)\rangle_H \phi_j(\omega,x)\mathbf{1}_{(r,+\infty)}\left(\beta_j(\omega)^2\right)\\ &+ \int_r^{k^2(\omega)} \langle y, \phi_\gamma(\omega,.)\rangle_H \phi_\gamma(\omega,x)d\gamma \mathbf{1}_{(-\infty,k^2(\omega))}(r),\end{aligned}$$

and $\forall y \in \mathcal{D}(R(\omega))$, $\forall r \in \mathbb{R}$,

$$\begin{aligned}\Pi_\omega(r,+\infty)(R(\omega)(y))(x) = &\sum_{j=1}^{N(\omega)} \beta_j(\omega)^2 \langle y, \phi_j(\omega,.)\rangle_H \phi_j(\omega,x)\mathbf{1}_{(r,+\infty)}\left(\beta_j(\omega)^2\right)\\ &+ \int_r^{k^2(\omega)} \gamma\langle y, \phi_\gamma(\omega,x)\rangle_H \phi_\gamma(\omega,x)d\gamma \mathbf{1}_{(-\infty,k^2(\omega))}(r).\end{aligned}$$

Let us describe these decompositions.

Discrete part of the decomposition $\forall j \in \{1,\dots,N(\omega)\}$, the jth eigenvector is given by [61]

$$\phi_j(\omega,x) = \begin{cases} A_j(\omega)\sin(\sigma_j(\omega)x/d) & \text{if} \quad 0 \le x \le d \\ A_j(\omega)\sin(\sigma_j(\omega))e^{-\zeta_j(\omega)\frac{x-d}{d}} & \text{if} \quad d \le x, \end{cases}$$

where

$$\sigma_j(\omega) = d\sqrt{n_1^2 k^2(\omega) - \beta_j^2(\omega)}, \quad \zeta_j(\omega) = d\sqrt{\beta_j(\omega)^2 - k^2(\omega)},$$

and

$$A_j(\omega) = \sqrt{\frac{2/d}{1 + \frac{\sin^2(\sigma_j(\omega))}{\zeta_j(\omega)} - \frac{\sin(2\sigma_j(\omega))}{2\sigma_j(\omega)}}}. \tag{2.8}$$

According to [61], $\sigma_1(\omega), \dots, \sigma_{N(\omega)}(\omega)$ are the solutions on $(0, n_1 k(\omega) d\theta)$ of the equation

$$\tan(y) = -\frac{y}{\sqrt{(n_1 k d\theta)^2 - y^2}}, \tag{2.9}$$

such that $0 < \sigma_1(\omega) < \dots < \sigma_{N(\omega)}(\omega) < n_1 k(\omega) d\theta$, and with $\theta = \sqrt{1 - 1/n_1^2}$. This last equation admits exactly one solution over each interval of the form $(\pi/2 + (j-1)\pi, \pi/2 + j\pi)$ for $j \in \{1, \dots, N(\omega)\}$, where

$$N(\omega) = \left[\frac{n_1 k(\omega) d}{\pi}\theta\right].$$

From (2.9), we get the following results about the localization of the solutions which is used to show the main result of Section 2.5.2.

Lemma 2.1 *Let $\alpha > 1/3$, we have as $N(\omega) \to +\infty$*

$$\sup_{j \in \{1, \dots, N(\omega) - [N(\omega)^\alpha] - 1\}} |\sigma_{j+1}(\omega) - \sigma_j(\omega) - \pi| = \mathcal{O}\left(N(\omega)^{\frac{1}{2} - \frac{3}{2}\alpha}\right).$$

$$\sup_{j \in \{1, \dots, N(\omega) - [N(\omega)^\alpha] - 2\}} \left|\sigma_{j+2}(\omega) - 2\sigma_{j+1}(\omega) + \sigma_j(\omega))\right| = \mathcal{O}\left(N(\omega)^{1 - 3\alpha}\right).$$

Continuous part of the decomposition For $\gamma \in (-\infty, k^2(\omega))$, we have [61]

$$\phi_\gamma(\omega, x) = \begin{cases} A_\gamma(\omega) \sin(\eta(\omega) x/d) & \text{if} \quad 0 \le x \le d \\ A_\gamma(\omega) \left(\sin(\eta(\omega)) \cos\left(\xi(\omega)\frac{x-d}{d}\right) + \frac{\eta(\omega)}{\xi(\omega)} \cos(\eta(\omega)) \sin\left(\xi(\omega)\frac{x-d}{d}\right)\right) & \text{if} \quad d \le x, \end{cases}$$

where

$$\eta(\omega) = d\sqrt{n_1^2 k^2(\omega) - \gamma}, \quad \xi(\omega) = d\sqrt{k^2(\omega) - \gamma},$$

and

$$A_\gamma(\omega) = \sqrt{\frac{d\xi(\omega)}{\pi\left(\xi^2(\omega)\sin^2(\eta(\omega)) + \eta^2(\omega)\cos^2(\eta(\omega))\right)}}. \tag{2.10}$$

It is easy to check that the function $\gamma \mapsto A_\gamma(\omega)$ is continuous on $\big(-\infty, k^2(\omega)\big)$ and

$$A_\gamma(\omega) \underset{\gamma \to -\infty}{\sim} \frac{1}{\sqrt{\pi}|\gamma|^{1/4}}. \tag{2.11}$$

We can remark that $\phi_\gamma(\omega, .)$ does not belong to H. Then, $\langle y, \phi_\gamma(\omega, .)\rangle_H$ is not defined in the classical way. In fact,

$$\langle y, \phi_\gamma(\omega, .)\rangle_H = \lim_{M \to +\infty} \int_0^M y(x)\phi_\gamma(\omega, x)dx \quad \text{ on } L^2\big(-\infty, k^2(\omega)\big).$$

Moreover, we have $\forall y \in H$

$$\|y\|_H^2 = \sum_{j=1}^{N(\omega)} \left|\langle y, \phi_j(\omega, .)\rangle_H\right|^2 + \int_{-\infty}^{k^2(\omega)} \left|\langle y, \phi_\gamma(\omega, .)\rangle_H\right|^2 d\gamma.$$

Then,

$$\begin{aligned} \Theta_\omega : H &\longrightarrow \mathcal{H}^\omega \\ y &\longrightarrow \Big(\big(\langle y, \phi_j(\omega, .)\rangle_H\big)_{j=1,\dots,N(\omega)}, \big(\langle y, \phi_\gamma(\omega, .)\rangle_H\big)_{\gamma \in (-\infty, k^2(\omega))}\Big) \end{aligned}$$

is an isometry, from H onto $\mathcal{H}^\omega = \mathbb{C}^{N(\omega)} \times L^2\big(-\infty, k^2(\omega)\big)$.

2.2.2 Modal Decomposition

In this section we apply the spectral decomposition introduced in Section 2.2.1 on a solution $\widehat{p}(\omega, x, z)$ of the equation (2.6). Consequently, we get the modal decomposition for $\widehat{p}(\omega, x, z)$ in the half-space $z > L_S$,

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega, z)\phi_j(\omega, x) + \int_{-\infty}^{k^2(\omega)} \widehat{p}_\gamma(\omega, z)\phi_\gamma(\omega, x)d\gamma,$$

where $\widehat{p}(\omega, z) = \Theta_\omega(\widehat{p}(\omega, ., z))$. For $j \in \{1, \dots, N(\omega)\}$, $\Theta_\omega \circ \Pi_\omega(\{j\})$ represents the projection over the jth propagating mode, and $\widehat{p}_j(\omega, z)$ is the amplitude of the jth propagating mode. $\Theta_\omega \circ \Pi_\omega(0, k^2(\omega))$ represents the projection over the radiating modes, and $\widehat{p}_\gamma(\omega, z)$ is the amplitude of the γth radiating mode for almost every $\gamma \in (0, k^2(\omega))$. Finally, $\Theta_\omega \circ \Pi_\omega(-\infty, 0)$ represents the projection over the evanescent modes and $\widehat{p}_\gamma(\omega, z)$ is the amplitude of the γth evanescent mode for almost every $\gamma \in (-\infty, 0)$.

Consequently, $\widehat{p}(\omega, z)$ satisfies

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega, z) + \beta_j^2(\omega)\widehat{p}_j(\omega, z) = 0,$$
$$\frac{d^2}{dz^2}\widehat{p}_\gamma(\omega, z) + \gamma\ \widehat{p}_\gamma(\omega, z) = 0$$

in $\mathcal{H}^\omega$ and the pressure field can be written as an expansion over the complete set of modes

$$\begin{aligned}\widehat{p}(\omega, x, z) = &\left[\sum_{j=1}^{N(\omega)} \frac{\widehat{a}_{j,0}(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega)z}\phi_j(\omega, x) + \int_0^{k^2(\omega)} \frac{\widehat{a}_{\gamma,0}(\omega)}{\gamma^{1/4}} e^{i\sqrt{\gamma}z}\phi_\gamma(\omega, x)d\gamma\right.\\ &\left.+ \int_{-\infty}^0 \frac{\widehat{c}_{\gamma,0}(\omega)}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|}z}\phi_\gamma(\omega, x)d\gamma\right] \mathbf{1}_{(L_S,+\infty)}(z)\\ &+ \left[\sum_{j=1}^{N(\omega)} \frac{\widehat{b}_{j,0}(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega)z}\phi_j(\omega, x) + \int_0^{k^2(\omega)} \frac{\widehat{b}_{\gamma,0}(\omega)}{\gamma^{1/4}} e^{-i\sqrt{\gamma}z}\phi_\gamma(\omega, x)d\gamma\right.\\ &\left.+ \int_{-\infty}^0 \frac{\widehat{d}_{\gamma,0}(\omega)}{|\gamma|^{1/4}} e^{\sqrt{|\gamma|}z}\phi_\gamma(\omega, x)d\gamma\right] \mathbf{1}_{(-\infty,L_S)}(z),\end{aligned} \tag{2.12}$$

under the assumption that $\left(\widehat{c}_{\gamma,0}(\omega)e^{-\sqrt{|\gamma|}L_S}/|\gamma|^{1/4}\right)_\gamma$ and $\left(\widehat{d}_{\gamma,0}(\omega)e^{\sqrt{|\gamma|}L_S}/|\gamma|^{1/4}\right)_\gamma$ belong to $L^2(-\infty, 0)$. Here, $\beta_j(\omega)$ are the modal wavenumbers.

In the previous decomposition, $\widehat{a}_{j,0}(\omega)$ (resp., $\widehat{b}_{j,0}(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating in the right half-space $z > L_S$ (resp., left half-space $z < L_S$), $\widehat{a}_{\gamma,0}(\omega)$ (resp., $\widehat{b}_{\gamma,0}(\omega)$) is the amplitude of the γth right-going (resp., left-going) mode radiating in the right half-space $z > L_S$ (resp., left half-space $z < L_S$), and $\widehat{c}_{\gamma,0}(\omega)$ (resp., $\widehat{d}_{\gamma,0}(\omega)$) is the amplitude of the γth right-going (resp., left-going) evanescent mode in the right half-space $z > L_S$ (resp., left half-space $z < L_S$).

We assume that the profile $\Psi(t, x)$ of the source term (2.2) is given, in the frequency domain, by

$$\widehat{\Psi}(\omega, x) = \widehat{f}(\omega)\left[\sum_{j=1}^{N(\omega)} \phi_j(\omega, x_0)\phi_j(\omega, x) + \int_{(-S,-\xi)\cup(\xi,k^2(\omega))} \phi_\gamma(\omega, x_0)\phi_\gamma(\omega, x)d\gamma\right], \tag{2.13}$$

where $x_0 \in (0, d)$. The bound S in the spectral decomposition of the source profile was introduced to have $\widehat{\Psi}(\omega, .) \in H$, and ξ was introduced for technical reasons. Note that S can be arbitrarily large and ξ can be arbitrarily small. Therefore, the spatial profile in (2.13) is an approximation of a Dirac distribution at x_0, which models a point source at x_0.

Applying Θ_ω on (2.5) and using (2.12), we get

$$\widehat{a}_{j,0}(\omega) = -\overline{\widehat{b}_{j,0}(\omega)} = \frac{\sqrt{\beta_j(\omega)}}{2}\widehat{f}(\omega)\phi_j(\omega,x_0)e^{-i\beta_j(\omega)L_S} \quad \forall j \in \{1,\dots,N(\omega)\},$$

$$\widehat{a}_{\gamma,0}(\omega) = -\overline{\widehat{b}_{\gamma,0}(\omega)} = \begin{cases} \frac{\gamma^{1/4}}{2}\widehat{f}(\omega)\phi_\gamma(\omega,x_0)e^{-i\sqrt{\gamma}L_S} & \text{for almost every} \quad \gamma \in (\xi, k^2(\omega)) \\ 0 & \text{for almost every} \quad \gamma \in (0,\xi), \end{cases}$$

$$\widehat{c}_{\gamma,0}(\omega) = -\frac{\gamma^{1/4}}{2}\widehat{f}(\omega)\phi_\gamma(\omega,x_0)e^{\sqrt{|\gamma|}L_S}, \quad \widehat{d}_{\gamma,0}(\omega) = \frac{\gamma^{1/4}}{2}\widehat{f}(\omega)\phi_\gamma(\omega,x_0)e^{-\sqrt{|\gamma|}L_S}$$

for almost every $\gamma \in (-S,-\xi)$, and

$$\widehat{c}_{\gamma,0}(\omega) = \widehat{d}_{\gamma,0}(\omega) = 0$$

for almost every $\gamma \in (-\infty,-S) \cup (-\xi,0)$.

2.3 Mode Coupling in Random Waveguides

In this section we study the expansion of $\widehat{p}(\omega,x,z)$ when a random section $[0,L/\epsilon]$ is inserted between two homogeneous waveguides (see Figure 2.2). In this section the medium parameters are given by

$$\frac{1}{K(x,z)} = \begin{cases} \frac{1}{K}\left(n^2(x) + \sqrt{\epsilon}V(x,z)\right) & \text{if} \quad x \in [0,d], \quad z \in [0,L/\epsilon] \\ \frac{1}{K}n^2(x) & \text{if} \quad \begin{cases} x \in [0,+\infty),\ z \in (-\infty,0) \cup (L/\epsilon,+\infty) \\ \text{or} \\ x \in (d,+\infty),\ z \in (-\infty,+\infty). \end{cases} \end{cases}$$

$$\rho(x,z) = \bar{\rho} \quad \text{if} \quad x \in [0,+\infty),\ z \in \mathbb{R}.$$

In the perturbed section, the pressure field can be decomposed using the resolution of the identity Π_ω of the unperturbed waveguide.

$$\widehat{p}(\omega,x,z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega,z)\phi_j(\omega,x) + \int_{-\infty}^{k^2(\omega)} \widehat{p}_\gamma(\omega,z)\phi_\gamma(\omega,x)d\gamma,$$

where $\widehat{p}(\omega,z) = \Theta_\omega(\widehat{p}(\omega,.,z))$. In what follows, we shall consider solutions of the form

$$\widehat{p}(\omega,x,z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega,z)\phi_j(\omega,x) + \int_{(-\infty,-\xi)\cup(\xi,k^2(\omega))} \widehat{p}_\gamma(\omega,z)\phi_\gamma(\omega,x)d\gamma$$

for technical reasons. This assumption lead us to simplified algebra in the proof of Theorem 2.1. In such a decomposition, the radiating and the evanescent part are separated by the small band $(-\xi,\xi)$ with $\xi \ll 1$. The goal is to isolate the transition mode 0 between the radiating and the evanescent part of the spectrum $Sp(R(\omega))$ given by (2.7). Moreover, we assume that $\epsilon \ll \xi$ and therefore we have two distinct scales. Let us remark that in Chapters 2 and 3, we shall consider in a first time the asymptotic ϵ goes to 0 and in a second time the asymptotic ξ goes to 0.

2.3.1 Coupled Mode Equations

In this section we give the coupled mode equations, which describes the coupling mechanism between the amplitudes of the three kinds of modes.

In the random section $[0, L/\epsilon]$ the pressure field $\widehat{p}(\omega, z)$ satisfies the following coupled equations in $\mathcal{H}^\omega$.

$$\begin{aligned} \frac{d^2}{dz^2}\widehat{p}_j(\omega,z) + \beta_j^2(\omega)\widehat{p}_j(\omega,z) + \sqrt{\epsilon}k^2(\omega)\sum_{l=1}^{N(\omega)} C_{jl}^\omega(z)\widehat{p}_l(\omega,z) \\ + \sqrt{\epsilon}k^2(\omega)\int_{(-\infty,-\xi)\cup(\xi,k^2(\omega))} C_{j\gamma'}^\omega(z)\widehat{p}_{\gamma'}(\omega,z)d\gamma' = 0, \\ \frac{d^2}{dz^2}\widehat{p}_\gamma(\omega,z) + \gamma\ \widehat{p}_\gamma(\omega,z) + \sqrt{\epsilon}k^2(\omega)\sum_{l=1}^{N(\omega)} C_{\gamma l}^\omega(z)\widehat{p}_l(\omega,z) \\ + \sqrt{\epsilon}k^2(\omega)\int_{(-\infty,-\xi)\cup(\xi,k^2(\omega))} C_{\gamma\gamma'}^\omega(z)\widehat{p}_{\gamma'}(\omega,z)d\gamma' = 0, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} C_{jl}^\omega(z) = \langle \phi_j(\omega,.), \phi_l(\omega,.)V(.,z)\rangle_H = \int_0^d \phi_j(\omega,x)\phi_l(\omega,x)V(x,z)dx, \\ C_{j\gamma}^\omega(z) = C_{\gamma j}(z) = \langle \phi_j(\omega,.), \phi_\gamma(\omega,.)V(.,z)\rangle_H = \int_0^d \phi_j(\omega,x)\phi_\gamma(\omega,x)V(x,z)dx, \\ C_{\gamma\gamma'}^\omega(z) = \langle \phi_\gamma(\omega,.), \phi_{\gamma'}(\omega,.)V(.,z)\rangle_H = \int_0^d \phi_\gamma(\omega,x)\phi_{\gamma'}(\omega,x)V(x,z)dx. \end{aligned} \tag{2.15}$$

We recall that $\widehat{p}(\omega,.,.) \in \mathcal{C}^0\big((0,+\infty), H_0^1(0,+\infty)\cap H^2(0,+\infty)\big)\cap\mathcal{C}^2\big((0,+\infty),H\big)$, then

$$\int_{-\infty}^{-\xi} \gamma^2|\widehat{p}_\gamma(\omega,z)|^2 d\gamma < +\infty. \tag{2.16}$$

In the previous coupled equation the coefficients $C^\omega(z)$ represent the coupling between the three kinds of modes, which are the propagating, radiating and evanescent modes.

Next, we introduce the amplitudes of the generalized right- and left-going modes $\widehat{a}(\omega,z)$ and $\widehat{b}(\omega,z)$, which are given by

$$\begin{aligned} \widehat{p}_j(\omega,z) &= \frac{1}{\sqrt{\beta_j(\omega)}}\Big(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} + \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\Big), \\ \frac{d}{dz}\widehat{p}_j(\omega,z) &= i\sqrt{\beta_j(\omega)}\Big(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} - \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\Big), \\ \widehat{p}_\gamma(\omega,z) &= \frac{1}{\gamma^{1/4}}\Big(\widehat{a}_\gamma(\omega,z)e^{i\sqrt{\gamma}z} + \widehat{b}_\gamma(\omega,z)e^{-i\sqrt{\gamma}z}\Big), \\ \frac{d}{dz}\widehat{p}_\gamma(\omega,z) &= i\gamma^{1/4}\Big(\widehat{a}_\gamma(\omega,z)e^{i\sqrt{\gamma}z} - \widehat{b}_\gamma(\omega,z)e^{-i\sqrt{\gamma}z}\Big) \end{aligned}$$

$\forall j \in \{1,\dots,N(\omega)\}$ and almost every $\gamma \in (\xi, k^2(\omega))$. Let

$$\mathcal{H}_\xi^\omega = \mathbb{C}^{N(\omega)} \times L^2(\xi, k^2(\omega)).$$

From (2.14), we obtain the coupled mode equation in $\mathcal{H}_\xi^\omega \times \mathcal{H}_\xi^\omega \times L^2(-\infty,-\xi)$ for the amplitudes

$(\widehat{a}(\omega,z),\widehat{b}(\omega,z),\widehat{p}(\omega,z))$:

$$\begin{aligned}\frac{d}{dz}\widehat{a}_j(\omega,z) &= \sqrt{\epsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)}\frac{C^\omega_{jl}(z)}{\sqrt{\beta_j\beta_l}}\left(\widehat{a}_l(\omega,z)e^{i(\beta_l-\beta_j)z}+\widehat{b}_l(\omega,z)e^{-i(\beta_l+\beta_j)z}\right)\\
&+\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_\xi^{k^2(\omega)}\frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j\sqrt{\gamma'}}}\left(\widehat{a}_{\gamma'}(\omega,z)e^{i\left(\sqrt{\gamma'}-\beta_j\right)z}+\widehat{b}_{\gamma'}(\omega,z)e^{-i\left(\sqrt{\gamma'}+\beta_j\right)z}\right)d\gamma'\\
&+\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_{-\infty}^{-\xi}\frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j}}\widehat{p}_{\gamma'}(\omega,z)d\gamma' e^{-i\beta_j z},\end{aligned} \tag{2.17}$$

$$\begin{aligned}\frac{d}{dz}\widehat{a}_\gamma(\omega,z) &= \sqrt{\epsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)}\frac{C^\omega_{\gamma l}(z)}{\sqrt{\sqrt{\gamma}\beta_l}}\left(\widehat{a}_l(\omega,z)e^{i\left(\beta_l-\sqrt{\gamma}\right)z}+\widehat{b}_l(\omega,z)e^{-i\left(\beta_l+\sqrt{\gamma}\right)z}\right)\\
&+\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_\xi^{k^2(\omega)}\frac{C^\omega_{\gamma\gamma'}(z)}{\gamma^{1/4}\gamma'^{1/4}}\left(\widehat{a}_{\gamma'}(\omega,z)e^{i\left(\sqrt{\gamma'}-\sqrt{\gamma}\right)z}+\widehat{b}_{\gamma'}(\omega,z)e^{-i\left(\sqrt{\gamma'}+\sqrt{\gamma}\right)z}\right)d\gamma'\\
&+\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_{-\infty}^{-\xi}\frac{C^\omega_{\gamma\gamma'}(z)}{\gamma^{1/4}}\widehat{p}_{\gamma'}(\omega,z)d\gamma' e^{-i\gamma z},\end{aligned} \tag{2.18}$$

$$\begin{aligned}\frac{d}{dz}\widehat{b}_j(\omega,z) &= -\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)}\frac{C^\omega_{jl}(z)}{\sqrt{\beta_j\beta_l}}\left(\widehat{a}_l(\omega,z)e^{i(\beta_l+\beta_j)z}+\widehat{b}_l(\omega,z)e^{-i(\beta_l-\beta_j)z}\right)\\
&-\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_\xi^{k^2(\omega)}\frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j\sqrt{\gamma'}}}\left(\widehat{a}_{\gamma'}(\omega,z)e^{i\left(\sqrt{\gamma'}+\beta_j\right)z}+\widehat{b}_{\gamma'}(\omega,z)e^{-i\left(\sqrt{\gamma'}-\beta_j\right)z}\right)d\gamma'\\
&-\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_{-\infty}^{-\xi}\frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j}}\widehat{p}_{\gamma'}(\omega,z)d\gamma' e^{-i\beta_j z},\end{aligned} \tag{2.19}$$

$$\begin{aligned}\frac{d}{dz}\widehat{b}_\gamma(\omega,z) &= -\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\sum_{l=1}^{N(\omega)}\frac{C^\omega_{\gamma l}(z)}{\sqrt{\sqrt{\gamma}\beta_l}}\left(\widehat{a}_l(\omega,z)e^{i\left(\beta_l+\sqrt{\gamma}\right)z}+\widehat{b}_l(\omega,z)e^{-i\left(\beta_l-\sqrt{\gamma}\right)z}\right)\\
&-\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_\xi^{k^2(\omega)}\frac{C^\omega_{\gamma\gamma'}(z)}{\gamma^{1/4}\gamma'^{1/4}}\left(\widehat{a}_{\gamma'}(\omega,z)e^{i\left(\sqrt{\gamma'}+\sqrt{\gamma}\right)z}+\widehat{b}_{\gamma'}(\omega,z)e^{-i\left(\sqrt{\gamma'}-\sqrt{\gamma}\right)z}\right)d\gamma'\\
&-\sqrt{\epsilon}\frac{ik^2(\omega)}{2}\int_{-\infty}^{-\xi}\frac{C^\omega_{\gamma\gamma'}(z)}{\gamma^{1/4}}\widehat{p}_{\gamma'}(\omega,z)d\gamma' e^{-i\sqrt{\gamma}z},\end{aligned} \tag{2.20}$$

$$\frac{d^2}{dz^2}\widehat{p}_\gamma(\omega,z)+\gamma\ \widehat{p}_\gamma(\omega,z)+\sqrt{\epsilon}g_\gamma(\omega,z)=0, \tag{2.21}$$

where

$$\begin{aligned}g_\gamma(\omega,z) &= k^2(\omega)\sum_{l=1}^{N(\omega)}\frac{C^\omega_{\gamma l}(z)}{\sqrt{\beta_l}}\left(\widehat{a}_l(\omega,z)e^{i\beta_l z}+\widehat{b}_l(\omega,z)e^{-i\beta_l z}\right)\\
&+k^2(\omega)\int_\xi^{k^2(\omega)}\frac{C^\omega_{\gamma\gamma'}(z)}{\gamma'^{1/4}}\left(\widehat{a}_{\gamma'}(\omega,z)e^{i\sqrt{\gamma'}z}+\widehat{b}_{\gamma'}(\omega,z)e^{-i\sqrt{\gamma'}z}\right)d\gamma'\\
&+k^2(\omega)\int_{-\infty}^{-\xi}C^\omega_{\gamma\gamma'}(z)\widehat{p}_{\gamma'}(\omega,z)d\gamma'.\end{aligned} \tag{2.22}$$

Let us note that in absence of random perturbations, the amplitudes $\widehat{a}(\omega,z)$ and $\widehat{b}(\omega,z)$ are constant.

We assume that a pulse is emitted at the source location L_S and propagates toward the randomly perturbed slab $[0, L/\epsilon]$. Using the previous section, the form of this incident field at $z = 0$ is given by

$$\begin{aligned}\widehat{p}(\omega, x, 0) = &\sum_{j=1}^{N(\omega)} \frac{\widehat{a}_{j,0}(\omega)}{\sqrt{\beta_j(\omega)}}\phi_j(\omega, x) + \int_{\xi}^{k^2(\omega)} \frac{\widehat{a}_{\gamma,0}(\omega)}{\gamma^{1/4}}\phi_\gamma(\omega, x)d\gamma \\ &+ \int_{-S}^{-\xi} \frac{\widehat{c}_{\gamma,0}(\omega)}{|\gamma|^{1/4}}\phi_\gamma(\omega, x)d\gamma.\end{aligned} \tag{2.23}$$

Consequently, by the continuity of the pressure field across the interfaces $z = 0$ and $z = L/\epsilon$, the coupled mode system is complemented with the boundary conditions

$$\widehat{a}(\omega, 0) = \widehat{a}_0(\omega) \quad \text{and} \quad \widehat{b}\left(\omega, \frac{L}{\epsilon}\right) = 0$$

in $\mathcal{H}^\omega_\xi$. For $j \in \{1, \ldots, N(\omega)\}$, $\widehat{a}_{j,0}(\omega)$ represents the initial amplitude of the jth propagating mode, and for $\gamma \in (\xi, k^2(\omega))$, $\widehat{a}_{\gamma,0}(\omega)$ represents the initial amplitude of the γth radiating mode at $z = 0$. Moreover, for $\gamma \in (-S, -\xi)$, $\widehat{c}_{\gamma,0}(\omega)$ represents the initial amplitude of the γth evanescent mode at $z = 0$. The second condition implies that no wave comes from the right homogeneous waveguide.

2.3.2 Energy Flux for the Propagating and Radiating Modes

In this section we study the energy flux for the propagating and radiating modes, and the influence of the evanescent modes on this flux.

We begin this section by introducing the radiation condition for the evanescent modes

$$\lim_{z \to +\infty} \|\Pi_\omega(-\infty, -\xi)(\widehat{p}(\omega, ., z))\|^2_H = 0.$$

This condition means that the energy carried by the evanescent modes decay as the propagation distance becomes large. From the radiation condition and (2.21), we get for almost every $\gamma \in (-\infty, -\xi)$

$$\begin{aligned}\widehat{p}_\gamma(\omega, z) = &\frac{\sqrt{\epsilon}}{2\sqrt{|\gamma|}} \int_0^{z \wedge L/\epsilon} g_\gamma(\omega, u)e^{\sqrt{|\gamma|}(u-z)}du + \frac{\sqrt{\epsilon}}{2\sqrt{|\gamma|}} \int_{z \wedge L/\epsilon}^{L/\epsilon} g_\gamma(\omega, u)e^{\sqrt{|\gamma|}(z-u)}du \\ &+ \phi_\gamma(\omega, x_0)e^{-\sqrt{|\gamma|}(z-L_S)}\mathbf{1}_{(-S,-\xi)}(\gamma)\end{aligned} \tag{2.24}$$

$\forall z \in [0, +\infty)$. According to (2.12), the relation (2.24) can be viewed as a perturbation of the form of the evanescent mode without a random perturbation. Using the same arguments as in [25, Chapter 20], we get $\forall z \in [0, L/\epsilon]$,

$$\frac{d}{dz}(\|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi}) = -\sqrt{\epsilon} Im\left(\int_{-\infty}^{-\xi} \overline{g_\gamma(\omega, z)}\widehat{p}_\gamma(\omega, z)d\gamma\right),$$

and

$$\begin{aligned}\|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} = &\|\widehat{a}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{b}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi} - \frac{\epsilon}{2}\int_{-\infty}^{-\xi} \frac{G_\gamma(\omega, z)}{\sqrt{|\gamma|}}d\gamma \\ &- \sqrt{\epsilon}\int_{-S}^{-\xi} \phi_\gamma(\omega, x_0)e^{\sqrt{|\gamma|}L_S} \int_0^z Im(\overline{g_\gamma(\omega, u)})e^{-\sqrt{|\gamma|}u}du\, d\gamma,\end{aligned} \tag{2.25}$$

where

$$G_\gamma(\omega, z) = \int_0^z \int_z^{L/\epsilon} Im(\overline{g_\gamma(\omega, u)}g_\gamma(\omega, v))e^{\sqrt{|\gamma|}(u-v)}dvdu.$$

Consequently, for $z = L/\epsilon$, we get

$$\begin{aligned}\|\widehat{a}(\omega, L/\epsilon)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, 0)\|^2_{\mathcal{H}^\omega_\xi} &= \|\widehat{a}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi} \\ &- \sqrt{\epsilon}\int_{-S}^{-\xi} \phi_\gamma(\omega, x_0) e^{\sqrt{|\gamma|}L_S} \int_0^{L/\epsilon} Im(\overline{g_\gamma(\omega, u)}) e^{-\sqrt{|\gamma|}u} du\, d\gamma.\end{aligned}$$

The second term on the right side of the previous relation has the factor $\phi_\gamma(\omega, x_0)e^{\sqrt{|\gamma|}L_S}$ which is the form of the evanescent mode at $z = 0$ without a random perturbation. Therefore, if L_S is far away from 0 and whatever the source (evanescent modes decay exponentially from L_S to 0) or if there is no excitation of modes $\gamma \in (-\infty, -\xi)$ by the source (that is when $S = \xi$), we can get the conservation of the global energy flux for the propagating and radiating modes:

$$\|\widehat{a}(\omega, L/\epsilon)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, 0)\|^2_{\mathcal{H}^\omega_\xi} = \|\widehat{a}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi}.$$

However, from (2.25) and even if there is no evanescent modes in (2.23), the local energy flux is not conserved. The energy related to the evanescent modes is given by the last two terms on the right side in (2.25). Let us estimate these two quantities. First,

$$\begin{aligned}\sup_{z\in[0,L/\epsilon]} \left| \int_{-\infty}^{-\xi} \frac{G_\gamma(\omega, z)}{\sqrt{|\gamma|}} d\gamma \right| &\leq K(\xi, d) \sup_{z\in[0,L]} \sup_{x\in[0,d]} \left| V\left(x, \frac{z}{\epsilon}\right)\right|^2 \\ &\times \sup_{z\in[0,L/\epsilon]} \|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega, z)\|^2_{L^1(-\infty,-\xi)}.\end{aligned}$$

Second,

$$\begin{aligned}&\sup_{z\in[0,L/\epsilon]} \Big| \int_{-S}^{-\xi} \phi_\gamma(\omega, x_0) e^{\sqrt{|\gamma|}L_S} \int_0^z Im(\overline{g_\gamma(\omega, u)}) e^{-\sqrt{|\gamma|}u} du\, d\gamma \Big| \\ &\qquad \leq K(\xi, d) \sup_{z\in[0,L]} \sup_{x\in[0,d]} \left| V\left(x, \frac{z}{\epsilon}\right)\right| \\ &\qquad \times \sup_{z\in[0,L/\epsilon]} \|\widehat{a}(\omega, z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega, z)\|_{L^1(-\infty,-\xi)}.\end{aligned}$$

In the two previous inequalities $K(\xi, d)$ represents a constant which can change between the different relations. However, it is difficult to get good a priori estimates about

$$\sup_{z\in[0,L/\epsilon]} \|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega, z)\|^2_{L^1(-\infty,-\xi)}. \tag{2.26}$$

For this reason, let us introduce the stopping "time"

$$L^\epsilon = \inf\left(L > 0, \quad \sup_{z\in[0,L/\epsilon]} \|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega, z)\|^2_{L^1(-\infty,-\xi)} \geq \frac{1}{\sqrt{\epsilon}}\right).$$

The role of this stopping "time" is to limit the size of the random section to ensure that the quantity (2.26) is not too large. Consequently, the energy carried by the evanescent modes over the section $[0, L/\epsilon]$ for $L \leq L^\epsilon$, is at most of order $\mathcal{O}\big(\epsilon^{1/4} \sup_{z\in[0,L/\epsilon]} \sup_{x\in[0,d]} |V(x, z)|^2\big)$, and according to (2.55) the local energy flux for the propagating and the radiating modes is conserved in the asymptotic $\epsilon \to 0$. More precisely, we can show that $\forall \eta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\sup_{z\in[0,L/\epsilon]} \left| \|\widehat{a}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{b}(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{a}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{b}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi} \right| > \eta,\, L \leq L^\epsilon\right) = 0. \tag{2.27}$$

In Section 2.4, we shall see, under the forward scattering approximation, that the condition $L \leq L^\epsilon$ is satisfied in the limit $\epsilon \to 0$, that is we have $\lim_{\epsilon\to 0} \mathbb{P}(L^\epsilon \leq L) = 0$.

2.3.3 Influence of the Evanescent Modes on the Propagating and Radiating Modes

We analyze, in this section, the influence of the evanescent modes on the coupling mechanism between the propagating and the radiating modes.

First of all, we recall that $\Theta_\omega \circ \Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,z)\big)$ represents the evanescent part of the pressure field $\widehat{p}(\omega,.,z)$, where Θ_ω and Π_ω are defined in Section 2.2.1. In this section we consider $F = L^1(-\infty,-\xi)$ equipped with the norm

$$\|y\|_F = \int_{-\infty}^{-\xi} |y_\gamma| d\gamma,$$

which is a Banach space. Substituting (2.22) into (2.24), we get

$$(Id - \sqrt{\epsilon}\Phi^\omega)\Big(\Theta_\omega \circ \Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big)\Big) = \sqrt{\epsilon}\,\tilde{p}(\omega,.) + \tilde{p}_0(\omega,.). \tag{2.28}$$

This equation holds in the Banach space $\big(\mathcal{C}([0,+\infty),F), \|.\|_{\infty,F}\big)$, where

$$\|y\|_{\infty,F} = \sup_{z\geq 0} \|y(z)\|_F \quad \forall y \in \mathcal{C}\big([0,+\infty),F\big).$$

In (2.28), Φ^ω is a linear bounded operator, from $\big(\mathcal{C}([0,+\infty),F), \|.\|_{\infty,F}\big)$ to itself, defined by

$$\begin{aligned}\Phi^\omega_\gamma(y)(z) = & \frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_0^{z\wedge L/\epsilon}\int_{-\infty}^{-\xi} C^\omega_{\gamma\gamma'}(u) y_{\gamma'}(u) d\gamma' e^{\sqrt{|\gamma|}(u-z)} du \\ & + \frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_{z\wedge L/\epsilon}^{L/\epsilon}\int_{-\infty}^{-\xi} C^\omega_{\gamma\gamma'}(u) y_{\gamma'}(u) d\gamma' e^{\sqrt{|\gamma|}(z-u)} du\end{aligned}$$

$\forall z \in [0,+\infty)$, and for almost every $\gamma \in (-\infty,-\xi)$

$$\begin{aligned}\tilde{p}_\gamma(\omega,z) = & \frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_0^{z\wedge L/\epsilon}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{\gamma l}(u)}{\sqrt{\beta_l}}\big(\widehat{a}_l(\omega,u)e^{i\beta_l u} + \widehat{b}_l(\omega,u)e^{-i\beta_l u}\big) \\ & + \int_\xi^{k^2(\omega)} \frac{C^\omega_{\gamma\gamma'}(u)}{\gamma'^{1/4}}\big(\widehat{a}_{\gamma'}(\omega,u)e^{i\sqrt{\gamma'}u} + \widehat{b}_{\gamma'}(\omega,u)e^{-i\sqrt{\gamma'}u}\big)\Big] d\gamma' e^{\sqrt{|\gamma|}(u-z)} du \\ + & \frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_{z\wedge L/\epsilon}^{L/\epsilon}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{\gamma l}(u)}{\sqrt{\beta_l}}\big(\widehat{a}_l(\omega,u)e^{i\beta_l u} + \widehat{b}_l(\omega,u)e^{-i\beta_l u}\big) \\ & + \int_\xi^{k^2(\omega)} \frac{C^\omega_{\gamma\gamma'}(u)}{\gamma'^{1/4}}\big(\widehat{a}_{\gamma'}(\omega,u)e^{i\sqrt{\gamma'}u} + \widehat{b}_{\gamma'}(\omega,u)e^{-i\sqrt{\gamma'}u}\big)\Big] d\gamma' e^{\sqrt{|\gamma|}(z-u)} du\end{aligned}$$

$\forall z \in [0,+\infty)$. Finally, for almost every $\gamma \in (-\infty,-\xi)$ and $\forall z \in [0,+\infty)$,

$$\tilde{p}_{\gamma,0}(\omega,z) = \phi_\gamma(\omega,x_0)e^{-\sqrt{|\gamma|}(z-L_S)}\mathbf{1}_{(-S,-\xi)}(\gamma).$$

We remark that $\Theta_\omega \circ \Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big) \in \mathcal{C}\big([0,+\infty),F\big)$ thanks to (2.16). Moreover, $\tilde{p}(\omega,.) \in \mathcal{C}\big([0,+\infty),F\big)$ since $\int_{-\infty}^{-\xi} \frac{A_\gamma(\omega)}{|\gamma|} d\gamma < +\infty$, where $A_\gamma(\omega)$ is defined by (2.10) and satisfies (2.11). We can check that the norm of the operator Φ^ω is bounded by

$$\|\Phi^\omega\| \leq K(\xi,d) \sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]} |V(x,z)|\,.$$

Consequently, using (2.55), $\lim_{\epsilon\to 0}\mathbb{P}(Id-\sqrt{\epsilon}\,\Phi^\omega \text{ is invertible}) = 1$. Then, the condition $(Id-\sqrt{\epsilon}\,\Phi^\omega$ is invertible) is satisfied in the asymptotic $\epsilon\to 0$. On the event $(Id-\sqrt{\epsilon}\,\Phi^\omega$ is invertible), we have

$$\begin{aligned}\Theta_\omega\circ\Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big) &= \big(Id-\sqrt{\epsilon}\,\Phi^\omega\big)^{-1}\big(\sqrt{\epsilon}\,\tilde{p}(\omega,.)+\tilde{p}_0(\omega,.)\big)\\ &= \sqrt{\epsilon}\,\tilde{p}(\omega,.)+\tilde{p}_0(\omega,.)+\sqrt{\epsilon}\Phi^\omega\big(\tilde{p}_0(\omega,.)\big)\\ &\quad+\sum_{j=1}^{+\infty}(\sqrt{\epsilon}\Phi^\omega)^j\big(\sqrt{\epsilon}\tilde{p}(\omega,.)+\sqrt{\epsilon}\Phi^\omega(\tilde{p}_0(\omega,.))\big).\end{aligned}\tag{2.29}$$

Moreover,

$$\begin{aligned}\|\Theta_\omega\circ\Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big)&-\sqrt{\epsilon}\,\tilde{p}(\omega,.)-\tilde{p}_0(\omega,.)-\sqrt{\epsilon}\Phi^\omega(\tilde{p}_0(\omega,.))\|_{\infty,F}\\ &\leq 2\epsilon\|\Phi^\omega\|\,\|\tilde{p}(\omega,.)\|_{\infty,F}+2\epsilon\|\Phi^\omega\|^2\,\|\tilde{p}_0(\omega,.)\|_{\infty,F}\\ &\leq K(\xi,d)\,\epsilon\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega},\end{aligned}$$

and therefore

$$\begin{aligned}\Theta_\omega\circ\Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big) &= \sqrt{\epsilon}\tilde{p}(\omega,.)+\tilde{p}_0(\omega,.)+\sqrt{\epsilon}\Phi^\omega\big(\tilde{p}_0(\omega,.)\big)\\ &\quad+\mathcal{O}\Big(\epsilon\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega}\Big)\end{aligned}$$

in $\mathcal{C}\big([0,+\infty),F\big)$. Now, we consider

$$\begin{aligned}\tilde{p}_{\gamma,2}(\omega,z) &= \frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_0^{z\wedge L/\epsilon}\Big[\sum_{l=1}^{N(\omega)}\frac{C_{\gamma l}(u)}{\sqrt{\beta_l}}\big(\widehat{a}_l(\omega,z\wedge L/\epsilon)e^{i\beta_l u}+\widehat{b}_l(\omega,z\wedge L/\epsilon)e^{-i\beta_l u}\big)\\ &\quad+\int_\xi^{k^2(\omega)}\frac{C_{\gamma\gamma'}(u)}{\gamma'^{1/4}}\big(\widehat{a}_{\gamma'}(\omega,z\wedge L/\epsilon)e^{i\sqrt{\gamma'}u}+\widehat{b}_{\gamma'}(\omega,z\wedge L/\epsilon)e^{-i\sqrt{\gamma'}u}\big)\Big]d\gamma' e^{\sqrt{|\gamma|}(u-z)}du\\ &\quad+\frac{k^2(\omega)}{2\sqrt{|\gamma|}}\int_{z\wedge L/\epsilon}^{L/\epsilon}\Big[\sum_{l=1}^{N(\omega)}\frac{C_{\gamma l}(u)}{\sqrt{\beta_l}}\big(\widehat{a}_l(\omega,z\wedge L/\epsilon)e^{i\beta_l u}+\widehat{b}_l(\omega,z\wedge L/\epsilon)e^{-i\beta_l u}\big)\\ &\quad+\int_\xi^{k^2(\omega)}\frac{C_{\gamma\gamma'}(u)}{\gamma'^{1/4}}\big(\widehat{a}_{\gamma'}(\omega,z\wedge L/\epsilon)e^{i\sqrt{\gamma'}u}+\widehat{b}_{\gamma'}(\omega,z\wedge L/\epsilon)e^{-i\sqrt{\gamma'}u}\big)\Big]d\gamma' e^{\sqrt{|\gamma|}(z-u)}du\end{aligned}$$

$\forall z\in[0,+\infty)$. Using (2.17), (2.18), (2.19), (2.20), and (2.29), we get

$$\begin{aligned}\|\tilde{p}(\omega,.)-\tilde{p}_2(\omega,.)\|_{\infty,F} &\leq K(\xi,d)\,\sqrt{\epsilon}\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\\ &\quad\times\Big(\sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{p}(\omega,z)\|_F\Big)\end{aligned}$$

and then

$$\begin{aligned}\Theta_\omega\circ\Pi_\omega(-\infty,-\xi)\big(\widehat{p}(\omega,.,.)\big) &= \sqrt{\epsilon}\,\tilde{p}_2(\omega,.)+\tilde{p}_0(\omega,.)+\sqrt{\epsilon}\Phi^\omega\big(\tilde{p}_0(\omega,.)\big)\\ &\quad+\mathcal{O}\Big(\epsilon\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{p}(\omega,z)\|_F\Big)\end{aligned}$$

in $\mathcal{C}\big([0,+\infty),F\big)$. Consequently, we can rewrite (2.17), (2.18), (2.19), and (2.20) in a closed form in $\mathcal{H}_\xi^\omega\times\mathcal{H}_\xi^\omega$. $\forall z\in[0,L/\epsilon]$, we get

$$\begin{aligned}\frac{d}{dz}\widehat{a}(\omega,z) &= \sqrt{\epsilon}\,\mathbf{H}^{aa}(\omega,z)\big(\widehat{a}(\omega,z)\big)+\sqrt{\epsilon}\,\mathbf{H}^{ab}(\omega,z)\big(\widehat{b}(\omega,z)\big)+\sqrt{\epsilon}\,\mathbf{R}^{a,L_S}(\omega,z)\\ &\quad+\epsilon\,\mathbf{G}^{aa}(\omega,z)\big(\widehat{a}(\omega,z)\big)+\epsilon\,\mathbf{G}^{ab}(\omega,z)\big(\widehat{b}(\omega,z)\big)+\epsilon\,\tilde{\mathbf{R}}^{a,L_S}(\omega,z)\\ &\quad+\mathcal{O}\Big(\epsilon^{3/2}\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega}+\|\widehat{p}(\omega,z)\|_F\Big),\end{aligned}$$

$$\frac{d}{dz}\widehat{b}(\omega,z) = \sqrt{\epsilon}\,\mathbf{H}^{ba}(\omega,z)\big(\widehat{a}(\omega,z)\big) + \sqrt{\epsilon}\,\mathbf{H}^{bb}(\omega,z)\big(\widehat{b}(\omega,z)\big) + \sqrt{\epsilon}\,\mathbf{R}^{b,L_S}(\omega,z)$$
$$+ \epsilon\,\mathbf{G}^{ba}(\omega,z)\big(\widehat{a}(\omega,z)\big) + \epsilon\,\mathbf{G}^{bb}(\omega,z)\big(\widehat{b}(\omega,z)\big) + \epsilon\tilde{\mathbf{R}}^{b,L_S}(\omega,z)$$
$$+ \mathcal{O}\Big(\epsilon^{3/2} \sup_{z\in[0,L/\epsilon]} \sup_{x\in[0,d]} |V(x,z)|^2 \sup_{z\in[0,L/\epsilon]} \|\widehat{a}(\omega,z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{b}(\omega,z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega,z)\|_F\Big).$$

Let us recall that these equations hold on the event $\big(Id - \sqrt{\epsilon}\,\Phi^\omega$ is invertible$\big)$ which satisfies $\lim_{\epsilon\to 0}\mathbb{P}(Id - \sqrt{\epsilon}\,\Phi^\omega$ is invertible$) = 1$. In these equations, $\mathbf{H}^{aa}(\omega,z)$, $\mathbf{H}^{ab}(\omega,z)$, $\mathbf{H}^{ba}(\omega,z)$, $\mathbf{H}^{bb}(\omega,z)$, $\mathbf{G}^{aa}(\omega,z)$, $\mathbf{G}^{ab}(\omega,z)$, $\mathbf{G}^{ba}(\omega,z)$ and $\mathbf{G}^{bb}(\omega,z)$ are operators from $\mathcal{H}^\omega_\xi$ to itself defined by:

$$\mathbf{H}^{aa}_j(\omega,z)(y) = \overline{\mathbf{H}^{bb}_j(\omega,z)}(y) = \frac{ik^2(\omega)}{2}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{jl}(z)}{\sqrt{\beta_j(\omega)\beta_l(\omega)}} y_l e^{i(\beta_l(\omega)-\beta_j(\omega))z} + \int_\xi^{k^2(\omega)} \frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j(\omega)\sqrt{\gamma'}}} y_{\gamma'} e^{i(\sqrt{\gamma'}-\beta_j(\omega))z} d\gamma'\Big], \tag{2.30}$$

$$\mathbf{H}^{aa}_\gamma(\omega,z)(y) = \overline{\mathbf{H}^{bb}_\gamma(\omega,z)}(y) = \frac{ik^2(\omega)}{2}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{\gamma l}(z)}{\sqrt{\sqrt{\gamma}\beta_l(\omega)}} y_l e^{i(\beta_l(\omega)-\sqrt{\gamma})z} + \int_\xi^{k^2(\omega)} \frac{C^\omega_{\gamma\gamma'}(z)}{\gamma^{1/4}\gamma'^{1/4}} y_{\gamma'} e^{i(\sqrt{\gamma'}-\sqrt{\gamma})z} d\gamma'\Big], \tag{2.31}$$

$$\mathbf{H}^{ab}_j(\omega,z)(y) = \overline{\mathbf{H}^{ba}_j(\omega,z)}(y) = \frac{ik^2(\omega)}{2}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{jl}(z)}{\sqrt{\beta_j(\omega)\beta_l(\omega)}} y_l e^{-i(\beta_l(\omega)+\beta_j(\omega))z} + \int_\xi^{k^2(\omega)} \frac{C^\omega_{j\gamma'}(z)}{\sqrt{\beta_j(\omega)\sqrt{\gamma'}}} y_{\gamma'} e^{-i(\sqrt{\gamma'}+\beta_j(\omega))z} d\gamma'\Big], \tag{2.32}$$

$$\mathbf{H}^{ab}_\gamma(\omega,z)(y) = \overline{\mathbf{H}^{ba}_\gamma(\omega,z)}(y) = \frac{ik^2(\omega)}{2}\Big[\sum_{l=1}^{N(\omega)} \frac{C^\omega_{\gamma l}(z)}{\sqrt{\sqrt{\gamma}\beta_l(\omega)}} y_l e^{-i(\beta_l(\omega)+\sqrt{\gamma})z} + \int_\xi^{k^2(\omega)} \frac{C_{\gamma\gamma'}(z)}{\gamma^{1/4}\gamma'^{1/4}} y_{\gamma'} e^{-i(\sqrt{\gamma'}+\sqrt{\gamma})z} d\gamma'\Big], \tag{2.33}$$

$$\begin{aligned} \mathbf{G}^{aa}_j(\omega,z)(y) = \overline{\mathbf{G}^{bb}_j(\omega,z)}(y) = \\ \frac{ik^4(\omega)}{4}\Big[\sum_{l=1}^{N(\omega)} \int_{-\infty}^{-\xi} \Big[\int_0^z \frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma' l}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\beta_l(\omega)}} e^{i\beta_l(\omega)u - \sqrt{|\gamma'|}(z-u)} du \\ + \int_z^{L/\epsilon} \frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma' l}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\beta_l(\omega)}} e^{i\beta_l(\omega)u - \sqrt{|\gamma'|}(u-z)} du\Big] d\gamma' e^{-i\beta_j(\omega)z} y_l\Big] \\ + \frac{ik^4(\omega)}{4}\Big[\int_\xi^{k^2(\omega)} \int_{-\infty}^{-\xi} \Big[\int_0^z \frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\sqrt{\gamma''}}} e^{i\sqrt{\gamma''}u - \sqrt{|\gamma'|}(z-u)} du \\ + \int_z^{L/\epsilon} \frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\sqrt{\gamma''}}} e^{i\sqrt{\gamma''}u - \sqrt{|\gamma'|}(u-z)} du\Big] d\gamma' e^{-i\beta_j(\omega)z} y_{\gamma''} d\gamma''\Big], \end{aligned} \tag{2.34}$$

$$
\begin{aligned}
&\mathbf{G}^{aa}_{\gamma}(\omega,z)(y) = \overline{\mathbf{G}^{bb}_{\gamma}(\omega,z)}(y) = \\
&\quad \frac{ik^4(\omega)}{4}\Big[\sum_{l=1}^{N(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\beta_l(\omega)}}e^{i\beta_l(\omega)u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\beta_l(\omega)}}e^{i\beta_l(\omega)u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\sqrt{\gamma}z}y_l\Big] \\
&\quad + \frac{ik^4(\omega)}{4}\Big[\int_{\xi}^{k^2(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\sqrt{\gamma''}}}e^{i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\sqrt{\gamma''}}}e^{i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\sqrt{\gamma}z}y_{\gamma''}d\gamma''\Big],
\end{aligned} \tag{2.35}
$$

$$
\begin{aligned}
&\mathbf{G}^{ab}_{j}(\omega,z)(y) = \overline{\mathbf{G}^{ba}_{j}(\omega,z)}(y) = \\
&\quad \frac{ik^4(\omega)}{4}\Big[\sum_{l=1}^{N(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{j\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\beta_l(\omega)}}e^{-i\beta_l(\omega)u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{j\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\beta_l(\omega)}}e^{-i\beta_l(\omega)u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\beta_j(\omega)z}y_l\Big] \\
&\quad + \frac{ik^4(\omega)}{4}\Big[\int_{\xi}^{k^2(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{j\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\sqrt{\gamma''}}}e^{-i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{j\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|\sqrt{\gamma''}}}e^{-i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\beta_j(\omega)z}y_{\gamma''}d\gamma''\Big],
\end{aligned} \tag{2.36}
$$

$$
\begin{aligned}
&\mathbf{G}^{ab}_{\gamma}(\omega,z)(y) = \overline{\mathbf{G}^{ba}_{\gamma}(\omega,z)}(y) = \\
&\quad \frac{ik^4(\omega)}{4}\Big[\sum_{l=1}^{N(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\beta_l(\omega)}}e^{-i\beta_l(\omega)u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma' l}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\beta_l(\omega)}}e^{-i\beta_l(\omega)u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\sqrt{\gamma}z}y_l\Big] \\
&\quad + \frac{ik^4(\omega)}{4}\Big[\int_{\xi}^{k^2(\omega)}\int_{-\infty}^{-\xi}\Big[\int_0^z \frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\sqrt{\gamma''}}}e^{-i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(z-u)}du \\
&\qquad + \int_z^{L/\epsilon}\frac{C^{\omega}_{\gamma\gamma'}(z)C^{\omega}_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|\sqrt{\gamma''}}}e^{-i\sqrt{\gamma''}u-\sqrt{|\gamma'|}(u-z)}du\Big]d\gamma' e^{-i\sqrt{\gamma}z}y_{\gamma''}d\gamma''\Big].
\end{aligned} \tag{2.37}
$$

The operators $\mathbf{H}^{aa}(\omega,z)$ and $\mathbf{H}^{ab}(\omega,z)$ represent the coupling between the propagating and the radiating modes with themselves, while the operators $\mathbf{G}^{aa}(\omega,z)$ and $\mathbf{G}^{ab}(\omega,z)$ represent the coupling between the evanescent modes with the propagating and the radiating modes. Moreover, $\mathbf{R}^{a,L_S}(\omega,z)$, $\tilde{\mathbf{R}}^{a,L_S}(\omega,z)$, $\mathbf{R}^{b,L_S}(\omega,z)$, and $\tilde{\mathbf{R}}^{b,L_S}(\omega,z)$ represent the influence of the evanescent modes produced by the source term on the propagating and the radiating modes. These terms are defined by

$$
\mathbf{R}^{a,L_S}_j(\omega,z) = \overline{\mathbf{R}^{b,L_S}_j(\omega,z)} = \frac{ik^2(\omega)}{2}\int_{-S}^{-\xi}\frac{C^{\omega}_{j\gamma'}(z)}{\sqrt{\beta_j(\omega)}}\phi_{\gamma'}(\omega,x_0)e^{-\sqrt{|\gamma'|}(z-L_S)}d\gamma' e^{-i\beta_j(\omega)z}, \tag{2.38}
$$

$$
\mathbf{R}^{a,L_S}_{\gamma}(\omega,z) = \overline{\mathbf{R}^{b,L_S}_{\gamma}(\omega,z)} = \frac{ik^2(\omega)}{2}\int_{-S}^{-\xi}\frac{C^{\omega}_{\gamma\gamma'}(z)}{|\gamma|^{1/4}}\phi_{\gamma'}(\omega,x_0)e^{-\sqrt{|\gamma'|}(z-L_S)}d\gamma' e^{-i\sqrt{\gamma}z}, \tag{2.39}
$$

$$\tilde{\mathbf{R}}_j^{a,L_S}(\omega,z) = \overline{\tilde{\mathbf{R}}_j^{b,L_S}(\omega,z)} =$$

$$\frac{ik^4(\omega)}{4}\int_{-\infty}^{-\xi}\int_{-S}^{-\xi}\Bigg[\int_0^z \frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|}}\phi_{\gamma''}(\omega,x_0)e^{-\sqrt{|\gamma''|}(u-L_S)}e^{-\sqrt{|\gamma'|}(z-u)}du$$
$$+\int_z^{L/\epsilon}\frac{C^\omega_{j\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\beta_j(\omega)|\gamma'|}}\phi_{\gamma''}(\omega,x_0)e^{-\sqrt{|\gamma''|}(u-L_S)}e^{-\sqrt{|\gamma'|}(u-z)}du\Bigg]\,d\gamma''d\gamma'\,e^{-i\beta_j(\omega)z}, \tag{2.40}$$

$$\tilde{\mathbf{R}}_\gamma^{a,L_S}(\omega,z) = \overline{\tilde{\mathbf{R}}_\gamma^{b,L_S}(\omega,z)} =$$

$$\frac{ik^4(\omega)}{4}\int_{-\infty}^{-\xi}\int_{-S}^{-\xi}\Bigg[\int_0^z \frac{C^\omega_{\gamma\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|}}\phi_{\gamma''}(\omega,x_0)e^{-\sqrt{|\gamma''|}(u-L_S)}e^{-\sqrt{|\gamma'|}(z-u)}du$$
$$+\int_z^{L/\epsilon}\frac{C^\omega_{\gamma\gamma'}(z)C^\omega_{\gamma'\gamma''}(u)}{\sqrt{\sqrt{\gamma}|\gamma'|}}\phi_{\gamma''}(\omega,x_0)e^{-\sqrt{|\gamma''|}(u-L_S)}e^{-\sqrt{|\gamma'|}(u-z)}du\Bigg]\,d\gamma''d\gamma'\,e^{-i\sqrt{\gamma}z}. \tag{2.41}$$

2.3.4 Forward Scattering Approximation

In this section we introduce the forward scattering approximation, which is widely used in the literature. In this approximation the coupling between forward- and backward-propagating modes is assumed to be negligible compared to the coupling between the forward-propagating modes. We refer to [30, 33] for justifications on the validity of this approximation.

The physical explanation is as follows. The coupling between a right-going propagating mode and a left-going propagating mode involves a coefficient of the form

$$\int_0^{+\infty}\mathbb{E}[C^\omega_{jl}(0)C^\omega_{jl}(z)]\cos\left((\beta_l(\omega)+\beta_j(\omega))z\right)dz,$$

and the coupling between two right-going propagating modes or two left-going propagating modes involves a coefficient of the form

$$\int_0^{+\infty}\mathbb{E}[C^\omega_{jl}(0)C^\omega_{jl}(z)]\cos\left((\beta_l(\omega)-\beta_j(\omega))z\right)dz$$

$\forall(j,l)\in\{1,\dots,N(\omega)\}^2$. Therefore, if we assume that

$$\int_0^{+\infty}\mathbb{E}[C^\omega_{jl}(0)C^\omega_{jl}(z)]\cos\left((\beta_l(\omega)+\beta_j(\omega))z\right)dz=0\quad\forall(j,l)\in\{1,\dots,N(\omega)\}^2.$$

There is no coupling between right-going and left-going propagating modes, which justifies the forward scattering approximation, but there is still coupling between right-going propagating modes which will be described in Section 2.4.

In our context the operator $R(\omega)$, introduced in Section 2.2.1, has a continuous spectrum and it becomes technically complex to apply a limit theorem for the rescaled process $(\widehat{a}(\omega,z/\epsilon),\widehat{b}(\omega,z/\epsilon))$. The reason is the following. This process is not bounded and the stopping times which are the first exit times of closed balls are not lower semicontinuous for the topology of $\mathcal{C}([0,L],\mathcal{H}^\omega_{\xi,w})$, where $\mathcal{H}^\omega_{\xi,w}$ stands for $\mathcal{H}^\omega_\xi$ equipped with the weak topology. In our context the continuous part $(\xi,k^2(\omega))$ of the spectrum imposes us to use the norm $\|.\|_{\mathcal{H}^\omega_\xi}$ to control some quantities. Moreover, according to Theorem 2.1, in which the energy of the limit process is not conserved, it seems not possible to show a limit theorem on $\mathcal{C}([0,L],(\mathcal{H}^\omega_\xi,\|.\|_{\mathcal{H}^\omega_\xi}))$ in view of (2.27). In [25] and [30] there is a finite number of propagating modes, then the weak topology and the strong topology are the same. In [33] or in Chapter 4 the number of propagating modes increases as ϵ goes to 0. However, in this last case, the problem can be corrected by considering the first exit times of a closed ball related to the weak topology and by considering the process in an appropriate finite-dimensional dual space.

In our context if we forget these technical problems, according to [25, 30] the forward scattering approximation should be valid in the asymptotic $\epsilon \to 0$ under the assumption that the power spectral density of the process V, i.e. the Fourier transform of its z-autocorrelation function, possesses a cut-off wavenumber. In other words, we can consider the case where

$$\int_0^{+\infty} \mathbb{E}[C^\omega_{jl}(0)C^\omega_{jl}(z)] \cos\left((\beta_l(\omega)+\beta_j(\omega))z\right)dz = 0 \quad \forall (j,l) \in \{1,\dots,N(\omega)\}^2.$$

Let us remark that the continuous part $(0, k^2(\omega))$ of the spectrum, which corresponds to the radiating modes, does not play any role in the previous assumption. The reason is that the radiating part of the process plays no role in the coupling mechanism as we can see in Theorems 2.1 and 2.2 below and therefore remains constant.

Finally, we shall consider the simplified equation on $[0, L/\epsilon]$,

$$\begin{aligned}\frac{d}{dz}\widehat{a}(\omega,z) &= \sqrt{\epsilon}\,\mathbf{H}^{aa}(\omega,z)\left(\widehat{a}(\omega,z)\right) + \sqrt{\epsilon}\,\mathbf{R}^{a,L_S}(\omega,z)\\ &+ \epsilon\,\mathbf{G}^{aa}(\omega,z)\left(\widehat{a}(\omega,z)\right) + \epsilon\,\tilde{\mathbf{R}}^{a,L_S}(\omega,z)\\ &+ \mathcal{O}\Big(\epsilon^{3/2}\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2 \sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega,z)\|_F\Big)\end{aligned}$$

in $\mathcal{H}^\omega_\xi$. We shall see in Section 2.4, under the forward scattering approximation, that

$$\lim_{\epsilon\to 0}\mathbb{P}\left(L^\epsilon \leq L\right) = 0 \quad \forall\, L>0,$$

where

$$L^\epsilon = \inf\Big(L>0, \quad \sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|^2_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega,z)\|^2_F \geq \frac{1}{\sqrt{\epsilon}}\Big).$$

Consequently, we can show that $\forall\,\eta>0$

$$\lim_{\epsilon\to 0}\mathbb{P}\Big(\sup_{z\in[0,L/\epsilon]}\Big|\|\widehat{a}(\omega,z)\|^2_{\mathcal{H}^\omega_\xi} - \|\widehat{a}_0(\omega)\|^2_{\mathcal{H}^\omega_\xi}\Big| > \eta\Big) = 0.$$

This result means that the local energy flux for the propagating and the radiating modes is conserved in the asymptotic $\epsilon \to 0$.

2.4 Coupled Mode Processes

In this section, we study the asymptotic behavior, as $\epsilon \to 0$ in first and $\xi \to 0$ in second, of the statistical properties of the coupling mechanism in terms of a diffusion process.

Let us define the rescaled process

$$\widehat{a}^\epsilon(\omega,z) = \widehat{a}\left(\omega,\frac{z}{\epsilon}\right) \quad \forall z\in[0,L].$$

This scaling corresponds to the size of the random section $[0, L/\epsilon]$. This process satisfies the rescaled coupled mode equations on $[0, L]$

$$\begin{aligned}\frac{d}{dz}\widehat{a}^\epsilon(\omega,z) &= \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\left(\widehat{a}^\epsilon(\omega,z)\right) + \frac{1}{\sqrt{\epsilon}}\mathbf{R}^{a,L_S}\left(\omega,\frac{z}{\epsilon}\right)\\ &+ \mathbf{G}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\left(\widehat{a}^\epsilon(\omega,z)\right) + \tilde{\mathbf{R}}^{a,L_S}\left(\omega,\frac{z}{\epsilon}\right)\\ &+ \mathcal{O}\Big(\sqrt{\epsilon}\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}|V(x,z)|^2\sup_{z\in[0,L]}\|\widehat{a}^\epsilon(\omega,z)\|_{\mathcal{H}^\omega_\xi} + \|\widehat{p}(\omega,z/\epsilon)\|_F\Big)\end{aligned} \tag{2.42}$$

in $\mathcal{H}_\xi^\omega$, with the initial condition $\widehat{a}^\epsilon(\omega,0)=\widehat{a}_0(\omega)$. We shall see that under the forward scattering approximation the condition $L^\epsilon > L$ is readily fulfilled in the asymptotic ϵ goes to 0.

Proposition 2.1 *$\forall L>0$,*

$$\lim_{\epsilon\to 0}\mathbb{P}\left(L^\epsilon\leq L\right)=0,$$

where

$$L^\epsilon=\inf\Big(L>0,\qquad \sup_{z\in[0,L/\epsilon]}\|\widehat{a}(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}+\|\widehat{p}(\omega,z)\|^2_F\geq\frac{1}{\sqrt{\epsilon}}\Big),$$

and

$$\lim_{M\to+\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{z\in[0,L]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}\geq M\right)=0.$$

This result means that the amplitude $\widehat{a}^\epsilon(\omega,z)$ is asymptotically uniformly bounded in the limit $\epsilon\to 0$ on $[0,L]$. More precisely, according to Section 2.3.2, we have $\forall\,\eta>0$

$$\lim_{\epsilon\to 0}\mathbb{P}\Big(\sup_{z\in[0,L]}\Big|\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}-\|\widehat{a}_0(\omega)\|^2_{\mathcal{H}_\xi^\omega}\Big|>\eta\Big)=0,$$

that is the local energy flux for the propagating and the radiating modes is conserved in the asymptotic $\epsilon\to 0$.

Proof Using Gronwall's inequality, $\forall L>0$ we get

$$\lim_{M\to+\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{z\in[0,L]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}\geq M,\,L\leq L^\epsilon\right)=0.$$

This result means that the process $\widehat{a}^\epsilon(\omega,.)$ is asymptotically uniformly bounded on $[0,L]$ and then L^ϵ is large compared to L in the asymptotic $\epsilon\to 0$. In fact, $\forall L>0$ and $\forall M>0$

$$\begin{aligned}\mathbb{P}\left(L^\epsilon\leq L\right)\leq{}&\mathbb{P}\left(L^\epsilon\leq L,\quad\sup_{z\in[0,L\wedge L^\epsilon]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}\leq M\right)\\&+\mathbb{P}\left(\sup_{z\in[0,L\wedge L^\epsilon]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}\geq M\right).\end{aligned}$$

Moreover,

$$\mathbb{P}\left(L^\epsilon\leq L,\quad\sup_{z\in[0,L\wedge L^\epsilon]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}\leq M\right)=0$$

for ϵ small enough, since for $L^\epsilon\leq L$

$$\begin{aligned}\epsilon^{-1/2}\leq{}&\sup_{z\in[0,L^\epsilon]}\|\widehat{a}^\epsilon(\omega,z)\|^2_{\mathcal{H}_\xi^\omega}+\|\widehat{p}(\omega,.)\|^2_F\\&\leq M+K(\xi,d)\epsilon\sup_{z\in[0,L/\epsilon]}\sup_{x\in[0,d]}\left|V\left(x,z\right)\right|^2M+2\|\tilde{p}_0(\omega,.)\|^2_{\infty,F}\end{aligned}$$

according to (2.29). ■

Let us introduce $\widehat{a}_1^\epsilon(\omega,.)$ the unique solution of the differential equation on $[0,L]$

$$\frac{d}{dz}\widehat{a}_1^\epsilon(\omega,z)=\frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\left(\widehat{a}_1^\epsilon(\omega,z)\right)+\mathbf{G}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\left(\widehat{a}_1^\epsilon(\omega,z)\right)\tag{2.43}$$

in $\mathcal{H}^{\omega}_{\xi}$, with initial condition $\widehat{a}^{\epsilon}_1(\omega,0)=\widehat{a}_0(\omega)$. Using Gronwall's inequality and (2.54) we can state that

$$\lim_{M\to+\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{z\in[0,L]}\|\widehat{a}^{\epsilon}_1(\omega,z)\|_{\mathcal{H}^{\omega}_{\xi}}\geq M\right)=0.$$

The relation between the solution of the full system (2.42) and the one of the simplified system (2.43) is given by the following proposition.

Proposition 2.2

$$\forall\eta>0 \textit{ and } \forall\mu>0,\quad \lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{z\in[\mu,L]}\|\widehat{a}^{\epsilon}(\omega,z)-\widehat{a}^{\epsilon}_1(\omega,z)\|_{\mathcal{H}^{\omega}_{\xi}}>\eta\right)=0.$$

Proposition 2.2 means that the information about the evanescent part of the source profile is lost in the asymptotic ϵ goes to 0. In fact, the coupling mechanism described by the system (2.42) implies that the information about the evanescent part of the source profile is transmitted to the propagating modes through the coefficients $\mathbf{R}^{a,L_S}(\omega,z)$ and $\tilde{\mathbf{R}}^{a,L_S}(\omega,z)$ defined by (2.38), (2.39), (2.40) and (2.41) page 46. In these expressions we have the term $\phi_{\gamma'}(\omega,x)e^{-\sqrt{|\gamma'|}(z-L_S)}$ which comes from the right-hand side of (2.24) page 41 and which is the form of evanescent modes without a random perturbation. This term is responsible for the loss of information about the evanescent part of the source profile because of its exponentially decreasing behavior.

Proof We begin by proving that $\forall L>0$, $\forall\eta>0$ and $\forall\mu>0$

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{z\in[\mu,L]}\|\widehat{a}^{\epsilon}(\omega,z)-\widehat{a}^{\epsilon}_1(\omega,z)\|^2_{\mathcal{H}^{\omega}_{\xi}}>\eta,\, L\leq L^{\epsilon}\right)=0.$$

In fact, $\mathbf{R}^{a,L_S}(\omega,z)$ decreases exponentially fast with the propagation distance. Moreover, $\tilde{\mathbf{R}}^{a,L_S}(\omega,z)$ can be treated as $\mathbf{G}^{aa}$ in the proof of Theorem 2.1 because $e^{-\sqrt{|\gamma'|}(u-L_S)}$ cannot be compensated by $e^{-i\beta_j(\omega)z}$ nor by $e^{-i\sqrt{\gamma}z}$. Moreover, using Proposition 2.1 we get the result. ■

Finally, we introduce the transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega,z)$ from $\mathcal{H}^{\omega}_{\xi}$ to itself, which is the unique operator solution of the differential equation

$$\frac{d}{dz}\mathbf{T}^{\xi,\epsilon}(\omega,z)=\frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^{\xi,\epsilon}(\omega,z)+\mathbf{G}^{aa}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^{\xi,\epsilon}(\omega,z) \tag{2.44}$$

with $\mathbf{T}^{\xi,\epsilon}(\omega,0)=Id$. Then,

$$\forall z\in[0,L],\quad \widehat{a}_1(\omega,z)=\mathbf{T}^{\xi,\epsilon}(\omega,z)(\widehat{a}_0(\omega)),$$

and we get the following result.

Proposition 2.3

$$\forall\eta>0 \textit{ and } \forall\mu>0,\quad \lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{z\in[\mu,L]}\|\widehat{a}^{\epsilon}(\omega,z)-\mathbf{T}^{\xi,\epsilon}(\omega,z)(\widehat{a}_0(\omega))\|^2_{\mathcal{H}^{\omega}_{\xi}}>\eta\right)=0.$$

2.4.1 Limit Theorem

This section presents the basic theoretical results of this chapter. Proofs are given in the appendix at the end of this chapter. In [30] and [39], the authors used the limit theorem stated in [48] since the number of propagating modes was fixed. However, in our configuration, in addition to the $N(\omega)$-discrete propagating modes the wave field consists of a continuum of radiating modes. The two following results are based on a diffusion-approximation result for the solution of an ordinary differential equation with random coefficients. This result is an extension of that stated in [48] to the case of processes with values in a Hilbert space.

Theorem 2.1 *$\forall L > 0$ and $\forall y \in \mathcal{H}_\xi^\omega = \mathbb{C}^{N(\omega)} \times L^2(\xi, k^2(\omega))$, the family $\big(\mathbf{T}^{\xi,\epsilon}(\omega,.)(y)\big)_{\epsilon\in(0,1)}$, solution of the differential equation (2.44), converges in distribution on $\mathcal{C}([0,L],\mathcal{H}_{\xi,w}^\omega)$ as $\epsilon \to 0$ to a limit denoted by $\mathbf{T}^\xi(\omega,.)(y)$. Here $\mathcal{H}_{\xi,w}^\omega$ stands for the Hilbert space $\mathcal{H}_\xi^\omega$ equipped with the weak topology. This limit is the unique diffusion process on $\mathcal{H}_\xi^\omega$, starting from y, associated to the infinitesimal generator*

$$\mathcal{L}_\xi^\omega = \mathcal{L}_1^\omega + \mathcal{L}_{2,\xi}^\omega + \mathcal{L}_{3,\xi}^\omega,$$

where

$$\begin{aligned}
\mathcal{L}_1^\omega =& \frac{1}{2}\sum_{\substack{j,l=1\\ j\neq l}}^{N(\omega)} \Gamma_{jl}^c(\omega)\left(T_j\overline{T_j}\partial_{T_l}\partial_{\overline{T_l}} + T_l\overline{T_l}\partial_{T_j}\partial_{\overline{T_j}} - T_jT_l\partial_{T_j}\partial_{T_l} - \overline{T_j}\,\overline{T_l}\partial_{\overline{T_j}}\partial_{\overline{T_l}}\right)\\
&+\frac{1}{2}\sum_{j,l=1}^{N(\omega)} \Gamma_{jl}^1(\omega)\left(T_j\overline{T_l}\partial_{T_j}\partial_{\overline{T_l}} + \overline{T_j}T_l\partial_{\overline{T_j}}\partial_{T_l} - T_jT_l\partial_{T_j}\partial_{T_l} - \overline{T_j}\,\overline{T_l}\partial_{\overline{T_j}}\partial_{\overline{T_l}}\right)\\
&+\frac{1}{2}\sum_{j=1}^{N(\omega)}\left(\Gamma_{jj}^c(\omega) - \Gamma_{jj}^1(\omega)\right)\left(T_j\partial_{T_j} + \overline{T_j}\partial_{\overline{T_j}}\right) + \frac{i}{2}\sum_{j=1}^{N(\omega)}\Gamma_{jj}^s(\omega)\left(T_j\partial_{T_j} - \overline{T_j}\partial_{\overline{T_j}}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{2,\xi}^\omega &= -\frac{1}{2}\sum_{j=1}^{N(\omega)}\left(\Lambda_j^{c,\xi}(\omega) + i\Lambda_j^{s,\xi}(\omega)\right)T_j\partial_{T_j} + \left(\Lambda_j^{c,\xi}(\omega) - i\Lambda_j^{s,\xi}(\omega)\right)\overline{T_j}\partial_{\overline{T_j}},\\
\mathcal{L}_{3,\xi}^\omega &= i\sum_{j=1}^{N(\omega)}\kappa_j^\xi(\omega)\left(T_j\partial_{T_j} - \overline{T_j}\partial_{\overline{T_j}}\right).
\end{aligned}$$

Here, we have considered the classical complex derivative with the following notation: If $v = v_1 + iv_2$, then $\partial_v = \frac{1}{2}(\partial_{v_1} - i\partial_{v_2})$ and $\partial_{\overline{v}} = \frac{1}{2}(\partial_{v_1} + i\partial_{v_2})$. We have used the following notations. $\forall (j,l) \in \{1,\dots,N(\omega)\}^2$ and $j \neq l$

$$\begin{aligned}
\Gamma_{jl}^c(\omega) &= \frac{k^4(\omega)}{2\beta_j(\omega)\beta_l(\omega)}\int_0^{+\infty}\mathbb{E}[C_{jl}^\omega(0)C_{jl}^\omega(z)]\cos\left((\beta_l(\omega)-\beta_j(\omega))z\right)dz,\\
\Gamma_{jj}^c(\omega) &= -\sum_{\substack{l=1\\ l\neq j}}^{N(\omega)}\Gamma_{jl}^c(\omega),\\
\Gamma_{jl}^s(\omega) &= \frac{k^4(\omega)}{2\beta_j(\omega)\beta_l(\omega)}\int_0^{+\infty}\mathbb{E}[C_{jl}^\omega(0)C_{jl}^\omega(z)]\sin\left((\beta_l(\omega)-\beta_j(\omega))z\right)dz,\\
\Gamma_{jj}^s(\omega) &= -\sum_{\substack{l=1\\ l\neq j}}^{N(\omega)}\Gamma_{jl}^s(\omega),
\end{aligned}$$

and $\forall (j,l) \in \{1,\dots,N(\omega)\}^2$,

$$\Gamma^1_{jl}(\omega) = \frac{k^4(\omega)}{2\beta_j(\omega)\beta_l(\omega)} \int_0^{+\infty} \mathbb{E}[C^\omega_{jj}(0)C^\omega_{ll}(z)]dz,$$

$$\Lambda^{c,\xi}_j(\omega) = \int_\xi^{k^2(\omega)} \frac{k^4(\omega)}{2\sqrt{\gamma'}\beta_j(\omega)} \int_0^{+\infty} \mathbb{E}[C^\omega_{j\gamma'}(0)C^\omega_{j\gamma'}(z)] \cos\left((\sqrt{\gamma'} - \beta_j(\omega))z\right) dz d\gamma',$$

$$\Lambda^{s,\xi}_j(\omega) = \int_\xi^{k^2(\omega)} \frac{k^4(\omega)}{2\sqrt{\gamma'}\beta_j(\omega)} \int_0^{+\infty} \mathbb{E}[C^\omega_{j\gamma'}(0)C^\omega_{j\gamma'}(z)] \sin\left((\sqrt{\gamma'} - \beta_j(\omega))z\right) dz d\gamma',$$

$$\kappa^\xi_j(\omega) = \int_{-\infty}^{-\xi} \frac{k^4(\omega)}{2\beta_j(\omega)\sqrt{|\gamma'|}} \int_0^{+\infty} \mathbb{E}[C^\omega_{j\gamma'}(0)C^\omega_{j\gamma'}(z)] \cos\left(\beta_j(\omega)z\right) e^{-\sqrt{|\gamma'|}z} dz d\gamma'.$$

The coupling coefficients $C^\omega(z)$ are defined by (2.15) page 39. We get the following result in the asymptotic $\xi \to 0$.

Theorem 2.2 *$\forall L > 0$ and $\forall y \in \mathcal{H}^\omega_0 = \mathbb{C}^{N(\omega)} \times L^2(0, k^2(\omega))$, the family $\left(\mathbf{T}^\xi(\omega,.)(y)\right)_{\xi\in(0,1)}$ converges in distribution on $\mathcal{C}([0,L], (\mathcal{H}^\omega_0, \|.\|_{\mathcal{H}^\omega_0}))$ as $\xi \to 0$ to a limit denoted by $\mathbf{T}^0(\omega,.)(y)$. This limit is the unique diffusion process on $\mathcal{H}^\omega_0$, starting from y, associated to the infinitesimal generator*

$$\mathcal{L}^\omega = \mathcal{L}^\omega_1 + \mathcal{L}^\omega_2 + \mathcal{L}^\omega_3,$$

where

$$\mathcal{L}^\omega_2 = -\frac{1}{2} \sum_{j=1}^{N(\omega)} \left(\Lambda^c_j(\omega) + i\Lambda^s_j(\omega)\right) T_j \partial_{T_j} + \left(\Lambda^c_j(\omega) - i\Lambda^s_j(\omega)\right) \overline{T_j} \partial_{\overline{T_j}},$$

$$\mathcal{L}^\omega_3 = i \sum_{j=1}^{N(\omega)} \kappa_j(\omega) \left(T_j \partial_{T_j} - \overline{T_j} \partial_{\overline{T_j}}\right).$$

Here, we have $\forall j \in \{1,\dots,N(\omega)\}$

$$\Lambda^c_j(\omega) = \lim_{\xi\to 0} \Lambda^{c,\xi}_j(\omega), \quad \Lambda^s_j(\omega) = \lim_{\xi\to 0} \Lambda^{s,\xi}_j(\omega), \quad \kappa_j(\omega) = \lim_{\xi\to 0} \kappa^\xi_j(\omega).$$

Theorems 2.1 and 2.2 describe the asymptotic behavior, as $\epsilon \to 0$ in first and $\xi \to 0$ in second, of the statistical properties of the transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega, L)$, in terms of a diffusion process.

The infinitesimal generator $\mathcal{L}^\omega$ is composed of three parts which represent different behaviors on the diffusion process. We can remark that the infinitesimal generator depends only on the $N(\omega)$-discrete coordinates. Therefore, the radiating part of the limit process remains constant in $L^2(0, k^2(\omega))$ during the propagation and does not play any role in the diffusion process of the propagating modes. The first operator $\mathcal{L}^\omega_1$ describes the coupling between the $N(\omega)$-propagating modes. This part is of the form of the infinitesimal generator obtained in [25, 30], and the total energy is conserved. The second operator $\mathcal{L}^\omega_2$ describes the coupling between the propagating modes with the radiating modes. This part implies a mode-dependent and frequency-dependent attenuation on the $N(\omega)$-propagating modes that we study in Section 2.5.1, and a mode-dependent and frequency-dependent phase modulation. The third operator $\mathcal{L}^\omega_3$ describes the coupling between the propagating and the evanescent modes, and implies a mode-dependent and frequency-dependent phase modulation. The purely imaginary part of the operator $\mathcal{L}^\omega$ does not remove energy from the propagating modes but gives an effective dispersion.

Moreover, let us remark that the convergence in Theorem 2.1 holds on $\mathcal{C}([0,L], (\mathcal{H}^\omega_\xi, \|.\|_{\mathcal{H}^\omega_\xi}))$ for the $N(\omega)$-discrete propagating mode amplitudes.

2.4.2 Mean Mode Amplitudes

In this section we study the asymptotic mean mode amplitudes. From Theorem 2.2, we get the following result about the mean mode amplitudes.

Proposition 2.4 $\forall y \in \mathcal{H}_0^\omega$, $\forall z \in [0, L]$, $\forall j \in \{1, \dots, N(\omega)\}$

$$\lim_{\xi \to 0} \lim_{\epsilon \to 0} \mathbb{E}\Big[\mathbf{T}_j^{\xi,\epsilon}(\omega, z)(y)\Big] = \mathbb{E}\Big[\mathbf{T}_j^0(\omega, z)(y)\Big]$$
$$= \exp\left[\left(\frac{\Gamma_{jj}^c(\omega) - \Gamma_{jj}^1(\omega) - \Lambda_j^c(\omega)}{2}\right) z + i\left(\frac{\Gamma_{jj}^s(\omega) - \Lambda_{jj}^s(\omega)}{2} + k_j(\omega)\right) z\right] y_j(\omega). \tag{2.45}$$

First, let us remark that the mean amplitude of the radiating part remains constant on $L^2(0, k^2(\omega))$. Second, $\forall j \in \{1, \dots, N(\omega)\}$, the coefficient $(\Gamma_{jj}^1(\omega) + \Lambda_j^c(\omega) - \Gamma_{jj}^c(\omega))/2$ is nonnegative. In fact, for $(j, l) \in \{1, \dots, N(\omega)\}^2$ such that $j \neq l$, $\Gamma_{jl}^c(\omega)$ and $\Gamma_{jj}^1(\omega)$ are nonnegative because they are proportional to the power spectral density of C_{jl}^ω and C_{jj}^ω at $\beta_l(\omega) - \beta_j(\omega)$ and 0 frequencies. Therefore, $-\Gamma_{jj}^c(\omega)$ is also nonnegative. Moreover, $\Lambda_j^c(\omega)$ is also nonnegative because it is proportional to the integral over $(0, k^2(\omega))$ of the power spectral density of $C_{j\gamma}^\omega$ at $\sqrt{\gamma} - \beta_j(\omega)$ frequency.

The exponential decay rate for the mean jth-propagating mode is given by

$$\left|\mathbb{E}\Big[\mathbf{T}_j^0(\omega, L)(y)\Big]\right| = |y_j| \exp\left[-\left(\frac{\Gamma_{jj}^1(\omega) - \Gamma_{jj}^c(\omega) + \Lambda_j^c(\omega)}{2}\right) L\right],$$

which depends on the effective coupling between the propagating modes, and the coupling between the propagating and the radiating modes. This exponential decay corresponds to a loss of coherence of the transmitted field.

2.5 Coupled Power Equations

This section is devoted to the analysis of the asymptotic mean mode powers of the propagating modes. More precisely, we study the asymptotic effects of the coupling between the propagating modes with the radiating modes. Let

$$\mathcal{T}_j^l(\omega, z) = \lim_{\xi \to 0} \lim_{\epsilon \to 0} \mathbb{E}\Big[\big|\mathbf{T}_j^{\xi,\epsilon}(\omega, L)(y^l)\big|^2\Big] = \mathbb{E}\Big[\big|\mathbf{T}_j^0(\omega, z)(y^l)\big|^2\Big], \tag{2.46}$$

be the asymptotic mean mode power of the jth propagating modes. $\mathcal{T}_j^l(\omega, L)$ is the expected power of the jth propagating mode at the propagation distance $z = L$. Here $y^l \in \mathcal{H}_0^\omega$ is defined by $y_j^l = \delta_{jl}$ and $y_\gamma^l = 0$ for $\gamma \in (0, k^2(\omega))$, and where δ_{jl} is the Kronecker symbol. The initial condition y^l means that an impulse equal to one charges only the lth propagating mode. From Theorem 2.2, we have the coupled power equations:

$$\frac{d}{dz}\mathcal{T}_j^l(\omega, z) = -\Lambda_j^c(\omega)\mathcal{T}_j^l(\omega, z) + \sum_{\substack{n=1 \\ n \neq j}}^{N(\omega)} \Gamma_{nj}^c(\omega)\left(\mathcal{T}_n^l(\omega, z) - \mathcal{T}_j^l(\omega, z)\right), \tag{2.47}$$

with initial conditions $\mathcal{T}_j^l(\omega, 0) = \delta_{jl}$. These equations describe the transfer of energy between the propagating modes and $\Gamma^c(\omega)$ is the energy transport matrix. In our context, we also have the coefficients $\Lambda_j^c(\omega)$ given by the coupling between the propagating modes with the radiating modes. These coefficients, defined in Theorem 2.2, are responsible for the radiative loss of energy in the ocean bottom (see Figure 2.3). This loss of energy is described more precisely in the following section.

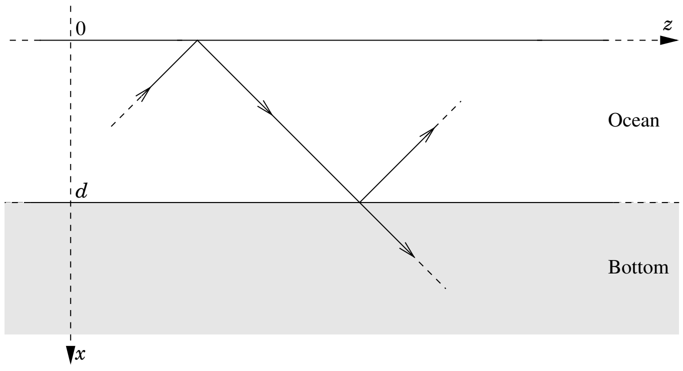


Figure 2.3: Illustration of the radiative loss in the shallow-water random waveguide model.

2.5.1 Exponential Decay of the Propagating Modes Energy

In this section, we assume that at least one of the coefficients $\Lambda^c(\omega)$ is positive. With this assumption, we show that the total energy carried by the propagating modes decays exponentially with the size L of the random section. In the opposite situation, that is when there is no radiative loss $\Lambda^c(\omega)=0$, it has been shown in [30] and [25, Chapter 20] that the energy of the propagating modes is conserved and for large L the asymptotic distribution of the energy becomes uniform over the propagating modes.

Let us define

$$\mathcal{S}_+^{N(\omega)}=\Big\{X\in\mathbb{R}^{N(\omega)},\quad X_j\geq 0\quad \forall j\in\{1,\dots,N(\omega)\}\text{ and }\|X\|^2_{2,\mathbb{R}^{N(\omega)}}=\langle X,X\rangle_{\mathbb{R}^{N(\omega)}}=1\Big\}$$

with $\langle X,Y\rangle_{\mathbb{R}^{N(\omega)}}=\sum_{j=1}^{N(\omega)}X_jY_j$ for $(X,Y)\in(\mathbb{R}^{N(\omega)})^2$, and

$$\Lambda^c_d(\omega)=diag\big(\Lambda^c_1(\omega),\dots,\Lambda^c_{N(\omega)}(\omega)\big).$$

Theorem 2.3 *Let us assume that the energy transport matrix $\Gamma^c(\omega)$ is irreducible. Then, we have*

$$\lim_{L\to+\infty}\frac{1}{L}\ln\left[\sum_{j=1}^{N(\omega)}\mathcal{T}^l_j(\omega,L)\right]=-\Lambda_\infty(\omega)$$

with

$$\Lambda_\infty(\omega)=\inf_{X\in\mathcal{S}_+^{N(\omega)}}\big\langle\big(-\Gamma^c(\omega)+\Lambda^c_d(\omega)\big)X,X\big\rangle_{\mathbb{R}^{N(\omega)}},\tag{2.48}$$

which is positive as soon as one of the coefficients $\Lambda^c_j(\omega)$ is positive.

This result means that the total energy carried by the expected powers of the propagating modes decays exponentially with the propagation distance, and the decay rate can be expressed in terms of a variational formula over a finite-dimensional space.

Proof The coupled power equations admit a probabilistic representation in terms of a jump Markov process. If we denote by $\big(Y^{N(\omega)}_t\big)_{t\geq 0}$ a jump Markov process with state space $\{1,\dots,N(\omega)\}$ and intensity matrix $\Gamma^c(\omega)$, then we have using the Feynman-Kac formula:

$$\mathcal{T}^l_j(\omega,z)=\mathbb{E}\left[\exp\Big(-\int_0^z\Lambda^c_{Y^{N(\omega)}_s}(\omega)ds\Big)\mathbf{1}_{\big(Y^{N(\omega)}_z=j\big)}\Big|Y^{N(\omega)}_0=l\right].\tag{2.49}$$

Moreover, we have supposed that $\Gamma^c(\omega)$ is irreducible. Then, $\big(Y_t^{N(\omega)}\big)_{t\geq 0}$ in an ergodic process with invariant measure $\mu_{N(\omega)}$, which is the uniform distribution over $\{1,\dots,N(\omega)\}$. That is, $\mu_{N(\omega)}(j) = 1/N(\omega)\ \forall j \in \{1,\dots,N(\omega)\}$. The self-adjoint generator of the jump Markov process $(Y_t^{N(\omega)})_{t\geq 0}$ is given by

$$\mathcal{L}^{N(\omega)}\phi(j) = \sum_{n=1}^{N(\omega)} \Gamma^c_{nj}(\omega)\left(\phi(n) - \phi(j)\right),$$

for every function ϕ from $\{1,\dots,N(\omega)\}$ to $\mathbb{R}$, and it is easy to check that $\mathcal{L}^{N(\omega)}\mu_{N(\omega)} = 0$. Let us consider the local times

$$l_T(j) = \int_0^T \mathbf{1}_{\left(Y_s^{N(\omega)}=j\right)} ds$$

for $j \in \{1,\dots,N(\omega)\}$ and $T > 0$, which corresponds to the time spent by the process $\big(Y_t^{N(\omega)}\big)_{t\geq 0}$ in the state j during the time interval $[0,T]$. According to [20], we have a large deviation principle for $\frac{1}{T}l_T$ viewed as a random process with values in $\mathcal{M}_1^{N(\omega)}$ which is the set of probability measures on $\{1,\dots,N(\omega)\}$. More precisely, we have

$$\begin{aligned}\lim_{L\to+\infty}\frac{1}{L}\ln\mathbb{E}\Big[\exp\Big(-L\,\langle\Lambda^c,\frac{1}{L}l_L\rangle_{\mathbb{R}^{N(\omega)}}\Big)\Big|Y_0^{N(\omega)}=l\Big]&\\ &= \lim_{L\to+\infty}\frac{1}{L}\ln\mathbb{E}\Big[\exp\Big(-\int_0^L \Lambda^c_{Y_s^{N(\omega)}}ds\Big)\Big|Y_0^{N(\omega)}=l\Big]\\ &= -\inf_{\mu\in\mathcal{M}_1^{N(\omega)}}\left(I(\mu) + \langle\Lambda^c(\omega),\mu\rangle\right)\end{aligned}$$

with

$$I(\mu) = \big\|\big(-\Gamma^c(\omega)\big)^{1/2}\sqrt{\mu}\big\|^2_{2,\mathbb{R}^{N(\omega)}} = \big\langle\big(-\Gamma^c(\omega)\big)\sqrt{\mu},\sqrt{\mu}\big\rangle_{\mathbb{R}^{N(\omega)}}.$$

Consequently,

$$\lim_{L\to+\infty}\frac{1}{L}\ln\left[\sum_{j=1}^{N(\omega)}\mathcal{T}_j^l(\omega,L)\right] = -\Lambda_\infty(\omega).$$

Let us assume that $\Lambda_\infty(\omega) = 0$. As $\mathcal{S}_+^{N(\omega)}$ is a compact space, there exists $X_0 \in \mathcal{S}_+^{N(\omega)}$ such that

$$\Lambda_\infty(\omega) = \big\langle\big(-\Gamma^c(\omega) + \Lambda_d^c(\omega)\big)X_0, X_0\big\rangle_{\mathbb{R}^{N(\omega)}} = 0.$$

Moreover, $-\Gamma^c(\omega)$ and $\Lambda_d^c(\omega)$ are two nonnegative matrices and 0 is a simple eigenvalue of $-\Gamma^c(\omega)$ by the Perron-Frobenius theorem. Then,

$$\big\langle(-\Gamma^c(\omega))X_0, X_0\big\rangle_{\mathbb{R}^{N(\omega)}} = 0 \Leftrightarrow X_0 = \sqrt{\mu_{N(\omega)}},$$

and

$$\big\langle\Lambda_d^c(\omega)X_0, X_0\big\rangle_{\mathbb{R}^{N(\omega)}} = 0 \Rightarrow \exists j\in\{1,\dots,N(\omega)\},\quad X_0(j) = 0.$$

Therefore,

$$\Lambda_\infty(\omega) > 0.$$

■

The expression (2.48) of $\Lambda_\infty(\omega)$ is not simple. However, we have the following inequalities.

$$\min_{j\in\{1,\dots,N(\omega)\}}\Lambda_j^c(\omega) \quad\leq\quad \Lambda_\infty(\omega) \quad\leq\quad \overline{\Lambda}(\omega) = \frac{1}{N(\omega)}\sum_{j=1}^{N(\omega)}\Lambda_j^c(\omega), \tag{2.50}$$

but the lower bound is not a sharp bound. In fact, this bound is equal to 0 if the vector $\Lambda^c(\omega)$ has only one coordinate equal to 0. To finish this section, let us investigate some special cases in which we can give a simple expression of $\Lambda_\infty(\omega)$.

First, we assume that $\forall j \in \{1, \dots, N(\omega)\}$, $\Lambda^c_j(\omega) = \Lambda(\omega) > 0$. In this case, using (2.50)

$$\Lambda_\infty(\omega) = \Lambda(\omega).$$

This means that if all the coefficients which represent the radiation losses are equal, the decay rate of the total energy of the propagating modes is given by this coefficient.

Second, we assume that the coupling matrix is small, that is we replace $\Gamma^c(\omega)$ by $\tau\Gamma^c(\omega)$ with $\tau \ll 1$. If $\forall j \in \{1, \dots, N(\omega)\}$, $\Lambda^c_j(\omega) > 0$ we have

$$\lim_{\tau\to 0} \Lambda^\tau_\infty(\omega) = \min_{j\in\{1,\dots,N(\omega)\}} \Lambda^c_j(\omega).$$

From (2.50), it is the smallest value that $\Lambda_\infty(\omega)$ can take. This result is consistent with the fact that the coupling process on the transfer of energy between propagating modes is negligible and the decay rate of the energy of a particular propagating mode j is given by its own decay coefficient $\Lambda_j(\omega)$. Then, for the total energy of propagating modes the decay rate is given by the minimum of those decay coefficients. Consequently, if there exists $\Lambda^c_{j_0}(\omega) = 0$, we have

$$\lim_{\tau\to 0} \Lambda^\tau_\infty(\omega) = 0.$$

The reason is the energy of the j_0th propagating mode stays approximately constant with a weak transfer of energy, and

$$\lim_{\tau\to 0} \frac{1}{\tau}\Lambda^\tau_\infty(\omega) = \inf_{X\in\tilde{V}} \big\langle\big(- \Gamma^c(\omega)\big)X, X\big\rangle_{\mathbb{R}^{N(\omega)}} > 0,$$

where

$$\tilde{V} = \Big\{X \in \mathcal{S}^{N(\omega)}_+, \quad supp X \subset \{1, \dots, N(\omega)\} \setminus supp(\Lambda^c(\omega))\Big\},$$

because $\sqrt{\mu_{N(\omega)}} \notin \tilde{V}$.

Now, we assume that the coupling matrix is large, that is we replace $\Gamma^c(\omega)$ by $\frac{1}{\tau}\Gamma^c(\omega)$ with $\tau \ll 1$. In this case, we have

$$\lim_{\tau\to 0} \Lambda^\tau_\infty(\omega) = \overline{\Lambda}(\omega).$$

From (2.50), it is the largest value that $\Lambda_\infty(\omega)$ can take. The strong coupling produces a uniform distribution of energy over the propagating modes and the decay rate becomes $\big\langle\Lambda^c(\omega), \mu_{N(\omega)}\big\rangle_{\mathbb{R}^{N(\omega)}} = \overline{\Lambda}(\omega)$ for each mode. A more convenient way to get this result is to use a probabilistic representation. In fact, we have

$$\begin{aligned}\mathcal{T}^l_j(\omega, z) &= \mathbb{E}\left[\exp\left(-\int_0^z \Lambda^c_{Y^{N(\omega)}_{s/\tau}}(\omega)\right)\mathbf{1}_{\left(Y^{N(\omega)}_z = j\right)}\Big| Y^{N(\omega)}_0 = l\right]\\ &= \mathbb{E}\left[\exp\left(-z\frac{\tau}{z}\int_0^{z/\tau} \Lambda^c_{Y^{N(\omega)}_s}(\omega)\right)\mathbf{1}_{\left(Y^{N(\omega)}_{z/\tau} = j\right)}\Big| Y^{N(\omega)}_0 = l\right],\end{aligned}$$

where $\big(Y^{N(\omega)}_t\big)_{t\geq 0}$ is a jump Markov process with state space $\{1, \dots, N(\omega)\}$ and intensity matrix $\Gamma^c(\omega)$. Using the ergodic properties of $\big(Y^{N(\omega)}_t\big)_{t\geq 0}$, we get that

$$\lim_{\tau\to 0} \mathcal{T}^{\tau,l}_j(\omega, L) = \frac{1}{N(\omega)} \exp\left(-\overline{\Lambda}(\omega)L\right).$$

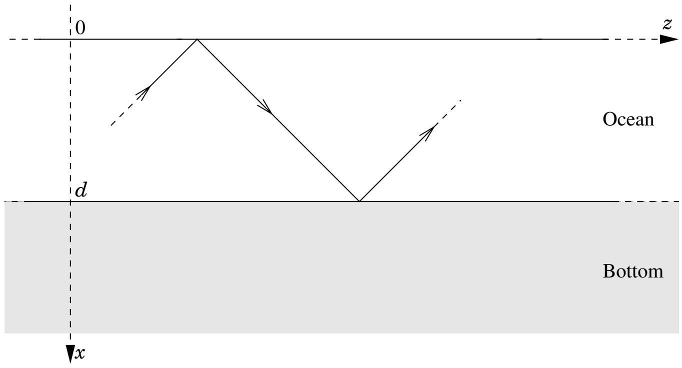


Figure 2.4: Illustration of negligible radiation losses in the shallow-water random waveguide model.

Finally, if we assume that the radiation losses are negligible, that is we replace $\Lambda^c(\omega)$ by $\tau\Lambda^c(\omega)$ with $\tau \ll 1$, we have

$$\lim_{\tau\to 0} \Lambda^\tau_\infty(\omega) = 0.$$

In fact, if the radiative loss is negligible, the coupling process becomes dominant, and we can show that

$$\forall L>0, \qquad \sup_{z\in[0,L]} \|\mathcal{T}^{\tau,l}_j(\omega,z) - \mathcal{T}^{0,l}_j(\omega,z)\|_{2,\mathbb{R}^{N(\omega)}} = \mathcal{O}(\tau),$$

where $\mathcal{T}^{0,l}(\omega,.)$ satisfies (2.47) without the coefficient $\Lambda^c(\omega)$. In this situation

$$\mathcal{T}^{0,l}_j(\omega,L) = \mathbb{P}\left(Y^{N(\omega)}_L = j \middle| Y^{N(\omega)}_0 = l\right),$$

and the total energy is conserved (see Figure 2.4), and

$$\lim_{\tau\to 0} \frac{1}{\tau}\Lambda^\tau_\infty(\omega) = \overline{\Lambda}(\omega) > 0.$$

As it was already observed in [30] the modal energy distribution converges as $L\to+\infty$ to a uniform distribution:

$$\lim_{L\to+\infty} \mathcal{T}^{0,l}_j(\omega,L) = \frac{1}{N(\omega)}.$$

2.5.2 High-Frequency Approximation to Coupled Power Equations

In this section we give, under the assumption that nearest neighbor coupling is the main power transfer mechanism, an approximation of the solution of the coupled power equations (2.47) in the high-frequency regime or in the limit of large number of propagating modes $N(\omega)\gg 1$. Let us note that the limit of a large number of propagating modes $N(\omega)\gg 1$ corresponds to the high-frequency regime $\omega\to+\infty$. Next, we analyze the energy carried by the propagating modes in this regime.

The coupled power equations can be approximated in the high-frequency regime by a diffusion equation. This approximation has been already obtained in [39] for instance, in which we can find further references about this topic. We can also refer to [44] for more discussions on this approximation. For an application of such a diffusion model to acoustic propagation in random sound channels we refer to [45], and for applications to time reversal of waves we refer to [33] and Chapters 3 and 4 of this manuscript .

Using the form of the covariance function (2.52) page 65, we find

$$\Gamma^c_{jl}(\omega) = \frac{ak^4(\omega)I_{j,l}(\omega)}{2\beta_j(\omega)\beta_l(\omega)(a^2 + (\beta_j(\omega) - \beta_l(\omega))^2)}$$

and

$$\Lambda^c_j(\omega) = \int_0^{k^2(\omega)} \frac{ak^4(\omega)I_{j,\gamma}(\omega)}{2\beta_j(\omega)\sqrt{\gamma}(a^2 + (\beta_j(\omega) - \sqrt{\gamma})^2)} d\gamma,$$

where

$$\begin{aligned} I_{jl} =& \frac{1}{4}A_j^2 A_l^2 \Big[S(\sigma_j - \sigma_l, \sigma_j - \sigma_l) + S(\sigma_j + \sigma_l, \sigma_j + \sigma_l) \\ & - S(\sigma_j - \sigma_l, \sigma_j + \sigma_l) - S(\sigma_j + \sigma_l, \sigma_j - \sigma_l) \Big], \\ I_{j\gamma} =& \frac{1}{4}A_j^2 A_\gamma^2 \Big[S(\sigma_j - \eta, \sigma_j - \eta) + S(\sigma_j + \eta, \sigma_j + \eta) \\ & - S(\sigma_j - \eta, \sigma_j + \eta) - S(\sigma_j + \eta, \sigma_j - \eta) \Big], \end{aligned}$$

with

$$S(v_1, v_2) = \int_0^d \int_0^d \gamma_0(x_1, x_2) \cos\big(\frac{v_1}{d}x_1\big) \cos\big(\frac{v_2}{d}x_2\big) dx_1 dx_2,$$

and where $A_j(\omega)$, $A_\gamma(\omega)$, $\sigma_j(\omega)$, $\eta(\omega)$, $\phi_j(\omega, x)$, and $\phi_\gamma(\omega, x)$ are defined in Section 2.2.1.

Band-Limiting Idealization

In this section, we introduce a band-limiting idealization hypothesis in which the power spectral density of the random fluctuations is assumed to be limited in both the transverse and the longitudinal directions.

We assume that the support of S lies in the square $\big[-\frac{3\pi}{2}, \frac{3\pi}{2}\big] \times \big[-\frac{3\pi}{2}, \frac{3\pi}{2}\big]$. Then,

$$I_{jl}(\omega) = \begin{cases} \frac{1}{4}A_j^2(\omega)A_l^2(\omega)S(\sigma_j(\omega) - \sigma_l(\omega), \sigma_j(\omega) - \sigma_l(\omega)) & \text{if} \quad |j - l| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$I_{j\gamma}(\omega) = \begin{cases} \frac{1}{4}A_j^2(\omega)A_l^2(\omega)S(\sigma_j(\omega) - \eta(\omega), \sigma_j(\omega) - \eta(\omega)) & \text{if} \quad |\sigma_j(\omega) - \eta(\omega)| \leq \frac{3\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

From this assumption we get $\forall\, 0 < \gamma < k^2(\omega)$ and $j \in \{1, \dots, N(\omega) - 2\}$,

$$\begin{aligned} \eta(\omega) - \sigma_j(\omega) \geq n_1 k(\omega) d \sqrt{1 - \frac{1}{n_1^2}} - \sigma_j(\omega) &\geq n_1 k(\omega) d\theta - (N(\omega) - 2)\pi \\ &\geq \pi \underbrace{\left(\frac{n_1 k(\omega) d}{\pi}\theta - N(\omega) \right)}_{\in[0,1)} + 2\pi. \end{aligned}$$

Then, for $j \in \{1, \dots, N(\omega) - 2\}$,

$$\inf_{0<\gamma<k^2} \eta(\omega) - \sigma_j(\omega) > \frac{3\pi}{2},$$

and

$$\Lambda^c_j(\omega) = 0, \quad \forall j \in \{1, \dots, N(\omega) - 2\}.$$

Consequently, the coupled power equations (2.47) become

$$\begin{aligned}
\frac{d}{dz}\mathcal{T}^l_N(z) &= -\Lambda^c_N \mathcal{T}^l_N(z) + \Gamma^c_{N-1\,N}\left(\mathcal{T}^l_{N-1}(z) - \mathcal{T}^l_N(z)\right),\\
\frac{d}{dz}\mathcal{T}^l_{N-1}(z) &= -\Lambda^c_{N-1}\mathcal{T}^l_{N-1}(z) + \Gamma^c_{N-1\,N-2}\left(\mathcal{T}^l_{N-2}(z) - \mathcal{T}^l_{N-1}(z)\right)\\
&\quad + \Gamma^c_{N-1\,N}\left(\mathcal{T}^l_{N}(z) - \mathcal{T}^l_{N-1}(z)\right),\\
\frac{d}{dz}\mathcal{T}^l_j(z) &= \Gamma^c_{j-1\,j}\left(\mathcal{T}^l_{j-1}(z) - \mathcal{T}^l_j(z)\right) + \Gamma^c_{j+1\,j}\left(\mathcal{T}^l_{j+1}(z) - \mathcal{T}^l_j(z)\right) \text{ for } j \in \{2,\dots,N-2\},\\
\frac{d}{dz}\mathcal{T}^l_1(z) &= \Gamma^c_{2\,1}\left(\mathcal{T}^l_2(z) - \mathcal{T}^l_1(z)\right),
\end{aligned} \tag{2.51}$$

with $\mathcal{T}^l_j(0) = \delta_{jl}$.

The band-limiting idealization hypothesis is tantamount to a nearest neighbor coupling. More precisely, this assumption implies that $\forall (j,l) \in \{1,\dots,N(\omega)\}^2$ the jth mode amplitude can exchange informations with the lth amplitude mode if they are direct neighbors, that is, if they satisfy $|j-l| \leq 1$.

High-Frequency Approximation

The evolution of the mean mode powers of the propagating modes can be described, in the high-frequency regime or in the limit of a large number of propagating modes $N(\omega) \gg 1$, by a diffusion model. This diffusive continuous model is equipped with boundary conditions which take into account the effect of the radiating modes at the bottom and the free surface of the waveguide (see Figure 2.3 page 54).

Let, $\forall \varphi \in \mathcal{C}^0([0,1])$, $\forall u \in [0,1]$, and $z \geq 0$,

$$\mathcal{T}^{N(\omega)}_\varphi(z,u) = \mathcal{T}^{[N(\omega)u]}_\varphi(\omega,z) = \sum_{j=1}^{N(\omega)} \varphi\Big(\frac{j}{N(\omega)}\Big)\mathcal{T}^{[N(\omega)u]}_j(\omega,z),$$

where $\varphi \mapsto \mathcal{T}^{N(\omega)}_\varphi(z,.)$ can be extended to an operator from $L^2(0,1)$ to itself. Here, $L^2(0,1)$ is equipped with the inner product defined as follows: $\forall(\varphi,\psi) \in L^2(0,1)^2$

$$\langle \varphi,\psi\rangle_{L^2(0,1)} = \int_0^1 \varphi(v)\psi(v)dv.$$

Theorem 2.4 *We have*

1. $\forall \varphi \in L^2(0,1)$ *and* $\forall z \geq 0$,

$$\lim_{\omega\to+\infty} \mathcal{T}^{N(\omega)}_\varphi(z,u) = \mathcal{T}_\varphi(z,u) \quad \textit{in } L^2(0,1),$$

where $\mathcal{T}_\varphi(z,u)$ *satisfies the partial differential equation :* $\forall (z,u) \in (0,+\infty)\times(0,1)$,

$$\frac{\partial}{\partial z}\mathcal{T}_\varphi(z,u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_\varphi\right)(z,u),$$

with the boundary conditions

$$\frac{\partial}{\partial u}\mathcal{T}_\varphi(z,0) = 0, \quad \mathcal{T}_\varphi(z,1) = 0, \quad \textit{and} \quad \mathcal{T}_\varphi(0,u) = \varphi(u),$$

$\forall z > 0$.

2. $\forall u \in [0,1]$, $\forall z \geq 0$, *and* $\forall \varphi \in \mathcal{C}^0([0,1])$ *such that* $\varphi(1) = 0$, *we have*

$$\lim_{\omega \to +\infty} \mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi(z,u).$$

Here,

$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$

with $a_0 = \frac{\pi^2 S_0}{2an_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d \int_0^d \gamma_0(x_1, x_2) \cos\left(\frac{\pi}{d} x_1\right) \cos\left(\frac{\pi}{d} x_2\right) dx_1 dx_2$. n_1 *is the index of refraction in the ocean section* $[0,d]$, $1/a = l_{z,x}$ *is the correlation length of the random inhomogeneities in the longitudinal direction, and* γ_0 *is the covariance function of the random inhomogeneities in the transverse direction.*

This theorem is a continuum approximation in the limit of a large number of propagating modes $N(\omega) \gg 1$. This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between the $N(\omega)$-discrete propagating modes, with a reflecting boundary condition at $x = 0$ (the top of the waveguide in Figure 2.2 page 33) and an absorbing boundary condition at $u = 1$ (the bottom of the waveguide in Figure 2.2) which represents the radiative loss (see Figure 2.3).

Exponential Decay in the High-Frequency Regime

In this high-frequency regime, we also observe that the energy carried by the continuum of propagating modes decays exponentially with the propagation distance. The exponential decay of the energy in the high-frequency regime is given by the following result.

Theorem 2.5 $\forall \varphi \in L^2(0,1) \setminus \{0\}$ *such that* $\varphi \geq 0$, *and* $\forall u \in [0,1)$,

$$\lim_{L \to +\infty} \frac{1}{L} \ln\left[\mathcal{T}_\varphi(L,u)\right] = -\Lambda_\infty,$$

where

$$\Lambda_\infty = \inf_{\varphi \in \mathcal{D}} \int_0^1 a_\infty(v) \varphi'(v)^2 dv > 0$$

and

$$\mathcal{D} = \left\{ \varphi \in \mathcal{C}^\infty([0,1]), \quad \|\varphi\|_{L^2(0,1)} = 1, \quad \frac{\partial}{\partial v}\varphi(0) = 0, \quad \varphi(1) = 0 \right\}.$$

This result means that the energy carried by each propagating mode decays exponentially with the propagation distance, and the decay rate can be expressed in terms of a variational formula. Consequently, the spatial inhomogeneities of the medium and the geometry of the shallow-water waveguide lead us to an exponential decay phenomenon caused by the radiative loss into the ocean bottom.

Proof We can see that the operator $P_\infty = \frac{\partial}{\partial v}\left(a_\infty(\cdot)\frac{\partial}{\partial v}\right)$ on $L^2([0,1])$, with domain

$$\mathcal{D}(P_\infty) = \left\{ \varphi \in H^2(0,1), \quad \frac{\partial}{\partial v}\varphi(0) = 0, \quad \varphi(1) = 0 \right\}$$

is self-adjoint. P_∞ has a compact resolvent $R_\lambda = (\lambda Id - P_\infty)^{-1}$ because $[0,1]$ is a compact set and then it has a point spectrum $(\lambda_j)_{j\geq 1}$ with eigenvectors denoted by $(\phi_{\infty,j})_{j\geq 1}$. Moreover, all the eigenspaces are finite-dimensional subspaces of $\mathcal{D}(P_\infty)$ and $\forall \varphi \in \mathcal{D}(P_\infty) \setminus \{0\}$

$$\langle P_\infty(\varphi), \varphi \rangle_{L^2(0,1)} < 0.$$

Let us organize the point spectrum in the nonincreasing way, $\cdots < \lambda_2 < \lambda_1 < 0$. We have

$$\mathcal{T}_\varphi(L,v) = \sum_{j\geq 1} \langle \varphi, \phi_{\infty,j} \rangle_{L^2(0,1)} e^{\lambda_j L} \phi_{\infty,j}(v).$$

Lemma 2.2 *λ_1 is a simple eigenvalue and one can choose $\phi_{\infty,1}$ such that $\phi_{\infty,1}(v) > 0$ $\forall v \in [0,1)$.*

Proof (of Lemma 2.2) This lemma is a consequence of the Krein-Rutman theorem, but not its strongest form [55]. Indeed, the set of nonnegative functions in $L^2([0,1])$ has an empty interior. However, using the smoothness of the eigenvectors, the proof also works in our case as we shall see it.

Using the maximum principle we know that if $\varphi \in L^2([0,1])$ such that $\varphi \geq 0$, we have $\mathcal{T}_\varphi(L,.) \geq 0$, and then $R_\lambda(\varphi) \geq 0$. Consequently, applying the Krein-Rutman theorem [55] to the resolvent operator R_λ with $\lambda > 0$ and which is a compact operator, the spectral radius $\rho(R_\lambda)$ is an eigenvalue, for which one can associate an eigenvector $\varphi_{\rho(R_\lambda)}$ such that $\forall v \in [0,1]$, $\varphi_{\rho(R_\lambda)}(v) \geq 0$. However, we have $\forall v \in [0,1)$, $\varphi_{\rho(R_\lambda)}(v) > 0$. In fact, let us assume that there exists $v_0 \in [0,1)$ such that $\varphi_{\rho(R_\lambda)}(v_0) = 0$, then $R_\lambda(\varphi_{R_\lambda})(v_0) = 0$. Moreover, $R_\lambda(\varphi_{R_\lambda}) = \rho(R_\lambda)\varphi_{\rho(R_\lambda)}$ is an eigenvector for P_∞, and then $\varphi_{\rho(R_\lambda)}$ is a smooth function on $[0,1]$. Therefore, according to the proof of Theorem 2.4 we have

$$\begin{aligned} R_\lambda(\varphi_{R_\lambda})(v_0) &= \int_0^{+\infty} e^{-\lambda t}\mathcal{T}_{\varphi_{\rho(R_\lambda)}}(t,v_0)dt \\ &= \int_0^{+\infty} e^{-\lambda t}\mathbb{E}^{\overline{\mathbb{P}}_{v_0}}\left[\varphi_{\rho(R_\lambda)}(x(t))\mathbf{1}_{(t<\tau_1)}\right]dt \\ &= \mathbb{E}^{\overline{\mathbb{P}}_{v_0}}\Big[\int_0^{\tau_1} e^{-\lambda t}\varphi_{\rho(R_\lambda)}(|x(t)|)dt\Big] = 0, \end{aligned}$$

where $\overline{\mathbb{P}}_{v_0}$ is the unique solution of the martingale problem associated to $\mathcal{L}_{\overline{a}_\infty} = \frac{\partial}{\partial v}\left(\overline{a}_\infty(\cdot)\frac{\partial}{\partial v}\right)$ and starting from v_0. Here, we have chosen $\overline{a}_\infty$ such that $\forall v \in [0,1]$, $\overline{a}_\infty(v) = \overline{a}_\infty(-v) = a_\infty(v)$, and the martingale problem associated to $\mathcal{L}_{\overline{a}_\infty}$ is well-posed. Moreover, $\tau_1 = \inf(t \geq 0, |x(t)| \geq 1)$. Consequently, $\overline{\mathbb{P}}_{v_0}\big(\int_0^{\tau_1} e^{-\lambda t}\varphi_{\rho(R_\lambda)}(|x(t)|)dt = 0\big) = 1$. However, we know that there exists $v_1 \in (0,1)$ such that $\varphi_{\rho(R_\lambda)}(v_1) > 0$, and then $v_1 < v_0 < 1$. Therefore, $\overline{\mathbb{P}}_{v_0}(\tau_1 < \tau_{v_1}) = 1$, and by the Markov property

$$\begin{aligned} 0 < \mathbb{E}^{\overline{\mathbb{P}}_{v_0}}\left[e^{-\tau_{v_1}}\mathbf{1}_{(\tau_{v_1}<+\infty)}\right] &= \mathbb{E}^{\overline{\mathbb{P}}_{v_0}}\left[e^{-\tau_{v_1}}\mathbf{1}_{(\tau_{v_1}<+\infty,\tau_1<\tau_{v_1})}\right] \\ &< \mathbb{E}^{\overline{\mathbb{P}}_1}\left[e^{-\tau_{v_1}}\mathbf{1}_{(\tau_{v_1}<+\infty)}\right] \\ &< \mathbb{E}^{\overline{\mathbb{P}}_{v_0}}\left[e^{-\tau_{v_1}}\mathbf{1}_{(\tau_{v_1}<+\infty)}\right], \end{aligned}$$

which is impossible. Therefore, $\forall v \in [0,1)$, $\varphi_{\rho(R_\lambda)} > 0$. Now, to see that the eigenvalue $\rho(R_\lambda)$ is simple, let $\varphi \in L^2(0,1) \setminus \{0\}$ such that $R_\lambda(\varphi) = \rho(R_\lambda)\varphi$, and let

$$\begin{array}{rcl} P_{R_\lambda} : \mathbb{R} & \longrightarrow & \mathcal{C}^0([0,1]) \\ t & \longmapsto & \varphi_{R_\lambda} - t\varphi, \end{array}$$

which is a continuous function. We recall that φ is a smooth function on $[0,1]$. Let us show that $\exists t \in \mathbb{R}$ such that $\varphi = t\,\varphi_{\rho(R_\lambda)}$, that is $0 \in P_{R_\lambda}(\mathbb{R})$. To do this let us assume that $0 \notin P_{R_\lambda}(\mathbb{R})$. By linearity one can assume that $\exists v_0 \in [0,1)$ such that $\varphi(v_0) > 0$. Let $\eta > 0$ be small enough to have $v_0 \in [0,1-\eta]$. Let $K_\eta^+ = \{\varphi \in \mathcal{C}^0([0,1-\eta]), \forall v \in [0,1-\eta], \varphi(v) \geq 0\}$, then the interior of K_η^+ for the sup norm on $[0,1]$ is $K_\eta^{++} = \{\varphi \in \mathcal{C}^0([0,1-\eta]), \forall v \in [0,1-\eta], \varphi(v) > 0\}$. Moreover, for t small enough $\varphi_{R_\lambda} - t\varphi \in K_\eta^{++}$, and $\varphi_{R_\lambda} - t\varphi \notin K_\eta^+$ for t large enough. Then $\exists t_0 \in \mathbb{R}$ such that $\varphi_{R_\lambda} - t_0\varphi \in K_\eta^+ \setminus K_\eta^{++}$. However, $\varphi_{R_\lambda} - t_0\varphi \geq 0$, but $\varphi_{R_\lambda} - t_0\varphi \neq 0$ because $0 \notin P_{R_\lambda}(\mathbb{R})$. Following the previous work we have

$$\rho(R_\lambda)(\varphi_{R_\lambda} - t_0\varphi) = R_\lambda(\varphi_{R_\lambda} - t_0\varphi) \in K_\eta^{++}.$$

Consequently, $\rho(R_\lambda) = 1/(\lambda - \lambda_1)$ implies that λ_1 is also a simple eigenvalue and one can choose

$$\phi_{\infty,1} = R_\lambda(\varphi_{R_\lambda}) = \rho(R_\lambda)\varphi_{R_\lambda} \in K_\eta^{++}.$$

That concludes the proof of Lemma 2.2. □

As a result, $\forall \varphi \in L^2(0,1) \setminus \{0\}$ such that $\varphi \geq 0$, $\forall v \in [0,1)$ we get

$$\lim_{L\to+\infty} \frac{1}{L} \ln\left[\mathcal{T}_\varphi(L,v)\right] = \lambda_1,$$

and

$$\lambda_1 = \sup_{\substack{\varphi\in\mathcal{D}(P_\infty)\\ \|\varphi\|_{L^2([0,1])}=1}} \left\langle P_\infty(\varphi), \varphi\right\rangle_{L^2([0,1])} = -\Lambda_\infty < 0.$$

■

In Theorem 2.5, we take $\varphi \in L^2(0,1) \setminus \{0\}$ such that $\varphi \geq 0$, which can be consider as being the initial repartition of energy over the continuum of modes. However, the result of Theorem 2.5 is also valid for any $\varphi \in L^2(0,1) \setminus \{0\}$ such that $\left\langle \varphi, \phi_{\infty,1}\right\rangle_{L^2(0,d)} > 0$.

2.5.3 High-Frequency Approximation to Coupled Power Equation with Negligible Radiation Losses

In the case of negligible radiation losses, we also get a continuous diffusive model for the coupled power equations in the high-frequency regime or in the limit of a large number of propagating modes $N(\omega) \gg 1$. This diffusive continuous model is equipped with boundary conditions which take into account the negligible effect of the radiation losses at the bottom and the free surface of the waveguide (see Figure 2.4 page 57).

Now, let us assume that the radiation losses are negligible, that is, $\Lambda^c(\omega) = \tau\tilde{\Lambda}^c(\omega)$ with $\tau \ll 1$. We have already remarked that, if the radiation losses are negligible, then the coupling process is predominant and we have

$$\forall L > 0, \quad \sup_{z\in[0,L]} \|\mathcal{T}_j^{\tau,l}(\omega,z) - \mathcal{T}_j^{0,l}(\omega,z)\|_{2,\mathbb{R}^{N(\omega)}} = \mathcal{O}(\tau),$$

where $\mathcal{T}^{0,l}(\omega,.)$ satisfies

$$\begin{aligned}
&\frac{d}{dz}\mathcal{T}_N^{0,l}(z) = \Gamma^c_{N-1\,N}\left(\mathcal{T}_{N-1}^{0,l}(z) - \mathcal{T}_N^{0,l}(z)\right),\\
&\frac{d}{dz}\mathcal{T}_j^{0,l}(z) = \Gamma^c_{j-1\,j}\left(\mathcal{T}_{j-1}^{0,l}(z) - \mathcal{T}_j^{0,l}(z)\right) + \Gamma^c_{j+1\,j}\left(\mathcal{T}_{j+1}^{0,l}(z) - \mathcal{T}_j^{0,l}(z)\right) \text{ for } j \in \{2,\dots,N-1\},\\
&\frac{d}{dz}\mathcal{T}_1^{0,l}(z) = \Gamma^c_{2\,1}\left(\mathcal{T}_2^{0,l}(z) - \mathcal{T}_1^{0,l}(z)\right),
\end{aligned}$$

with $\mathcal{T}_j^{0,l}(0) = \delta_{jl}$.

High Frequency Approximation

Let, $\forall \varphi \in \mathcal{C}^0([0,1])$, $\forall u \in [0,1]$, and $z \geq 0$,

$$\mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi^{[N(\omega)u]}(z) = \sum_{j=1}^{N(\omega)} \varphi\Big(\frac{j}{N(\omega)}\Big)\mathcal{T}_j^{[N(\omega)u]}(z),$$

where $\varphi \mapsto \mathcal{T}_\varphi^{N(\omega)}(z,.)$ can be extended into an operator from $L^2(0,1)$ to itself.

Theorem 2.6 *We have*

1. $\forall \varphi \in L^2(0,1)$ *and* $\forall z \geq 0$,
$$\lim_{\omega \to +\infty} \mathcal{T}_{\varphi}^{N(\omega)}(z,u) = \mathcal{T}_{\varphi}(z,u) \quad \text{in } L^2(0,1),$$
where $\mathcal{T}_{\varphi}(z,u)$ *satisfies the partial differential equation :* $\forall (z,u) \in (0,+\infty) \times (0,1)$,
$$\frac{\partial}{\partial z}\mathcal{T}_{\varphi}(z,u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_{\varphi}\right)(z,u),$$
with the boundary conditions
$$\frac{\partial}{\partial u}\mathcal{T}_{\varphi}(z,0) = 0, \quad \frac{\partial}{\partial v}\mathcal{T}_{\varphi}(z,1) = 0, \quad \text{and} \quad \mathcal{T}_{\varphi}(0,u) = \varphi(u),$$
$\forall z > 0$.

2. $\forall u \in [0,1)$, $\forall z \geq 0$, *and* $\forall \varphi \in \mathcal{C}^0([0,1])$ *such that* $\varphi(1) = 0$, *we have*
$$\lim_{\omega \to +\infty} \mathcal{T}_{\varphi}^{N(\omega)}(z,u) = \mathcal{T}_{\varphi}(z,u).$$

Here,
$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$
with $a_0 = \frac{\pi^2 S_0}{2a n_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d \int_0^d \gamma_0(x_1,x_2)\cos\left(\frac{\pi}{d}x_1\right)\cos\left(\frac{\pi}{d}x_2\right)dx_1 dx_2$. n_1 *is the index of refraction in the ocean section* $[0,d]$, $1/a = l_{z,x}$ *is the correlation length of the random inhomogeneities in the longitudinal direction, and* γ_0 *is the covariance function of the random inhomogeneities in the transverse direction.*

This theorem is a continuum approximation in the limit of a large number of propagating modes in the case where the radiation losses are negligible.This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between the $N(\omega)$-discrete propagating modes, with two reflecting boundary conditions at $u = 0$ (the top of the waveguide in Figure 2.2 page 33) and $u = 1$ (the bottom of the waveguide in Figure 2.2). Here, the two reflecting boundary conditions mean that there is no radiative loss anymore (see Figure 2.4).

Asymptotic behavior of $\mathcal{T}(L,v)$ as $L \to +\infty$

In the case where the radiation losses are negligible, we have seen in Section 2.5.1 that the decay rate satisfies $\lim_{\tau \to 0} \Lambda_\infty^\tau(\omega) = 0$ and $\mathcal{T}^{0,l}(\omega, L)$ converge to the uniform distribution over $\{1,\dots,N(\omega)\}$ as $L \to +\infty$ [30]. In the high-frequency regime we have the following continuous version.

Theorem 2.7 $\forall \varphi \in L^2(0,1)$ *and* $\forall u \in [0,1]$,
$$\lim_{L \to +\infty} \mathcal{T}_{\varphi}(L,u) = \int_0^1 \varphi(v)dv,$$
that is, the energy carried by the continuum of propagating modes converges exponentially fast to the uniform distribution over $[0,1]$ *as* $L \to +\infty$.

As a result, the energy is conserved and the modal energy distribution converges to a uniform distribution as $L \to +\infty$.

Proof We can see that the operator $P_\infty = \frac{\partial}{\partial v}(a_\infty(\cdot)\frac{\partial}{\partial v})$ on $L^2([0,1])$, with domain

$$\mathcal{D}(P_\infty) = \left\{\varphi \in H^2(0,1), \quad \frac{\partial}{\partial v}\varphi(0) = 0, \quad \frac{\partial}{\partial v}\varphi(1) = 0\right\}$$

is self-adjoint. Moreover, P_∞ has a compact resolvant because $[0,1]$ is a compact set and then it has a point spectrum $(\lambda_j)_{j\geq 0}$ with eigenvectors denoted by $(\phi_{\infty,j})_{j\geq 0}$. Moreover, all the eigenspaces are finite-dimensional subspaces of $\mathcal{D}(P_\infty)$ and $\forall \varphi \in \mathcal{D}(P_\infty) \setminus \{0\}$

$$\left\langle P_\infty(\varphi), \varphi\right\rangle_{L^2(0,1)} \leq 0.$$

Let us remark that $\lambda_0 = 0$ is a simple eigenvalue with eigenvector $\phi_{\infty,0} = 1$. Then, the spectrum is include in $(-\infty, 0]$ and we have the following decomposition

$$\mathcal{T}_\varphi(z,v) = \int_0^1 \varphi(v)dv + \sum_{j\geq 1} \left\langle \varphi, \phi_{\infty,j}\right\rangle_{L^2(0,1)} e^{\lambda_j z}\phi_{\infty,j}(v).$$

Therefore, $\forall u \in [0,1]$,

$$\lim_{L\to+\infty} \mathcal{T}_\varphi(L,u) = \int_0^1 \varphi(v)dv,$$

with exponential rate $\lambda_1 < 0$. ■

Conclusion

In Chapter 2 we have analyzed the propagation of waves in a shallow-water acoustic waveguide with random perturbations. In such a waveguide, the wave field can be decomposed into three kinds of modes, which are the propagating, the radiating, and the evanescent modes, and the random perturbations produce a coupling between these modes.

We have shown that the evolution of the propagating mode amplitudes can be described as a diffusion process (Theorems 2.1 and 2.2). This diffusion takes into account the main coupling mechanisms: The coupling with the evanescent modes induces a mode-dependent and frequency-dependent phase modulation on the propagating modes, the coupling with the radiating modes, in addition to a mode-dependent and frequency-dependent phase modulation, induces a mode-dependent and frequency-dependent attenuation on the propagating modes. In other words, the propagating modes lose energy in the form of radiation into the bottom of the waveguide and their total energy decays exponentially with the propagation distance. We can express the decay rate in terms of a variational formula over a finite-dimensional space (Theorem 2.3).

Under the assumption that nearest neighbor coupling is the main power transfer mechanism, the evolution of the mean mode powers of the propagating modes can be described, in the high frequency regime or in the limit of a large number of propagating modes, by a continuous diffusive model with boundary conditions which take into account the effect of the radiation losses at the bottom and the free surface of the waveguide. In this regime, we observe that the energy carried by the continuum of propagating modes also decay exponentially with the propagation distance. The exponential decay rate can be expressed in terms of a variational formula (Theorem 2.5).

The diffusive systems obtained in Chapter 2 will be used in Chapter 3 of this manuscript to analyze pulse propagation and refocusing during time-reversal experiments in underwater acoustics.

2.6 Appendix

2.6.1 Gaussian Random Field

This section is a short remainder about some properties of Gaussian random fields that we shall use in the proofs of Theorems 2.1 and 4.1, and in Sections 2.3.2 and 2.3.3. All the results exposed in this section can be shown using the standard properties of Gaussian random fields presented in [1] and [2] for instance.

In this thesis, the random perturbations of the medium parameters are modeled using a random process denoted by $(V(x,t), x\in[0,d], t\geq 0)$. Throughout this manuscript the process V is a continuous real-valued zero-mean Gaussian field with a covariance function given by

$$\mathbb{E}\left[V(x,t)V(y,s)\right]=\gamma_0(x,y)e^{-a|t-s|}\quad \forall(x,y)\in[0,d]^2 \text{ and } \forall(s,t)\in[0,+\infty)^2. \tag{2.52}$$

Here, $a>0$; $\gamma_0:[0,d]\times[0,d]\to\mathbb{R}$ is a Lipschitz function, which is the kernel of a nonnegative operator, that is, there exists a nonnegative operator Q_{γ_0} from $L^2(0,d)$ to itself such that $\forall(\varphi,\psi)\in L^2(0,d)^2$

$$\langle Q_{\gamma_0}(\varphi),\psi\rangle_{L^2(0,d)}=\int_0^d\int_0^d\gamma_0(x,y)\varphi(x)\psi(y)dxdy.$$

Consequently, one can consider the process $(V(.,t))_{t\geq 0}$ as being a continuous zero-mean Gaussian field with values in $L^2(0,d)$ and covariance operator Q_{γ_0}. In other words, $\forall n\in\mathbb{N}^*$, $\forall(\varphi_1,\dots,\varphi_n)\in L^2(0,d)^n$, and $\forall(t_1\dots,t_n)\in[0,+\infty)^n$

$$(V_{\varphi_1}(t_1),\dots,V_{\varphi_n}(t_n))=\left(\langle V(.,t_1),\varphi_1\rangle_{L^2(0,d)},\dots,\langle V(.,t_n),\varphi_n\rangle_{L^2(0,d)}\right)$$

is a real-valued zero-mean Gaussian vector such that $\forall(j,l)\in\{1,\dots,n\}^2$

$$\mathbb{E}\left[V_{\varphi_j}(t_j)V_{\varphi_l}(t_l)\right]=\langle Q_{\gamma_0}(\varphi_j),\varphi_l\rangle_{L^2(0,d)}e^{-a|t_j-t_l|}. \tag{2.53}$$

With this point of view we have the following proposition.

Proposition 2.5 *We have*

1. $(V(.,t))_{t\geq 0}$ *is a continuous zero-mean stationary Gaussian field with values in* $L^2(0,d)$ *and autocorrelation function given by* (2.53). *Then, we have* $\forall n\in\mathbb{N}^*$ *and* $\forall t\geq 0$,

$$\mathbb{E}\left[\left(\int_0^d|V(x,t)|^2dx\right)^n\right]=\mathbb{E}\left[\left(\int_0^d|V(x,0)|^2dx\right)^n\right]<+\infty. \tag{2.54}$$

2. *We have the following Markov property. Let*

$$\mathcal{F}_t=\sigma(V(.,s),s\leq t)$$

be the σ*-algebra generated by* $(V(.,s),s\leq t)$. *We have*

$$\left(V(.,t+h)\Big|\mathcal{F}_t\right)=\left(V(.,t+h)\Big|\sigma(V(.,t))\right),$$

where the equality holds in law, and this law is the one of a Gaussian field with mean

$$\mathbb{E}[V(.,t+h)|\mathcal{F}_t]=e^{-ah}V(.,t)$$

and covariance, $\forall(\varphi,\psi)\in L^2(0,d)^2$,

$$\begin{aligned}\mathbb{E}\Big[V_\varphi(t+h)V_\psi(t+h)-\mathbb{E}[V_\varphi(t+h)|\mathcal{F}_t]\mathbb{E}[V_\psi(t+h)|\mathcal{F}_t]\Big|\mathcal{F}_t\Big]&\\ =\langle Q_{\gamma_0}(\varphi),\psi\rangle_{L^2(0,d)}\left(1-e^{-2ah}\right).&\end{aligned}$$

The Markov property of the random process $(V(.,t))_{t\geq 0}$ is a direct consequence of the exponential form of the autocorrelation function (2.53) with respect to the variable t [1]. This property will be used in the proof of Theorems 2.1 and 4.1, which are based on the perturbed-test-function method.

Now, we are interested in some estimation on the supremum of $V(x,t)$ with respect to the two variables x and t. To this end, let us introduce some notations [2]. Let $\epsilon > 0$ be a small parameter and $L > 0$. We consider the following pseudo-metric on the square $[0,d]\times[0,L/\epsilon]$ defined by

$$m((x,t),(y,s)) = \mathbb{E}\Big[(V(x,t)-V(y,s))^2\Big]^{1/2} \leq K_{\gamma_0}\big[|t-s|+|x-y|\big].$$

Let us remark that $[0,d]\times[0,L/\epsilon]$ associated to the pseudo-metric m is a compact set. From Theorem 1.3.3 in [2], we have

$$\begin{aligned}\mathbb{E}\left[\sup_{\substack{x\in[0,d]\\ t\in[0,L/\epsilon]}}\big|V(x,t)\big|\right] &\leq K\int_0^{diam([0,d]\times[0,L/\epsilon])/2} H^{1/2}(r)dr\\ &\leq K_1\int_0^{\sup_{x\in[0,d]}\gamma_0(x,x)}\sqrt{\ln\left(K_2\frac{dL}{r^2\epsilon}\right)}dr,\end{aligned}$$

where $H(r)=\ln(N(r))$, and $N(r)$ denotes the smallest number of balls, for the pseudo-metric m, with radius r to cover the square $[0,d]\times[0,L/\epsilon]$. Here, *diam* stands for the diameter with respect to the pseudo-metric m. Consequently, we have the following proposition.

Proposition 2.6 $\forall \mu > 0$ *and* $\forall K > 0$,

$$\lim_{\epsilon\to 0}\mathbb{P}\Big(\epsilon^{\mu}\sup_{x\in[0,d]}\sup_{t\in[0,L/\epsilon]}\big|V(x,t)\big|\geq K\Big)=0. \tag{2.55}$$

Moreover, according to Theorem 2.1.1 in [2], one can show that the limit (2.55) is obtained exponentially fast as $\epsilon\to 0$.

2.6.2 Proof of Theorem 2.1

The proof of this theorem is in two parts. The process $\left(\mathbf{T}^{\xi,\epsilon}(z)\right)_{z\geq 0}$ is not adapted with respect to the filtration $\mathcal{F}^{\epsilon}_z=\mathcal{F}_{z/\epsilon}$. Then, the first part of the proof consists in simplifying the problem and introducing a new process for which the martingale approach can be used. The first part of the proof follows the ideas of [36]. The second part of proof of this theorem is based on a martingale approach using the perturbed-test-function method and follows the ideas developed in [16].

Then, let us introduce $\tilde{\mathbf{T}}^{\xi,\epsilon}(.)$ the unique solution of the differential equation

$$\frac{d}{dz}\tilde{\mathbf{T}}^{\xi,\epsilon}(z)=\frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\frac{z}{\epsilon}\right)\tilde{\mathbf{T}}^{\xi,\epsilon}(z)+\langle\mathbf{G}^{aa}\rangle\tilde{\mathbf{T}}^{\xi,\epsilon}(z), \tag{2.56}$$

with $\mathbf{T}^{\xi,\epsilon}(0)=Id$ and where $\langle\mathbf{G}^{aa}\rangle$ is defined, $\forall y\in\mathcal{H}_\xi$, by

$$\langle\mathbf{G}^{aa}\rangle_j(y)=\int_{-\infty}^{-\xi}\frac{ik^4}{2\beta_j\sqrt{|\gamma'|}}\int_0^{+\infty}\mathbb{E}[C_{j\gamma'}(0)C_{j\gamma'}(z)]\cos(\beta_j z)e^{-\sqrt{|\gamma'|}z}dzd\gamma' y_j$$

$\forall j\in\{1,\dots,N\}$ and $\langle\mathbf{G}^{aa}\rangle_\gamma(y)=0$ for $\gamma\in(\xi,k^2)$. We have the following proposition that describes the relation between the two processes $\mathbf{T}^{\xi,\epsilon}(z)$ and $\tilde{\mathbf{T}}^{\xi,\epsilon}(z)$.

Proposition 2.7

$$\forall y \in \mathcal{H}_\xi \text{ and } \forall \eta > 0, \quad \lim_{\epsilon \to 0} \mathbb{P}\left(\sup_{z\in[0,L]} \|\mathbf{T}^{\xi,\epsilon}(z)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi} > \eta \right) = 0.$$

Let us remark that the new process $\big(\tilde{\mathbf{T}}^{\xi,\epsilon}(z)\big)_{z\geq 0}$ is adapted to the filtration $\mathcal{F}^\epsilon_z$ and

$$\|\tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi} = \|y\|^2_{\mathcal{H}_\xi} \quad \forall z \geq 0.$$

Let $r_y = \|y\|_{\mathcal{H}_\xi}$,

$$\mathcal{B}_{r_y,\mathcal{H}_\xi} = \left\{ \lambda \in \mathcal{H}_\xi, \|\lambda\|_{\mathcal{H}_\xi} = \sqrt{\langle \lambda, \lambda\rangle_{\mathcal{H}_\xi}} \leq r_y \right\}$$

the closed ball with radius r_y, and $\{g_n, n \geq 1\}$ a dense subset of $\mathcal{B}_{r_y,\mathcal{H}_\xi}$. We equip $\mathcal{B}_{r_y,\mathcal{H}_\xi}$ with the distance $d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}$ defined by

$$d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}(\lambda, \mu) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \left| \langle \lambda - \mu, g_n\rangle_{\mathcal{H}_\xi} \right|$$

$\forall(\lambda, \mu) \in (\mathcal{B}_{r_y,\mathcal{H}_\xi})^2$, and then $(\mathcal{B}_{\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}})$ is a compact metric space.

Using a particular tightness criteria, we prove the tightness of the family $(\tilde{\mathbf{T}}^{\xi,\epsilon}(.))_{\epsilon\in(0,1)}$ on $\mathcal{C}([0, +\infty), (\mathcal{B}_{r_y,\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}))$, which is a polish space. In a second part, we shall characterize all subsequence limits as solutions of a well-posed martingale problem in the Hilbert space $\mathcal{H}_\xi$.

We have the following version of the Arzelà-Ascoli theorem [14, 35] for processes with values in a complete separable metric space.

Theorem 2.8 *A set* $B \subset \mathcal{C}([0, +\infty), (\mathcal{B}_{r_y,\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}))$ *has a compact closure if and only if*

$$\forall T > 0, \quad \lim_{\eta\to 0} \sup_{g\in A} m_T(g, \eta) = 0,$$

with

$$m_T(g, \eta) = \sup_{\substack{(s,t)\in[0,T]^2 \\ |t-s|\leq \eta}} d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}(g(s), g(t)).$$

From this result, we obtain the classical tightness criterion.

Theorem 2.9 *A family of probability measure* $(\mathbb{P}^\epsilon)_{\epsilon\in(0,1)}$ *on* $\mathcal{C}([0, +\infty), (\mathcal{B}_{r_y,\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}))$ *is tight if and only if*

$$\forall T > 0, \eta' > 0 \quad \lim_{\eta\to 0} \sup_{\epsilon\in(0,1)} \mathbb{P}^\epsilon(g\,;\ m_T(g, \eta) > \eta') = 0.$$

From the definition of the metric $d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}$, the tightness criterion becomes the following.

Theorem 2.10 *A family of processes* $(X^\epsilon)_{\epsilon\in(0,1)}$ *is tight on* $\mathcal{C}([0, +\infty), (\mathcal{B}_{r_y,\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}))$ *if and only if* $(\langle X^\epsilon, \lambda\rangle_{\mathcal{H}_\xi})_{\epsilon\in(0,1)}$ *is tight on* $\mathcal{C}([0, +\infty), \mathbb{C})$ $\forall \lambda \in \mathcal{H}_\xi$.

This last theorem looks like the tightness criterion of Mitoma and Fouque [47, 23].

For any $\lambda \in \mathcal{H}_\xi$, we set $\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(z)(y) = \langle \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y), \lambda\rangle_{\mathcal{H}_\xi}$. According to Theorem 2.10, the family $(\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y))_\epsilon$ is tight on $\mathcal{C}([0, +\infty), (\mathcal{B}_{r_y,\mathcal{H}_\xi}, d_{\mathcal{B}_{r_y,\mathcal{H}_\xi}}))$ if and only if the family $(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(.)(y))_\epsilon$ is tight on $\mathcal{C}([0, +\infty), \mathbb{C})$ $\forall \lambda \in \mathcal{H}_\xi$. Furthermore, $(\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y))_\epsilon$ is a family of continuous processes. Then, it is sufficient to prove that $\forall \lambda \in \mathcal{H}_\xi$, $(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(.)(y))_\epsilon$ is tight on $\mathcal{D}([0, +\infty), \mathbb{C})$, which is the set of cad-lag functions with values in $\mathbb{C}$.

Proof (of Proposition 2.7) Differentiating the square norm and using the fact that $\mathbf{H}^{aa}(z)$ is skew Hermitian, we get

$$\begin{aligned}
&\|\mathbf{T}^{\xi,\epsilon}(z)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi} \\
&\qquad \leq 2\Big| \int_0^z \Big\langle (\mathbf{G}^{aa}\Big(\frac{z}{\epsilon}\Big) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(z)(y), \mathbf{T}^{\xi,\epsilon}(z)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big| \\
&\qquad\quad + 2\|\langle \mathbf{G}^{aa}\rangle\| \int_0^z \|\mathbf{T}^{\xi,\epsilon}(u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(u)(y)\|^2_{\mathcal{H}_\xi}.
\end{aligned}$$

Let $\eta' > 0$, we will split the interval $[0, z/\epsilon]$ into intervals of length $\eta'/\sqrt{\epsilon}$. The idea is that over these intervals the fast dynamic of $\mathbf{G}^{aa}$ averages out while $\mathbf{T}^{\xi,\epsilon}$ does not move significantly. We have

$$\begin{aligned}
&\Big|\epsilon \int_0^{z/\epsilon} \Big\langle (\mathbf{G}^{aa}(u)) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big| \\
&\quad \leq \Big|\epsilon \int_0^{\left[\frac{z}{\sqrt{\epsilon}\eta'}\right]\frac{\eta'}{\sqrt{\epsilon}}} \Big\langle (\mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big| \\
&\quad + \Big|\epsilon \int_{\left[\frac{z}{\sqrt{\epsilon}\eta'}\right]\frac{\eta'}{\sqrt{\epsilon}}}^{z/\epsilon} \Big\langle (\mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big|,
\end{aligned}$$

with

$$\begin{aligned}
&\Big|\epsilon \int_{\left[\frac{z}{\sqrt{\epsilon}\eta'}\right]\frac{\eta'}{\sqrt{\epsilon}}}^{z/\epsilon} \Big\langle (\mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big| \\
&\qquad \leq \left[\epsilon^{1/4}\sqrt{\eta'}\left(\int_0^L \Big\|\mathbf{G}^{aa}\Big(\frac{u}{\epsilon}\Big)\Big\|^2 du\right)^{1/2} + \sqrt{\epsilon}\eta'\|\langle \mathbf{G}^{aa}\rangle\|\right] \\
&\qquad\quad \times \sup_{z\in[0,L]} \|\mathbf{T}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi} \|\mathbf{T}^{\xi,\epsilon}(z)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi}
\end{aligned}$$

since $0 \leq z - \left[\frac{z}{\sqrt{\epsilon}\eta'}\right]\sqrt{\epsilon}\eta' \leq \sqrt{\epsilon}\eta'$, and

$$\begin{aligned}
&\Big|\epsilon \int_0^{\left[\frac{z}{\sqrt{\epsilon}\eta'}\right]\frac{\eta'}{\sqrt{\epsilon}}} \Big\langle (\mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big| \\
&\leq \sqrt{\epsilon} \sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1} \Big|\sqrt{\epsilon} \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \Big\langle (\mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y), \mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi} du\Big|.
\end{aligned}$$

Moreover,

$$\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y) = \mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y) + \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{u} \sqrt{\epsilon}\mathbf{H}^{aa}(v)\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y) + \epsilon\mathbf{G}^{aa}(v)\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y)dv$$

and

$$\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y) = \tilde{\mathbf{T}}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y) + \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{u} \sqrt{\epsilon}\mathbf{H}^{aa}(v)\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon v)(y) + \epsilon\langle \mathbf{G}^{aa}\rangle\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon v)(y)dv.$$

Therefore, we have

$$\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y),\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi}du$$

$$\begin{aligned}
&=\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y),\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)\Big\rangle_{\mathcal{H}_\xi}du\\
&\quad+\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{u}\epsilon\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{H}^{aa}(v)\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y),\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi}\\
&\qquad+\epsilon^{3/2}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{G}^{aa}(v)\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y),\mathbf{T}^{\xi,\epsilon}(\epsilon u)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon u)(y)\Big\rangle_{\mathcal{H}_\xi}\\
&\qquad+\epsilon\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y),\mathbf{H}^{aa}(v)(\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon v)(y))\Big\rangle_{\mathcal{H}_\xi}\\
&\quad+\epsilon^{3/2}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y),\mathbf{G}^{aa}(v)\mathbf{T}^{\xi,\epsilon}(\epsilon v)(y)-\langle\mathbf{G}^{aa}\rangle\tilde{\mathbf{T}}^{\xi,\epsilon}(\epsilon v)(y)\Big\rangle_{\mathcal{H}_\xi}du.
\end{aligned}$$

Consequently, by the Gronwall's inequality

$$\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\le B(\epsilon,\eta')e^{2\|\langle G^{aa}\rangle\|L},$$

where

$$\begin{aligned}
B(\epsilon,\eta')&=2\left[\epsilon^{1/4}\sqrt{\eta'}\left(\int_0^L\|\mathbf{G}^{aa}\left(\frac{u}{\epsilon}\right)\|^2du\right)^{1/2}+\sqrt{\epsilon}\eta'\|\langle\mathbf{G}^{aa}\rangle\|\right]\\
&\qquad\times\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi}\|\mathbf{T}^{\xi,\epsilon}(z)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi}\\
&\quad+2\sqrt{\epsilon}\sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{u}\Big(2\epsilon\big[\|\mathbf{G}^{aa}(u)\|+\|\langle G^{aa}\rangle\|\big]\|\mathbf{H}^{aa}(v)\|\\
&\quad+\epsilon^{3/2}\big[\|\mathbf{G}^{aa}(u)\|+\|\langle G^{aa}\rangle\|\big]^2\Big)\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi}\|\mathbf{T}^{\xi,\epsilon}(z)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|_{\mathcal{H}_\xi}dv\,du\\
&\quad+\Big|\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y),\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)\Big\rangle_{\mathcal{H}_\xi}du\Big|,
\end{aligned}$$

and

$$\mathbb{P}\Big(\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}>\eta\Big)\le\mathbb{P}\Big(B(\epsilon,\eta')\ge\eta e^{-2\|\langle G^{aa}\rangle\|L}\Big).$$

Setting $\eta''=\eta e^{-2\|\langle G^{aa}\rangle\|L}$, we have

$$\begin{aligned}
\mathbb{P}\Big(B(\epsilon,\eta')\ge\eta''\Big)&\le\mathbb{P}\Big(B(\epsilon,\eta')\ge\eta'',\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\le M\Big)\\
&\quad+\mathbb{P}\Big(\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\ge M\Big).
\end{aligned}$$

We already know that the process $\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y)$ is bounded. Moreover,

$$\mathbb{P}\Big(B(\epsilon,\eta')\ge\eta'',\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\le M\Big)\le\frac{1}{\eta''}\mathbb{E}\left[B(\epsilon,\eta')\mathbf{1}_{\left(\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\le M\right)}\right]$$

with

$$\mathbb{E}\left[B(\epsilon,\eta')\mathbf{1}_{\left(\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\leq M\right)}\right]\leq K[\eta'^2+\epsilon^{1/4}\sqrt{\eta'}+\sqrt{\epsilon}(\eta'+\eta'^2)]$$
$$+2\sqrt{\epsilon}\sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1}\mathbb{E}\Big[\mathbf{1}_{\left(\sup_{z\in[0,L]}\|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi}\leq M\right)}$$
$$\times\Big|\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\Big\langle(\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle)\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y),\mathbf{T}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)-\tilde{\mathbf{T}}^{\xi,\epsilon}(m\eta'\sqrt{\epsilon})(y)\Big\rangle_{\mathcal{H}_\xi}du\Big|\Big]$$
$$\leq K[\eta'^2+\epsilon^{1/4}\sqrt{\eta'}+\sqrt{\epsilon}(\eta'+\eta'^2)]$$
$$+2\sqrt{\epsilon}K\sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1}\mathbb{E}\left[\left\|\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle\,du\right\|^2\right]^{1/2},$$

since $\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{u}dv\,du=\frac{\eta'}{\epsilon}$ and

$$\mathbb{E}\left[\left\|\mathbf{G}^{aa}\left(\frac{u}{\epsilon}\right)\right\|^2\right]\leq K\,\mathbb{E}\left[\Big(\int_0^d|V(x,0)|^2dx\Big)^2\right]\tag{2.57}$$

for $u\in[0,L]$. As a result, it remains us to estimate only one term.

Lemma 2.3

$$\lim_{\epsilon\to 0}\sqrt{\epsilon}\sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1}\mathbb{E}\left[\left\|\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle\,du\right\|^2\right]^{1/2}=0.$$

Proof (of Lemma 2.3) Let us remark that we have the following decomposition. For $j\in\{1,\dots,N\}$, almost every $\gamma\in(\xi,k^2)$, and $\forall y\in\mathcal{H}_\xi$,

$$\mathbf{G}^{aa}_j(z)(y)=\sum_{l=1}^N\mathbf{G}^{aa}_{jl}(z)y_l+\int_\xi^{k^2}\mathbf{G}^{aa}_{j\gamma'}(z)y_{\gamma'}d\gamma',$$
$$\mathbf{G}^{aa}_\gamma(z)(y)=\sum_{l=1}^N\mathbf{G}^{aa}_{\gamma l}(z)y_l+\int_\xi^{k^2}\mathbf{G}^{aa}_{\gamma\gamma'}(z)y_{\gamma'}d\gamma'.$$

Letting

$$\mathbf{P}=\sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\mathbf{G}^{aa}(u)-\langle\mathbf{G}^{aa}\rangle\,du,$$

we have $(j,l)^2 \in \{1,\dots,N\}^2$ such that $j \neq l$, and almost every $\gamma \in (\xi, k^2)$

$$\mathbf{P}_{jj} = \sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}_{jj}(u) - \left\langle \mathbf{G}^{aa}\right\rangle_{jj} du,$$

$$\mathbf{P}_{jl} = \sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}_{jl}(u)du,$$

$$\mathbf{P}_{j\gamma'} = \sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}_{j\gamma'}(u)du,$$

$$\mathbf{P}_{\gamma l} = \sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}_{\gamma l}(u)du,$$

$$\mathbf{P}_{\gamma\gamma'} = \sqrt{\epsilon}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}_{\gamma\gamma'}(u)du,$$

and

$$\frac{1}{2}\|\mathbf{P}\|^2 \leq \sum_{j,l=1}^{N}|\mathbf{P}_{jl}|^2 + \sum_{j=1}^{N}\int_\xi^{k^2}|\mathbf{P}_{j\gamma'}|^2 d\gamma' + \int_\xi^{k^2}\sum_{l=1}^{N}|\mathbf{P}_{\gamma l}|^2 d\gamma + \int_\xi^{k^2}\int_\xi^{k^2}|\mathbf{P}_{\gamma\gamma'}|^2 d\gamma'\, d\gamma.$$

Moreover,

$$\begin{aligned}\mathbb{E}\Big[V(x_1,z_1)V(x_2,z_2)V(x_3,z_3)V(x_4,z_4)\Big] =& E\Big[V(x_1,z_1)V(x_2,z_2)\Big]E\Big[V(x_3,z_3)V(x_4,z_4)\Big]\\ &+ E\Big[V(x_1,z_1)V(x_3,z_3)\Big]E\Big[V(x_2,z_2)V(x_4,z_4)\Big]\\ &+ E\Big[V(x_1,z_1)V(x_4,z_4)\Big]E\Big[V(x_2,z_2)V(x_3,z_3)\Big]\\ &= \gamma_0(x_1,x_2)\gamma_0(x_3,x_4)e^{-a|z_1-z_2|}e^{-a|z_3-z_4|}\\ &+ \gamma_0(x_1,x_3)\gamma_0(x_2,x_4)e^{-a|z_1-z_3|}e^{-a|z_2-z_4|}\\ &+ \gamma_0(x_1,x_4)\gamma_0(x_2,x_3)e^{-a|z_1-z_4|}e^{-a|z_2-z_3|},\end{aligned}$$

which is the fourth order moment of a Gaussian field. To compute the expectation of the square norm of $\mathbf{P}$ we must know these moments. Following that decomposition, the square norm of $\mathbf{P}$ can be decomposed in three parts. First, after a long computation, the two parts corresponding to the two last terms of the previous decomposition are dominated by $\sqrt{\epsilon}$ uniformly in m. Then, we focus our attention on the part corresponding to the first part of the previous decomposition. For $\mathbb{E}[|\mathbf{P}_{j\gamma'}|^2]$, $\mathbb{E}[|\mathbf{P}_{\gamma l}|^2]$, and $\mathbb{E}[|\mathbf{P}_{jl}|^2]$ with $j \neq l$, we get after a long computation terms of the form

$$\epsilon\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} e^{i(\sqrt{\gamma'}-\beta_j)u_1}e^{-i(\sqrt{\gamma'}-\beta_j)u_2}du_1\,du_2 = \mathcal{O}(\epsilon),$$

$$\epsilon\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} e^{i(\beta_l-\sqrt{\gamma})u_1}e^{-i(\beta_l-\sqrt{\gamma})u_2}du_1\,du_2 = \mathcal{O}(\epsilon),$$

$$\epsilon\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}}\int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} e^{i(\beta_l-\beta_j)u_1}e^{-i(\beta_l-\beta_j)u_2}du_1\,du_2 = \mathcal{O}(\epsilon).$$

For $\mathbb{E}[|\mathbf{P}_{\gamma\gamma'}|^2]$ we separate the integral into two parts.

$$\int_\xi^{k^2}\int_\xi^{k^2}\mathbb{E}[|\mathbf{P}_{\gamma\gamma'}|^2]d\gamma'\,d\gamma = \int_{I_{\geq\mu}}\mathbb{E}[|\mathbf{P}_{\gamma\gamma'}|^2]d\gamma'\,d\gamma + \int_{I_{<\mu}}\mathbb{E}[|\mathbf{P}_{\gamma\gamma'}|^2]d\gamma'\,d\gamma,$$

where $\mu > 0$ and

$$I_{\geq\mu} = \Big\{(\gamma,\gamma') \in (\xi,k^2)^2, \quad |\sqrt{\gamma}-\sqrt{\gamma'}| \geq \mu\Big\},$$
$$I_{<\mu} = \Big\{(\gamma,\gamma') \in (\xi,k^2)^2, \quad |\sqrt{\gamma}-\sqrt{\gamma'}| < \mu\Big\}.$$

Consequently,

$$\int_{I_{<\mu}} \mathbb{E}[|\mathbf{P}_{\gamma\gamma'}|^2] d\gamma'\, d\gamma \leq K \int_{I_{<\mu}} d\gamma'\, d\gamma,$$

and on $I_{\geq\mu}$ we get terms of the form

$$\epsilon \int_{I_{\geq\mu}} \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} e^{i(\sqrt{\gamma'}-\sqrt{\gamma})u_1} e^{-i(\sqrt{\gamma'}-\sqrt{\gamma})u_2} du_1\, du_2 d\gamma' d\gamma = \mathcal{O}(\epsilon).$$

Now, it remains us to study $\mathbb{E}[|\mathbf{P}_{jj}|^2]$. After a long computation, the terms of order one produced by $\mathbf{G}^{aa}_{jj}$ are compensated by the terms of order one given by $\left\langle \mathbf{G}^{aa}\right\rangle_j$. Moreover, the other terms are dominated by $\sqrt{\epsilon}$.

As a result, we get

$$\overline{\lim_{\epsilon\to 0}}\, \sqrt{\epsilon} \sum_{m=0}^{\left[\frac{L}{\sqrt{\epsilon}\eta'}\right]-1} \mathbb{E}\left[\left\| \sqrt{\epsilon} \int_{m\frac{\eta'}{\sqrt{\epsilon}}}^{(m+1)\frac{\eta'}{\sqrt{\epsilon}}} \mathbf{G}^{aa}(u) - \langle \mathbf{G}^{aa}\rangle\, du \right\|^2\right]^{1/2} \leq K\sqrt{\int_{I_{<\mu}} d\gamma'\, d\gamma}$$

and one can conclude the proof of Lemma 2.3 by letting $\mu \to 0$.□

From the previous lemma, we finally get, $\forall \eta' > 0$

$$\overline{\lim_{\epsilon\to 0}}\, \mathbb{P}\Big(\sup_{z\in[0,L]} \|\mathbf{T}^{\xi,\epsilon}(z)(y) - \tilde{\mathbf{T}}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi} > \eta\Big) \leq \frac{e^{2\|\langle G^{aa}\rangle\|L}}{\eta} K\eta'^2,$$

since using the Gronwall's inequality and (2.57) we have

$$\lim_{M\to+\infty} \overline{\lim_{\epsilon\to 0}}\, \mathbb{P}\Big(\sup_{z\in[0,L]} \|\mathbf{T}^{\xi,\epsilon}(z)(y)\|^2_{\mathcal{H}_\xi} \geq M\Big) = 0.$$

Consequently, we conclude the proof of Proposition 2.7 by letting $\eta' \to 0$. ■

According to Proposition 2.7, to study the convergence in distribution of the process $\left(\mathbf{T}^{\xi,\epsilon}(.)(y)\right)_\epsilon$ it suffices to study the convergence for $\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y)\right)_\epsilon$. Moreover, we shall consider the complex case for more convenient manipulations. Letting $\lambda \in \mathcal{H}_\xi$, we consider the equation

$$\frac{d}{dt}\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y) = \frac{1}{\sqrt{\epsilon}} H_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{t}{\epsilon}\right), \frac{t}{\epsilon}\right) + G_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right),$$

with $H_\lambda = \langle H,\lambda\rangle_{\mathcal{H}_\xi}$, $G_\lambda = \langle\langle \mathbf{G}^{aa}\rangle(.),\lambda\rangle_{\mathcal{H}_\xi}$, where, for $j \in \{1,\dots,N\}$ and almost every $\gamma \in (\xi,k^2)$

$$H_j(\mathbf{T},C,s) = \frac{ik^2}{2}\Big[\sum_{l=1}^N \frac{C_{jl}}{\sqrt{\beta_j\beta_l}} e^{i(\beta_l-\beta_j)s}\mathbf{T}_l + \int_\xi^{k^2} \frac{C_{j\gamma'}}{\sqrt{\beta_j\sqrt{\gamma'}}} e^{i(\sqrt{\gamma'}-\beta_j)s}\mathbf{T}_{\gamma'} d\gamma'\Big],$$

$$H_\gamma(\mathbf{T},C,s) = \frac{ik^2}{2}\Big[\sum_{l=1}^N \frac{C_{\gamma l}}{\sqrt{\sqrt{\gamma}\beta_l}} e^{i(\beta_l-\sqrt{\gamma})s}\mathbf{T}_l + \int_\xi^{k^2} \frac{C_{\gamma\gamma'}}{\gamma^{1/4}\gamma'^{1/4}} e^{i(\sqrt{\gamma'}-\sqrt{\gamma})s}\mathbf{T}_{\gamma'} d\gamma'\Big].$$

The proof of Theorem 2.1 is based on the perturbed-test-function approach. Using the notion of a pseudogenerator, we prove tightness and characterize all subsequence limits.

Pseudogenerator

We recall the techniques developed by Kurtz and Kushner [41]. Let $\mathcal{M}^\epsilon$ be the set of all $\mathcal{F}^\epsilon$-measurable functions $f(t)$ for which $\sup_{t\leq T}\mathbb{E}\left[|f(t)|\right] < +\infty$ and where $T>0$ is fixed. The $p-\lim$ and the pseudogenerator are defined as follows. Let f and f^δ in $\mathcal{M}^\epsilon$ $\forall\delta>0$. We say that $f = p-\lim_\delta f^\delta$ if

$$\sup_{t,\delta}\mathbb{E}[|f^\delta(t)|] < +\infty \quad \text{and} \quad \lim_{\delta\to 0}\mathbb{E}[|f^\delta(t)-f(t)|] = 0 \quad \forall t.$$

The domain of $\mathcal{A}^\epsilon$ is denoted by $\mathcal{D}(\mathcal{A}^\epsilon)$. We say that $f\in\mathcal{D}(\mathcal{A}^\epsilon)$ and $\mathcal{A}^\epsilon f = g$ if f and g are in $\mathcal{D}(\mathcal{A}^\epsilon)$ and

$$p-\lim_{\delta\to 0}\left[\frac{\mathbb{E}^\epsilon_t[f(t+\delta)]-f(t)}{\delta} - g(t)\right] = 0,$$

where $\mathbb{E}^\epsilon_t$ is the conditional expectation given $\mathcal{F}^\epsilon_t$ and $\mathcal{F}^\epsilon_t = \mathcal{F}_{t/\epsilon}$. A useful result about $\mathcal{A}^\epsilon$ is given by the following theorem.

Theorem 2.11 *Let $f\in\mathcal{D}(\mathcal{A}^\epsilon)$. Then*

$$M^\epsilon_f(t) = f(t) - \int_0^t \mathcal{A}^\epsilon f(u)du$$

is an $(\mathcal{F}^\epsilon_t)$-martingale.

Tightness

We consider the classical complex derivative with the following notation: If $v=\alpha+i\beta$, then $\partial_v = \frac{1}{2}(\partial_\alpha - i\partial_\beta)$ and $\partial_{\overline{v}} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$.

Proposition 2.8 *$\forall\lambda\in\mathcal{H}_\xi$, the family $\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(.)(y)\right)_{\epsilon\in(0,1)}$ is tight on $\mathcal{D}([0,+\infty),\mathbb{C})$.*

Proof According to Theorem 4 in [41], we need to show the three following lemmas. Let $\lambda\in\mathcal{H}_\xi$, f be a smooth function, and $f^\epsilon_0(t) = f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)$. We have,

$$\begin{aligned}\mathcal{A}^\epsilon f^\epsilon_0(t) &= \partial_v f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)\left[\frac{1}{\sqrt{\epsilon}}H_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{t}{\epsilon}\right),\frac{t}{\epsilon}\right) + G_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)\right]\\ &+ \partial_{\overline{v}} f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)\overline{\left[\frac{1}{\sqrt{\epsilon}}H_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{t}{\epsilon}\right),\frac{t}{\epsilon}\right) + G_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)\right]}.\end{aligned}$$

Let

$$\begin{aligned}f^\epsilon_1(t) &= \frac{1}{\sqrt{\epsilon}}\partial_v f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)\int_t^{+\infty}\mathbb{E}^\epsilon_t\left[H_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{u}{\epsilon}\right),\frac{u}{\epsilon}\right)\right]du\\ &+ \frac{1}{\sqrt{\epsilon}}\partial_{\overline{v}} f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)\int_t^{+\infty}\mathbb{E}^\epsilon_t\left[\overline{H_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{u}{\epsilon}\right),\frac{u}{\epsilon}\right)}\right]du.\end{aligned}$$

Lemma 2.4 $\forall T>0$, $\lim_\epsilon \sup_{0\leq t\leq T}|f^\epsilon_1(t)| = 0$ *almost surely, and* $\sup_{t\geq 0}\mathbb{E}\left[|f^\epsilon_1(t)|\right] = \mathcal{O}(\sqrt{\epsilon})$.

Proof (of Lemma 2.4) Using the Markov property of the Gaussian field V, we have

$$f^\epsilon_1(t) = \frac{ik^2\sqrt{\epsilon}}{2}\partial_v f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)F^\epsilon_{1,\lambda}(t) - \frac{ik^2\sqrt{\epsilon}}{2}\partial_{\overline{v}} f\left(\tilde{\mathbf{T}}^{\xi,\epsilon}_\lambda(t)(y)\right)\overline{F^\epsilon_{1,\lambda}(t)}$$

with

$$F^{\epsilon}_{1,\lambda}(t) = \sum_{j=1}^{N}\left[\sum_{l=1}^{N}\frac{C_{jl}\left(\frac{t}{\epsilon}\right)}{\sqrt{\beta_j\beta_l}}e^{i(\beta_l-\beta_j)\frac{t}{\epsilon}}\ \tilde{\mathbf{T}}^{\xi,\epsilon}_{l}(t)(y)\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.$$
$$\left.+\int_{\xi}^{k^2}\frac{C_{j\gamma'}\left(\frac{t}{\epsilon}\right)}{\sqrt{\beta^{\epsilon}_j\sqrt{\gamma'}}}e^{i(\sqrt{\gamma'}-\beta_j)\frac{t}{\epsilon}}\tilde{\mathbf{T}}^{\xi,\epsilon}_{\gamma'}(t)(y)\frac{a+i(\sqrt{\gamma'}-\beta_j)}{a^2+(\sqrt{\gamma'}-\beta_j)^2}d\gamma'\right]\overline{\lambda_j}$$
$$+\int_{\xi}^{k^2}\left[\sum_{l=1}^{N}\frac{C_{\gamma l}\left(\frac{t}{\epsilon}\right)}{\sqrt{\sqrt{\gamma}\beta_l}}e^{i(\beta_l-\sqrt{\gamma})\frac{t}{\epsilon}}\ \tilde{\mathbf{T}}^{\xi,\epsilon}_{l}(t)(y)\frac{a+i(\beta_l-\sqrt{\gamma})}{a^2+(\beta_l-\sqrt{\gamma})^2}\right.$$
$$\left.+\int_{\xi}^{k^2}\frac{C_{\gamma\gamma'}\left(\frac{t}{\epsilon}\right)}{\gamma^{1/4}\gamma'^{1/4}}e^{i(\sqrt{\gamma'}-\sqrt{\gamma})\frac{t}{\epsilon}}\tilde{\mathbf{T}}^{\xi,\epsilon}_{\gamma'}(t)(y)\frac{a+i(\sqrt{\gamma'}-\sqrt{\gamma})}{a^2+(\sqrt{\gamma'}-\sqrt{\gamma})^2}d\gamma'\right]\overline{\lambda_\gamma}d\gamma.$$

Using (2.54), we easily get

$$\mathbb{E}\left[|f^{\epsilon}_1(t)|\right]\leq\sqrt{\epsilon}K(f,\lambda).$$

and

$$|f^{\epsilon}_1(t)|\leq K(\lambda,f)\sqrt{\epsilon}\sup_{0\leq t\leq T/\epsilon}\sup_{x\in[0,d]}|V(x,t)|\,.$$

Then, we can conclude with (2.55).□

Lemma 2.5 $\{\mathcal{A}^{\epsilon}\left(f^{\epsilon}_0+f^{\epsilon}_1\right)(t),\epsilon\in(0,1),0\leq t\leq T\}$ *is uniformly integrable* $\forall T>0$.

Proof (of Lemma 2.5) A computation gives us

$$\mathcal{A}^{\epsilon}\left(f^{\epsilon}_0+f^{\epsilon}_1\right)(t)=\tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y),C\left(\frac{t}{\epsilon}\right)\otimes C\left(\frac{t}{\epsilon}\right),\frac{t}{\epsilon}\right),$$

where

$$C(T)\otimes C(T)_{q_1\,q_2\,q_3\,q_4}=C_{q_1\,q_2}(T)C_{q_3\,q_4}(T)$$

for $(q_1,q_2,q_3,q_4)\in\left(\{1,\dots,N\}\cup(\xi,k^2)\right)^4$, with

$$\tilde{F}_{\lambda}\left(\mathbf{T},C,s\right)=\partial_v f(\mathbf{T})\left[\tilde{F}^{1,\epsilon}_{\lambda}(\mathbf{T},C,s)+G_{\lambda}\left(\mathbf{T}\right)\right]+\partial_{\overline{v}}f(\mathbf{T})\overline{\left[\tilde{F}^{1}_{\lambda}(\mathbf{T},C,s)+G_{\lambda}\left(\mathbf{T}\right)\right]}$$
$$+\partial^2_v f(\mathbf{T})\tilde{F}^2_{\lambda}(\mathbf{T},C,s)+\partial^2_{\overline{v}}f(\mathbf{T})\overline{\tilde{F}^2_{\lambda}(\mathbf{T},C,s)}$$
$$+\partial_{\overline{v}}\partial_v f(\mathbf{T})\tilde{F}^3_{\lambda}(\mathbf{T},C,s)+\partial_v\partial_{\overline{v}}f(\mathbf{T})\overline{\tilde{F}^3_{\lambda}(\mathbf{T},C,s)},$$

and

$$\tilde{F}^1_{\lambda}(\mathbf{T},C,s)=$$
$$-\frac{k^4}{4}\sum_{j=1}^{N}\left[\sum_{l,l'=1}^{N}\frac{C_{jll'}}{\sqrt{\beta_j\beta_l^2\beta_{l'}}}e^{i(\beta_{l'}-\beta_j)s}\mathbf{T}_{l'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.$$
$$+\sum_{l=1}^{N}\int_{\xi}^{k^2}\frac{C_{jll\gamma''}}{\sqrt{\beta_j\beta_l^2\sqrt{\gamma''}}}e^{i(\sqrt{\gamma''}-\beta_j)s}\mathbf{T}_{\gamma''}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}d\gamma''$$
$$+\int_{\xi}^{k^2}\sum_{l'=1}^{N}\frac{C_{j\gamma'\gamma'l'}}{\sqrt{\beta_j\gamma'\beta_{l'}}}e^{i(\beta_{l'}-\beta_j)s}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma'}-\beta_j)}{a^2+(\sqrt{\gamma'}-\beta_j)^2}d\gamma'$$
$$\left.+\int_{\xi}^{k^2}\int_{\xi}^{k^2}\frac{C_{j\gamma'\gamma'\gamma''}}{\sqrt{\beta_j\gamma'\sqrt{\gamma''}}}e^{i(\sqrt{\gamma''}-\beta_j)s}\mathbf{T}_{\gamma''}\frac{a+i(\sqrt{\gamma'}-\beta_j)}{a^2+(\sqrt{\gamma'}-\beta_j)^2}d\gamma'd\gamma''\right]\overline{\lambda_j}$$

$$-\frac{k^4}{4}\int_\xi^{k^2}\left[\sum_{l,l'=1}^{N}\frac{C_{\gamma ll'}}{\sqrt{\sqrt{\gamma}\beta_l^2\beta_{l'}}}e^{i(\beta_{l'}-\sqrt{\gamma})s}\mathbf{T}_{l'}\frac{a+i(\beta_l-\sqrt{\gamma})}{a^2+(\beta_l-\sqrt{\gamma})^2}\right.$$
$$+\sum_{l=1}^{N}\int_\xi^{k^2}\frac{C_{\gamma l l\gamma''}}{\sqrt{\sqrt{\gamma}\beta_l^2\sqrt{\gamma''}}}e^{i(\sqrt{\gamma''}-\sqrt{\gamma})s}\mathbf{T}_{\gamma''}\frac{a+i(\beta_l-\sqrt{\gamma})}{a^2+(\beta_l-\sqrt{\gamma})^2}d\gamma''$$
$$+\int_\xi^{k^2}\sum_{l'=1}^{N}\frac{C_{\gamma\gamma'\gamma'l'}}{\sqrt{\sqrt{\gamma}\gamma'\beta_{l'}}}e^{i(\beta_{l'}-\sqrt{\gamma})s}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma'}-\sqrt{\gamma})}{a^2+(\sqrt{\gamma'}-\sqrt{\gamma})^2}d\gamma'$$
$$\left.+\int_\xi^{k^2}\int_\xi^{k^2}\frac{C_{\gamma\gamma'\gamma'\gamma''}}{\sqrt{\sqrt{\gamma}\gamma'\sqrt{\gamma''}}}e^{i(\sqrt{\gamma''}-\sqrt{\gamma})s}\mathbf{T}_{\gamma''}\frac{a+i(\sqrt{\gamma'}-\beta_j)}{a^2+(\sqrt{\gamma'}-\sqrt{\gamma})^2}d\gamma' d\gamma''\right]\overline{\lambda_\gamma}d\gamma,$$

$$\tilde{F}_\lambda^2(\mathbf{T},C,s)$$
$$=-\frac{k^4}{4}\sum_{j,j'=1}^{N}\left[\sum_{l,l'=1}^{N}\frac{C_{jlj'l'}}{\sqrt{\beta_j\beta_l\beta_{j'}\beta_{l'}}}e^{i(\beta_l-\beta_j+\beta_{l'}-\beta_{j'})s}\mathbf{T}_l\mathbf{T}_{l'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.$$
$$+\sum_{l=1}^{N}\int_\xi^{k^2}\frac{C_{jlj'\gamma_2'}}{\sqrt{\beta_j\beta_l\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\beta_l-\beta_j+\sqrt{\gamma_2'}-\beta_{j'})s}\mathbf{T}_l\mathbf{T}_{\gamma_2'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}d\gamma_2'$$
$$+\int_\xi^{k^2}\sum_{l'=1}^{N}\frac{C_{j\gamma_1'j'l'}}{\sqrt{\beta_j\sqrt{\gamma_1'}\beta_{j'}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\beta_j+\beta_{l'}-\beta_{j'})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'$$
$$\left.+\int_\xi^{k^2}\int_\xi^{k^2}\frac{C_{j\gamma_1'j'\gamma_2'}}{\sqrt{\beta_j\sqrt{\gamma_1'}\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\beta_j+\sqrt{\gamma_2'}-\beta_{j'})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{\gamma_2'}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1' d\gamma_2'\right]\overline{\lambda_j\lambda_{j'}}$$
$$-\frac{k^4}{4}\sum_{j=1}^{N}\int_\xi^{k^2}\left[\sum_{l,l'=1}^{N}\frac{C_{jl\gamma_2l'}}{\sqrt{\beta_j\beta_l\sqrt{\gamma_2}\beta_{l'}}}e^{i(\beta_l-\beta_j+\beta_{l'}-\sqrt{\gamma_2})s}\mathbf{T}_l\mathbf{T}_{l'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.$$
$$+\sum_{l=1}^{N}\int_\xi^{k^2}\frac{C_{jl\gamma_2\gamma_2'}}{\sqrt{\beta_j\beta_l\sqrt{\gamma}_2\gamma_2'}}e^{i(\beta_l-\beta_j+\sqrt{\gamma_2'}-\sqrt{\gamma_2})s}\mathbf{T}_l\mathbf{T}_{\gamma_2'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}d\gamma_2'$$
$$+\int_\xi^{k^2}\sum_{l'=1}^{N}\frac{C_{j\gamma_1'\gamma_2l'}}{\sqrt{\beta_j\sqrt{\gamma_1'\gamma_2}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\beta_j+\beta_{l'}-\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'$$
$$\left.+\int_\xi^{k^2}\int_\xi^{k^2}\frac{C_{j\gamma_1'\gamma_2\gamma_2'}}{\sqrt{\beta_j\sqrt{\gamma_1'\gamma_2\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\beta_j+\sqrt{\gamma_2'}-\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{\gamma_2'}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1' d\gamma_2'\right]\overline{\lambda_j\lambda_{\gamma_2}}d\gamma_2$$
$$-\frac{k^4}{4}\int_\xi^{k^2}\sum_{j'=1}^{N}\left[\sum_{l,l'=1}^{N}\frac{C_{\gamma_1lj'l'}}{\sqrt{\sqrt{\gamma_1}\beta_l\beta_{j'}\beta_{l'}}}e^{i(\beta_l-\sqrt{\gamma_1}+\beta_{l'}-\beta_{j'})s}\mathbf{T}_l\mathbf{T}_{l'}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}\right.$$
$$+\sum_{l=1}^{N}\int_\xi^{k^2}\frac{C_{\gamma_1lj'\gamma_2'}}{\sqrt{\sqrt{\gamma_1}\beta_l\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\beta_l-\sqrt{\gamma_1}+\sqrt{\gamma_2'}-\beta_{j'})s}\mathbf{T}_l\mathbf{T}_{\gamma_2'}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}d\gamma_2'$$
$$+\int_\xi^{k^2}\sum_{l'=1}^{N}\frac{C_{\gamma_1\gamma_1'j'l'}}{\sqrt{\sqrt{\gamma_1\gamma_1'}\beta_{j'}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}+\beta_{l'}-\beta_{j'})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'$$
$$\left.+\int_\xi^{k^2}\int_\xi^{k^2}\frac{C_{\gamma_1\gamma_1'j'\gamma_2'}}{\sqrt{\sqrt{\gamma_1\gamma_1'}\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}+\sqrt{\gamma_2'}-\beta_{j'})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{\gamma_2'}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1' d\gamma_2'\right]\overline{\lambda_{\gamma_1}\lambda_{j'}}$$

$$
\begin{aligned}
&-\frac{k^4}{4}\int_{\xi}^{k^2}\int_{\xi}^{k^2}\left[\sum_{l,l'=1}^{N}\frac{C_{\gamma_1 l\gamma_2 l'}}{\sqrt{\sqrt{\gamma_1}\beta_l\sqrt{\gamma_2}\beta_{l'}}}e^{i(\beta_l-\sqrt{\gamma_1}+\beta_{l'}-\sqrt{\gamma_2})s}\mathbf{T}_l\mathbf{T}_{l'}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}\right.\\
&+\sum_{l=1}^{N}\int_{\xi}^{k^2}\frac{C_{\gamma_1 l\gamma_2\gamma_2'}}{\sqrt{\sqrt{\gamma_1}\beta_l\sqrt{\gamma_2\gamma_2'}}}e^{i(\beta_l-\sqrt{\gamma_1}+\sqrt{\gamma_2'}-\sqrt{\gamma_2})s}\mathbf{T}_l\mathbf{T}_{\gamma_2'}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}d\gamma_2'\\
&+\int_{\xi}^{k^2}\sum_{l'=1}^{N}\frac{C_{\gamma_1\gamma_1'\gamma_2 l'}}{\sqrt{\sqrt{\gamma_1\gamma_1'\gamma_2}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}+\beta_{l'}-\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{l'}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'\\
&\left.+\int_{\xi}^{k^2}\int_{\xi}^{k^2}\frac{C_{\gamma_1\gamma_1'\gamma_2\gamma_2'}}{(\gamma_1'\gamma_2\gamma_2')^{1/4}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}+\sqrt{\gamma_2'}-\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\mathbf{T}_{\gamma_2'}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'd\gamma_2'\right]\overline{\lambda_{\gamma_1}\lambda_{\gamma_2}}d\gamma_1 d\gamma_2,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{F}_{\lambda}^{3}(\mathbf{T},C,s)\\
&=\frac{k^4}{4}\sum_{j,j'=1}^{N}\left[\sum_{l,l'=1}^{N}\frac{C_{jlj'l'}}{\sqrt{\beta_j\beta_l\beta_{j'}\beta_{l'}}}e^{i(\beta_l-\beta_j-\beta_{l'}+\beta_{j'})s}\mathbf{T}_l\overline{\mathbf{T}_{l'}}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.\\
&+\sum_{l=1}^{N}\int_{\xi}^{k^2}\frac{C_{jlj'\gamma_2'}}{\sqrt{\beta_j\beta_l\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\beta_l-\beta_j-\sqrt{\gamma_2'}+\beta_{j'})s}\mathbf{T}_l\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}d\gamma_2'\\
&+\int_{\xi}^{k^2}\sum_{l'=1}^{N}\frac{C_{j\gamma_1'j'l'}}{\sqrt{\beta_j\sqrt{\gamma_1'}\beta_{j'}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\beta_j-\beta_{l'}+\beta_{j'})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{l'}}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'\\
&\left.+\int_{\xi}^{k^2}\int_{\xi}^{k^2}\frac{C_{j\gamma_1'j'\gamma_2'}}{\sqrt{\beta_j\sqrt{\gamma_1'}\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\beta_j-\sqrt{\gamma_2'}+\beta_{j'})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'd\gamma_2'\right]\overline{\lambda_j}\lambda_{j'}\\
&+\frac{k^4}{4}\sum_{j=1}^{N}\int_{\xi}^{k^2}\left[\sum_{l,l'=1}^{N}\frac{C_{jl\gamma_2 l'}}{\sqrt{\beta_j\beta_l\sqrt{\gamma_2}\beta_{l'}}}e^{i(\beta_l-\beta_j-\beta_{l'}+\sqrt{\gamma_2})s}\mathbf{T}_l\overline{\mathbf{T}_{l'}}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\right.\\
&+\sum_{l=1}^{N}\int_{\xi}^{k^2}\frac{C_{jl\gamma_2\gamma_2'}}{\sqrt{\beta_j\beta_l\sqrt{\gamma_2\gamma_2'}}}e^{i(\beta_l-\beta_j-\sqrt{\gamma_2'}+\sqrt{\gamma_2})s}\mathbf{T}_l\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}d\gamma_2'\\
&+\int_{\xi}^{k^2}\sum_{l'=1}^{N}\frac{C_{j\gamma_1'\gamma_2 l'}}{\sqrt{\beta_j\sqrt{\gamma_1'\gamma_2}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\beta_j-\beta_{l'}+\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{l'}}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'\\
&\left.+\int_{\xi}^{k^2}\int_{\xi}^{k^2}\frac{C_{j\gamma_1'\gamma_2\gamma_2'}}{\sqrt{\beta_j\sqrt{\gamma_1'\gamma_2\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\beta_j-\sqrt{\gamma_2'}+\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\sqrt{\gamma_1'}-\beta_j)}{a^2+(\sqrt{\gamma_1'}-\beta_j)^2}d\gamma_1'd\gamma_2'\right]\overline{\lambda_j}\lambda_{\gamma_2}d\gamma_2\\
&+\frac{k^4}{4}\int_{\xi}^{k^2}\sum_{j'=1}^{N}\left[\sum_{l,l'=1}^{N}\frac{C_{\gamma_1 lj'l'}}{\sqrt{\sqrt{\gamma_1}\beta_l\beta_{j'}\beta_{l'}}}e^{i(\beta_l-\sqrt{\gamma_1}-\beta_{l'}+\beta_{j'})s}\mathbf{T}_l\overline{\mathbf{T}_{l'}}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}\right.\\
&+\sum_{l=1}^{N}\int_{\xi}^{k^2}\frac{C_{\gamma_1 lj'\gamma_2'}}{\sqrt{\sqrt{\gamma_1}\beta_l\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\beta_l-\sqrt{\gamma_1}-\sqrt{\gamma_2'}+\beta_{j'})s}\mathbf{T}_l\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}d\gamma_2'\\
&+\int_{\xi}^{k^2}\sum_{l'=1}^{N}\frac{C_{\gamma_1\gamma_1'j'l'}}{\sqrt{\sqrt{\gamma_1\gamma_1'}\beta_{j'}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}-\beta_{l'}+\beta_{j'})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{l'}}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'\\
&\left.+\int_{\xi}^{k^2}\int_{\xi}^{k^2}\frac{C_{\gamma_1\gamma_1'j'\gamma_2'}}{\sqrt{\sqrt{\gamma_1\gamma_1'}\beta_{j'}\sqrt{\gamma_2'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}-\sqrt{\gamma_2'}+\beta_{j'})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'd\gamma_2'\right]\overline{\lambda_{\gamma_1}}\lambda_{j'}
\end{aligned}
$$

$$+\frac{k^4}{4}\int_\xi^{k^2}\int_\xi^{k^2}\left[\sum_{l,l'=1}^N \frac{C_{\gamma_1 l\gamma_2 l'}}{\sqrt{\sqrt{\gamma_1}\beta_l\sqrt{\gamma_2}\beta_{l'}}}e^{i(\beta_l-\sqrt{\gamma_1}-\beta_{l'}+\sqrt{\gamma_2})s}\mathbf{T}_l\overline{\mathbf{T}_{l'}}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}\right.$$
$$+\sum_{l=1}^N\int_\xi^{k^2}\frac{C_{\gamma_1 l\gamma_2\gamma_2'}}{\sqrt{\sqrt{\gamma_1}\beta_l\sqrt{\gamma_2\gamma_2'}}}e^{i(\beta_l-\sqrt{\gamma_1}-\sqrt{\gamma_2'}+\sqrt{\gamma_2})s}\mathbf{T}_l\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\beta_l-\sqrt{\gamma_1})}{a^2+(\beta_l-\sqrt{\gamma_1})^2}d\gamma_2'$$
$$+\int_\xi^{k^2}\sum_{l'=1}^N\frac{C_{\gamma_1\gamma_1'\gamma_2 l'}}{\sqrt{\sqrt{\gamma_1\gamma_1'\gamma_2}\beta_{l'}}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}-\beta_{l'}+\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{l'}}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'$$
$$\left.+\int_\xi^{k^2}\int_\xi^{k^2}\frac{C_{\gamma_1\gamma_1'\gamma_2\gamma_2'}}{(\gamma_1\gamma_1'\gamma_2\gamma_2')^{1/4}}e^{i(\sqrt{\gamma_1'}-\sqrt{\gamma_1}-\sqrt{\gamma_2'}+\sqrt{\gamma_2})s}\mathbf{T}_{\gamma_1'}\overline{\mathbf{T}_{\gamma_2'}}\frac{a+i(\sqrt{\gamma_1'}-\sqrt{\gamma_1})}{a^2+(\sqrt{\gamma_1'}-\sqrt{\gamma_1})^2}d\gamma_1'd\gamma_2'\right]\overline{\lambda_{\gamma_1}}\lambda_{\gamma_2}d\gamma_1 d\gamma_2.$$

This expression combined with (2.54) gives us, $\sup_{\epsilon,t}\mathbb{E}\left[\,\left|\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon\right)(t)\right|^2\,\right]<+\infty$. □

Lemma 2.6

$$\lim_{M\to+\infty}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{0\leq t\leq T}|\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)|\geq M\right)=0.$$

Proof (of Lemma 2.6) We recall that $\|\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\|_{\mathcal{H}_\xi}=\|y\|_{\mathcal{H}_\xi}$ and then

$$|\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)|\leq\|\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\|_{\mathcal{H}_\xi}\|\lambda\|_{\mathcal{H}_\xi}=\|y\|_{\mathcal{H}_\xi}\|\lambda\|_{\mathcal{H}_\xi}.$$

□

This last lemma completes the proof of Proposition 2.8. ■

Martingale problem

In this section, we shall characterize all subsequence limits by showing they are solution of a well-posed martingale problem. To do that, we consider a converging subsequence of $(\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y))_{\epsilon\in(0,1)}$ which converges to a limit $\mathbf{T}^\xi(.)(y)$. For the sake of simplicity we denote by $(\tilde{\mathbf{T}}^{\xi,\epsilon}(.)(y))_{\epsilon\in(0,1)}$ the subsequence.

Convergence Result

Proposition 2.9 *$\forall\lambda\in\mathcal{H}_\xi$ and $\forall f$ smooth test function,*

$$\begin{aligned}
&f(\mathbf{T}_\lambda^\xi(t)(y))\\
&\quad-\int_0^t \partial_v f(\mathbf{T}_\lambda^\xi(s)(y))\left\langle J^\xi(\mathbf{T}^\xi(s)(y)),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial_{\overline{v}}f(\mathbf{T}_\lambda^\xi(s)(y))\overline{\left\langle J^\xi(\mathbf{T}^\xi(s)(y)),\lambda\right\rangle_{\mathcal{H}_\xi}}\\
&\quad+\partial_v^2 f(\mathbf{T}_\lambda^\xi(s)(y))\left\langle K(\mathbf{T}^\xi(s)(y))(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial_{\overline{v}}^2 f(\mathbf{T}_\lambda^\xi(s)(y))\overline{\left\langle K(\mathbf{T}^\xi(s)(y))(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}}\\
&\quad+\partial_{\overline{v}}\partial_v f(\mathbf{T}_\lambda^\xi(s)(y))\left\langle L(\mathbf{T}^\xi(s)(y))(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial_v\partial_{\overline{v}} f(\mathbf{T}_\lambda^\xi(s)(y))\overline{\left\langle L(\mathbf{T}^\xi(s)(y))(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}}ds
\end{aligned}$$

is a martingale, where

$$J^\xi(\mathbf{T})_j=\left[\frac{\Gamma_{jj}^c-\Gamma_{jj}^1-\Lambda_j^{c,\xi}}{2}+i\left(\frac{\Gamma_{jj}^s-\Lambda_j^{s,\xi}}{2}+\kappa_j^\xi\right)\right]\mathbf{T}_j,$$

$$K(\mathbf{T})(\lambda)_j=-\frac{1}{2}\sum_{l=1}^N\Gamma_{jl}^1\mathbf{T}_j\mathbf{T}_l\overline{\lambda_l}-\frac{1}{2}\sum_{\substack{l=1\\l\neq j}}^N\left(\Gamma_{jl}^c+i\Gamma_{jl}^s\right)\mathbf{T}_j\mathbf{T}_l\overline{\lambda_l},$$

$$L(\mathbf{T})(\lambda)_j=\frac{1}{2}\sum_{l=1}^N\Gamma_{jl}^1\mathbf{T}_j\overline{\mathbf{T}_l}\lambda_l+\frac{1}{2}\sum_{\substack{l=1\\l\neq j}}^N\Gamma_{jl}^c\mathbf{T}_l\overline{\mathbf{T}_l}\lambda_j,$$

and

$$J^{\xi}(\mathbf{T})_{\gamma} = K(\mathbf{T})(\lambda)_{\gamma} = L(\mathbf{T})(\lambda)_{\gamma} = 0$$

for almost every $\gamma \in (\xi, k^2)$*, and for* $(\mathbf{T}, \lambda) \in \mathcal{H}^2_{\xi}$*.*

Proof (of Proposition 2.9) Let

$$f_2^{\epsilon}(t) = \int_t^{+\infty} \mathbb{E}_t^{\epsilon}\left[\tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(\frac{u}{\epsilon}\right)\otimes C\left(\frac{u}{\epsilon}\right), \frac{u}{\epsilon}\right)\right] - \tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \mathbb{E}[C(0)\otimes C(0)], \frac{u}{\epsilon}\right) du.$$

Lemma 2.7

$$\sup_{t\geq 0} \mathbb{E}\left[|f_2^{\epsilon}(t)|\right] = \mathcal{O}\left(\epsilon\right)$$

and

$$\mathcal{A}^{\epsilon}\left(f_0^{\epsilon} + f_1^{\epsilon} + f_2^{\epsilon}\right)(t) = \tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \mathbb{E}[C(0)\otimes C(0)], \frac{t}{\epsilon}\right) + A(\epsilon, t),$$

where $\sup_{t\geq 0}\mathbb{E}\left[|A(\epsilon,t)|\right] = \mathcal{O}(\sqrt{\epsilon})$*.*

Proof (of Lemma 2.7) Using a change of variable we get $f_2^{\epsilon}(t) = \epsilon B(\epsilon, t)$ with

$$B(\epsilon, t) = \int_0^{+\infty} \mathbb{E}_t^{\epsilon}\left[\tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), C\left(u+\frac{t}{\epsilon}\right)\otimes C\left(u+\frac{t}{\epsilon}\right), u+\frac{t}{\epsilon}\right)\right] - \tilde{F}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \mathbb{E}[C(0)\otimes C(0)], u+\frac{t}{\epsilon}\right) du.$$

By a computation, we get that $\sup_{\epsilon,t\geq 0}\mathbb{E}\left[|B(\epsilon,t)|\right] < +\infty$, and after a long but straightforward computation we get the second part of the lemma. $\square$

Next, let $\tilde{G}_{\lambda}(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \frac{t}{\epsilon}) = \tilde{F}_{\lambda}(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \mathbb{E}[C(0)\otimes C(0)], \frac{t}{\epsilon})$ and

$$f_3^{\epsilon}(t) = -\int_0^t \Big[\tilde{G}_{\lambda}(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), \frac{u}{\epsilon}) - \lim_{T\to+\infty}\frac{1}{T}\int_0^T \tilde{G}_{\lambda}(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), s)ds\Big]du.$$

Lemma 2.8 $\forall T' > 0$*, we have*

$$\lim_{\epsilon\to 0}\sup_{0\leq t\leq T'}\mathbb{E}\left[|f_3^{\epsilon}(t)|\right] = 0.$$

Proof (of Lemma 2.8) Using a change of variable, we get

$$f_3^{\epsilon}(t) = -\epsilon\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), u\right) - \lim_{T\to+\infty}\frac{1}{T}\int_0^T \tilde{G}_{\lambda}\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y), s\right)ds\right]du.$$

Let $\mu > 0$, we have

$$\Big|\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_{\lambda}\left(\mathbf{T}^{\xi,\epsilon}(t)(y), u\right) - \lim_{T\to+\infty}\frac{1}{T}\int_0^T \tilde{G}_{\lambda}\left(\mathbf{T}^{\xi,\epsilon}(t), s\right)ds\right]du\Big| \leq K(\mu, T', \xi, y) + \frac{K(T',\xi,y)}{\epsilon}\sum_{j=1}^4\int_{I^j_{<\mu}} d\gamma_1\dots d\gamma_j,$$

where

$$I^j_{<\mu} = \Big\{(\gamma_l)_{l\in\{1,\dots,j\}} \in (\xi, k^2)^j, \exists (q_l)_{l\in\{1,\dots,4-j\}} \in \{\beta_1,\dots,\beta_N\}^{4-j} \text{ and } (\mu_l)_{l\in\{1,\dots,4\}} \in \{-1,1\}^4, \text{ with } \Big|\sum_{l=1}^j \mu_l\sqrt{\gamma_l} + \sum_{l=1}^{4-j}\mu_{l+j}q_l\Big| < \mu\Big\}.$$

Finally,

$$\overline{\lim_{\epsilon\to 0}} \sup_{0\le t\le T'} \mathbb{E}\left[\epsilon\left|\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_\lambda\left(\mathbf{T}^{\xi,\epsilon}(t)(y),u\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda\left(\mathbf{T}^{\xi,\epsilon}(t)(y),s\right)ds\right]du\right|\right]$$
$$\le K(T',\xi,y)\sum_{j=1}^4\int_{I^j_{<\mu}}d\gamma_1\dots d\gamma_j,$$

and then by letting $\mu\to 0$ we get the announced result. □

Let $f^\epsilon(t)=f_0^\epsilon(t)+f_1^\epsilon(t)+f_2^\epsilon(t)+f_3^\epsilon(t)$. A computation gives

$$\begin{aligned}\mathcal{A}^\epsilon f^\epsilon(t)&=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y),s\right)ds+C(\epsilon,t)\\&=\tilde{G}^\infty_\lambda\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)+C(\epsilon,t),\end{aligned}$$

where, $\forall\mu>0$,

$$\overline{\lim_{\epsilon\to 0}}\sup_{0\le t\le T'}\mathbb{E}\left[|C(\epsilon,t)|\right]\le K(T',\xi,y)\sum_{j=1}^4\int_{I^j_{<\mu}}d\gamma_1\dots d\gamma_j,$$

using the boundness condition (2.54). Moreover, for $(\mathbf{T},\lambda)\in\mathcal{H}^2_\xi$, $\tilde{G}^\infty$ is defined as follow

$$\begin{aligned}\tilde{G}^\infty_\lambda(\mathbf{T})=&\partial_v f\left(\mathbf{T}\right)\left\langle J^\xi(\mathbf{T}),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial_{\overline{v}}f\left(\mathbf{T}\right)\overline{\left\langle J^\xi(\mathbf{T}),\lambda\right\rangle_{\mathcal{H}_\xi}}\\&+\partial_v^2 f\left(\mathbf{T}\right)\left\langle\tilde{K}\left(\mathbf{T}\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial^2_{\overline{v}}f\left(\mathbf{T}\right)\overline{\left\langle\tilde{K}\left(\mathbf{T}\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}}\\&+\partial_{\overline{v}}\partial_v f\left(\mathbf{T}\right)\left\langle\tilde{L}\left(\mathbf{T}\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}+\partial_v\partial_{\overline{v}}f\left(\mathbf{T}\right)\overline{\left\langle\tilde{L}\left(\mathbf{T}\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}},\end{aligned}$$

where

$$\begin{aligned}\tilde{K}(\mathbf{T})(\lambda)_j&=-\frac{k^4}{4}\sum_{\beta_j+\beta_{j'}=\beta_l+\beta_{l'}}\frac{\mathbf{C}_{jlj'l'}}{\sqrt{\beta_j\beta_l\beta_{j'}\beta_{l'}}}\mathbf{T}_l\mathbf{T}_{l'}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\overline{\lambda_{j'}}\\\tilde{L}(\mathbf{T})(\lambda)_j&=\frac{k^4}{4}\sum_{\beta_j-\beta_{j'}=\beta_l-\beta_{l'}}\frac{\mathbf{C}_{jlj'l'}}{\sqrt{\beta_j\beta_l\beta_{j'}\beta_{l'}}}\mathbf{T}_l\overline{\mathbf{T}_{l'}}\frac{a+i(\beta_l-\beta_j)}{a^2+(\beta_l-\beta_j)^2}\lambda_{j'}\end{aligned}$$

for $j\in\{1,\dots,N\}$, with

$$\mathbf{C}=\mathbb{E}\big[C(0)\otimes C(0)\big],$$

and

$$\tilde{K}(\mathbf{T})(\lambda)_\gamma=\tilde{L}(\mathbf{T})(\lambda)_\gamma=0$$

for almost every $\gamma\in(\xi,k^2)$.

We assume that the following nondegeneracy condition holds. The wavenumbers β_j are distinct along with their sums and differences. This assumption is also considered in [25], [30] and [39]. As a result we get

$$\begin{aligned}
\tilde{G}_\lambda^\infty\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right) &= \partial_v f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\left\langle J^\xi(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)),\lambda\right\rangle_{\mathcal{H}_\xi} \\
&+ \partial_{\overline{v}} f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\overline{\left\langle J^\xi(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)),\lambda\right\rangle_{\mathcal{H}_\xi}} \\
&+ \partial_v^2 f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\left\langle K\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi} \\
&+ \partial_{\overline{v}}^2 f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\overline{\left\langle K\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}} \\
&+ \partial_{\overline{v}}\partial_v f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\left\langle L\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi} \\
&+ \partial_v\partial_{\overline{v}} f\left(\tilde{\mathbf{T}}_\lambda^{\xi,\epsilon}(t)(y)\right)\overline{\left\langle L\left(\tilde{\mathbf{T}}^{\xi,\epsilon}(t)(y)\right)(\lambda),\lambda\right\rangle_{\mathcal{H}_\xi}}.
\end{aligned} \tag{2.58}$$

By Theorem 2.11, $\left(M_{f^\epsilon}^\epsilon(t)\right)_{t\geq 0}$ is an $(\mathcal{F}_t^\epsilon)$-martingale. Then, for every bounded continuous function h, every sequence $0 < s_1 < \cdots < s_n \leq s < t$, and every family $(\lambda_j)_{j\in\{1,\ldots,n\}}$ with values in $\mathcal{H}_\xi^n$ we have

$$\mathbb{E}\left[h\left(\tilde{\mathbf{T}}_{\lambda_j}^{\xi,\epsilon}(s_j)(y), 1\leq j\leq n\right)\left(f^\epsilon(t) - f^\epsilon(s) - \int_s^t \mathcal{A}^\epsilon f^\epsilon(u)du\right)\right] = 0.$$

Finally, using (2.58) and Lemmas 2.4, 2.7, and 2.8, we can conclude the proof of Proposition 2.9. ■

Uniqueness In order to prove uniqueness, we decompose $\mathbf{T}^\xi(.)(y)$ into real and imaginary parts. Then, let us consider the new process

$$\mathbf{Y}^\xi(t) = \begin{bmatrix}\mathbf{Y}^{1,\xi}(t)\\ \mathbf{Y}^{2,\xi}(t)\end{bmatrix}, \text{ where } \mathbf{Y}^{1,\xi}(t) = Re(\mathbf{T}^\xi(t)(y)) \text{ and } \mathbf{Y}^{2,\xi}(t) = Im(\mathbf{T}^\xi(t)(y)).$$

This new process takes its values in $\mathcal{G}_\xi \times \mathcal{G}_\xi$, where $\mathcal{G}_\xi = \mathbb{R}^N \times L^2((\xi,k^2),\mathbb{R})$. $\mathcal{G}_\xi\times\mathcal{G}_\xi$ is equipped with the inner product defined by

$$\langle \mathbf{T},\mathbf{S}\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} = \sum_{j=1}^N \mathbf{T}_j^1\mathbf{S}_j^1 + \mathbf{T}_j^2\mathbf{S}_j^2 + \int_\xi^{k^2} \mathbf{T}_\gamma^1\mathbf{S}_\gamma^1 + \mathbf{T}_\gamma^2\mathbf{S}_\gamma^2 d\gamma$$

$\forall(\mathbf{T},\mathbf{S})\in\mathcal{G}_\xi\times\mathcal{G}_\xi$. We also use the notation $\mathbf{Y}_\lambda^\xi(t) = \left\langle \mathbf{Y}^\xi(t),\lambda\right\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi}$ with $\lambda\in\mathcal{G}_\xi\times\mathcal{G}_\xi$. We introduce the operator Υ on $\mathcal{G}_\xi\times\mathcal{G}_\xi$ given by

$$\begin{aligned}\Upsilon : \mathcal{G}_\xi\times\mathcal{G}_\xi &\longrightarrow \mathcal{G}_\xi\times\mathcal{G}_\xi,\\ \begin{bmatrix}\mathbf{T}^1\\ \mathbf{T}^2\end{bmatrix} &\longmapsto \begin{bmatrix}\mathbf{T}^2\\ -\mathbf{T}^1\end{bmatrix}.\end{aligned}$$

By Proposition 2.9, we get the following result.

Proposition 2.10 $\forall\lambda\in\mathcal{G}_\xi\times\mathcal{G}_\xi$, $\forall f\in\mathcal{C}_b^\infty(\mathbb{R})$

$$\begin{aligned}f(\mathbf{Y}_\lambda^\xi(t)) - \int_0^t &\langle B^\xi(\mathbf{Y}^\xi(s)),\lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} f'(\mathbf{Y}_\lambda^\xi(s))\\ &+ \frac{1}{2}\langle A(\mathbf{Y}^\xi(s))(\lambda),\lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} f''(\mathbf{Y}_\lambda^\xi(s))ds\end{aligned}$$

is a martingale, where

$$A(\mathbf{Y})(\lambda) = A_1(\mathbf{Y})(\lambda) + A_2(\mathbf{Y})(\lambda) + A_3(\mathbf{Y})(\lambda),$$

with, for $j \in \{1, \dots, N\}$,

$$\begin{aligned}
B^\xi(\mathbf{Y})_j &= \left[\frac{\Gamma^c_{jj} - \Lambda^{c,\xi}_j}{2}\right]\mathbf{Y}_j - \left[\frac{\Gamma^s_{jj} - \Lambda^{s,\xi}_j}{2} + \kappa^\xi_j\right]\Upsilon_j(\mathbf{Y}) \\
A_1(\mathbf{Y})(\lambda)_j &= \Upsilon_j(\mathbf{Y})\sum_{l=1}^{N}\Gamma^1_{jl}[\Upsilon^1_l(\mathbf{Y})\lambda^1_l + \Upsilon^2_l(\mathbf{Y})\lambda^2_l] \\
A_2(\mathbf{Y})(\lambda)_j &= -\mathbf{Y}_j\sum_{\substack{l=1\\ l\neq j}}^{N}\Gamma^c_{jl}[\mathbf{Y}^1_l\lambda^1_l + \mathbf{Y}^2_l\lambda^2_l] + \Upsilon_j(\mathbf{Y})\sum_{\substack{l=1\\ l\neq j}}^{N}\Gamma^c_{jl}[\Upsilon^1_l(\mathbf{Y})\lambda^1_l + \Upsilon^2_l(\mathbf{Y})\lambda^2_l] \\
A_3(\mathbf{Y})(\lambda)_j &= \lambda_j\sum_{\substack{l=1\\ l\neq j}}^{N}\Gamma^c_{jl}[(\mathbf{Y}^1_l)^2 + (\mathbf{Y}^2_l)^2],
\end{aligned}$$

and

$$B^\xi_\gamma(\mathbf{Y}) = A_\gamma(\mathbf{Y})(\lambda) = A_\gamma(\mathbf{Y})(\lambda) = A_\gamma(\mathbf{Y})(\lambda) = 0$$

for almost every $\gamma \in (\xi, k^2)$*, and for* $(\mathbf{Y}, \lambda) \in (\mathcal{G}_\xi \times \mathcal{G}_\xi)^2$.

Proof (of Proposition 2.10) Using Proposition 2.9,

$$f(\mathbf{Y}^\xi_\lambda(t)) - \int_0^t Re(\langle J^\xi(\mathbf{T}^\xi(s)(y)), \lambda\rangle_{\mathcal{H}_\xi})f'(\mathbf{Y}^\xi_\lambda(s)) \\ + \frac{1}{2}Re(\langle (L+K)(\mathbf{T}^\xi(s)(y))(\lambda), \lambda\rangle_{\mathcal{H}_\xi})f''(\mathbf{Y}^\xi_\lambda(s))ds$$

is a martingale. Let us remark that we also denote by λ the function $\lambda^1 + i\lambda^2$, and

$$Re\Big(\langle \mathbf{T}^\xi(t)(y), \lambda\rangle_{\mathcal{H}_\xi}\Big) = \langle \mathbf{Y}^\xi(t), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} \text{ and } Im\Big(\langle \mathbf{T}^\xi(t)(y), \lambda\rangle_{\mathcal{H}_\xi}\Big) = \langle \Upsilon(\mathbf{Y}^\xi(t)), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi}.$$

Then, we have

$$\begin{aligned}
Re(\langle J^\xi(\mathbf{T}^\xi(s)(y)), \lambda\rangle_{\mathcal{H}_\xi}) &= \langle B^\xi(\mathbf{Y}^\xi(s)), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} \\
Re(\langle (L+K)(\mathbf{T}^\xi(s)(y))(\lambda), \lambda\rangle_{\mathcal{H}_\xi}) &= \langle A(\mathbf{Y}^\xi(s))(\lambda), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi}.
\end{aligned}$$

■

As a consequence of Proposition 2.10, $\forall\lambda \in \mathcal{G}\times\mathcal{G}$, letting successively $f \in \mathcal{C}^\infty_b(\mathbb{R})$ such that $f(s) = s$ and $f(s) = s^2$ if $|s| \leq r_y\|\lambda\|_{\mathcal{G}\times\mathcal{G}}$, we get that

$$\langle M^\xi(t), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi} = M^\xi_\lambda(t) = \langle \mathbf{Y}^\xi(t) - \int_0^t B^\xi(\mathbf{Y}^\xi(s))ds, \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi}$$

is a continuous martingale with quadratic variation given by

$$< M^\xi_\lambda > (t) = \int_0^t \langle A(\mathbf{Y}^\xi(s))(\lambda), \lambda\rangle_{\mathcal{G}_\xi\times\mathcal{G}_\xi}ds.$$

Proposition 2.11 $\forall f \in \mathcal{C}_b^2(\mathcal{G}_\xi \times \mathcal{G}_\xi)$,

$$M_f^\xi(t) = f(\mathbf{Y}^\xi(t)) - \int_0^t L^\xi f(\mathbf{Y}^\xi(s))ds \tag{2.59}$$

is a continuous martingale, where $\forall \mathbf{Y} \in \mathcal{G}_\xi \times \mathcal{G}_\xi$

$$L^\xi f(\mathbf{Y}) = \frac{1}{2} trace\left(A(\mathbf{Y})D^2 f(\mathbf{Y})\right) + \left\langle B^\xi(\mathbf{Y}), Df(\mathbf{Y})\right\rangle_{\mathcal{G}_\xi \times \mathcal{G}_\xi}.$$

Moreover, the martingale problem associated to the generator L^ξ *is well-posed.*

Proof (of Proposition 2.11) We begin with the following lemma.

Lemma 2.9

$$\begin{aligned} A &: \mathcal{G}_\xi \times \mathcal{G}_\xi \longrightarrow L_1^+\left(\mathcal{G}_\xi \times \mathcal{G}_\xi\right), \\ B^\xi &: \mathcal{G}_\xi \times \mathcal{G}_\xi \longrightarrow \mathcal{G}_\xi \times \mathcal{G}_\xi, \end{aligned}$$

where $L_1^+\left(\mathcal{G}_\xi \times \mathcal{G}_\xi\right)$ *is a set of nonnegative operators with finite trace.*

Proof $\forall (\mathbf{Y}, \lambda) \in (\mathcal{G}_\xi \times \mathcal{G}_\xi)^2$, we have

$$\begin{aligned} \langle A(\mathbf{Y})(\lambda), \lambda\rangle_{\mathcal{G}_\xi \times \mathcal{G}_\xi} &= Re(\langle (L+K)(\mathbf{T})(\lambda), \lambda\rangle_{\mathcal{H}_\xi}) \\ &= Re\Big(\sum_{j,l=1}^N \Gamma_{jl}^1 [\mathbf{T}_j \overline{\lambda_j} - \overline{\mathbf{T}_j}\lambda_j]\,\overline{[\mathbf{T}_l \overline{\lambda_l} - \overline{\mathbf{T}_l}\lambda_l]}\Big) \\ &\quad + \sum_{\substack{j,l=1 \\ j\neq l}}^N \Gamma_{jl}^c \Big|\mathbf{T}_j \overline{\lambda_l} - \overline{\mathbf{T}_l}\lambda_j\Big|^2. \end{aligned}$$

with $\mathbf{T} = \mathbf{Y}^1 + i\mathbf{Y}^2$ and $\lambda = \lambda^1 + i\lambda^2$. First, $\forall (j,l) \in \{1,\dots,N\}^2$ such that $j \neq l$, Γ_{jl}^c is nonnegative because it is proportional to the power spectral density of C_{jl} at $\beta_l - \beta_j$ frequency. Second, the matrix Γ^1 is nonnegative since $\forall X \in \mathbb{C}^N$, we have

$${}^t X \Gamma^1 X = \frac{k^4}{2} \sum_{j,l=1}^N \int_0^{+\infty} \mathbb{E}[C_{jj}(0)C_{ll}(z)]dz \tilde{X}_j \tilde{X}_l = \frac{k^4}{2}\int_0^{+\infty} \mathbb{E}[C_{\tilde{X}}(0)C_{\tilde{X}}(z)]dz \geq 0$$

because it is proportional to the power spectral density of $C_{\tilde{X}}(z) = \sum_j C_{jj}(z)\tilde{X}_j$ at 0 frequency, and with $\tilde{X}_j = X_j/\beta_j$, $\forall j \in \{1,\dots,N\}$. Moreover,

$$trace(A(\mathbf{Y})) = \sum_{j=1}^N \Gamma_{jj}^1 [(\mathbf{Y}_j^1)^2 + (\mathbf{Y}_j^2)^2] \leq \sup_{j\in\{1,\dots,N\}} \Gamma_{jj}^1 \, \|\mathbf{Y}\|^2_{\mathcal{G}_\xi \times \mathcal{G}_\xi}.$$

□

Consequently, following the proof of Theorem 4.1.4 in [63], (2.59) is a martingale. However, B^ξ and A are not bounded functions but this problem can be compensated by the fact that the process $\mathbf{Y}^\xi(.)$ takes its values in $\mathcal{B}_{r_y, \mathcal{G}_\xi \times \mathcal{G}_\xi}$.

Moreover, from this lemma there exists a linear operator σ from $\mathcal{G}_\xi \times \mathcal{G}_\xi$ to $L_2(\mathcal{G}_\xi \times \mathcal{G}_\xi)$, which is the set of Hilbert-Schmidt operators from $\mathcal{G}_\xi \times \mathcal{G}_\xi$ to itself, such that $A(\mathbf{Y}) = \sigma(\mathbf{Y}) \circ \sigma^*(\mathbf{Y})$. According to Theorem 3.2.2 and 4.4.1 in [63], the martingale problem associated to L^ξ is well-posed because $\forall \mathbf{Y} \in \mathcal{G}_\xi \times \mathcal{G}_\xi$

$$\|\sigma(\mathbf{Y})\| \leq K(N)\|\mathbf{Y}\|_{\mathcal{G}_\xi \times \mathcal{G}_\xi}.$$

■

Let us recall that the process $\mathbf{Y}^\xi(.)$ is an element of $\mathcal{C}([0,+\infty),(\mathcal{B}_{r_y,\mathcal{G}_\xi\times\mathcal{G}_\xi},d_{\mathcal{B}_{r_y,\mathcal{G}_\xi\times\mathcal{G}_\xi}}))$, and we cannot assert that $\mathbf{Y}^\xi(.)$ is uniquely determined. In fact, we need to know if its law is supported by $\mathcal{C}([0,+\infty),(\mathcal{G}_\xi\times\mathcal{G}_\xi,\|.\|_{\mathcal{G}_\xi\times\mathcal{G}_\xi}))$. Letting

$$f(\mathbf{Y})=\|\Pi(\xi,k^2)\otimes\Pi(\xi,k^2)(\mathbf{Y}-y)\|^2_{\mathcal{G}_\xi\times\mathcal{G}_\xi},$$

where

$$\begin{aligned}\Pi(\xi,k^2)\otimes\Pi(\xi,k^2):\mathcal{G}_\xi\times\mathcal{G}_\xi &\longrightarrow \mathcal{G}_\xi\times\mathcal{G}_\xi,\\ \begin{bmatrix}\mathbf{Y}^1\\ \mathbf{Y}^2\end{bmatrix} &\longmapsto \begin{bmatrix}\Pi(\xi,k^2)(\mathbf{Y}^1)\\ \Pi(\xi,k^2)(\mathbf{Y}^2)\end{bmatrix}.\end{aligned}$$

As $\mathbf{Y}^\xi(.)$ is a solution on $\mathcal{C}([0,+\infty),(\mathcal{B}_{r_y,\mathcal{G}_\xi\times\mathcal{G}_\xi},d_{\mathcal{B}_{r_y,\mathcal{G}_\xi\times\mathcal{G}_\xi}}))$ of the martingale associated to L^ξ, we get

$$\mathbb{E}[f(\mathbf{Y}^\xi(t))]=0\quad\forall t\geq 0,$$

and therefore $\Pi(\xi,k^2)\otimes\Pi(\xi,k^2)(\mathbf{Y}^\xi(.))=\Pi(\xi,k^2)\otimes\Pi(\xi,k^2)(Re(y),Im(y))$. Consequently, the process $\mathbf{Y}^\xi(.)$ is strongly continuous since the weak and the strong topologies are the same on $\mathbb{R}^N$. Finally, $\mathbf{Y}^\xi(.)$ is uniquely characterized as being the unique solution of the martingale problem associated to L^ξ and starting from $(Re(y),Im(y))$, and that concludes the proof of Theorem 2.1.

2.6.3 Proof of Theorem 2.2

Let $\mathcal{H}_0=\mathbb{C}^N\times L^2(0,k^2)$ and $y\in\mathcal{H}_0$. We begin by showing the tightness of the process $(\mathbf{T}^\xi(.)(y^\xi))_\xi$, which is the unique solution of the martingale problem associated to $\mathcal{L}_\xi$ and starting from $y^\xi=\Pi(\xi,+\infty)(y)$. As the radiating part $\Pi(0,k^2)(\mathbf{T}^\xi(.)(y^\xi))$ of the process $\mathbf{T}^\xi(.)(y^\xi)$ is constant equal to $\Pi(\xi,k^2)(y^\xi)$, to prove the tightness of $(\mathbf{T}^\xi(.)(y^\xi))_\xi$ is suffices to show the tightness of the finite-dimensional process $(\Pi(k^2,+\infty)(\mathbf{T}^\xi(.)(y^\xi)))_\xi$. Let $\mathbb{E}^\xi_t$ be the conditional expectation given $\sigma(\mathbf{T}^\xi(u)(y^\xi),0\leq u\leq t)$. Then, $\forall t\geq 0$, $\forall h\in(0,1)$ and $\forall s\in[0,h]$, we have

$$\begin{aligned}\mathbb{E}^\xi_t\Big[\|\mathbf{T}^\xi(t+s)(y^\xi)-&\mathbf{T}^\xi(t)(y^\xi)\|^2_{\mathbb{C}^N}\Big]\\ &\leq \mathbb{E}^\xi_t\Big[\|\mathbf{Y}^{1,\xi}(t+s)-\mathbf{Y}^{1,\xi}(t)\|^2_{\mathbb{R}^N}\Big]+\mathbb{E}^\xi_t\Big[\|\mathbf{Y}^{2,\xi}(t+s)-\mathbf{Y}^{2,\xi}(t)\|^2_{\mathbb{R}^N}\Big]\\ &\leq \sum_{\substack{j=1\\ l=1,2}}^{N}\mathbb{E}^\xi_t\Big[(\mathbf{Y}^{l,\xi}_j(t+s)-\mathbf{Y}^{l,\xi}_j(t))^2\Big]\\ &\leq \sum_{\substack{j=1\\ l=1,2}}^{N}\mathbb{E}^\xi_t\Big[\Big(\int_t^{t+s}L^\xi f^l_j(\mathbf{Y}^\xi(u))du\Big)^2\Big]+\mathbb{E}^\xi_t\Big[\Big(M^\xi_{f^l_j}(t+s)-M^\xi_{f^l_j}(t)\Big)^2\Big],\end{aligned}$$

with $\forall\mathbf{Y}\in\mathcal{G}_0\times\mathcal{G}_0$, $f^l_j(\mathbf{Y})=\mathbf{Y}^l_j$. Therefore, using that the process $\mathbf{T}^\xi(.)(y^\xi)$ takes its values in $\mathcal{B}_{r_y,\mathcal{H}_\xi}$, we first get

$$\mathbb{E}^\xi_t\Big[\Big(\int_t^{t+s}L^\xi f^l_j(\mathbf{Y}^\xi(u))du\Big)^2\Big]\leq K\,h^2,$$

and second,

$$\mathbb{E}^\xi_t\Big[\Big(M^\xi_{f^l_j}(t+s)-M^\xi_{f^l_j}(t)\Big)^2\Big]=\mathbb{E}^\xi_t\Big[<M^\xi_{f^l_j}>_{t+s}-<M^\xi_{f^l_j}>_t\Big]\leq K\,h$$

with
$$< M^{\xi}_{f^l_j} >_t = \int_0^t L^{\xi}(f^l_j)^2(\mathbf{Y}^{\xi}(u)) - 2f^l_j(\mathbf{Y}^{\xi}(u))L^{\xi}f^l_j(\mathbf{Y}^{\xi}(u))\,du.$$
Consequently, the process $(\mathbf{T}^{\xi}(.)(y^{\xi}))_{\xi}$ is tight on $\mathcal{C}([0,+\infty),(\mathcal{H}_0,\|.\|_{\mathcal{H}_0}))$. Now, to characterize all limits of converging subsequences, let us denote by $\mathbf{T}^0(.)(y)$ such a limit point. First, for every smooth function f on $\mathcal{H}_0$, for every bounded continuous function h, and every sequence $0 < s_1 < \cdots < s_n \leq s < t$, we have

$$\mathbb{E}\left[h\left(\mathbf{T}^{\xi}(s_j)(y^{\xi}), 1\leq j\leq n\right)\left(f(\mathbf{T}^{\xi}(t)(y^{\xi})) - f(\mathbf{T}^{\xi}(s)(y^{\xi})) - \int_s^t \mathcal{L}_{\xi}f(\mathbf{T}^{\xi}(u)(y^{\xi}))du\right)\right] = 0.$$

Second,

$$\sup_{\mathbf{T}\in\mathcal{B}_{r_y,\mathcal{H}_0}}\left|\mathcal{L}f(\mathbf{T}) - \mathcal{L}_{\xi}f(\mathbf{T})\right| \leq K \sup_{j\in\{1,\dots,N\}} \left|\Lambda^{c,\xi}_j - \Lambda^c_j\right| + \left|\Lambda^{s,\xi}_j - \Lambda^s_j\right| + \left|\kappa^{\xi}_j - \kappa_j\right|.$$

Consequently, $\mathbf{T}^0(.)(y)$ is a solution of the martingale problem associated to $\mathcal{L}$ and starting from y. However, following the proof of the uniqueness in Theorem 2.1, this martingale problem is well-posed and therefore $\mathbf{T}^{\xi}(.)(y^{\xi})$ converges in distribution to the unique solution of the martingale problem associated to $\mathcal{L}$ and starting from y.

2.6.4 Proof of Theorem 2.4

The proof of this theorem follows ideas developed in [59, Chapter 11]. In order to prove this theorem we use a probabilistic representation of $\mathcal{T}^l_j(\omega,z)$ by using the Feynman-Kac formula. To this end, we introduce the jump Markov process $(X^N_t)_{t\geq 0}$ with state space $\{-(N-1)/N,\dots,0,\dots,(N-1)/N\}$ and generator given by

$$\mathcal{L}^N\phi\left(\frac{l}{N}\right) = \Gamma^c_{l\,l+1}\left(\phi\left(\frac{l-1}{N}\right) - \phi\left(\frac{l}{N}\right)\right) + \Gamma^c_{l+2\,l+1}\left(\phi\left(\frac{l+1}{N}\right) - \phi\left(\frac{l}{N}\right)\right)$$

for $l\in\{1,\dots,N-2\}$,

$$\mathcal{L}^N\phi\left(\frac{l}{N}\right) = \Gamma^c_{|l|+2\,|l|+1}\left(\phi\left(\frac{l-1}{N}\right) - \phi\left(\frac{l}{N}\right)\right) + \Gamma^c_{|l|\,|l|+1}\left(\phi\left(\frac{l+1}{N}\right) - \phi\left(\frac{l}{N}\right)\right)$$

for $l\in\{-(N-2),\dots,-1\}$,

$$\mathcal{L}^N\phi(0) = \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{1}{N}\right) - \phi(0)\right) + \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{-1}{N}\right) - \phi(0)\right),$$

and

$$\mathcal{L}^N\phi\left(\frac{\pm(N-1)}{N}\right) = \Gamma^c_{N-1\,N}\left(\phi\left(\frac{\pm(N-2)}{N}\right) - \phi\left(\frac{\pm(N-1)}{N}\right)\right).$$

Using the Feynman-Kac formula, we get for $(j,l)\in\{1,\dots,N(\omega)\}^2$

$$\mathcal{T}^l_j(\omega,L) = \mathbb{E}_{\frac{l-1}{N}}\left[e^{-\Lambda^c_N\int_0^L \mathbf{1}_{\left(|X^N_v|=\frac{N-1}{N}\right)}dv - \Lambda^c_{N-1}\int_0^L \mathbf{1}_{\left(|X^N_v|=\frac{N-2}{N}\right)}dv}\,\mathbf{1}_{\left(|X^N_L|+\frac{1}{N}=\frac{j}{N}\right)}\right].$$

Let f be a bounded continuous function on $[0,1]$, we consider $\mathcal{T}^l(\omega,L)$ as a family of bounded measures on $[0,1]$ by setting

$$\mathcal{T}^l_f(\omega,L) = \mathbb{E}_{\frac{l-1}{N}}\left[e^{-\Lambda^c_N\int_0^L \mathbf{1}_{\left(|X^N_v|=\frac{N-1}{N}\right)}dv - \Lambda^c_{N-1}\int_0^L \mathbf{1}_{\left(|X^N_v|=\frac{N-2}{N}\right)}dv}\,f\left(|X^N_L| + \frac{1}{N}\right)\right].$$

In the first part of the proof, we consider the case $v \in [0,1)$ and in a second part we shall treat the case $v = 1$.

Let $u \in [0,1)$ such that $l(N)/N \to u$. We begin by introduce some notations. Throughout the proof we denote by $\tau^{(l)}_{j/N}$ the lth passage in j/N, for $j \in \{-(N-1),\dots,N-1\}$. To avoid the unboundness in $\mathcal{L}^N$ of the reflecting barriers $\mathcal{L}^N\phi(\pm(N-1)/N)$, we introduce the stopping time

$$\tau^\alpha_N = \tau^{(1)}_{(N-[N^\alpha])/N} \wedge \tau^{(1)}_{-(N-[N^\alpha])/N}$$

with $\alpha \in (0,1)$. Let $X^{N,\tau}_t = X^N_{t\wedge\tau^\alpha_N}$, $\forall t \geq 0$, be the stopped process and $d(N) = (l(N)-1)/N$. We denote by $\mathbb{P}^N_{d(N)}$ the law of $(X^N_t)_{t\geq 0}$ starting from $d(N)$ and by $\mathbb{P}^{N,\tau}_{d(N)}$ the law of $(X^{N,\tau}_t)_{t\geq 0}$ starting from $d(N)$. Let

$$\mathcal{L}_{\overline{a}_\infty} = \frac{\partial}{\partial v}\left(\overline{a}_\infty(\cdot)\frac{\partial}{\partial v}\right),$$

where $\overline{a}_\infty(\cdot) \in \mathcal{C}^1(\mathbb{R})$ is an extension over $\mathbb{R}$ of $a_\infty(\cdot)$, which is defined on $[-1,1]$, and such that the martingale problem associated to $\mathcal{L}_{\overline{a}_\infty}$ and starting from u is well posed. We denote by $\overline{\mathbb{P}}_u$ this unique solution. Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$,

$$M_\varphi(t) = \varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathcal{L}_{\overline{a}_\infty}\varphi(x(s))ds,$$

and $\tau_r = \inf(u \geq 0,\, |x(t)| \geq r)$ for $r \in (0,1)$.

Lemma 2.10 *$\forall\varphi \in \mathcal{C}^\infty_0(\mathbb{R})$. $\forall\alpha \in (2/3,1)$,*

$$\lim_{N\to+\infty} \sup_{v\in\left[-\frac{N-[N^\alpha]}{N},-\frac{1}{N}\right]\cup\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]} |\mathcal{L}^N\varphi(v) - \mathcal{L}_{\overline{a}_\infty}\varphi(v)| = 0,$$

where $\mathcal{L}^N\varphi(v)$ is defined as follows. $\forall j \in \{1,\dots,N-2\}$,

$$\mathcal{L}^N\varphi(v) = \Gamma^c_{j\,j+1}\left(\varphi\left(\frac{j-1}{N}\right) - \varphi\left(\frac{j}{N}\right)\right) + \Gamma^c_{j+1\,j+2}\left(\varphi\left(\frac{j+1}{N}\right) - \varphi\left(\frac{j}{N}\right)\right),$$

for $v \in [j/N,(j+1)/N]$, and

$$\mathcal{L}^N\varphi(v) = \Gamma^c_{j\,j+1}\left(\varphi\left(\frac{-j+1}{N}\right) - \varphi\left(\frac{-j}{N}\right)\right) + \Gamma^c_{j+1\,j+2}\left(\varphi\left(\frac{-j-1}{N}\right) - \varphi\left(\frac{-j}{N}\right)\right),$$

if $v \in [-(j+1)/N,-j/N]$.

Proof (of Lemma 2.10) We shall restrict the proof of this lemma to the proof of

$$\lim_{N\to+\infty} \sup_{v\in\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]} |\mathcal{L}^N\varphi(v) - \mathcal{L}_{\overline{a}_\infty}\varphi(v)| = 0,$$

since the other case is completely similar by symmetry. We start the proof of this technical lemma by proving Lemma 2.1 page 36.

Proof (of Lemma 2.1) Let $h_e(v) = \frac{v}{\sqrt{(n_1 kd\theta)^2 - v^2}}$ and $g(v) = \arctan(v)$. We recall that $\forall j \in \{1,\dots,N\}$, $\tan(\sigma_j) = -h_e(\sigma_j)$ First,

$$\begin{aligned}|\sigma_{j+1} - \sigma_j - \pi| &\le |g(\tan(\sigma_{j+1} - (j+1)\pi)) - g(\tan(\sigma_j - j\pi))| \\ &\le K|\tan(\sigma_{j+1}) - \tan(\sigma_j)| \\ &\le K|h_e(\sigma_{j+1}) - h_e(\sigma_j)| \\ &\le K \sup_{v\in[\sigma_j,\sigma_{j+1}]} h_e'(v)\end{aligned}$$

where $h_e'(v) = \frac{(n_1 kd\theta)^2}{((n_1 kd\theta)^2 - v^2)^{\frac{3}{2}}}$ which is a positive and increasing function. Moreover,

$$\sigma_{N-[N^\alpha]} \le (N - [N^\alpha])\pi$$

and then

$$\sup_{j\in\{1,\dots,N-[N^\alpha]\}} |\sigma_{j+1} - \sigma_j - \pi| = \mathcal{O}\left(N^{\frac{1}{2}-\frac{3}{2}\alpha}\right).$$

Second, in the same way we have

$$\begin{aligned}\sigma_{j+2} - &2\sigma_{j+1} + \sigma_j \\ &= g(\tan(\sigma_{j+2} - (j+2)\pi)) - 2g(\tan(\sigma_{j+1} - (j+1)\pi)) + g(\tan(\sigma_j - j\pi)) \\ &= -(g(h_e(\sigma_{j+2})) - 2g(h_e(\sigma_{j+1})) + g(h_e(\sigma_j)))\end{aligned}$$

and

$$\begin{aligned}g(h_e(\sigma_{j+2})) - 2g(h_e(\sigma_{j+1})) + g(h_e(\sigma_j)) &= [g'(h_e(\sigma_{j+1})) - g'(h_e(\sigma_j))].[h_e(\sigma_{j+2}) - h_e(\sigma_{j+1})] \\ &+ g'(h_e(\sigma_j))[h_e(\sigma_{j+2}) - 2h_e(\sigma_{j+1}) + h_e(\sigma_j)] \\ &+ \int_{h_e(\sigma_{j+1})}^{h_e(\sigma_{j+2})} (h_e(\sigma_{j+2}) - t)\, g''(t)\ dt \\ &- \int_{h_e(\sigma_j)}^{h_e(\sigma_{j+1})} (h_e(\sigma_{j+1}) - t)\, g''(t)\ dt.\end{aligned}$$

Moreover,

$$|g'(h_e(\sigma_{j+1})) - g'(h_e(\sigma_j))|.|h_e(\sigma_{j+2}) - h_e(\sigma_{j+1})| \le K\, N^{1-3\alpha}.$$

$$\begin{aligned}h_e(\sigma_{j+2}) - 2h_e(\sigma_{j+1}) + h_e(\sigma_j) &= \int_{\sigma_{j+1}}^{\sigma_{j+2}} h_e'(t) - h_e'(t-\pi)\, dt \\ &+ \int_{\sigma_{j+1}}^{\sigma_{j+2}} h_e'(t-\pi)\, dt - \int_{\sigma_j}^{\sigma_{j+1}} h_e'(t)\, dt,\end{aligned}$$

with

$$\left|\int_{\sigma_{j+1}}^{\sigma_{j+2}} h_e'(t) - h_e'(t-\pi)\, dt\right| \le K\, h_e''(\sigma_{N-[N^\alpha]}) = \mathcal{O}(N^{\frac{1}{2}-\frac{5}{2}\alpha}),$$

because $h_e''(v) = \frac{3(n_1 kd\theta)^2 v}{((n_1 kd\theta)^2 - v^2)^{5/2}}$, and

$$\begin{aligned}&\Big|\int_{\sigma_{j+1}}^{\sigma_{j+2}} h_e'(t-\pi)\, dt - \int_{\sigma_j+\pi}^{\sigma_{j+1}+\pi} h_e'(t-\pi)\, dt\Big| \\ &\qquad \le h_e'(\sigma_{N-[N^\alpha]})\,(|\sigma_{j+2} - \sigma_{j+1} - \pi| + |\sigma_{j+2} - \sigma_{j+1} - \pi|) \\ &\qquad \le KN^{1-3\alpha}.\end{aligned}$$

Finally,

$$\Big|\int_{h_e(\sigma_{j+1})}^{h_e(\sigma_{j+2})} (h_e(\sigma_{j+2})-t)\,g''(t)\;dt - \int_{h_e(\sigma_j)}^{h_e(\sigma_{j+1})} (h_e(\sigma_{j+1})-t)\,g''(t)\;dt\Big|$$
$$\le K\left((h_e(\sigma_{j+2})-h_e(\sigma_{j+1}))^2+(h_e(\sigma_{j+1})-h_e(\sigma_j))^2\right)$$
$$\le K\,N^{1-3\alpha},$$

and

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-2\}} |\sigma_{j+2}-2\sigma_{j+1}+\sigma_j| = \mathcal{O}(N^{1-3\alpha}).$$

This completes the proof of Lemma 2.1 since we can take $\alpha > 1/3$ and we have $N^{\frac{1}{2}-\frac{5}{2}\alpha} \le N^{1-3\alpha}$. □

From this lemma, we immediately get

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-1\}} \big|S(\sigma_{j+1}-\sigma_j,\sigma_{j+1}-\sigma_j)-S(\pi,\pi)\big|$$
$$\le K \sup_{j\in\{1,\dots,N-[N^\alpha]-1\}} |\sigma_{j+1}-\sigma_j-\pi| = \mathcal{O}\left(N^{\frac{1}{2}-\frac{3}{2}\alpha}\right).$$

Before showing that

$$\sup_{j\in\{1,\dots,N-[N^\alpha]\}} \Big|A_j^2-\frac{2}{d}\Big| = \mathcal{O}(N^{\alpha-1}),$$

where A_j is defined by (2.8), we prove that

$$\sup_{j\in\{1,\dots,[N^\alpha]\}} |\sigma_j - j\pi| = \mathcal{O}\left(\frac{1}{N^{1-\alpha}}\right).$$

In fact, $\forall j\in\{1,\dots,N^\alpha\}$

$$|\sigma_j-j\pi| = \big|g\big(\tan(\sigma_j-j\pi)\big)-\arctan(\tan(0))\big|$$
$$\le K\big|\tan(\sigma_j)\big|$$
$$\le K\,h_e(\sigma_{[N^\alpha]}).$$

Moreover, $\sigma_{[N^\alpha]}\le N^\alpha\pi$ and then

$$h(\sigma_{[N^\alpha]}) = \mathcal{O}\left(\frac{1}{N^{1-\alpha}}\right).$$

Consequently,

$$\sup_{j\in\{1,\dots,N-[N^\alpha]\}} \Big|A_j^2-\frac{2}{d}\Big| \le K \sup_{j\in\{1,\dots,N-[N^\alpha]\}} \left|\frac{\sin^2(\sigma_j)}{\zeta_j}-\frac{\sin(2\sigma_j)}{2\sigma_j}\right| \le \frac{K}{N^{1-\alpha}}$$

because

$$\sup_{j\in\{1,\dots,N-[N^\alpha]\}} \Big|\frac{\sin^2(\sigma_j)}{\zeta_j}-\frac{\sin(2\sigma_j)}{2\sigma_j}\Big| \le \frac{1}{\sqrt{(n_1kd\theta)^2-\sigma_{N-[N^\alpha]}^2}}$$
$$+\sup_{j\in\{[N^\alpha]+1,\dots,N-[N^\alpha]\}}\left|\frac{\sin(2\sigma_j)}{2\sigma_j}\right| + \sup_{j\in\{1,\dots,[N^\alpha]\}}\left|\frac{\sin(2\sigma_j)}{2\sigma_j}\right|$$
$$\le \frac{K}{N^{1/2+\alpha/2}}+\frac{1}{2\sigma_{[N^\alpha]+1}}+\sup_{j\in\{1,\dots,[N^\alpha]\}}\left|\frac{\sin(2\sigma_j)-\sin(2j\pi)}{2\sigma_j}\right|$$
$$\le K\left(\frac{1}{N^{1/2+\alpha/2}}+\frac{1}{N^\alpha}+\frac{1}{\sigma_1}\sup_{j\in\{1,\dots,[N^\alpha]\}}|\sigma_j-j\pi|\right).$$

Now, let us introduce

$$B_j^N = \frac{a}{2n_1^2\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\frac{\frac{1}{4}A_j^2A_{j+1}^2S(\sigma_{j+1}-\sigma_j,\sigma_{j+1}-\sigma_j)}{a^2+(\beta_j-\beta_{j+1})^2}.$$

Then, for $j\in\{1,\dots,N-2\}$ and $v\in\left[\frac{j}{N},\frac{j+1}{N}\right]$.

$$\mathcal{L}^N\varphi(v) = \left(\frac{n_1kd\theta}{N\pi}\right)^2\left(\frac{N\pi}{n_1d\theta}\right)^2\Big[\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)B_{j+1}^N + \Big(\varphi\Big(\frac{[Nv]-1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)B_j^N\Big].$$

Consequently, from the following decomposition

$$\begin{aligned}
N^2\Big[\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)B_{j+1}^N + \Big(\varphi\Big(\frac{[Nv]-1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)B_j^N\Big] \\
-\frac{n_1^2d^2\theta^2}{\pi^2}\mathcal{L}_{a_\infty}\varphi(v) \\
= N^2\Big[\varphi\Big(\frac{[Nv]+1}{N}\Big)-2\varphi\Big(\frac{[Nv]}{N}\Big)+\varphi\Big(\frac{[Nv]-1}{N}\Big)\Big]\Big[B_j^N-\frac{n_1^2d^2\theta^2}{\pi^2}a_\infty(v)\Big] \\
+N\Big[\varphi\Big(\frac{[Nv]+1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big]\Big[N(B_{j+1}^N-B_j^N)-\frac{n_1^2d^2\theta^2}{\pi^2}\frac{d}{dv}a_\infty(v)\Big] \\
+\frac{n_1^2d^2\theta^2}{\pi^2}a_\infty(v)\Big[N^2\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-2\varphi\Big(\frac{[Nv]}{N}\Big)+\varphi\Big(\frac{[Nv]-1}{N}\Big)\Big)-\varphi''(v)\Big] \\
+\frac{n_1^2d^2\theta^2}{\pi^2}\frac{d}{dv}a_\infty(v)\Big[N\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)-\varphi'(v)\Big],
\end{aligned}$$

and because it is easy to show that

$$\sup_{v\in\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]}\Big|N\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-\varphi\Big(\frac{[Nv]}{N}\Big)\Big)-\varphi'(v)\Big| = \mathcal{O}\Big(\frac{1}{N}\Big)$$

$$\sup_{v\in\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]}\Big|N^2\Big(\varphi\Big(\frac{[Nv]+1}{N}\Big)-2\varphi\Big(\frac{[Nv]}{N}\Big)+\varphi\Big(\frac{[Nv]-1}{N}\Big)\Big)-\varphi''(v)\Big| = \mathcal{O}\Big(\frac{1}{N}\Big),$$

it suffices to show the two following points

- $$\lim_N \sup_{j\in\{1,\dots,N-[N^\alpha]-1\}}\sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\Big|B_j^N-\frac{n_1^2d^2\theta^2}{\pi^2}a_\infty(v)\Big| = 0.$$

- $$\lim_N \sup_{j\in\{1,\dots,N-[N^\alpha]-1\}}\sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\Big|N\Big(B_j^N-B_{j+1}^N\Big)-\frac{n_1^2d^2\theta^2}{\pi^2}\frac{d}{dv}a_\infty(v)\Big| = 0.$$

We decompose the proof of these two points into two sublemmas.

Lemma 2.11

$$\lim_N \sup_{j\in\{1,\dots,N-[N^\alpha]-1\}}\sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\Big|B_j^N-\frac{n_1^2d^2\theta^2}{\pi^2}a_\infty(v)\Big| = 0$$

Proof (of Lemma 2.11) $\forall j \in \{1,\ldots,N-[N^\alpha]-1\}$ and $\forall v \in \left[\frac{j}{N},\frac{j+1}{N}\right]$, we have the following inequalities,

$$\frac{1}{1-\frac{\sigma_j^2}{n_1^2k^2d^2}} \leq \frac{1}{\sqrt{\left(1-\frac{\sigma_j^2}{n_1^2k^2d^2}\right)\left(1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}\right)}} \leq \frac{1}{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}.$$

Moreover, for $l \in \{j, j+1\}$

$$\begin{aligned}\Big|\frac{1}{1-(\theta v)^2} - \frac{1}{1-\frac{\sigma_l^2}{n_1^2k^2d^2}}\Big| &\leq \Big|\frac{j+1}{N}\theta - \frac{j-1}{n_1kd}\pi\Big|\frac{2\theta}{(1-\theta^2)^2}\\ &\leq \frac{K}{N}.\end{aligned}$$

Consequently,

$$\sup_{j\in\{1,\ldots,N-[N^\alpha]-1\}}\ \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\Big|\frac{1}{1-(\theta v)^2} - \frac{1}{\sqrt{\left(1-\frac{\sigma_j^2}{n_1^2k^2d^2}\right)\left(1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}\right)}}\Big| = \mathcal{O}\left(\frac{1}{N}\right).$$

Next,

$$\begin{aligned}&\Big|n_1k\Big(\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}} - \sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}\Big) - \frac{\pi}{d}\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}\Big|\\
&\qquad\leq K\Big|n_1k\Big(\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}} - \sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}\Big) - \frac{1}{d}(\sigma_{j+1}-\sigma_j)\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}\Big|\\
&\qquad\quad + K\Big|(\sigma_{j+1}-\sigma_j-\pi)\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}\Big|\\
&\qquad\leq \frac{K}{N}\frac{1}{(1-\theta^2)^{3/2}}(\sigma_{j+1}-\sigma_j)^2 + K\big|\sigma_{j+1}-\sigma_j-\pi\big|\frac{\theta}{\sqrt{1-\theta^2}} \leq KN^{\frac{1}{2}-\frac{3}{2}\alpha}\end{aligned}$$

and then

$$\sup_{j\in\{1,\ldots,N-[N^\alpha]-1\}}\Big|\frac{1}{a^2+(\beta_j-\beta_{j+1})^2} - \frac{1}{a^2+\frac{\pi^2}{d^2}\frac{\frac{\sigma_j^2}{n_1^2k^2d^2}}{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}\Big| = \mathcal{O}\Big(N^{\frac{1}{2}-\frac{3}{2}\alpha}\Big).$$

Moreover $\forall j \in \{1,\ldots,N-[N^\alpha]-1\}$ and $\forall v \in \left[\frac{j}{N},\frac{j+1}{N}\right]$, we have

$$\Big|\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}} - \frac{\theta v}{\sqrt{1-(\theta v)^2}}\Big| \leq K\Big|\frac{\sigma_j}{n_1kd} - \theta v\Big| \leq \frac{K}{N},$$

and finally

$$\sup_{j\in\{1,\ldots,N-[N^\alpha]-1\}}\ \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\Big|\frac{1}{a^2+(\beta_j-\beta_{j+1})^2} - \frac{1}{a^2+\frac{\pi^2}{d^2}\frac{(\theta v)^2}{1-(\theta x)^2}}\Big| = \mathcal{O}\Big(N^{\frac{1}{2}-\frac{3}{2}\alpha}\Big).$$

This concludes the proof of Lemma 2.11.□

Lemma 2.12

$$\lim_N \sup_{j\in\{1,\dots,N-[N^\alpha]-1\}} \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]} \left|N\left(B_j^N - B_{j+1}^N\right) - \frac{n_1^2 d^2\theta^2}{\pi^2}\frac{d}{dv}a_\infty(v)\right| = 0.$$

Proof (of Lemma 2.12) We separate the proof of this lemma into two step. First, for $j \in \{1,\dots,N-[N^\alpha]-2\}$ let

$$C_j^N = N\left(\frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}\sqrt{1-\frac{\sigma_{j+2}^2}{n_1^2k^2d^2}}} - \frac{1}{\sqrt{1-\frac{\sigma_{j}^2}{n_1^2k^2d^2}}\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\right).$$

We can write $\forall v \in \left[\frac{j}{N},\frac{j+1}{N}\right]$

$$\begin{aligned}
C_j^N - \frac{2\theta^2 v}{(1-(\theta v)^2)^2} &= \frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}} N\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}} \frac{w}{(1-w^2)^{\frac{3}{2}}}dw - \frac{2\theta^2 v}{(1-(\theta v)^2)^2} \\
&= \frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\Big(N\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}} \frac{w}{(1-w^2)^{\frac{3}{2}}}dw - N\Big(\frac{\sigma_{j+2}}{n_1kd}-\frac{\sigma_j}{n_1kd}\Big)\frac{\theta v}{(1-(\theta v)^2)^{\frac{3}{2}}}\Big) \\
&+ N\Big(\frac{\sigma_{j+2}}{n_1kd}-\frac{\sigma_j}{n_1kd}\Big)\frac{\theta v}{(1-(\theta v)^2)^{\frac{3}{2}}}\Big(\frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}} - \frac{1}{\sqrt{1-(\theta v)^2}}\Big) \\
&+ \Big(N\Big(\frac{\sigma_{j+2}}{n_1kd}-\frac{\sigma_j}{n_1kd}\Big) - 2\theta\Big)\frac{\theta v}{(1-(\theta v)^2)^2}.
\end{aligned}$$

We can check that the function $v \mapsto \frac{\theta v}{(1-(\theta v)^2)^2}$ is bounded on $[0,1]$ and

$$\Big|N\Big(\frac{\sigma_{j+2}}{n_1kd}-\frac{\sigma_j}{n_1kd}\Big) - 2\theta\Big| \le \frac{N}{n_1kd}\Big|\sigma_{j+2}-\sigma_j-2\pi\Big| + 2\theta\Big|\frac{N\pi}{n_1dk\theta}-1\Big| \le K\,N^{\frac{1}{2}-\frac{3}{2}\alpha}.$$

Moreover, $v \mapsto \frac{\theta v}{(1-(\theta v)^2)^2}$ is bounded on $[0,1]$ and

$$\Big|\frac{1}{\sqrt{1-(\theta v)^2}} - \frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2}}}\Big| \le \frac{K}{N}\frac{\theta}{(1-\theta^2)^{3/2}}.$$

Finally, $0 \le \frac{1}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}} \le \frac{1}{\sqrt{1-\theta^2}}$ and

$$\begin{aligned}
\Big|N\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}} \frac{w}{(1-w^2)^{\frac{3}{2}}}dw - N\Big(\frac{\sigma_{j+2}}{n_1kd}-\frac{\sigma_j}{n_1kd}\Big)\frac{\theta v}{(1-(\theta v)^2)^{\frac{3}{2}}}\Big| \\
&\le N\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}} |w-\theta v| dw \frac{2\theta^2+1}{(1-\theta^2)^{\frac{5}{2}}} \\
&\le KN\Big[\Big(\theta v - \frac{\sigma_j}{n_1kd}\Big)^2 + \Big(\theta v - \frac{\sigma_{j+2}}{n_1kd}\Big)^2\Big] \\
&\le \frac{K}{N}.
\end{aligned}$$

Consequently,

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-1\}}\ \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\left|C_j^N-\frac{2\theta^2 v}{(1-(\theta v)^2)^2}\right|=\mathcal{O}(N^{\frac12-\frac32\alpha}).$$

Second, for $j\in\{1,\dots,N-[N^\alpha]-1\}$ and $v\in[\frac{j}{N},\frac{j+1}{N}]$, we have

$$\begin{aligned}
&\Big|N\Big(\frac{1}{a^2+(\beta_{j+1}-\beta_{j+2})^2}-\frac{1}{a^2+(\beta_j-\beta_{j+1})^2}\Big)+\frac{\frac{\pi^2}{d^2}\frac{2\theta^2 v}{(1-(\theta v)^2)^2}}{\Big(a^2+\frac{\pi^2}{d^2}\frac{(\theta v)^2}{1-(\theta v)^2}\Big)^2}\Big|\\
&\quad\le\Big|N\Big(\frac{1}{a^2+(\beta_{j+1}-\beta_{j+2})^2}-\frac{1}{a^2+(\beta_j-\beta_{j+1})^2}\Big)\\
&\qquad\qquad-N\big((\beta_{j+1}-\beta_{j+2})-(\beta_j-\beta_{j+1})\big)\frac{-2(\beta_j-\beta_{j+1})}{\big(a^2+(\beta_j-\beta_{j+1})^2\big)^2}\Big|\\
&\quad+\Big|N\big((\beta_{j+1}-\beta_{j+2})-(\beta_j-\beta_{j+1})\big)\frac{-2(\beta_j-\beta_{j+1})}{\big(a^2+(\beta_j-\beta_{j+1})^2\big)^2}+\frac{\frac{\pi^2}{d^2}\frac{2\theta^2 v}{(1-(\theta v)^2)^2}}{\Big(a^2+\frac{\pi^2}{d^2}\frac{(\theta v)^2}{1-(\theta v)^2}\Big)^2}\Big|.
\end{aligned}$$

For the first term on the right of the previous inequality, we have

$$\begin{aligned}
&\Big|N\Big(\frac{1}{a^2+(\beta_{j+1}-\beta_{j+2})^2}-\frac{1}{a^2+(\beta_j-\beta_{j+1})^2}\Big)\\
&\qquad\qquad-N\big((\beta_{j+1}-\beta_{j+2})-(\beta_j-\beta_{j+1})\big)\frac{-2(\beta_j-\beta_{j+1})}{\big(a^2+(\beta_j-\beta_{j+1})^2\big)^2}\Big|\\
&\quad\le K\,N\big((\beta_{j+1}-\beta_{j+2})-(\beta_j-\beta_{j+1})\big)^2,
\end{aligned}$$

and we shall see just below that

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-2\}}|\beta_{j+2}-2\beta_{j+1}-\beta_j|=\mathcal{O}\Big(\frac{1}{N}\Big).$$

Now, for the second term we have previously get

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-1\}}\ \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\left|\beta_j-\beta_{j+1}-\frac{\pi}{d}\frac{\theta v}{\sqrt{1-(\theta v)^2}}\right|=\mathcal{O}(N^{\frac12-\frac32\alpha}).$$

Then, to finish the proof of this lemma it suffices to show that

$$\sup_{j\in\{1,\dots,N-[N^\alpha]-2\}}\ \sup_{v\in\left[\frac{j}{N},\frac{j+1}{N}\right]}\left|N\big(\beta_j-2\beta_{j+1}+\beta_{j+2}\big)+\frac{\frac{\pi}{d}\theta}{(1-(\theta v)^2)^{\frac32}}\right|=\mathcal{O}(N^{2-3\alpha}).$$

To show this relation we shall use the following decompositions. For $l\in\{j,j+1\}$

$$\begin{aligned}
&\sqrt{1-\frac{\sigma_l^2}{n_1^2k^2d^2}}-\sqrt{1-\frac{\sigma_{l+1}^2}{n_1^2k^2d^2}}-\frac{1}{n_1kd}(\sigma_{l+1}-\sigma_l)\frac{\frac{\sigma_l}{n_1kd}}{\sqrt{1-\frac{\sigma_l^2}{n_1^2k^2d^2}}}\\
&\qquad\qquad=\int_{\frac{\sigma_l}{n_1kd}}^{\frac{\sigma_{l+1}}{n_1kd}}\left(\frac{\sigma_{l+1}}{n_1kd}-w\right)\frac{1}{(1-w^2)^{\frac32}}dw,
\end{aligned}$$

and

$$Nn_1k\Big(\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}-2\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}+\sqrt{1-\frac{\sigma_{j+2}^2}{n_1^2k^2d^2}}\Big)+\frac{\frac{\pi}{d}\theta}{(1-(\theta v)^2)^{\frac{3}{2}}}$$
$$=\frac{N}{d}\Big((\sigma_{j+1}-\sigma_j)\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}-(\sigma_{j+2}-\sigma_{j+1})\frac{\frac{\sigma_{j+1}}{n_1kd}}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\Big)+\frac{\frac{\pi}{d}\theta}{(1-(\theta v)^2)^{\frac{3}{2}}}$$
$$+Nn_1k\Big(\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+1}}{n_1kd}}\Big(\frac{\sigma_{j+1}}{n_1kd}-w\Big)\frac{1}{(1-w^2)^{\frac{3}{2}}}dw-\int_{\frac{\sigma_{j+1}}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}}\Big(\frac{\sigma_{j+2}}{n_1kd}-w\Big)\frac{1}{(1-w^2)^{\frac{3}{2}}}dw\Big).$$

First, using Lemma 2.1 we have

$$\int_{\frac{\sigma_j}{n_1kd}}^{\frac{\sigma_{j+1}}{n_1kd}}\Big(\frac{\sigma_{j+1}}{n_1kd}-w\Big)\frac{1}{(1-w^2)^{\frac{3}{2}}}dw-\int_{\frac{\sigma_{j+1}}{n_1kd}}^{\frac{\sigma_{j+2}}{n_1kd}}\Big(\frac{\sigma_{j+2}}{n_1kd}-w\Big)\frac{1}{(1-w^2)^{\frac{3}{2}}}dw=\mathcal{O}(N^{\frac{1}{2}-\frac{3}{2}\alpha-2}).$$

Second, we have

$$\frac{N}{d}\Big((\sigma_{j+1}-\sigma_j)\frac{\frac{\sigma_j}{n_1k}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2}}}-(\sigma_{j+2}-\sigma_{j+1})\frac{\frac{\sigma_{j+1}}{n_1k}}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2}}}\Big)+\frac{\frac{\pi}{d}\theta}{(1-(\theta v)^2)^{\frac{3}{2}}}$$
$$=\frac{N}{d}(\sigma_{j+1}-\sigma_j-\pi)\Big[\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}-\frac{\frac{\sigma_{j+1}}{n_1kd}}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\Big]$$
$$-\frac{N}{d}(\sigma_{j+2}-2\sigma_{j+1}+\sigma_j)\frac{\frac{\sigma_{j+1}}{n_1kd}}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}$$
$$+\frac{\pi}{d}\Big(N\Big[\frac{\frac{\sigma_j}{n_1kd}}{\sqrt{1-\frac{\sigma_j^2}{n_1^2k^2d^2}}}-\frac{\frac{\sigma_{j+1}}{n_1kd}}{\sqrt{1-\frac{\sigma_{j+1}^2}{n_1^2k^2d^2}}}\Big]+\frac{\theta}{(1-(\theta v)^2)^{\frac{3}{2}}}\Big),$$

where, according to Lemma 2.1, the first and the third term are $\mathcal{O}(N^{\frac{1}{2}-\frac{3}{2}\alpha})$, and the second term is $\mathcal{O}(N^{2-3\alpha})$. That concludes the proof of Lemma 2.12 for $\alpha\in(2/3,1)$.□

Consequently, thanks to Lemma 2.11 and Lemma 2.12, we get

$$\sup_{v\in\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]}|\mathcal{L}^N\varphi(v)-\mathcal{L}_{a_\infty}\varphi(v)|=\mathcal{O}(N^{(2-3\alpha)\vee(\alpha-1)}),$$

this concludes the proof of Lemma 2.10.□

Lemma 2.13 $\mathbb{P}^{N,\tau}_{d(N)}$ *is tight on* $\mathcal{D}([0,+\infty),\mathbb{R})$.

Proof (of Lemma 2.13) Let $\mathcal{M}_t=\sigma(x(u),\,0\le u\le t)$. According to Theorem 3 in [41, Chapter 3], we have to show the two following points. First,

$$\lim_{K\to+\infty}\overline{\lim_{N}}\,\mathbb{P}^{N,\tau}_{d(N)}\left(\sup_{t\ge0}|x(t)|\ge K\right)=0.$$

The first point is satisfied since we have $\forall N$, $\mathbb{P}^{N,\tau}_{d(N)}\Big(\sup_{t\ge0}|x(t)|\le1\Big)=1$. Second, for each N, $h\in(0,1)$, $s\in[0,h]$ and $t\ge0$,

$$\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}((x(t+s)-x(t))^2|\mathcal{M}_t)\le K\,h.$$

Concerning the second point, letting $\varphi \in \mathcal{C}_b^\infty(\mathbb{R})$ such that $\varphi(s) = s$ if $|s| \leq 1$, we have

$$\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\big((x(t+s)-x(t))^2|\mathcal{M}_t\big) \leq 2\,\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\big((M_\varphi^N(t+s)-M_\varphi^N(t))^2|\mathcal{M}_t\big) + 2\,\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\left(\int_t^{t+s}\mathcal{L}^N\varphi(x(w))dw\right)^2\Big|\mathcal{M}_t\right),$$

with

$$M_\varphi^N(t) = \varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathcal{L}^N\varphi(x(s))ds,$$

which is a $(\mathcal{M}_t)_{t\geq 0}$-martingale under $\mathbb{P}^N_{d(N)}$ and we know that

$$\mathbb{P}^{N,\tau}_{d(N)}\left(\sup_{t\geq 0}|x(t)| \leq \frac{N-[N^\alpha]}{N}\right) = 1.$$

Moreover, by Lemma 2.10

$$\sup_N \sup_{v\in\left[-\frac{N-[N^\alpha]}{N},-\frac{1}{N}\right]\cup\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]} |\mathcal{L}^N\varphi(v)| < +\infty$$

and the fact that $\mathcal{L}^N\varphi(0) = 0$, we get

$$\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\left(\int_t^{t+s}\mathcal{L}^N\varphi(x(w))dw\right)^2\Big|\mathcal{M}_t\right) \leq Ch^2.$$

We recall that

$$< M_\varphi^N >_t = \int_0^t \left(\mathcal{L}^N\varphi^2 - 2\varphi\mathcal{L}^N\varphi\right)(x(s))ds.$$

Then, using the martingale property of $(M_\varphi^N(t))_{t\geq 0}$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\big((M_\varphi^N(t+s)-M_\varphi^N(t))^2|\mathcal{M}_t\big) &= \mathbb{E}^{\mathbb{P}^{N}_{d(N)}}\left((M_\varphi^N((t+s)\wedge\tau_N^\alpha)-M_\varphi^N(t\wedge\tau_N^\alpha))^2|\mathcal{M}_t\right)\\
&= \mathbb{E}^{\mathbb{P}^{N}_{d(N)}}\left(M_\varphi^N((t+s)\wedge\tau_N^\alpha)^2-M_\varphi^N(t\wedge\tau_N^\alpha)^2|\mathcal{M}_t\right)\\
&= \mathbb{E}^{\mathbb{P}^{N}_{d(N)}}\left(< M_\varphi^N >_{(t+s)\wedge\tau_N^\alpha} - < M_\varphi^N >_{t\wedge\tau_N^\alpha} |\mathcal{M}_t\right)\\
&= \mathbb{E}^{\mathbb{P}^{N}_{d(N)}}\left(\int_{t\wedge\tau_N^\alpha}^{(t+s)\wedge\tau_N^\alpha}\left(\mathcal{L}^N\varphi^2 - 2\varphi\mathcal{L}^N\varphi\right)(x(w))dw\Big|\mathcal{M}_t\right)\\
&\leq C\,h.
\end{aligned}$$

In fact, by Lemma 2.10 we have

$$\sup_N \sup_{v\in\left[-\frac{N-[N^\alpha]}{N},-\frac{1}{N}\right]\cup\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]} |\mathcal{L}^N\varphi(v)| < +\infty,$$

$$\sup_N \sup_{v\in\left[-\frac{N-[N^\alpha]}{N},-\frac{1}{N}\right]\cup\left[\frac{1}{N},\frac{N-[N^\alpha]}{N}\right]} |\mathcal{L}^N\varphi^2(v)| < +\infty,$$

in addition to $\mathcal{L}^N\varphi(0) = 0$ and $\sup_N \mathcal{L}^N\varphi^2(0) = \frac{\Gamma^c_{1\,2}}{N^2} < +\infty$. □

Lemma 2.14 *Let $\mathbb{Q}_u$ be a limit point of the relatively compact sequence $\left(\mathbb{P}^{N,\tau}_{d(N)}\right)_N$. Then, $\forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ and $\forall r \in (0,1)$, $(M_\varphi(t\wedge\tau_r))_{t\geq 0}$ is a $(\mathcal{M})_t$-martingale under $\mathbb{Q}_u$.*

Proof (of Lemma 2.14) Let $\left(\mathbb{P}^{N',\tau}_{d(N')}\right)_{N'}$ be a converging subsequence. Throughout this proof we will take N for N' to simplify the notations. Let $0 \leq t_1 < t_2$ and Φ be a bounded continuous $\mathcal{M}_{t_1}$-measurable function. We have

$$\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(M^N_\varphi(t_2 \wedge \tau_r)\Phi\right) = \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(M^N_\varphi(t_1 \wedge \tau_r)\Phi\right).$$

Furthermore, $\forall t \geq 0$

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r} \mathcal{L}^N\varphi(x(s))ds\Phi\right) &= \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r} \mathcal{L}^N\varphi(x(s))\mathbf{1}_{\left(x(s)\in I^\alpha_N\right)}ds\Phi\right)\\ &+ \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r} \mathcal{L}^N\varphi(x(s))\mathbf{1}_{(x(s)=0)}ds\Phi\right),\end{aligned}$$

with $I^\alpha_N = [-(N-[N^\alpha])/N, -1/N] \cup [1/N, (N-[N^\alpha])/N]$. Using Lemma 2.10

$$\lim_N \left|\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r}\left(\mathcal{L}^N\varphi(x(s)) - \mathcal{L}_{\overline{a}_\infty}\varphi(x(s))\right)\mathbf{1}_{\left(x(s)\in I^\alpha_N\right)}ds\Phi\right)\right| = 0.$$

Consequently, we have to prove the two following points:

- $\lim_N \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(M_\varphi(t\wedge\tau_r)\Phi\right) = \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t\wedge\tau_r)\Phi\right)$.

- $\lim_N \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r}\mathbf{1}_{(x(s)=0)}ds\right) = 0$.

We prove the first point as follows. The problem is to apply the mapping theorem to the functional $M_\varphi(t\wedge\tau_r)$ and to do this we must have $\mathbb{Q}_u(D_{M_\varphi(t\wedge\tau_r)}) = 0$, where $D_{M_\varphi(t\wedge\tau_r)}$ is the set of discontinuities of $M_\varphi(t\wedge\tau_r)$ for the Skorokhod topology. While $M_\varphi(t)$ is continuous for this topology, it is not necessarily true for τ_r. However, we can follow the proof of Lemma 11.1.3 in [59] and then use a family of stopping times for which we can apply the mapping theorem.

We know that the size of the jumps of $(X^N_t)_t$ is constant equal to $1/N$, therefore we have $\mathbb{Q}_u(\mathcal{C}([0,+\infty),\mathbb{R})) = 1$ (see Theorem 13.4 in [14] for instance). Then

$$\mathbb{Q}_u(D_{M_\varphi(t\wedge\tau_r)}) = \mathbb{Q}_u(D_{M_\varphi(t\wedge\tau_r)} \cap \mathcal{C}([0,+\infty),\mathbb{R})).$$

We recall that the Skorokhod topology on $\mathcal{C}([0,+\infty),\mathbb{R})$ coincides with the usual topology defined on this space. Therefore, $D_{M_\varphi(t\wedge\tau_r)} \cap \mathcal{C}([0,+\infty),\mathbb{R})$ is the set of discontinuities of $M_\varphi(t\wedge\tau_r)$ under the topology of $\mathcal{C}([0,+\infty),\mathbb{R})$, and τ_r restrict to $\mathcal{C}([0,+\infty),\mathbb{R})$ is lower semi-continuous. Consequently, according to the proof of lemmas 11.1.2 in [59], there exists a sequence $(r_n)_n$ such that $r_n \nearrow r$ and

$$\mathbb{Q}_u\left((\tau_{r_n} < +\infty) \cap D_{\tau_{r_n}} \cap \mathcal{C}([0,+\infty),\mathbb{R})\right) = \mathbb{Q}_u\left((\tau_{r_n} < +\infty) \cap D_{\tau_r}\right) = 0.$$

Then, $\mathbb{Q}_u(D_{M_\varphi(t\wedge\tau_{r_n})}) = 0$ and we can apply the mapping theorem to $M_\varphi(t\wedge\tau_{r_n})$, i.e

$$\lim_N \mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(M_\varphi(t\wedge\tau_{r_n})\Phi\right) = \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t\wedge\tau_{r_n})\Phi\right).$$

Finally, we obtain

$$\mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t_2\wedge\tau_{r_n})\Phi\right) = \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t_1\wedge\tau_{r_n})\Phi\right),$$

and

$$\lim_n \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t\wedge\tau_{r_n})\Phi\right) = \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t\wedge\tau_r)\Phi\right)$$

because $\tau_{r_n} \nearrow \tau_r$. Consequently,

$$\mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t_2 \wedge \tau_r)\Phi\right) = \mathbb{E}^{\mathbb{Q}_u}\left(M_\varphi(t_1 \wedge \tau_r)\Phi\right).$$

For the second point, we have

$$\mathbb{E}^{\mathbb{P}^{N,\tau}_{d(N)}}\left(\int_0^{t\wedge\tau_r} \mathbf{1}_{(x(s)=0)}ds\right) = \mathbb{E}_{d(N)}\left[\int_0^{t\wedge\tau_r} \mathbf{1}_{(X^N_s=0)}ds\right] \leq \mathbb{E}_0\left[\int_0^t \mathbf{1}_{(X^N_s=0)}ds\right],$$

since the stopped process spends less time in 0 than the original process and the last inequality is given by the Markov property. We denote by N^0_t the number of returns in 0 during the time interval $[0,t]$ and by $(Y_j)_{j\geq 0}$ the renewal process associated with the return times in 0, $(\sigma_0^{(i)})_{i\geq 1}$, of the process $(X^N_t)_t$, with $Y_0 = \sigma_0^{(0)} = 0$. Moreover, for $\alpha' \in (0,1)$

$$\begin{aligned}\mathbb{E}_0\left[\int_0^t \mathbf{1}_{(X^N_s=0)}ds\right] &\leq t\mathbb{P}_0\left(N^0_t \geq [N^{1+\alpha'}]\right) + \mathbb{E}_0\left[\sum_{j=0}^{[N^{1+\alpha'}]}\int_{\sigma_0^{(j)}}^{\sigma_0^{(j+1)}} \mathbf{1}_{(X^N_s=0)}ds\right] \\ &\leq \frac{t}{[N^{1+\alpha'}]}\mathbb{E}_0[N^0_t] + \frac{[N^{1+\alpha'}]+1}{\Gamma^c_{21}},\end{aligned}$$

since $\left(\int_{\sigma_0^{(j)}}^{\sigma_0^{(j+1)}} \mathbf{1}_{(X^N_s=0)}ds\right)_j$ is an i.i.d sequence with mean $1/\Gamma^c_{21}$. We recall that $N^0_t + 1$ is a stopping time for $(Y_j)_{j\geq 1}$. Then,

$$\mathbb{E}_0\left[\sigma_0^{(N^0_t+1)}\right] = \mathbb{E}_0\left[\sum_{j=1}^{N^0_t+1} Y_j\right] = \left(\mathbb{E}_0\left[N^0_t\right]+1\right)\mathbb{E}_0\left[\sigma_0^{(1)}\right].$$

Furthermore,

$$\begin{aligned}\mathbb{E}_0\left[\sigma_0^{(N^0_t+1)}\right] &= \mathbb{E}_0\left[\sigma_0^{(N^0_t+1)}\left(\mathbf{1}_{\left(X^N_t=0\right)} + \mathbf{1}_{\left(X^N_t\neq 0\right)}\right)\right] \\ &= \mathbb{E}_0\left[\inf\left(s > T^N_t,\, X^N_{t+s}=0\right)\mathbf{1}_{\left(X^N_t=0\right)}\right] + \mathbb{E}_0\left[\inf\left(s>0,\, X^N_{t+s}=0\right)\mathbf{1}_{\left(X^N_t\neq 0\right)}\right]\end{aligned}$$

where $T^N_t = \inf\left(s>0,\, X^N_{t+s}\neq 0\right)$. Then, using the Markov property we get

$$\begin{aligned}\mathbb{E}_0\left[\inf\left(s > T^N_t,\, X^N_{t+s}=0\right)\mathbf{1}_{\left(X^N_t=0\right)}\right] &= \left(t + \mathbb{E}_0\left[\sigma_0^{(1)}\right]\right)\mathbb{P}_0(X^N_t=0) \\ &\leq t\mathbb{P}_0(X^N_t=0) + \frac{2N-1}{\Gamma^c_{12}}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_0\left[\inf\left(s>0,\, X^N_{t+s}=0\right)\mathbf{1}_{\left(X^N_t\neq 0\right)}\right] &= \sum_{\substack{j=-(N-1)\\ j\neq 0}}^{N-1}\mathbb{E}_0\left[\inf\left(s>0,\, X^N_{t+s}=0\right)\mathbf{1}_{\left(X^N_t=j\right)}\right] \\ &= \sum_{\substack{j=-(N-1)\\ j\neq 0}}^{N-1}\left(t + \mathbb{E}_j\left[\tau_0^{(1)}\right]\right)\mathbb{P}_0(X^N_t=j) \\ &= \sum_{j=1}^{N}\sum_{l=1}^{j}\frac{N-l}{\Gamma^c_{l\,l+1}}\mathbb{P}_0\left(|X^N_t|=j\right) + t\mathbb{P}_0(X^N_t\neq 0) \\ &\leq K\frac{N-1}{N^2}\mathbb{E}_0\left[|X^N_t|\mathbf{1}_{\left(X^N_t\neq 0\right)}\right] + t\mathbb{P}_0(X^N_t\neq 0),\end{aligned}$$

where K is a constant independent of N. Consequently,

$$\mathbb{E}_0[N_t^0] \leq \tilde{K}\frac{\Gamma^c_{1\,2}}{2N-1} - \frac{2N-1}{\Gamma^c_{1\,2}} = \mathcal{O}(N),$$

and

$$\mathbb{E}_0\left[\int_0^t \mathbf{1}_{(X_s^N=0)}ds\right] = \mathcal{O}\left(\frac{1}{N^{\alpha'\wedge(1-\alpha')}}\right).$$

□

From Lemma 2.14, we have $\forall r \in (0,1)$, $\mathbb{Q}_u = \overline{\mathbb{P}}_u$ on $\mathcal{M}_{\tau_r}$. From this relation and the fact that $\mathbb{Q}_u(\mathcal{C}([0,+\infty),\mathbb{R})) = \overline{\mathbb{P}}_u(\mathcal{C}([0,+\infty),\mathbb{R})) = 1$, $\mathbb{Q}_u = \overline{\mathbb{P}}_u$ on $\mathcal{M}_{\tau_1}$ since $\tau_r \nearrow \tau_1$ as $r \nearrow 1$.

Let $f \in \mathcal{C}^\infty([0,1])$ with compact support included in $[0,1)$ and let $\left(\mathbb{P}^{N',\tau}_{d(N')}\right)_{N'}$ be a converging subsequence as in the previous proof. We have

$$\mathcal{T}_f^{l(N')}(\omega,t) = \mathbb{E}_{d(N')}\left[f\left(|X_t^{N'}| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right] + r(N'), \tag{2.60}$$

with

$$r(N) = \mathbb{E}_{d(N)}\left[e^{-\Lambda^c_N\int_0^t \mathbf{1}_{\left(|X_v^N|=\frac{N-1}{N}\right)}dv - \Lambda^c_{N-1}\int_0^t \mathbf{1}_{\left(|X_v^N|=\frac{N-2}{N}\right)}dv}\right.$$
$$\left.\times f\left(|X_t^N| + \frac{1}{N}\right)\left(\mathbf{1}_{(\tau^\alpha_N\leq t<\tau^0_N+\lambda)} + \mathbf{1}_{(t\geq\tau^0_N+\lambda)}\right)\right],$$

where $\tau_N^0 = \tau^{(1)}_{(N-1)/N} \wedge \tau^{(1)}_{-(N-1)/N}$ and $\lambda \in (0,t)$. Using Lemma 2.13 and Lemma 2.14, we can study the first term on the right in (2.60).

$$\begin{aligned}\mathbb{E}_{d(N')}\left[f\left(|X_t^{N'}| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right] &= \mathbb{E}^{\mathbb{P}^{N'}_{d(N')}}\left[f\left(|x(t)| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right]\\ &= \mathbb{E}^{\mathbb{P}^{N',\tau}_{d(N')}}\left[f\left(|x(t)| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right],\end{aligned}$$

since $(t < \tau_N^\alpha) \in \mathcal{M}_{\tau_N^\alpha}$ and $\mathbb{P}^N_{d(N)} = \mathbb{P}^{N,\tau}_{d(N)}$ on $\mathcal{M}_{\tau^\alpha_N}$. Moreover,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}^{N',\tau}_{d(N')}}\left[f\left(|x(t)| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right]\\ &\quad= \mathbb{E}^{\mathbb{P}^{N',\tau}_{d(N')}}\left[f\left(|x(t)| + \frac{1}{N'}\right)\right] - f\left(\frac{N'-[N'^\alpha]+1}{N'}\right)\mathbb{P}^{N',\tau}_{d(N')}\left(t \geq \tau^\alpha_{N'}\right)\\ &\quad= \mathbb{E}^{\mathbb{P}^{N',\tau}_{d(N')}}\left[f(|x(t)|)\right] + o(1).\end{aligned}$$

Consequently,

$$\lim_{N'}\mathbb{E}_{d(N')}\left[f\left(|X_t^{N'}| + \frac{1}{N'}\right)\mathbf{1}_{(t<\tau^\alpha_{N'})}\right] = \mathbb{E}^{\mathbb{Q}_u}[f(|x(t)|)].$$

However,

$$\mathbb{E}^{\mathbb{Q}_u}\left[f(|x(t)|)\mathbf{1}_{(\tau_1\leq t)}\right] = 0.$$

In fact, let $\tau_s = \inf(t \geq 0, \forall v > t, x(v) = x(t))$ be the first time for which the process becomes constant. From the Portmanteau theorem

$$1 = \overline{\lim_{N'}}\,\mathbb{P}^{N',\tau}_{d(N')}\left(\overline{(\tau_s \leq \tau_1)}\right) \leq \mathbb{Q}_u\left(\overline{(\tau_s \leq \tau_1)}\right),$$

where $\overline{A}$ denote the closure under the Skorokhod topology of a subset A of $\mathcal{D}([0,+\infty),\mathbb{R})$. Moreover, we have

$$\begin{aligned}\overline{(\tau_s\leq\tau_1)\cap(\tau_1\leq t)}&\cap(x(0)=u)\cap\mathcal{C}([0,+\infty),\mathbb{R})\\&=(\tau_s\leq\tau_1)\cap(\tau_1\leq t)\cap(x(0)=u)\cap\mathcal{C}([0,+\infty),\mathbb{R}).\end{aligned}$$

Then,

$$\begin{aligned}\mathbb{Q}_u\big(|x(t)|\in supp(f),\tau_1\leq t\big)&\leq\mathbb{Q}_u\big(|x(t)|\in supp(f),\tau_s\leq\tau_1\leq t\big)\\&\leq\mathbb{Q}_u\big(|x(t)|\in supp(f),|x(t)|=1\big)=0,\end{aligned}$$

and

$$\lim_{N'}\mathbb{E}_{d(N')}\left[f\left(|X_t^{N'}|+\frac{1}{N'}\right)\mathbf{1}_{(t<\tau^{\alpha}_{N'})}\right]=\mathbb{E}^{\mathbb{Q}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right]=\mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right].$$

Finally, by the following lemma we get

$$\lim_{N'}\mathcal{T}_f^{l(N')}(\omega,t)=\mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right].$$

We can remark that this limit does not depend on the subsequence (N'). The following lemma represents the loss of energy from the propagating modes produced by the coupling between the propagating and the radiating modes. Moreover, this lemma implies the absorbing condition at the boundary 1 in Theorem 2.4, which implies the dissipation behavior in Theorem 2.5.

Lemma 2.15 $\lim_{N'}r(N')=0$.

Proof

$$\begin{aligned}|r(N')|\leq\|f\|_\infty\Bigg(&\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)ds}\mathbf{1}_{(t\geq\tau^0_{N'}+\lambda)}\right]\\&+\mathbb{P}_{d(N')}\left(|X_t^{N'}|+\frac{1}{N'}\in supp(f),\tau^{\alpha}_{N'}\leq t<\tau^0_{N'}+\lambda\right)\Bigg).\end{aligned}$$

First, let $\alpha'\in(3/4,1)$ and N_t^N the number of passages in $(N-1)/N$ during the time interval $[0,t]$.

$$\begin{aligned}\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)ds}\mathbf{1}_{(t\geq\tau^0_{N'}+\lambda)}\right]\leq\mathbb{E}_{d(N')}\Bigg[&e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)ds}\\&\times\left(\mathbf{1}_{(t\geq\tau^{(1)}_{(N'-1)/N'}+\lambda)}+\mathbf{1}_{(t\geq\tau^{(1)}_{-(N'-1)/N'}+\lambda)}\right)\Bigg].\end{aligned}$$

We shall work only with $\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)ds}\mathbf{1}_{(t\geq\tau^{(1)}_{(N'-1)/N'}+\lambda)}\right]$ but the same proof works for the other term.

$$\begin{aligned}\mathbb{E}_{d(N')}&\left[e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)ds}\mathbf{1}_{(t\geq\tau^{(1)}_{(N'-1)/N'}+\lambda)}\right]\\&\leq\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t\mathbf{1}\left(X_s^{N'}=\frac{N'-1}{N'}\right)ds}\mathbf{1}_{(N_t^{N'}\geq[N^{\alpha'}]+1)}\right]\\&+\mathbb{P}_{d(N')}\left(N_t^{N'}\leq[N^{\alpha'}],t-\tau^{(1)}_{(N'-1)/N'}\geq\lambda\right).\end{aligned}$$

On $(N_t^{N'} \geq [N'^{\alpha'}]+1)$, we have

$$e^{-\Lambda^c_{N'}\int_0^t \mathbf{1}_{\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)}ds} \leq e^{-\Lambda^c_{N'}\sum_{j=1}^{N_t^{N'}-1}\int_{\tau^{(j)}_{(N'-1)/N'}}^{\tau^{(j+1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}$$
$$\leq \prod_{j=1}^{[N'^{\alpha'}]} e^{-\Lambda^c_{N'}\int_{\tau^{(j)}_{(N'-1)/N'}}^{\tau^{(j+1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}.$$

We denote by $\sigma^{(1)}_{(N-1)/N}$ the time of the first return in $(N-1)/N$, then

$$\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}\mathbf{1}_{(N_t^{N'}\geq[N'^{\alpha'}]+1)}\right]$$
$$\leq \prod_{j=1}^{[N'^{\alpha'}]}\mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_{\tau^{(j)}_{(N'-1)/N'}}^{\tau^{(j+1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}\right]$$
$$\leq \left(\mathbb{E}_{\frac{N'-1}{N'}}\left[e^{-\Lambda^c_{N'}\int_0^{\sigma^{(1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}\right]\right)^{[N'^{\alpha'}]}$$

since $\left(\int_{\tau^{(j)}_{(N-1)/N}}^{\tau^{(j+1)}_{(N-1)/N}} \mathbf{1}_{\left(X_s^{N}=\frac{N-1}{N}\right)}ds\right)_j$ is an i.i.d sequence. Moreover, we can check that $\Lambda^c_N \geq CN^{3/2}$ and then

$$\mathbb{E}_{\frac{N'-1}{N'}}\left[e^{-\Lambda^c_{N'}\int_0^{\sigma^{(1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}\right] \leq \mathbb{E}_{\frac{N'-1}{N'}}\left[e^{-C\,N'^{3/2}\int_0^{\sigma^{(1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)}ds}\right].$$

In fact, a computation gives

$$\Lambda^c_N = \frac{ak^4A_N^2}{16\pi\beta_N}\int_{n_1kd\theta}^{n_1kd}\frac{\eta\sqrt{\eta^2-(n_1kd\theta)^2}}{\sqrt{(n_1kd)^2-\eta^2}\left(1+(\beta_N-\frac{1}{d}\sqrt{(n_1kd)^2-\eta^2})^2\right)}$$
$$\times\frac{S(\eta-\sigma_N,\eta-\sigma_N)}{(\eta^2-(n_1kd\theta)^2)\sin^2(\eta)+\eta^2\cos^2(\eta)}d\eta.$$

However, we recall that the support of S lies in the square $\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]\times\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$, then we can restrict the integration over $\left[n_1kd\theta, n_1kd\theta+\frac{3\pi}{2}\right]$. Moreover,

$$\left(\beta_N-\sqrt{(n_1k)^2-\eta^2/d^2}\right)^2 = (\eta-\sigma)^2\frac{y^2}{1-y^2},\quad \text{for some } y\in\left[\frac{\sigma_N}{n_1kd},\frac{\eta}{n_1kd}\right]$$
$$\leq\left(\frac{3\pi}{2}\right)^2\frac{\eta^2/(n_1kd)^2}{1-\eta^2/(n_1kd)^2}\leq K,$$

where K stands for a constant independent of N, because $\theta<1$ and $k\gg 1$. Therefore,

$$\Lambda^c_N \geq K'\int_{n_1kd\theta}^{n_1kd\theta+\frac{3\pi}{2}}\eta\sqrt{\eta^2-(n_1kd\theta)^2}d\eta \geq K''N^{3/2},$$

where we assume that the function S has a positive minimum, and then $K'' > 0$.

Now, let us remark that $\forall v \in [0, \ln(N^{1/4})]$

$$e^{-v} \leq 1 - \frac{1}{N^{1/4}} v$$

and

$$\mathbb{E}_{\frac{N'-1}{N'}}\left[\int_0^{\sigma^{(1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)} ds\right] = \frac{1}{\Gamma^c_{N'-1\,N'}}.$$

Then, we get

$$\mathbb{E}_{\frac{N'-1}{N'}}\left[e^{-K'' N'^{3/2}\int_0^{\sigma^{(1)}_{(N'-1)/N'}} \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)} ds}\right] \leq 1 - \frac{K''}{N'^{3/4}}\left(1 - \frac{1}{\ln(N'^{1/4})}\right)$$

and

$$\mathbb{E}_{d(N')}\left[e^{-\Lambda_{N'}\int_0^t \mathbf{1}_{\left(X_s^{N'}=\frac{N'-1}{N'}\right)} ds}\mathbf{1}_{(N_t^{N'}\geq [N'^{\alpha'}]+1)}\right] \leq e^{[N'^{\alpha'}]\ln\left[1-\frac{K''}{N'^{3/4}}\left(1-\frac{1}{\ln(N'^{1/4})}\right)\right]}.$$

Moreover,

$$\begin{aligned}
\mathbb{P}_{d(N)}\left(N_t^N \leq [N^{\alpha'}], t - \tau^{(1)}_{(N-1)/N} \geq \lambda\right) &\leq \mathbb{P}_{d(N)}\left(\tau^{([N^{\alpha'}])}_{(N-1)/N} - \tau^{(1)}_{(N-1)/N} \geq \lambda\right) \\
&\leq \frac{1}{\lambda}\mathbb{E}_{d(N)}\left(\tau^{([N^{\alpha'}])}_{(N-1)/N} - \tau^{(1)}_{(N-1)/N}\right) \\
&\leq \frac{N^{\alpha'}}{\lambda}\mathbb{E}_{(N-1)/N}\left[\sigma^{(1)}_{(N-1)/N}\right] \\
&\leq \frac{K}{N^{1-\alpha'}}.
\end{aligned}$$

Consequently,

$$\lim_{N'} \mathbb{E}_{d(N')}\left[e^{-\Lambda^c_{N'}\int_0^t \mathbf{1}_{\left(|X_s^{N'}|=\frac{N'-1}{N'}\right)} ds}\mathbf{1}_{(t\geq \tau^0_{N'}+\lambda)}\right] = 0.$$

Second, let $c_f \in (0,1)$ such that $supp(f) \subset [0, c_f - 1/N']$ and $x \in [0, c_f)$, then

$$\begin{aligned}
\mathbb{P}_{d(N)}\left(|X_t^N| + \frac{1}{N} \in supp(f), \tau_N^\alpha \leq t < \tau_N^0 + \lambda\right) &\leq \mathbb{P}_{d(N)}\left(X_t^N \in [-c_f, c_f], \tau_N^\alpha \leq t < \tau_N^0 + \lambda\right) \\
\leq \mathbb{P}_{d(N)}\left(X_t^N \in [-c_f, c_f], X^N_{\tau_N^\alpha} = \frac{N-[N^\alpha]}{N}, \tau_N^\alpha \leq t < \tau_N^0 + \lambda\right) \\
+ \mathbb{P}_{d(N)}\left(X_t^N \in [-c_f, c_f], X^N_{\tau_N^\alpha} = -\frac{N-[N^\alpha]}{N}, \tau_N^\alpha \leq t < \tau_N^0 + \lambda\right).
\end{aligned}$$

We shall treat only the case where $X^N_{\tau_N^\alpha} = (N - [N^\alpha])/N$, but the following proof works also in the other case. Let $\tilde{c}_f \in (c_f, 1)$, $\rho \in (0,1)$ such that $[\tilde{c}_f - \rho, \tilde{c}_f + \rho] \subset (c_f, 1)$ and $\lambda' \in (0,1)$. Using the strong Markov property we have

$$\begin{aligned}
&\mathbb{P}_{d(N)}\left(X_t^N \in [-c_f, c_f], X^N_{\tau_N^\alpha} = \frac{N-[N^\alpha]}{N}, \tau_N^\alpha \leq t < \tau_N^0 + \lambda\right) \\
&\quad \leq \mathbb{P}_{\frac{N-[N^\alpha]}{N}}\left(\tau^{(1)}_{(N-1)/N} > \lambda'\right) + \mathbb{P}^N_{\frac{[N\tilde{c}_f]}{N}}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right),
\end{aligned}$$

where $\tau_{\tilde{c}_f\pm\rho} = \inf(t \geq 0, |x(t) - \tilde{c}_f| \geq \rho)$. First, a computation gives

$$\mathbb{P}_{\frac{N-[N^\alpha]}{N}}\left(\tau^{(1)}_{(N-1)/N} > \lambda'\right) \leq \frac{1}{\lambda'}\mathbb{E}_{\frac{N-[N^\alpha]}{N}}\left[\tau^{(1)}_{(N-1)/N}\right] = \frac{1}{\lambda'}\sum_{l=N-[N^\alpha]}^{N-2}\frac{N+1+l}{\Gamma^c_{l+1\,l+2}} \leq \frac{K}{N^{1-\alpha}}.$$

Second, the sequence $(r(N'))_{N'}$ is bounded. Let $(r(N''))_{N''}$ be a converging subsequence. We recall that $\mathbb{P}^N_{c(N)} = \mathbb{P}^{N,\tau}_{c(N)}$ on $\mathcal{M}_{\tau^\alpha_N}$, where $c(N) = [N\tilde{c}_f]/N$, and by Lemma 2.13 the sequence $\left(\mathbb{P}^{N'',\tau}_{c(N'')}\right)_{N''}$ is tight. Let $\left(\mathbb{P}^{N''',\tau}_{c(N''')}\right)_{N'''}$ be a converging subsequence to $\mathbb{Q}_{\tilde{c}_f}$. Moreover, $\tau_{\tilde{c}_f\pm\rho} \leq \tau^\alpha_N$ and therefore $\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right) \in \mathcal{M}_{\tau^\alpha_N}$. Consequently, by the Portmanteau theorem

$$\begin{aligned}\overline{\lim_{N'''}}\,\mathbb{P}^{N'''}_{c(N''')}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right) &= \overline{\lim_{N'''}}\,\mathbb{P}^{N''',\tau}_{c(N''')}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right)\\ &\leq \overline{\lim_{N'''}}\,\mathbb{P}^{N''',\tau}_{c(N''')}\left(\overline{\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right)}\right)\\ &\leq \mathbb{Q}_{\tilde{c}_f}\left(\overline{\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right)}\right).\end{aligned}$$

We recall that $\mathbb{Q}_{\tilde{c}_f}\left(\mathcal{C}([0,+\infty),\mathbb{R})\right) = 1$ and we can show that

$$\overline{\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right)} \cap \mathcal{C}([0,+\infty),\mathbb{R}) = \left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right) \cap \mathcal{C}([0,+\infty),\mathbb{R}).$$

Then,

$$\overline{\lim_{N'''}}\,\mathbb{P}^{N''',\tau}_{c(N''')}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda'\right) \leq \mathbb{Q}_{\tilde{c}_f}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right),$$

and

$$\overline{\lim_{N'''}}\,r(N''') \leq \mathbb{Q}_{\tilde{c}_f}\left(\tau_{\tilde{c}_f\pm\rho} \leq \lambda + \lambda'\right).$$

Finally, $\lim_{N'''} r(N''') = 0$ and the limit of all subsequences $(r(N''))_{N''}$ of $(r(N'))_{N'}$ is 0. □

To finish, $\left(\mathcal{T}^{l(N)}_f(\omega,t)\right)_N$ is a bounded sequence. Let $\left(\mathcal{T}^{l(N')}_f(\omega,t)\right)_{N'}$ be a converging subsequence. By the previous work, there exists an another subsequence such that

$$\lim_{N''}\mathcal{T}^{l(N'')}_f(\omega,t) = \mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right],$$

where the limit does not depend on the particular subsequence, then all subsequence limits of $\left(\mathcal{T}^{l(N)}_f(\omega,t)\right)_N$ are equal to $\mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right]$. Consequently,

$$\lim_{N}\mathcal{T}^{l(N)}_f(\omega,t) = \mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right].$$

Now, we have to show that this equality holds even for a sequence $(l(N))_N$ such that $l(N)/N \to u = 1$, i.e $\lim_N \mathcal{T}^{l(N)}_f(\omega,t) = 0$. To do this, we write for $\lambda \in (0,t)$,

$$\begin{aligned}\mathcal{T}^{l(N)}_f(\omega,t) \leq \|f\|_\infty\Bigg(&\mathbb{P}_{d(N)}\left(t < \tau^{(1)}_{(N-1)/N)} + \lambda\right)\\ &+ \mathbb{E}_{d(N)}\left[e^{-\Lambda_N\int_0^t \mathbf{1}_{\left(X^N_u = \frac{N-1}{N}\right)}du}\,\mathbf{1}_{\left(t\geq\tau^{(1)}_{(N-1)/N}+\lambda\right)}\right]\Bigg).\end{aligned}$$

We have already shown in Lemma 2.15 that the second term on the right in the previous inequality goes to 0. The proof did not depend of the sequence $(d(N))_N$. Moreover, we have

$$\mathbb{P}_{d(N)}\left(t < \tau^{(1)}_{(N-1)/N)} + \lambda\right) \leq \frac{1}{t-\lambda}\mathbb{E}_{d(N)}\left[\tau^{(1)}_{(N-1)/N)}\right],$$

and

$$\mathbb{E}_{d(N)}\left[\tau^{(1)}_{(N-1)/N)}\right]=\sum_{j=l(N)-1}^{N-2}\frac{N+1+j}{\Gamma_{j+1,j+2}}\leq K\left(1-\frac{l(N)}{N}\right).$$

Consequently, we have $\forall u\in[0,1]$ and $\forall(l(N))_N$ such that $l(N)/N\to u$,

$$\lim_N \mathcal{T}_f^{l(N)}(\omega,t)=\mathbb{E}^{\overline{\mathbb{P}}_u}\left[f(|x(t)|)\mathbf{1}_{(t<\tau_1)}\right], \tag{2.61}$$

where the limit satisfies the required conditions. Finally, from the decomposition used in the proof of Theorem 2.5, we have $\forall\varphi\in L^2(0,1)$ and $\tilde{\varphi}$ a smooth function with compact support

$$\|\mathcal{T}_\varphi^N(L,.)-\mathcal{T}_\varphi(L,.)\|_{L^2(0,1)}\leq 2\|\varphi-\tilde{\varphi}\|_{L^2(0,1)}+\|\mathcal{T}_{\tilde{\varphi}}^N(L,.)-\mathcal{T}_{\tilde{\varphi}}(L,.)\|_{L^2(0,1)}.$$

Using the density of the smooth functions with compact support in $L^2(0,1)$ for $\|.\|_{L^2(0,1)}$ and the dominated convergence theorem we get the first point of Theorem 2.4. The second point is a direct consequence of the probabilistic representation (2.61) and the density for the sup norm over $[0,1]$ in $\{\varphi\in\mathcal{C}^0([0,1]),\ \varphi(1)=0\}$ of the smooth functions with compact support included in $[0,1)$.

2.6.5 Proof of Theorem 2.6

As in the proof of Theorem 2.4, we use a probabilistic representation of $\mathcal{T}_j^{0,l}(z)$ by using the Feynman-Kac formula. However, we introduce the jump Markov process which is a symmetric version with respect to reflecting barrier $(N-1)/N$ of that used in the proof of Theorem 2.4.

Let $(X_t^N)_{t\geq 0}$ be a jump Markov process with state space $\{-(N-1)/N,\dots,(N-1)/N,\dots,3(N-1)/N\}$ and generator given by

$$\mathcal{L}^N\phi\left(\frac{l}{N}\right)=\Gamma^c_{|l|+2\,|l|+1}\left(\phi\left(\frac{l-1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)+\Gamma^c_{|l|\,|l|+1}\left(\phi\left(\frac{l+1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)$$

for $l\in\{-(N-2),\dots,-1\}$,

$$\mathcal{L}^N\phi\left(\frac{l}{N}\right)=\Gamma^c_{l\,l+1}\left(\phi\left(\frac{l-1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)+\Gamma^c_{l+2\,l+1}\left(\phi\left(\frac{l+1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)$$

for $l\in\{1,\dots,N-2\}$,

$$\begin{aligned}\mathcal{L}^N\phi\left(\frac{l}{N}\right)&=\Gamma^c_{|l-2(N-1)|+2\ |l-2(N-1)|+1}\left(\phi\left(\frac{l-1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)\\&\quad+\Gamma^c_{|l-2(N-1)|\ |l-2(N-1)|+1}\left(\phi\left(\frac{l+1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)\end{aligned}$$

for $l\in\{N,\dots,2N-3\}$,

$$\begin{aligned}\mathcal{L}^N\phi\left(\frac{l}{N}\right)&=\Gamma^c_{l+2-2(N-1)\ l+1-2(N-1)}\left(\phi\left(\frac{l+1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)\\&\quad+\Gamma^c_{l-2(N-1)\ l+1-2(N-1)}\left(\phi\left(\frac{l-1}{N}\right)-\phi\left(\frac{l}{N}\right)\right)\end{aligned}$$

for $l\in\{2N-1,\dots,3N-2\}$,

$$\mathcal{L}^N\phi\left(-\frac{N-1}{N}\right)=\Gamma^c_{N-1\,N}\left(\phi\left(-\frac{N-2}{N}\right)-\phi\left(-\frac{N-1}{N}\right)\right),$$

$$\mathcal{L}^N\phi\left(\frac{3N-3}{N}\right)=\Gamma^c_{N-1\,N}\left(\phi\left(\frac{3N-4}{N}\right)-\phi\left(\frac{3N-3}{N}\right)\right),$$

$$\mathcal{L}^N\phi(0) = \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{1}{N}\right) - \phi(0)\right) + \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{-1}{N}\right) - \phi(0)\right),$$

$$\mathcal{L}^N\phi\left(\frac{N-1}{N}\right) = \frac{\Gamma^c_{N-1\,N}}{2}\left(\phi\left(\frac{N-2}{N}\right) - \phi\left(\frac{N-1}{N}\right)\right) + \frac{\Gamma^c_{N-1\,N}}{2}\left(\phi\left(\frac{N}{N}\right) - \phi\left(\frac{N-1}{N}\right)\right),$$

$$\mathcal{L}^N\phi\left(\frac{2N-2}{N}\right) = \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{2N-3}{N}\right) - \phi\left(\frac{2N-2}{N}\right)\right) + \frac{\Gamma^c_{2\,1}}{2}\left(\phi\left(\frac{2N-1}{N}\right) - \phi\left(\frac{2N-2}{N}\right)\right).$$

We recall that $\mathcal{T}^{0,l}(z)$ can be viewed as a probability measure on $[0,1]$ by setting

$$\mathcal{T}^{0,l}_f(z) = \sum_{j=1}^{N} f\left(\frac{j}{N}\right)\mathcal{T}^{0,l}_j(z)$$

for all bounded continuous function f on $[0,1]$. Let $0 < r \ll 1$ and f be a smooth function with support included in $[0,1-r)$. In order to make the link between $\mathcal{T}^{0,l}(z)$ and the process X^N, let us introduce an extension of f by setting

$$f^{N,s}(v) = \begin{cases} f\left(-v + 1/N\right) & \text{if} \quad v \in \big[-(N-1)/N, 0\big] \\ f\left(v + 1/N\right) & \text{if} \quad v \in \big[0, (N-1)/N\big] \\ f\left(-v + (2N-1)/N\right) & \text{if} \quad v \in \big[(N-1)/N, 2(N-1)/N\big] \\ f\left(v - (2N-3)/N\right) & \text{if} \quad v \in \big[2(N-1)/N, (3N-3)/N\big]. \end{cases}$$

With these two functions we get the following representation. $\forall l \in \{1,\dots,N\}$,

$$\mathcal{T}^{0,l}_f(z) = \mathbb{E}_{\frac{l-1}{N}}\left[f^{N,s}(X^N_z)\right].$$

Moreover, we have

$$\mathcal{T}^{0,l}_f(z) = \mathbb{E}_{\frac{l-1}{N}}\left[f^{s}(X^N_z)\right] + \mathcal{O}\left(\frac{1}{N}\right) = \mathbb{E}_{\frac{l-1}{N}}\left[f^{s}(g_r(X^N_z))\right] + \mathcal{O}\left(\frac{1}{N}\right),$$

where

$$g_r(v) = \begin{cases} v & \text{if} \quad v \in (-(1-r), 1-r) \cup (1+r, 3-r) \\ v_s & \text{elsewhere}, \end{cases}$$

with $v_s \in (1-r, 1-r/2)$, and where

$$f^s(v) = \begin{cases} f(-v) & \text{if} \quad v \in [-1,0] \\ f(v) & \text{if} \quad v \in [0,1] \\ f(-v+2) & \text{if} \quad v \in [1,2] \\ f(v-2) & \text{if} \quad v \in [2,3]. \end{cases}$$

Let $u \in [0,1)$ such that $l(N)/N \to u$. One can assume $u \in [0,1-r)$ by changing r if necessary. As in the proof of Theorem 2.4, we have the following lemma.

Lemma 2.16 $\forall \varphi \in \mathcal{C}^\infty_0(\mathbb{R})$.

$$\lim_{N\to+\infty} \sup_{v\in I_N} \left|\mathcal{L}^N\varphi\left(\frac{[Nv]}{N}\right) - \mathcal{L}_{a_{r,\infty}}\varphi(v)\right| = 0,$$

where

$$I_N = \left[-\frac{N-1-[Nr]}{N}, -\frac{1}{N}\right] \cup \left[\frac{1}{N}, \frac{N-1-[Nr]}{N}\right] \cup \left[\frac{N-1+[Nr]}{N}, \frac{2N-3}{N}\right] \cup \left[\frac{2N-1}{N}, \frac{3N-3-[Nr]}{N}\right],$$

and $a_{r,\infty}$ is a $\mathcal{C}^1$-extended version of a_∞ such that

$$a_{r,\infty}(v) = \begin{cases} a_\infty(-v) & \text{if} \quad v \in (-(1-r), 0] \\ a_\infty(v) & \text{if} \quad v \in [0, 1-r) \\ a_\infty(-v+2) & \text{if} \quad v \in (1+r, 2] \\ a_\infty(v-2) & \text{if} \quad v \in [2, 3-r), \end{cases}$$

and the martingale problem associated to $\mathcal{L}_{a_{r,\infty}}$ and starting from u is well-posed.

Lemma 2.17 *The law of the process $(g_r(X^N))_N$ starting from $d(N) = (l(N)-1)/N$ is tight on $\mathcal{D}([0,+\infty), \mathbb{R})$.*

Proof (of Lemma 2.17) Let $\mathcal{F}_t^N = \sigma(X_s^N,\ s \le t)$. According to Theorem 3 in [41, Chapter 3]. We have to show only the two following points. First, we have

$$\lim_{K\to+\infty} \overline{\lim_N}\, \mathbb{P}_{d(N)}\left(\sup_{t\ge 0}|g_r(X_t^N)| \ge K\right) = 0,$$

since $\forall N$, $\sup_{t\ge0}|g_r(X_t^N)| \le 3$. Second, we have for each N, $h \in (0,1)$, $s \in [0,h]$ and $t \ge 0$,

$$\mathbb{E}_{d(N)}\big((g_r(X_{t+s}^N) - g_r(X_t^N))^2|\mathcal{F}_t^N\big) \le K\,h.$$

In fact, we have

$$\begin{aligned}\mathbb{E}_{d(N)}\big((g_r(X_{t+s}^N) - g_r(X_t^N))^2|\mathcal{F}_t^N\big) &\le 2\,\mathbb{E}_{d(N)}\big((M_{g_r}^N(t+s) - M_{g_r}^N(t))^2|\mathcal{F}_t^N\big) \\ &\quad + 2\,\mathbb{E}_{d(N)}\left(\left(\int_t^{t+s} \mathcal{L}^N g_r(X_w^N)dw\right)^2 \Big|\mathcal{F}_t^N\right),\end{aligned}$$

with

$$M_{g_r}^N(t) = g_r(X_t^N) - g_r(X_0^N) - \int_0^t \mathcal{L}^N g_r(X_s^N)ds,$$

which is a $(\mathcal{F}_t^N)_{t\ge0}$-martingale. We also have

$$\sup_N \sup_{v\in\left[-\frac{N-1}{N}, 3\frac{N-1}{N}\right]\setminus\left\{0, 2\frac{N-1}{N}\right\}} |\mathcal{L}^N g_r(v)| < +\infty$$

since by Lemma 2.16

$$\sup_N \sup_{v\in I_N\cup\{v_s\}} \left|\mathcal{L}^N g_r(v)\right| < +\infty.$$

Moreover, $\mathcal{L}^N g_r(0) = \mathcal{L}^N g_r(2(N-1)/N) = 0$. Then, we get

$$\mathbb{E}_{d(N)}\left(\left(\int_t^{t+s} \mathcal{L}^N g_r(X_w^N)dw\right)^2 \Big|\mathcal{F}_t^N\right) \le Ch^2.$$

We recall that

$$< M_{g_r}^N >_t = \int_0^t \left(\mathcal{L}^N {g_r}^2 - 2g_r\mathcal{L}^N g_r\right)(X_s^N)ds.$$

Consequently, by the martingale property of $(M_{g_r}^N(t))_{t\ge0}$,

$$\begin{aligned}\mathbb{E}_{d(N)}\big((M_{g_r}^N(t+s) - M_{g_r}^N(t))^2|\mathcal{F}_t^N\big) &= \mathbb{E}_{d(N)}\left((M_{g_r}^N(t+s) - M_{g_r}^N(t))^2|\mathcal{F}_t^N\right) \\ &= \mathbb{E}_{d(N)}\left(M_{g_r}^N(t+s)^2 - M_{g_r}^N(t)^2|\mathcal{F}_t^N\right) \\ &= \mathbb{E}_{d(N)}\left(< M_{g_r}^N >_{t+s} - < M_{Id}^N >_t |\mathcal{F}_t^N\right) \\ &= \mathbb{E}_{d(N)}\left(\int_t^{t+s} \left(\mathcal{L}^N g_r^2 - 2g_r\mathcal{L}^N g_r\right)(X_w^N)dw \Big|\mathcal{F}_t^N\right) \\ &\le C\,h.\end{aligned}$$

In fact, in addition to the previous arguments, we also have

$$\sup_N \sup_{v\in I_N\cup\{v_s\}} |\mathcal{L}^N g_r^2(v)| < +\infty,$$

$\sup_N \mathcal{L}^N g_r^2(0) = \frac{\Gamma^c_{1\,2}}{N^2} < +\infty$, and $\sup_N \mathcal{L}^N g_r^2(2(N-1)/2) = 2\frac{\Gamma^c_{1\,2}}{N^2} < +\infty$. That concludes the proof Lemma 2.17. □

Now, let us introduce some notations. $\forall j \in \mathbb{N}^*$, let

$$\begin{aligned}\tau_r^{(j)} &= \inf\left(t > \tau_{r,c}^{(j-1)}, \quad x(t) \in [-1,-(1-r)) \cup (1-r,1+r) \cup (3-r,3]\right)\\ \tau_{r,c}^{(j)} &= \inf\left(t > \tau_r^{(j)}, \quad x(t) \in (-(1-r),1-r) \cup (1+r,3-r)\right),\end{aligned}$$

with $\tau_{r,c}^{(0)} = 0$. Using the previous lemma, there exists (N') such that

$$\lim_{N'\to+\infty} \mathbb{E}_{d(N')}\left[f^s(g_r(X_z^{N'}))\right] = \mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\right].$$

Moreover,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\right] &= \sum_{j\geq 1} \mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\mathbf{1}_{(\tau_{r,c}^{(j-1)}\leq z<\tau_r^{(j)})}\right]\\ &= \sum_{j\geq 1} \mathbb{E}^{\mathbb{Q}_u}\left[\mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\mathbf{1}_{(\tau_{r,c}^{(j-1)}\leq z<\tau_r^{(j)})}\Big|\mathcal{M}_{\tau_{r,c}^{(j-1)}}\right]\right],\end{aligned}$$

where $\mathcal{M}_t = \sigma(x(s),\ 0 \leq s \leq t)$. With the following lemma we can identify each excursion between $\tau_{r,c}^{(j-1)}$ and $\tau_r^{(j)}$.

Lemma 2.18 $\forall j \in \mathbb{N}^*$, *the conditional law* $\mathbb{Q}_u(\,\cdot\,|\mathcal{M}_{\tau_{r,c}^{(j-1)}})$ *coincide up to the stopping time* $\tau_r^{(j)}$ *with the conditional law* $\overline{\mathbb{P}}_u^r(\,\cdot\,|\mathcal{M}_{\tau_{r,c}^{(j-1)}})$, *where* $\overline{\mathbb{P}}_u^r$ *is the unique solution of the martingale problem associated to* $\mathcal{L}_{a_{r,\infty}}$ *and starting from* u.

Proof (of Lemma 2.18) This proof is a conditional version of Lemma 2.14. Moreover, this lemma follows from Lemma 2.16 and the fact that we are studying excursions between $\tau_{r,c}^{(j-1)}$ and $\tau_r^{(j)}$. By Lemma 2.16, in addition to $g_r(X_z^N) = X_z^N$ for $\tau_{r,c}^{(j-1)} \leq z < \tau_r^{(j)}$,

$$\lim_N \mathbb{E}_{d(N)}\left[\int_{\tau_{r,c}^{(j-1)}}^{t\wedge\tau_r^{(j)}} \left|\mathcal{L}^N\varphi(X_s^N) - \mathcal{L}_{a_{r,\infty}}\varphi(X_s^N)\right| ds\Big|\mathcal{M}_{\tau_{r,c}^{(j-1)}}\right] = 0,$$

and we also have

$$\begin{aligned}\mathbb{E}_0\left[\int_0^t \mathbf{1}_{(X_s^N=0)}ds\right] &= \mathcal{O}\left(\frac{1}{N^{\alpha'\wedge(1-\alpha')}}\right),\\ \mathbb{E}_{2(N-1)/N}\left[\int_0^t \mathbf{1}_{(X_s^N=2(N-1)/N)}ds\right] &= \mathcal{O}\left(\frac{1}{N^{\alpha'\wedge(1-\alpha')}}\right)\end{aligned}$$

by symmetry of the process X^N. As in the proof of Lemma 2.14, we get that $\forall\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$

$$\varphi(x(t\wedge\tau_r^{(j)})) - \varphi(x(\tau_{r,c}^{(j-1)})) - \int_{\tau_{r,c}^{(j-1)}}^{t\wedge\tau_r^{(j)}} \mathcal{L}_{a_{r,\infty}}\varphi(x(s))ds$$

is a martingale under the conditional law $\mathbb{Q}_u(\,\cdot\,|\mathcal{M}_{\tau_{r,c}^{(j-1)}})$. Finally, from the uniqueness of the martingale problem associated to $\mathcal{L}_{a_{r,\infty}}$, $\mathbb{Q}_u(\,\cdot\,|\mathcal{M}_{\tau_{r,c}^{(j-1)}})$ coincide up to the stopping time $\tau_r^{(j)}$ with $\overline{\mathbb{P}}_u^r(\,\cdot\,|\mathcal{M}_{\tau_{r,c}^{(j-1)}})$ (see Theorem 6.2.2 in [59]). That concludes the proof of Lemma 2.18.□

From the previous lemma, $\forall j \in \mathbb{N}^*$, we have

$$\mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\mathbf{1}_{(\tau_{r,c}^{(j-1)}\leq z<\tau_r^{(j)})}\right] = \mathbb{E}^{\overline{\mathbb{P}}^r_u}\left[f^s(x(z))\mathbf{1}_{(\tau_{r,c}^{(j-1)}\leq z<\tau_r^{(j)})}\right]$$

and then

$$\lim_{N'\to+\infty} \mathbb{E}_{d(N')}\left[f^s(g_r(X_z^{N'}))\right] = \mathbb{E}^{\mathbb{Q}_u}\left[f^s(x(z))\right] = \mathbb{E}^{\overline{\mathbb{P}}^r_v}\left[f^s(x(z))\right],$$

where the limit does not depend to (N'). Consequently,

$$\lim_{N\to+\infty} \mathcal{T}_f^{0,l(N)}(z) = \mathbb{E}^{\overline{\mathbb{P}}^r_u}\left[f^s(x(z))\right] = \mathcal{T}_f(z,u),$$

with

$$\frac{\partial}{\partial z}\mathcal{T}_f(z,u) = \mathcal{L}_{a_{r,\infty}}\mathcal{T}_f(z,u) = \mathcal{L}_{a_\infty}\mathcal{T}_f(z,u).$$

For the boundary conditions, first let $h \in (0,1)$ such that $0 < h \ll 1$, we have

$$\frac{1}{h}(\mathcal{T}_f(z,h) - \mathcal{T}_f(z,-h)) = \frac{1}{h}\lim_{N\to+\infty}\left(\mathbb{E}_{\frac{[Nh]}{N}}\left[f^s(X_z^N)\right] - \mathbb{E}_{-\frac{[Nh]}{N}}\left[f^s(X_z^N)\right]\right) = 0,$$

because of the symmetry of the process X^N and f^s, and therefore,

$$2\frac{\partial}{\partial u}\mathcal{T}_f(z,0) = 0.$$

Second, in the same way, let $h \in (0,1)$ such that $h \ll 1$. Moreover, one can assume $r < h$ by changing r if necessary. Then, we have

$$\frac{1}{h}(\mathcal{T}_f(z,1-h) - \mathcal{T}_f(z,1+h)) = \frac{1}{h}\lim_{N\to+\infty}\left(\mathbb{E}_{\frac{[N(1-h)]}{N}}\left[f^s(X_z^N)\right] - \mathbb{E}_{\frac{[N(1+h)]}{N}}\left[f^s(X_z^N)\right]\right) = 0,$$

and therefore,

$$2\frac{\partial}{\partial u}\mathcal{T}_f(z,1) = 0.$$

As a result, using the density of the smooth functions with compact support in $L^2(0,1)$ for $\|.\|_{L^2(0,1)}$ and the dominated convergence theorem we get the first point of Theorem 2.6. The second point is a consequence of the maximum principle and the density for the sup norm over $[0,1]$ in $\{\varphi \in \mathcal{C}^0([0,1]),\ \varphi(1) = 0\}$ of the smooth functions with compact support included in $[0,1)$. ■

Chapter 3

Pulse Propagation and Time Reversal in Shallow-Water Acoustic Random Waveguides

Introduction

This chapter is devoted to the study of the propagation and the time reversal of a broadband pulse in the random waveguide model introduced in Chapter 2.

Acoustic pulse propagation in shallow-water waveguides has numerous domains of applications. One of the most important applications is submarine detection with active or passive sonars. Pulse propagation in random media has been studied in different contexts, in one-dimensional random media in [17] and [25, Chapter 8], in three-dimensional randomly layered media in [25, Chapter 14], and in random waveguides in [25, Chapter 20] and [30]. In these cases, it has been observed that the amplitude of the coherent wave decays with the propagation distance, since the coherent energy is converted into small incoherent wave fluctuations.

The time-reversal experiments of M. Fink and his group in Paris have attracted considerable attention because of the surprising effect of enhanced spatial focusing and time compression in random media. The refocusing properties have numerous applications, in detection, destruction of kidney stones, and wireless communication for instance. Time-reversal experiments have been intensively analyzed experimentally and theoretically. This experiment is carried out in two steps. In the first step (see Figure 3.1 (a)), a source sends a pulse into a medium. The wave propagates and is recorded by a device called a time-reversal mirror. A time-reversal mirror is a device that can receive a signal, record it, and resend it time-reversed into the medium. In the second step (see Figure 3.1 (b)), the wave emitted by the time-reversal mirror has the property of refocusing near the original source location, and it has been observed experimentally that random inhomogeneities enhance refocusing [19, 22, 42]. Time-reversal refocusing in one-dimensional media has been studied in [18, 25], in three-dimensional randomly layered media in [26], in the paraxial approximation in [10, 15, 49], and in random waveguides in [30, 25, 33].

The pulse propagation and the time reversal of a broadband pulse, in the case of a waveguide with a bounded cross-section and Dirichlet boundary conditions, is carried out in [25, Chapter 20]. However, it does not take into account radiation losses. In this chapter, the waveguide model introduced in Chapter 2 permits us to decompose the wave field into three kinds of modes: the propagating modes, the evanescent modes, and the radiating modes. However, in this chapter, for the sake of simplicity we do not consider the effect

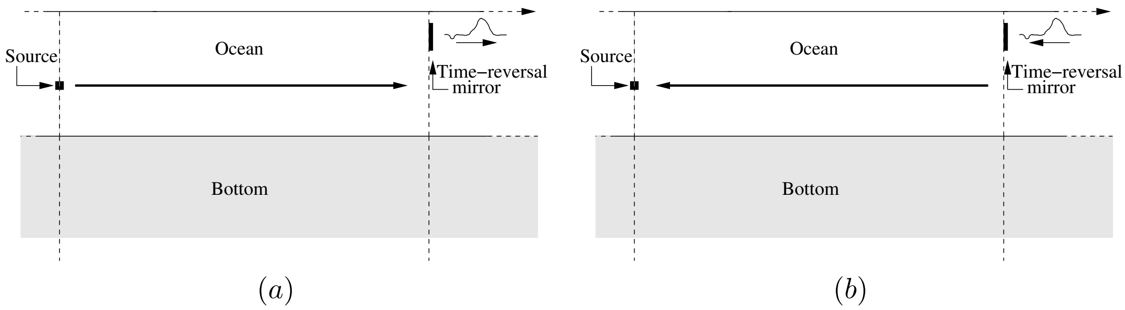


Figure 3.1: Representation of the time-reversal experiment. In (a) we represent the first step of the experiment, and in (b) we represent the second step of the experiment.

of the evanescent modes on the propagating and the radiating modes. We have seen, in Chapter 2, that the presence of evanescent modes induces an effective dispersion, and we know that dispersion effects can be compensated for by time reversal [25]. We have seen that the presence of radiating modes produces an effective diffusion, and we anticipate that diffusive effects cannot be fully compensated for by time reversal. Therefore, it is interesting to understand these effects. The main result of this chapter is the analysis of the influence of the radiation losses on the refocused wave in the time-reversal experiment. In Propositions 3.11 and 3.12, we show that the radiative loss affects the quality of the time-reversal refocusing. First, the amplitude of the refocused wave decays exponentially with the propagation distance. Second, the width of the main focal spot increases and converges to an asymptotic value, which is significantly larger than the diffraction limit $\lambda_{oc}/(2\theta)$ obtained in Proposition 3.6 (where λ_{oc} is the carrier wavelength in the ocean section $[0, d]$ with index of refraction n_1, and $\theta = \sqrt{1 - 1/n_1^2}$).

This Chapter is in two parts. The first part concerns the propagation of a broadband pulse and the second part concerns the time-reversal experiment. In Section 3.1 we recall the waveguide model introduced in Chapter 2. In Section 3.2 we recall the mode decomposition associated to this model with the simplification that we neglect the effect of the evanescent modes. Section 3.3 concerns the study of the propagation of a broadband pulse. In this section we show that the coherent transmitted wave is a sequence of modal waves with different arrival times and different modal speeds. The amplitude of each modal wave is exponentially damped and the rates depend on the effective coupling between the propagating modes and the radiation losses. The study of the incoherent wave fluctuations requires the analysis of the product of two transfer operators at two nearby frequencies. Then, we derive an effective system of transport equations which takes into account the effect of the radiation losses. Applying this result to the study of the intensity of the incoherent wave fluctuations, we observe that it is exponentially damped and becomes uniform across the waveguide section $[0, d]$ when the propagation distance is large. In Section 3.4 we study the time-reversal experiment in which the spatial random inhomogeneities may have changed during the two steps of the experiment. In this case, both the amplitude and the statistical stability of the refocused wave depend on the degree of correlation between the two realizations of the random medium. Moreover, we describe the refocused transverse profile in terms of the solution of the continuous diffusive model introduced in Section 2.5.2. Consequently, we show that the quality of the time-reversal refocusing is degraded by the radiative loss.

3.1 Waveguide Model

In this chapter we conserve the setting of Chapter 2, which is illustrated in Figure 2.2 page 33, but with some simplifications. We consider a two-dimensional linear acoustic waveguide model. The conservation equations of mass and linear momentum are given by

$$\begin{aligned} \rho(x,z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{F}_q^\epsilon, \\ \frac{1}{K(x,z)}\frac{\partial p}{\partial t} + \nabla.\mathbf{u} &= 0, \end{aligned} \tag{3.1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ is the density of the medium, K is the bulk modulus, and the source is modeled by the forcing term $\mathbf{F}_q^\epsilon(t,x,z)$ given by

$$\mathbf{F}_q^\epsilon(t,x,z) = \Psi_q^\epsilon(t,x)\delta(z - L_S)\mathbf{e}_z.$$

Here, $\Psi_q^\epsilon(t,x)$ is the profile of the source. The third coordinate z represents the propagation axis along the waveguide. The transverse section of the waveguide is the semi-infinite interval $[0,+\infty)$, and $x \in [0,+\infty)$ represents the transverse coordinate. Let $d > 0$, the medium parameters are given by

$$\frac{1}{K(x,z)} = \begin{cases} \frac{1}{\bar{K}}\left(n^2(x) + \sqrt{\epsilon}V(x,z)\right) & \text{if} \quad x \in [0,d], \quad z \in [0, L/\epsilon] \\ \frac{1}{\bar{K}} n^2(x) & \text{if} \quad \begin{cases} x \in [0,+\infty),\ z \in (-\infty,0) \cup (L/\epsilon,+\infty) \\ \text{or} \\ x \in (d,+\infty),\ z \in (-\infty,+\infty). \end{cases} \end{cases}$$

$$\rho(x,z) = \bar{\rho} \quad \text{if} \quad x \in [0,+\infty),\ z \in \mathbb{R},$$

and where the process V is described in Section 2.6.1. We consider the Pekeris waveguide model. This kind of model has been studied for half a century [51] and in this model the index of refraction $n(x)$ is given by

$$n(x) = \begin{cases} n_1 > 1 & \text{if} \quad x \in [0,d) \\ 1 & \text{if} \quad x \in [d,+\infty). \end{cases}$$

This profile can model an ocean with a constant sound speed, where d represents the ocean depth. Such conditions can be found during the winter in Earth's mid latitudes and in water shallower than about 30 meters.

From the conservation equations (3.1), we derive the wave equation for the pressure field,

$$\Delta p - \frac{1}{c(x,z)^2}\frac{\partial^2 p}{\partial t^2} = \nabla.\mathbf{F}_q^\epsilon, \tag{3.2}$$

where $c(x,z) = \sqrt{K(x,z)/\rho(x,z)}$, $\Delta = \partial_x^2 + \partial_z^2$, and $c = \sqrt{\bar{K}/\bar{\rho}}$. In underwater acoustics the density of air is very small compared to the density of water, then it is natural to use a pressure-release condition. The pressure is very weak outside the waveguide, and by continuity, the pressure is zero at the free surface $x = 0$. This consideration leads us to consider the Dirichlet boundary conditions

$$p(t,0,z) = 0 \quad \forall (t,z) \in [0,+\infty) \times \mathbb{R}.$$

In addition to the classical scales which are the wavelength, the correlation length, the standard deviation, and the propagation distance, we also consider the bandwidth of the pulse. This scale plays a key role in the pulse propagation and the time-reversal experiment.

In this chapter, the source profile $\Psi_q^\epsilon(t,x)$ is given, in the frequency domain, by

$$\begin{aligned}\widehat{\Psi}_q^\epsilon(\omega,x) = \frac{1}{\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right) \\ \times\left[\sum_{j=1}^{N(\omega)}\phi_j(\omega,x_0)\phi_j(\omega,x)+\int_{(-S,-\xi)\cup(\xi,k^2(\omega))}\phi_\gamma(\omega,x_0)\phi_\gamma(\omega,x)d\gamma\right],\end{aligned} \tag{3.3}$$

with $q>0$. The restriction $q>0$ allows us to freeze the number of propagating and radiating modes and gives simpler expressions of the transmitted field. Here, we have used the decomposition with respect to the resolution of the identity Π_ω associated to the operator $R(\omega)$ and introduced in Section 2.2.1. We refer to Section 2.2.1 for a summary of the spectral analysis of this operator. The Fourier transform and the inverse Fourier transform, with respect to time, are defined by

$$\widehat{f}(\omega)=\int f(t)e^{i\omega t}dt, \quad f(t)=\frac{1}{2\pi}\int\widehat{f}(\omega)e^{-i\omega t}d\omega.$$

We also recall that S can be arbitrarily large and ξ can be arbitrarily small. Consequently, the spatial profile (3.3) is an approximation of a Dirac distribution at x_0, which models a point source at x_0. Moreover, $\frac{1}{\epsilon^q}\widehat{f}(\frac{\omega-\omega_0}{\epsilon^q})$ is the Fourier transform of $f(\epsilon^q t)e^{-i\omega_0 t}$, which is a pulse with bandwidth of order ϵ^q and carrier frequency ω_0. In this chapter, we study the broadband case, that is for $q\in(0,1)$. In the broadband case the pulse width is of order $1/\epsilon^q$, which is much smaller that the propagation distance, and therefore the propagating modes are separated in time by the modal dispersion. In the broadband case, the transmitted wave can be described by a front stabilization theory (Section 3.3.2), and the statistical stability of the time-reversal refocusing can be study in a simple way (Section 3.4.6).

The case $q=1$, that we shall not treat in this chapter, corresponds to the narrowband case. In this case the order of the pulse width is comparable to the propagation distance, and consequently the modes overlap during the propagation.

However, for the sake of simplicity, we shall consider the case $q=1/2$ and the analysis that follows could be carried out $\forall q\in(0,1)$.

According to (2.12) page 37, the evanescent part of the wave field decreases exponentially fast with the propagation distance. For more convenient manipulations in the study of the time-reversal experiment we assume that the source location L_S is sufficiently far away from 0 so that the evanescent modes generated by the source are negligible. With this assumption and using 2.23 page 41, we can assume that the incident pulse coming from the left is given, at $z=0$, by:

$$p_{inc}^{\xi,\epsilon}(t,x,0)=\frac{1}{2\pi}\int\left[\sum_{j=1}^{N(\omega)}\frac{\widehat{a}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j(\omega)}}\phi_j(\omega,x)+\int_\xi^{k^2(\omega)}\frac{\widehat{a}_{\gamma,0}^\epsilon(\omega)}{\gamma^{1/4}}\phi_\gamma(\omega,x)d\gamma\right]e^{-i\omega t}d\omega,$$

where

$$\widehat{a}_{j,0}^\epsilon(\omega)=\frac{\sqrt{\beta_j(\omega)}}{2\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right)\phi_j(\omega,x_0)e^{-i\beta_j(\omega)L_S}=\frac{1}{2\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right)\tilde{a}_j(\omega) \tag{3.4}$$

$\forall j\in\{1,\dots,N(\omega)\}$,

$$\widehat{a}_{\gamma,0}^\epsilon(\omega)=\frac{\gamma^{1/4}}{2\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right)\phi_\gamma(\omega,x_0)e^{-i\sqrt{\gamma}L_S}=\frac{1}{2\epsilon^q}\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^q}\right)\tilde{a}_\gamma(\omega) \tag{3.5}$$

for almost every $\gamma\in(\xi,k^2(\omega))$. Let us remark that this assumption is not restrictive and all the results of this chapter are valid for any $L_S<0$. Indeed, according to Proposition 2.2 page 50, in the asymptotic $\epsilon\to 0$, the information about the evanescent part of the source profile are lost during the propagation in the random section $[0,L/\epsilon]$, and therefore they play no role in the pulse propagation and in the time-reversal experiment.

3.2 Mode Coupling in Random Waveguides

In this section, we study the Fourier transform $\widehat{p}(\omega, x, z)$ of the pressure $p(t, x, z)$ when a random section $[0, L/\epsilon]$ is inserted between two homogeneous waveguides. In the half-space $z \geq 0$, by taking the Fourier transform in (3.2), we get the perturbed time harmonic wave equation

$$\partial_z^2 \widehat{p}(\omega, x, z) + \partial_x^2 \widehat{p}(\omega, x, z) + k^2(\omega)(n^2(x) + \sqrt{\epsilon}\tilde{V}(x, z))\widehat{p}(\omega, x, z) = 0, \tag{3.6}$$

where $k(\omega) = \frac{\omega}{c}$ is the wavenumber, and where

$$\tilde{V}(x, z) = \begin{cases} V(x, z) & \text{if} \quad x \in [0, d], \quad z \in [0, L/\epsilon] \\ 0 & \text{elsewhere.} \end{cases}$$

Moreover, we consider Dirichlet boundary conditions $\widehat{p}(\omega, 0, z) = 0 \ \forall z \in \mathbb{R}$. As in Chapter 2, we are interested in smooth solutions such that

$$\widehat{p}(\omega, ., .) \in \mathcal{C}^0\Big([0, +\infty), \mathcal{D}\big(R(\omega)\big)\Big) \cap \mathcal{C}^2\Big([0, +\infty), H\Big),$$

with $H = L^2(0, +\infty)$, in order to consider (3.6) as an operational differential equation. Here,

$$R(\omega) = \frac{\partial^2}{\partial x^2} + k^2(\omega) n^2(x)$$

is the Pekeris operator of the unperturbed waveguide, with domain $\mathcal{D}\big(R(\omega)\big) = H_0^1(0, +\infty) \cap H^2(0, +\infty)$. We recall that H is equipped with the inner product defined by

$$\forall (h_1, h_2) \in H \times H, \quad \langle h_1, h_2 \rangle_H = \int_0^{+\infty} h_1(x)\overline{h_2(x)} dx.$$

We refer to Section 2.2.1 for a summary of the spectral analysis of this operator. In the perturbed section $[0, L/\epsilon]$, a solution of (3.6) can be decomposed using the resolution of the identity Π_ω associated to $R(\omega)$,

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega, z)\phi_j(\omega, x) + \int_{-\infty}^{k^2(\omega)} \widehat{p}_\gamma(\omega, z)\phi_\gamma(\omega, x) d\gamma,$$

where $\widehat{p}(\omega, z) = \Theta_\omega(\widehat{p}(\omega, ., z))$. The operator Θ_ω is defined in Section 2.2.1. However, in what follows, we shall consider solutions of the form

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N(\omega)} \widehat{p}_j(\omega, z)\phi_j(\omega, x) + \int_{\xi}^{k^2(\omega)} \widehat{p}_\gamma(\omega, z)\phi_\gamma(\omega, x) d\gamma \tag{3.7}$$

to simplify the study of the time-reversal experiment. This assumption is tantamount to neglecting the coupling mechanism with the evanescent modes. Furthermore, as it has been observed in Chapter 2 or in [25], this mechanism implies mode-dependent and frequency-dependent phase modulations, that is dispersion, but does not remove any energy from the propagating modes in the pulse propagation. Dispersion is compensated by the time-reversal mechanism and therefore plays no role in this experiment [25]. This assumption leads us to simplified algebra in the proof of Theorem 3.1 page 114. Moreover, we assume that $\epsilon \ll \xi$ and therefore we have two distinct scales. We shall consider in a first time the asymptotic ϵ goes to 0 and in a second time the asymptotic ξ goes to 0.

3.2.1 Coupled Mode Equations

In this section we give the coupled mode equations, which describes the coupling mechanism between the amplitudes of the two kinds of modes. According to the decomposition (3.7), we consider the coupling between the propagating modes with the radiating modes.

In the random section $[0, L/\epsilon]$, $\widehat{p}(\omega, z)$ satisfies the following coupled equation in $\mathcal{H}_\xi^\omega = \mathbb{C}^{N(\omega)} \times L^2(\xi, k^2(\omega))$.

$$\begin{aligned} \frac{d^2}{dz^2}\widehat{p}_j(\omega,z) + \beta_j^2(\omega)\widehat{p}_j(\omega,z) + \sqrt{\epsilon}k^2(\omega)\sum_{l=1}^{N(\omega)} C_{jl}^\omega(z)\widehat{p}_l(\omega,z) \\ + \sqrt{\epsilon}k^2(\omega)\int_\xi^{k^2(\omega)} C_{j\gamma'}^\omega(z)\widehat{p}_{\gamma'}(\omega,z)d\gamma' = 0, \\ \frac{d^2}{dz^2}\widehat{p}_\gamma(\omega,z) + \gamma\ \widehat{p}_\gamma(\omega,z) + \sqrt{\epsilon}k^2(\omega)\sum_{l=1}^{N(\omega)} C_{\gamma l}^\omega(z)\widehat{p}_l(\omega,z) \\ + \sqrt{\epsilon}k^2(\omega)\int_\xi^{k^2(\omega)} C_{\gamma\gamma'}^\omega(z)\widehat{p}_{\gamma'}(\omega,z)d\gamma' = 0, \end{aligned} \tag{3.8}$$

where the coupling coefficients $C^\omega(z)$ are defined by (2.15) page 39, and they represent the coupling between the propagating and radiating modes.

Next, we introduce the amplitudes of the generalized right- and left-going modes $\widehat{a}(\omega, z)$ and $\widehat{b}(\omega, z)$, which are given by

$$\begin{aligned} \widehat{p}_j(\omega,z) &= \frac{1}{\sqrt{\beta_j(\omega)}}\Big(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} + \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\Big), \\ \frac{d}{dz}\widehat{p}_j(\omega,z) &= i\sqrt{\beta_j(\omega)}\Big(\widehat{a}_j(\omega,z)e^{i\beta_j(\omega)z} - \widehat{b}_j(\omega,z)e^{-i\beta_j(\omega)z}\Big), \\ \widehat{p}_\gamma(\omega,z) &= \frac{1}{\gamma^{1/4}}\Big(\widehat{a}_\gamma(\omega,z)e^{i\sqrt{\gamma}z} + \widehat{b}_\gamma(\omega,z)e^{-i\sqrt{\gamma}z}\Big), \\ \frac{d}{dz}\widehat{p}_\gamma(\omega,z) &= i\gamma^{1/4}\Big(\widehat{a}_\gamma(\omega,z)e^{i\sqrt{\gamma}z} - \widehat{b}_\gamma(\omega,z)e^{-i\sqrt{\gamma}z}\Big) \end{aligned}$$

$\forall j \in \{1, \dots, N(\omega)\}$ and almost every $\gamma \in (\xi, k^2(\omega))$. From (3.8), we obtain the coupled mode equation in $\mathcal{H}_\xi^\omega \times \mathcal{H}_\xi^\omega$ for the amplitudes $(\widehat{a}, \widehat{b})$,

$$\frac{d}{dz}\widehat{a}(\omega,z) = \sqrt{\epsilon}\,\mathbf{H}^{aa}(\omega,z)\big(\widehat{a}(\omega,z)\big) + \sqrt{\epsilon}\,\mathbf{H}^{ab}(\omega,z)\big(\widehat{b}(\omega,z)\big) \tag{3.9}$$

$$\frac{d}{dz}\widehat{b}(\omega,z) = \sqrt{\epsilon}\,\mathbf{H}^{ba}(\omega,z)\big(\widehat{a}(\omega,z)\big) + \sqrt{\epsilon}\,\mathbf{H}^{bb}(\omega,z)\big(\widehat{b}(\omega,z)\big), \tag{3.10}$$

where $\mathbf{H}^{aa}(\omega, z)$, $\mathbf{H}^{ab}(\omega, z)$, $\mathbf{H}^{ba}(\omega, z)$, and $\mathbf{H}^{bb}(\omega, z)$ are defined by (2.30)-(2.33) page 45. This system is complemented with the boundary conditions

$$\widehat{a}(\omega,0) = \widehat{a}_0^\epsilon(\omega) \quad \text{ and } \quad \widehat{b}\left(\omega, \frac{L}{\epsilon}\right) = 0$$

in $\mathcal{H}_\xi^\omega$, and where $\widehat{a}_0^\epsilon(\omega)$ is defined by (3.4) and (3.5). For $j \in \{1, \dots, N(\omega)\}$, $\widehat{a}_{j,0}(\omega_0)$ represents the initial amplitude of the jth propagating mode, and for $\gamma \in (\xi, k^2(\omega))$, $\widehat{a}_{\gamma,0}(\omega)$ represents the initial amplitude of the γth radiating mode at $z = 0$. Moreover, the second condition means that no wave is coming from the right homogeneous waveguide. According to Section 2.3.2 this system leads us to the local and global conservation relations

$$\begin{aligned} \|\widehat{a}(\omega,z)\|_{\mathcal{H}_\xi^\omega}^2 - \|\widehat{b}(\omega,z)\|_{\mathcal{H}_\xi^\omega}^2 &= \|\widehat{a}(\omega,0)\|_{\mathcal{H}_\xi^\omega}^2 - \|\widehat{b}(\omega,0)\|_{\mathcal{H}_\xi^\omega}^2 \quad \forall z \in [0, L/\epsilon]\,, \\ \|\widehat{a}\left(\omega, L/\epsilon\right)\|_{\mathcal{H}_\xi^\omega}^2 + \|\widehat{b}(\omega,0)\|_{\mathcal{H}_\xi^\omega}^2 &= \|\widehat{a}(\omega,0)\|_{\mathcal{H}_\xi^\omega}^2. \end{aligned}$$

3.2.2 Propagator and Forward Scattering Approximation

In this section we introduce the forward scattering approximation, which is widely used in the literature. In this approximation the coupling between forward- and backward-propagating modes is assumed to be negligible compared to the coupling between the forward-propagating modes. We refer to Section 2.3.4 for the physical explanation and to [30, 33] for justifications on the validity of this approximation.

Let us define the rescaled processes

$$\widehat{a}^{\epsilon}(\omega, z) = \widehat{a}_j\left(\omega, \frac{z}{\epsilon}\right) \quad \text{and} \quad \widehat{b}^{\epsilon}(\omega, z) = \widehat{b}\left(\omega, \frac{z}{\epsilon}\right) \text{ for } z \in [0, L].$$

These scalings correspond to the size of the random section $[0, L/\epsilon]$, and they satisfy the rescaled coupled mode equation

$$\begin{aligned} \frac{d}{dz}\widehat{a}^{\epsilon}(\omega, z) &= \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega, \frac{z}{\epsilon}\right)\left(\widehat{a}^{\epsilon}(\omega, z)\right) + \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{ab}\left(\omega, \frac{z}{\epsilon}\right)\left(\widehat{b}^{\epsilon}(\omega, z)\right) \\ \frac{d}{dz}\widehat{b}^{\epsilon}(\omega, z) &= \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{ba}\left(\omega, \frac{z}{\epsilon}\right)\left(\widehat{a}^{\epsilon}(\omega, z)\right) + \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{bb}\left(\omega, \frac{z}{\epsilon}\right)\left(\widehat{b}^{\epsilon}(\omega, z)\right), \end{aligned} \tag{3.11}$$

with the two-point boundary conditions

$$\widehat{a}^{\epsilon}(\omega, 0) = \widehat{a}_0^{\epsilon}(\omega) \quad \text{and} \quad \widehat{b}^{\epsilon}(\omega, L) = 0$$

in $\mathcal{H}_{\xi}^{\omega}$. We can rewrite (3.11) in a vector form as

$$\frac{d}{dz}X^{\epsilon}(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}\left(\omega, \frac{z}{\epsilon}\right)\left(X^{\epsilon}(\omega, z)\right).$$

where

$$X^{\epsilon}(\omega, z) = \begin{bmatrix} \widehat{a}^{\epsilon}(\omega, z) \\ \widehat{b}^{\epsilon}(\omega, z) \end{bmatrix} \quad \text{and} \quad \mathbf{H}(\omega, z) = \begin{bmatrix} \mathbf{H}^{aa}(\omega, z) & \mathbf{H}^{ab}(\omega, z) \\ \overline{\mathbf{H}^{ab}(\omega, z)} & \overline{\mathbf{H}^{aa}(\omega, z)} \end{bmatrix}.$$

Now, we introduce the propagator matrix $\mathbf{P}^{\epsilon}(\omega, z)$, that is, the solution of the differential equation

$$\frac{d}{dz}\mathbf{P}(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}\left(\omega, \frac{z}{\epsilon}\right)\mathbf{P}^{\epsilon}(\omega, z) \quad \text{with} \quad \mathbf{P}^{\epsilon}(\omega, 0) = Id.$$

Therefore, we get

$$\begin{bmatrix} \widehat{a}^{\epsilon}(\omega, z) \\ \widehat{b}^{\epsilon}(\omega, z) \end{bmatrix} = \mathbf{P}^{\epsilon}(\omega, z)\begin{bmatrix} \widehat{a}^{\epsilon}(\omega, 0) \\ \widehat{b}^{\epsilon}(\omega, 0) \end{bmatrix},$$

and by the symmetry of $\mathbf{H}(\omega, z)$ we have a particular form for the propagator, which is

$$\mathbf{P}^{\epsilon}(\omega, z) = \begin{bmatrix} \mathbf{P}_{\epsilon}^{a}(\omega, z) & \mathbf{P}_{\epsilon}^{b}(\omega, z) \\ \overline{\mathbf{P}_{\epsilon}^{b}(\omega, z)} & \overline{\mathbf{P}_{\epsilon}^{a}(\omega, z)} \end{bmatrix}.$$

Here, $\mathbf{P}_{\epsilon}^{a}(\omega, z)$ and $\mathbf{P}_{\epsilon}^{b}(\omega, z)$ are operators which represent, respectively, the coupling between right-going modes and the coupling between right-going and left-going modes.

In what follows, we shall consider the forward scattering approximation already discussed in Section 2.3.4, that is, we assume that the power spectral density of the process V, i.e. the Fourier transform of its z-autocorrelation function, possesses a cut-off wavenumber. In other words, we consider the case where

$$\int_0^{+\infty} \mathbb{E}[C_{jl}^{\omega}(0)C_{jl}^{\omega}(z)]\cos\left((\beta_l(\omega) + \beta_j(\omega))z\right)dz = 0 \quad \forall (j, l) \in \{1, \ldots, N(\omega)\}^2.$$

Under this approximation, we can neglect the left-going propagating modes in the asymptotic $\epsilon \to 0$. Consequently, we can consider the simplified coupled amplitude equation on $[0, L]$

$$\frac{d}{dz}\widehat{a}^\epsilon(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega, \frac{z}{\epsilon}\right)(\widehat{a}^\epsilon(\omega, z)) \quad \text{with} \quad \widehat{a}^\epsilon(\omega, 0) = \widehat{a}^\epsilon_0(\omega),$$

which implies the conservation relation

$$\|\widehat{a}^\epsilon(\omega, z)\|^2_{\mathcal{H}^\omega_\xi} = \|\widehat{a}(\omega, 0)\|^2_{\mathcal{H}^\omega_\xi} \quad \forall z \in [0, L].$$

Finally, we introduce the transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega, z)$, which is the solution of

$$\frac{d}{dz}\mathbf{T}^{\xi,\epsilon}(\omega, z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{aa}\left(\omega, \frac{z}{\epsilon}\right)\mathbf{T}^{\xi,\epsilon}(\omega, z) \quad \text{with} \quad \mathbf{T}^{\xi,\epsilon}(\omega, 0) = Id. \tag{3.12}$$

From this equation, one can check that the transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega, z)$ is unitary since $\mathbf{H}^{aa}$ is skew-Hermitian and

$$\forall z \geq 0, \quad \widehat{a}^\epsilon(\omega, z) = \mathbf{T}^{\xi,\epsilon}(\omega, z)(\widehat{a}^\epsilon_0(\omega)).$$

3.2.3 Limit Theorem

This section presents a simplified version of results introduced in Section 2.4.1. In [30] and [39], for the study of the pulse propagation and the time-reversal experiment the authors used the limit theorem stated in [48] since the number of propagating modes was fixed. However, in our configuration, in addition to the $N(\omega)$-discrete propagating modes we have a continuum of radiating modes on the interval $(\xi, k^2(\omega))$. The two following results are based on a diffusion-approximation result for the solution of an ordinary differential equation with random coefficients. This result is an extension of that stated in [48] to the case of processes with values in a Hilbert space.

Theorem 3.1 *$\forall y \in \mathcal{H}^\omega_\xi = \mathbb{C}^{N(\omega)} \times L^2(\xi, k^2(\omega))$, the family of processes $\left(\mathbf{T}^{\xi,\epsilon}(\omega, .)(y)\right)_{\epsilon\in(0,1)}$ converges in distribution, as $\epsilon \to 0$ on $\mathcal{C}([0, +\infty), \mathcal{H}^\omega_{\xi,w})$, to a limit denoted by $\mathbf{T}^\xi(\omega, .)(y)$. Here $\mathcal{H}^\omega_{\xi,w}$ stands for the Hilbert space $\mathcal{H}^\omega_\xi$ equipped with the weak topology. This limit is the unique diffusion process on $\mathcal{H}^\omega_\xi$, starting from y, associated to the infinitesimal generator*

$$\mathcal{L}^\omega_\xi = \mathcal{L}^\omega_1 + \mathcal{L}^\omega_{2,\xi},$$

where $\mathcal{L}^\omega_1$ and $\mathcal{L}^\omega_{2,\xi}$ are defined in Theorem 2.1 page 51.

We can get the following result in the asymptotic $\xi \to 0$.

Theorem 3.2 *$\forall y \in \mathcal{H}^\omega_0$, The family of processes $\left(\mathbf{T}^\xi(\omega, .)(y)\right)_{\xi\in(0,1)}$ converges in distribution, as $\xi \to 0$ on $\mathcal{C}([0, +\infty), (\mathcal{H}^\omega_0, \|.\|_{\mathcal{H}^\omega_0}))$, to a limit denoted by $\mathbf{T}^0(\omega, .)(y)$. Here $\mathcal{H}^\omega_0 = \mathbb{C}^{N(\omega)} \times L^2(0, k^2(\omega))$. This limit is the unique diffusion process on $\mathcal{H}^\omega_0$, starting from y, associated to the infinitesimal generator*

$$\mathcal{L}^\omega = \mathcal{L}^\omega_1 + \mathcal{L}^\omega_2,$$

where $\mathcal{L}^\omega_2$ is defined in Theorem 2.2 page 52.

The infinitesimal generator $\mathcal{L}^\omega$ is composed of two parts which induce different behaviors on the diffusion process and we recall their interpretation. The first operator $\mathcal{L}^\omega_1$ describes the coupling between the $N(\omega)$-discrete propagating modes. This part is of the form of the infinitesimal generator obtained in [25, 30], and for which the total energy is conserved. The second operator $\mathcal{L}^\omega_2$ describes the coupling between the propagating modes with the radiating modes. This part implies a mode-dependent and frequency-dependent attenuation on the $N(\omega)$-propagating modes already studied in Section 2.5.1, and a mode-dependent and frequency-dependent phase modulation. We use these results in the following section which concerns the study of the pulse propagation.

Moreover, let us remark that the convergence in Theorem 2.1 holds on $\mathcal{C}([0,L],(\mathcal{H}^{\omega}_{\xi},\|.\|_{\mathcal{H}^{\omega}_{\xi}}))$ for the $N(\omega)$-discrete propagating mode amplitudes.

3.3 Pulse Propagation in Random Waveguides

In this section we study the pulse propagation in the broadband case $q=1/2$. The analysis in the case of waveguides with bounded cross-section (see Figure 2.1 page 32) is carried out in [25, Chapter 20].

Using the modal decomposition, the transmitted field at time t and $z=L/\epsilon$ is given by

$$\begin{aligned}
p_{tr}\left(t,x,\frac{L}{\epsilon}\right) &= \frac{1}{2\pi}\int \widehat{p}(\omega,x,L/\epsilon)e^{-i\omega t}d\omega \\
&= \frac{1}{4\pi\sqrt{\epsilon}}\int \widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right)\left[\sum_{j=1}^{N(\omega)}\frac{1}{\sqrt{\beta_j(\omega)}}\mathbf{T}^{\xi,\epsilon}_j(\omega,L)(\tilde{a}(\omega))\phi_j(\omega,x)e^{i\beta_j(\omega)\frac{L}{\epsilon}}\right. \\
&\qquad \left.+\int_{\xi}^{k^2(\omega)}\frac{1}{\gamma^{1/4}}\mathbf{T}^{\xi,\epsilon}_{\gamma}(\omega,L)(\tilde{a}(\omega))\phi_{\gamma}(\omega,x)e^{i\sqrt{\gamma}\frac{L}{\epsilon}}d\gamma\right]e^{-i\omega t}d\omega,
\end{aligned}$$

where $\tilde{a}(\omega)$ is defined by (3.4) and (3.5). We observe the transmitted wave in a time window of order $1/\sqrt{\epsilon}$, which is of the order of the pulse width, and centered at time t_0/ϵ, which is of the order of the travel time for a distance of order $1/\epsilon$. Let us assume, throughout this chapter, that $\widehat{f}$ has a compact support included in $(-h_c,h_c)$, and then by making the change of variable $\omega=\omega_0+\sqrt{\epsilon}h$ we get

$$\begin{aligned}
p^{\xi,\epsilon}_{tr}(t_0,t,x,L) &= p_{tr}\left(\frac{t_0}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,\frac{L}{\epsilon}\right) \\
&= e^{-i\omega_0\left(\frac{t_0}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\right)}\frac{1}{4\pi}\int \widehat{f}(h)e^{-ih\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)} \\
&\times\left[\sum_{j=1}^{N(\omega_0)}\frac{1}{\sqrt{\beta_j(\omega_0+\sqrt{\epsilon}h)}}\mathbf{T}^{\xi,\epsilon}_j(\omega_0+\sqrt{\epsilon}h,L)(\tilde{a}(\omega_0+\sqrt{\epsilon}h))\phi_j(\omega_0+\sqrt{\epsilon}h,x)e^{i\beta_j(\omega_0+\sqrt{\epsilon}h)\frac{L}{\epsilon}}\right. \\
&\left.+\int_{\xi}^{k^2(\omega_0+\sqrt{\epsilon}h)}\frac{1}{\gamma^{1/4}}\mathbf{T}^{\xi,\epsilon}_{\gamma}(\omega_0+\sqrt{\epsilon}h,L)(\tilde{a}(\omega_0+\sqrt{\epsilon}h))\phi_{\gamma}(\omega_0+\sqrt{\epsilon}h,x)e^{i\sqrt{\gamma}\frac{L}{\epsilon}}d\gamma\right]dh.
\end{aligned}$$

Here, ϵ is small enough to have $N(\omega_0+\epsilon^q h)=N(\omega_0)$. In this section we consider the case $q=1/2$, but the same analysis can be carried out for any $q\in(0,1)$. In this case the pulse width, which is of order $1/\sqrt{\epsilon}$, is much smaller than the propagation distance. The transmitted wave can be decomposed as follows.

$$p^{\xi,\epsilon}_{tr}(t_0,t,x,L)e^{i\omega_0\left(\frac{t_0}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\right)} = p^{1,\xi,\epsilon}_{tr}(t_0,t,x,L)+p^{2,\xi,\epsilon}_{tr}(t_0,t,x,L),$$

with

$$p_{tr}^{1,\xi,\epsilon}(t_0,t,x,L) = \frac{1}{4\pi}\int \widehat{f}(h)e^{-ih\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)}\sum_{j=1}^{N(\omega_0)}\frac{1}{\sqrt{\beta_j(\omega_0+\sqrt{\epsilon}h)}}e^{i\beta_j(\omega_0+\sqrt{\epsilon}h)\frac{L}{\epsilon}}$$
$$\times \mathbf{T}_j^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h,L)(\tilde{a}(\omega_0+\sqrt{\epsilon}h))\phi_j(\omega_0+\sqrt{\epsilon}h,x)dh, \tag{3.13}$$
$$p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L) = \frac{1}{4\pi}\int \widehat{f}(h)e^{-ih\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)}\int_{\xi}^{k^2(\omega_0+\sqrt{\epsilon}h)}\frac{1}{\gamma^{1/4}}e^{i\sqrt{\gamma}\frac{L}{\epsilon}}$$
$$\times \mathbf{T}_\gamma^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h,L)(\tilde{a}(\omega_0+\sqrt{\epsilon}h))\phi_\gamma(\omega_0+\sqrt{\epsilon}h,x)d\gamma dh,$$

where $p_{tr}^{1,\xi,\epsilon}(t_0,t,x,L)$ is the projection of the transmitted wave over the propagating modes, and $p_{tr}^{2,\xi,\epsilon}(t_0,t,x,L)$ is the projection of the transmitted wave over the radiating modes.

3.3.1 Broadband Pulse in Homogeneous Waveguides

In this section, we study the transmitted wave through a homogeneous waveguide. In the homogeneous case, that is, when the transfer operator $\mathbf{T}^{\xi,\epsilon}(\omega,z) = Id$, we have

$$p_{tr,hom}^{1,\xi,\epsilon}(t_0,t,x,L) = \frac{1}{4\pi}\int \widehat{f}(h)e^{-ih\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)}\sum_{j=1}^{N(\omega_0)} e^{i\beta_j(\omega_0+\sqrt{\epsilon}h)\left(-L_S+\frac{L}{\epsilon}\right)}$$
$$\times \phi_j(\omega_0+\sqrt{\epsilon}h,x_0)\phi_j(\omega_0+\sqrt{\epsilon}h,x)dh,$$
$$p_{tr,hom}^{2,\xi,\epsilon}(t_0,t,x,L) = \frac{1}{4\pi}\int \widehat{f}(h)e^{-ih\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)}\int_{\xi}^{k^2(\omega_0+\sqrt{\epsilon}h)} e^{i\sqrt{\gamma}\left(-L_S+\frac{L}{\epsilon}\right)}$$
$$\times \phi_\gamma(\omega_0+\sqrt{\epsilon}h,x_0)\phi_\gamma(\omega_0+\sqrt{\epsilon}h,x)d\gamma dh.$$

First of all, let us remark that by integration by parts we get $p_{tr,hom}^{2,\xi,\epsilon}(t_0,t,x,L) = \mathcal{O}(\epsilon)$ uniformly in t, and uniformly in x on each bounded subset of $[0,+\infty)$. Consequently, the amplitude of the radiating part of the wave is very small. This amplitude is smaller than the error obtained when we make the approximation $\omega_0+\sqrt{\epsilon}h \to \omega_0$ for the propagating part $p_{tr,hom}^{1,\xi,\epsilon}$ of the transmitted wave, and smaller than the error produced by the diffusion approximation, which are of order $\mathcal{O}(\sqrt{\epsilon})$.

The propagating part $p_{tr,hom}^{1,\xi,\epsilon}$ can be treated in the same way as in [25, Chapter 20]. Let us remark that the coefficients $\beta_j(\omega)$ are smooth in ω. This fact can be shown by using the implicit function theorem on (2.9) page 36. Consequently, we can consider the following expansion

$$\beta_j(\omega_0+\sqrt{\epsilon}h) = \beta_j(\omega_0) + \sqrt{\epsilon}h\beta_j'(\omega_0) + \epsilon\frac{h^2}{2}\beta_j''(\omega_0) + \mathcal{O}(\epsilon^{3/2}). \tag{3.14}$$

Therefore,

$$p_{tr,hom}^{1,\xi,\epsilon}(t_0,t,x,L) = \frac{1}{2}\sum_{j=1}^{N(\omega_0)} e^{i\beta_j(\omega_0)\left(-L_S+\frac{L}{\epsilon}\right)}\phi_j(\omega_0,x_0)\phi_j(\omega_0,x)$$
$$\times \frac{1}{2\pi}\int \widehat{f}(h)e^{ih\left(\frac{\beta_j'(\omega_0)L-t_0}{\sqrt{\epsilon}}-t\right)}e^{i\beta_j''(\omega_0)L\frac{h^2}{2}}dh$$
$$+\mathcal{O}(\sqrt{\epsilon}),$$

and because of the fast phase $e^{i\frac{\beta_j'(\omega_0)L-t_0}{\sqrt{\epsilon}}h}$, we have for any $\xi>0$

$$\lim_{\epsilon\to 0} e^{i\omega_0\left(\frac{t_j}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\right)}e^{-i\beta_j(\omega_0)\left(-L_S+\frac{L}{\epsilon}\right)}p_{tr,hom}^{\xi,\epsilon}(t_j,t,x,L) = p_{tr,hom,j}^{1,\xi,\epsilon}(t,x,L).$$

Here,

$$p^{1,\xi,\epsilon}_{tr,hom,j}(t,x,L) = \frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)K^{\omega_0}_{j,L} * f(t),$$

with

$$t_j = \beta'_j(\omega_0)L \quad \text{and} \quad \widehat{K^{\omega_0}_{j,L}}(\omega) = e^{i\beta''_j(\omega_0)L\frac{\omega^2}{2}}.$$

Moreover, $\forall t_0 \neq t_j$

$$\lim_{\epsilon\to 0} p^{\xi,\epsilon}_{tr,hom}(t_0,t,x,L) = 0.$$

As a result, the radiating modes play no role on the shape of the transmitted wave for $\epsilon \ll 1$. Consequently, in a homogeneous waveguide, we can observe for $\epsilon \ll 1$ a train of separated waves with arrival times t_j, $j \in \{1,\dots,N(\omega_0)\}$. The wave with arrival time t_j corresponds to the jth-propagating mode, travels with the group velocity $1/\beta'_j(\omega_0)$, and is dispersed by the convolution kernel $K^{\omega_0}_{j,L}(t)$. Moreover, the total energy of the transmitted wave is given by

$$\begin{aligned}\sum_{j=1}^{N(\omega_0)} \iint \big|p_{tr,hom,j}(t,x,L)\big|^2 dxdt &= \int |K^{\omega_0}_{j,L} * f(t)|^2 dt \cdot \frac{1}{4}\sum_{j=1}^{N(\omega_0)} |\phi_j(\omega_0,x_0)|^2 \\ &= \int |f(t)|^2 dt \cdot \frac{1}{4}\sum_{j=1}^{N(\omega_0)} |\phi_j(\omega_0,x_0)|^2,\end{aligned}$$

which is not equal to the total energy of the incident pulse. In fact, the total energy of the incident pulse in the asymptotic $\epsilon \to 0$ is given by

$$\iint |p^0_{inc}(t,x,0)|^2 dxdt = \int |f(t)|^2 dt \cdot \frac{1}{4}\left[\sum_{j=1}^{N(\omega_0)} |\phi_j(\omega_0,x_0)|^2 + \int_{\xi}^{k^2(\omega_0)} |\phi_\gamma(\omega_0,x_0)|^2\right],$$

where $p^0_{inc}(t,x,0) = \lim_\epsilon p^{\xi,\epsilon}_{inc}(t/\sqrt{\epsilon},x,0)$, and therefore, the missing energy was converted into radiative waves with small amplitudes. As the convergence is obtained in the space of continuous functions, equipped with the supremum norm over the compact sets, this energy cannot be detected.

3.3.2 Broadband Pulse in Random Waveguides

Now, we are interested in the transmitted wave through a randomly perturbed waveguide. First, let us investigate the radiating part $p^{2,\xi,\epsilon}_{tr}$ of the transmitted wave. Using the perturbed-test-function method we get

$$\mathbb{E}\big[p^{2,\xi,\epsilon}_{tr}(t_0,t,x,L)\big] = p^{2,\xi,\epsilon}_{tr,hom}(t_0,t,x,L) + \mathcal{O}(\sqrt{\epsilon}).$$

Then, $p^{2,\xi,\epsilon}_{tr,hom}(t_0,t,x,L)$ is an approximation of the mean transmitted wave $\mathbb{E}\big[p^{2,\xi,\epsilon}_{tr}(t_0,t,x,L)\big]$, but we know that $p^{2,\xi,\epsilon}_{tr,hom}(t_0,t,x,L) = \mathcal{O}(\epsilon)$. Consequently, the amplitude of the radiating part of the transmitted wave is very small and it does not play any role in the pulse propagation.

Now, let us consider

$$e^{-i\beta_j(\omega_0)\left(-L_S+\frac{L}{\epsilon}\right)} p^{1,\xi,\epsilon}_{tr}(t_j,t,x,L) = p^{\xi,\epsilon}_{tr,j}(t,x,L),$$

which is the transmitted wave observed in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_j/ϵ, which is of the order the travel time for a distance of order $1/\epsilon$. Let us note that $t_j = \beta'_j(\omega_0)L$ is the arrival time for the jth modal wave in the homogeneous case.

According to the analysis developed in [17] and [25, Chapter 20] for instance, we can state the following proposition.

Proposition 3.1 *The jth-transmitted wave, observed around time t_j, $p^{\xi,\epsilon}_{tr,j}(t,x,L)$ converges in distribution as $\epsilon \to 0$ and as a continuous process in the three variables (t,x,L) to*

$$p^{\xi}_{tr,j}(t,x,L) = \frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)e^{iW^j_L}\tilde{K}^{\omega_0,\xi}_{j,L} * f(t),$$

where

$$\widehat{\tilde{K}^{\omega_0,\xi}_{j,L}}(\omega) = e^{\frac{1}{2}\left(\Gamma^c_{jj}(\omega_0)+i\Gamma^s_{jj}(\omega_0)-\Lambda^{c,\xi}_j(\omega_0)-i\Lambda^{s,\xi}_j(\omega_0)\right)L+i\beta''_j(\omega_0)\omega^2\frac{L}{2}},$$

and $(W^j)_j$ is a $N(\omega_0)$-dimensional Brownian motion with covariance matrix $\Gamma^1(\omega_0)$. Moreover, $p^{\xi}_{tr,j}(t,x,L)$ converges almost surely and uniformly in (t,x,L) as $\xi \to 0$ to

$$p_{tr,j}(t,x,L) = \frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)e^{iW^j_L}\tilde{K}^{\omega_0}_{j,L} * f(t),$$

where

$$\widehat{\tilde{K}^{\omega_0}_{j,L}}(\omega) = e^{\frac{1}{2}\left(\Gamma^c_{jj}(\omega_0)+i\Gamma^s_{jj}(\omega_0)-\Lambda^{c}_j(\omega_0)-i\Lambda^{s}_j(\omega_0)\right)L+i\beta''_j(\omega_0)\omega^2\frac{L}{2}}.$$

Here, $\Gamma^c_{jj}(\omega_0)$, $\Gamma^s_{jj}(\omega_0)$, $\Lambda^{c,\xi}(\omega_0)$, $\Lambda^{s,\xi}_j(\omega_0)$, $\Lambda^c(\omega_0)$, and $\Lambda^s_j(\omega_0)$ are defined in Section 2.4.1.

As in [25, Chpater 20], it possible to observe coherent transmitted waves only around times t_j, $j \in \{1,\dots,N(\omega_0)\}$. The transmitted wave is composed of a sequence of transmitted waves which are separated from each other. Each pulse corresponds to a single mode. $\forall j \in \{1,\dots,N(\omega_0)\}$, the jth modal wave travels with the group velocity $1/\beta'_j(\omega_0)$. This result means that we have stabilization of the transmitted wave up to a random phase; that is one can observe deterministic intensity around the arrival times $t_0 = t_j$ $\forall j \in \{1,\dots,N(\omega_0)\}$. The random phase is characterized in terms of a Brownian motion. The pulse intensities decrease exponentially with the propagation distance and the pulse spreads dispersively through $\tilde{K}^{\omega_0}_{j,L}$. Moreover, there is no diffusion for the deterministic pulse profile.

Consequently, the coherent waves are given by

$$\mathbb{E}[p_{tr,j}(t,x,L)] = \frac{1}{2}\phi_j(\omega_0,x)\phi_j(\omega_0,x_0)e^{-\Gamma^1_{jj}(\omega_0)\frac{L}{2}}\tilde{K}^{\omega_0}_{j,L} * f(t),$$

where $e^{-\Gamma^1_{jj}(\omega_0)\frac{L}{2}}$ is given by the averaging of the random phase. Moreover, the intensity of each coherent wave observed around times t_j is deterministic and given by

$$\mathbb{E}\Big[\iint |p_{tr,j}(t,x,L)|^2 dxdt\Big] = \int |f(t)|^2 dt \cdot \frac{1}{4}e^{\left(\Gamma^c_{jj}(\omega_0)-\Lambda^c_j(\omega_0)\right)L}|\phi_j(\omega_0,x_0)|^2,$$

where the damping term $e^{\left(\Gamma^c_{jj}(\omega_0)-\Lambda^c_j(\omega_0)\right)L}$ is responsible for a mode-dependent attenuation. We refer to Section 2.4.2 for a discussion about the nonnegativity of $\Gamma^c_{jj}(\omega_0)$ and $\Lambda^c_j(\omega_0)$.

3.3.3 Incoherent Fluctuations in the Broadband Case

We have seen that one can observe coherent waves with deterministic intensity only around the times $t_j = \beta'_j(\omega_0)L$, $j \in \{1,\dots,N(\omega_0)\}$. In this section we study the transmitted wave at time $t_0 \neq t_j$ $\forall j \in \{1,\dots,N(\omega_0)\}$. This analysis has already been carried out in [25, Chapter 20] in the case of waveguides with bounded cross-section. We observe the mean transmitted intensity in a time window of order $1/\sqrt{\epsilon}$, which is of the order of the pulse width, and centered at time t_0/ϵ, which is of the order of the travel time for a distance of order $1/\epsilon$. The mean transmitted intensity is given by

$$\begin{aligned}\left|p^{\xi,\epsilon}_{tr}(t_0,t,x,L)\right|^2 =& \frac{1}{16\pi^2}\iint \widehat{f}(h)\overline{\widehat{f}(h')}e^{-i(h-h')\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)} \\ &\times \left\langle \mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h)(\tilde{a}(\omega_0+\sqrt{\epsilon}h)),\lambda^{\epsilon}_x(\omega_0+\sqrt{\epsilon}h)\right\rangle_{\mathcal{H}^{\omega_0+\sqrt{\epsilon}h}_{\xi}} \\ &\times \overline{\left\langle \mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h')(\tilde{a}(\omega_0+\sqrt{\epsilon}h')),\lambda^{\epsilon}_x(\omega_0+\sqrt{\epsilon}h')\right\rangle_{\mathcal{H}^{\omega_0+\sqrt{\epsilon}h'}_{\xi}}}dhdh',\end{aligned}$$

where

$$\lambda^{\epsilon}_{x,j}(\omega) = \frac{1}{\sqrt{\beta_j(\omega)}}\phi_j(\omega,x)e^{-i\beta_j(\omega)\frac{L}{\epsilon}} \quad \text{and} \quad \lambda^{\epsilon}_{x,\gamma}(\omega) = \frac{1}{\gamma^{1/4}}\phi_\gamma(\omega,x)e^{-i\sqrt{\gamma}\frac{L}{\epsilon}}$$

in $\mathcal{H}^{\omega}_{\xi}$. The expansion (3.14) gives us terms of the form $e^{\frac{i}{\sqrt{\epsilon}}(\beta'_j(\omega_0)L-t_0)(h-h')}$, and then we make the change of variable $h' = h - \sqrt{\epsilon}s$. These terms mean that the coherent field can be observed only around the times t_j, $j \in \{1,\dots,N(\omega)\}$. Therefore,

$$\begin{aligned}\left|p^{\xi,\epsilon}_{tr}(t_0,t,x,L)\right|^2 = &\frac{\sqrt{\epsilon}}{16\pi^2}\iint \widehat{f}(h)\overline{\widehat{f}(h-\sqrt{\epsilon}s)}e^{-i\sqrt{\epsilon}s\left(t+\frac{t_0}{\sqrt{\epsilon}}\right)} \\ &\times \langle \mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h)(\tilde{a}(\omega_0+\sqrt{\epsilon}h)), \lambda^{\epsilon}_x(\omega_0+\sqrt{\epsilon}h)\rangle_{\mathcal{H}^{\omega_0+\sqrt{\epsilon}h}_{\xi}} \\ &\times \overline{\langle \mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h-\epsilon s)(\tilde{a}(\omega_0+\sqrt{\epsilon}h-\epsilon s)), \lambda^{\epsilon}_x(\omega_0+\sqrt{\epsilon}h-\epsilon s)\rangle}_{\mathcal{H}^{\omega_0+\sqrt{\epsilon}h-\epsilon s}_{\xi}} dh ds.\end{aligned}$$

One can remark that in the asymptotic $\epsilon \to 0$ the mean transmitted intensity does not depend on t anymore, which means that the transmitted intensity becomes locally stationary.

Following [25, Chapter 20], in order to analyze the incoherent fluctuations we need to study the statistics, as $\epsilon \to 0$ and $\xi \to 0$, of the the product of two transfer operators $\overline{\mathbf{T}^{\xi,\epsilon}(\omega+\epsilon s)} \otimes \mathbf{T}^{\xi,\epsilon}(\omega)$ at two nearby frequencies. In the following proposition we summarize the results that we need in this section. Following [50], it is possible to show a functional limit theorem for the process $\mathbf{V}^{\xi,\epsilon}(\omega,s) = \overline{\mathbf{T}^{\xi,\epsilon}(\omega+\epsilon s)} \otimes \mathbf{T}^{\xi,\epsilon}(\omega)$ with values in a space of distributions. $\mathbf{V}^{\xi,\epsilon}$ represents the product of two transfer operators at two nearby frequencies, and where $\otimes$ is defined by $\forall(\lambda,\mu) \in \mathcal{H}^{\omega}_{\xi} \times \mathcal{H}^{\omega+\epsilon s}_{\xi}$,

$$(\lambda \otimes \mu)_{rs} = \lambda_r \mu_s$$

for $(r,s) \in \big(\{1,\dots,N(\omega)\} \cup (\xi, k^2(\omega))\big) \times \big(\{1,\dots,N(\omega+\epsilon s)\} \cup (\xi, k^2(\omega+\epsilon s))\big)$, and

$$\mathcal{H}^{\omega}_{\xi} \otimes \mathcal{H}^{\omega+\epsilon s}_{\xi} = \left\{\lambda \otimes \mu, \quad (\lambda,\mu) \in \mathcal{H}^{\omega}_{\xi} \times \mathcal{H}^{\omega+\epsilon s}_{\xi}\right\}.$$

Proposition 3.2 *$\forall(y^1,y^2) \in \mathcal{H}^{\omega}_{\xi} \times \mathcal{H}^{\omega+h_c}_{\xi}$ and $\forall\lambda \in \mathcal{H}^{\omega}_{\xi} \otimes \mathcal{H}^{\omega+h_c}_{\xi}$, the autocorrelation function of the transfer operator at two nearby frequencies as $\epsilon \to 0$ is given by*

$$\begin{aligned}\lim_{\epsilon\to 0} &\langle \mathbf{V}^{\xi,\epsilon}(\omega,s,L)(y^1,y^2), \lambda\rangle_{\mathcal{H}^{\omega}_{\xi}\otimes\mathcal{H}^{\omega+\epsilon s}_{\xi}} \\ &= \sum_{j,l=1}^{N(\omega)} \widehat{\mathcal{W}^{\xi,l}_j}(\omega,s,L)e^{-is\beta'_j(\omega)L}\overline{y^1_l}y^2_l\overline{\lambda_{jj}} + \sum_{\substack{j,m=1\\ j\neq m}}^{N(\omega)} e^{Q^{\xi}_{jm}(\omega)L}\overline{y^1_j}y^2_m\overline{\lambda_{jm}} \\ &+ \sum_{j=1}^{N(\omega)}\int_{\xi}^{k^2(\omega)} e^{\frac{1}{2}(\Gamma^c_{jj}(\omega)-\Gamma^1_{jj}(\omega)-\Lambda^{c,\xi}_j(\omega))L-\frac{i}{2}(\Gamma^s_{jj}(\omega)-\Lambda^{s,\xi}_j(\omega))L}\overline{y^1_j}y^2_{\gamma'}\overline{\lambda_{j\gamma'}}d\gamma' \\ &+ \int_{\xi}^{k^2(\omega)}\sum_{m=1}^{N(\omega)} e^{\frac{1}{2}(\Gamma^c_{mm}(\omega)-\Gamma^1_{mm}(\omega)-\Lambda^{c,\xi}_m(\omega))L+\frac{i}{2}(\Gamma^s_{mm}(\omega)-\Lambda^{s,\xi}_m(\omega))L}\overline{y^1_\gamma}y^2_m\overline{\lambda_{\gamma m}}d\gamma \\ &+ \int_{\xi}^{k^2(\omega)}\int_{\xi}^{k^2(\omega)} \overline{y^1_\gamma}y^2_{\gamma'}\overline{\lambda_{\gamma\gamma'}}d\gamma d\gamma'.\end{aligned}$$

Here,

$$\begin{aligned}Q^{\xi}_{jm}(\omega) =& \frac{1}{2}\left[\Gamma^c_{jj}(\omega)+\Gamma^c_{mm}(\omega)-(\Gamma^1_{jj}(\omega)+\Gamma^1_{mm}(\omega)-2\Gamma^1_{jm}(\omega))-(\Lambda^{c,\xi}_j(\omega)+\Lambda^{c,\xi}_m(\omega))\right] \\ &+\frac{i}{2}\left[\Gamma^s_{mm}(\omega)-\Gamma^s_{jj}(\omega)-(\Lambda^{s,\xi}_m(\omega)-\Lambda^{s,\xi}_j(\omega))\right],\end{aligned}$$

and $\widehat{\mathcal{W}_j^{\xi,l}}(\omega,s,L)$ *stands for the Fourier transform of the distribution* $\mathcal{W}^{\xi,l}(\omega,.,z)$ *which satisfies the system of transport equations*

$$\frac{\partial}{\partial z}\mathcal{W}_j^{\xi,l}(\omega,r,z)+\beta_j'(\omega)\frac{\partial}{\partial r}\mathcal{W}_j^{\xi,l}(\omega,r,z) = -\Lambda_j^{c,\xi}(\omega)\mathcal{W}_j^{\xi,l}(\omega,r,z)+\sum_{n=1}^{N(\omega)}\Gamma_{nj}^c(\omega)\big(\mathcal{W}_n^{\xi,l}(\omega,r,z)-\mathcal{W}_j^{\xi,l}(\omega,r,z)\big),$$

with initial conditions $\mathcal{W}_j^{\xi,l}(\omega,.,0)=\delta(.)\delta_{jl}$. *Here,* $\Gamma_{jj}^c(\omega_0)$, $\Gamma_{jj}^s(\omega_0)$, $\Lambda^{c,\xi}(\omega_0)$, $\Lambda_j^{s,\xi}(\omega_0)$, $\Lambda^c(\omega_0)$, *and* $\Lambda_j^s(\omega_0)$ *are defined in Section 2.4.1.*

Let us note that the matrix $Q^\xi(\omega)$ has coefficients with negative real part and $\mathcal{W}_j^{\xi,l}(\omega,r,z)$ are measures. The system of transport equations, for a waveguide with bounded cross-section and without radiation losses, has been already obtained in [30]. In our context, the system of coupled transport equations takes into account the radiative loss. The system of transport equations describes the coupling between the $N(\omega)$-propagating modes. These equations are a generalization of the coupled power equations affected by the modal dispersion. In other words it is a space and time version of the coupled power equations with transport velocity equal to the group velocity $1/\beta_j'(\omega)$ for the jth mode.

Following Section 20.6.2 in [25, Chapter 20], we introduce a probabilistic representation of the system of coupled transport equations. Let $(Y_t^{N(\omega)})_{t\geq 0}$ be a jump Markov process with state space $\{1,\dots,N(\omega)\}$ and intensity matrix $\Gamma^c(\omega)$. Then, $\forall\varphi\in\mathcal{S}$, where $\mathcal{S}$ is the set of infinitely differentiable functions which are rapidly decreasing at infinity, we have the probabilistic representation

$$\mathcal{W}_j^{\xi,l}(\omega,L)(\varphi)=\mathbb{E}\Big[e^{-\int_0^L\Lambda^{c,\xi}_{Y_v^{N(\omega)}}(\omega)dv}\varphi\Big(\int_0^L\beta'_{Y_v^{N(\omega)}}(\omega)dv\Big)\mathbf{1}_{(Y_L^{N(\omega)}=l)}\big|Y_0^{N(\omega)}=j\Big].$$

Consequently, $\lim_{\xi\to 0}\mathcal{W}_j^{\xi,l}(\omega,L)(\varphi)=\tilde{\mathcal{W}}^{\varphi,l}(\omega,j,0,L)$, where

$$\tilde{\mathcal{W}}^{\varphi,l}(\omega,j,r,L)=\mathbb{E}\Big[e^{-\int_0^L\Lambda^{c}_{Y_v^{N(\omega)}}(\omega)dv}\varphi\Big(r+\int_0^L\beta'_{Y_v^{N(\omega)}}(\omega)dv\Big)\mathbf{1}_{(Y_L^{N(\omega)}=l)}\big|Y_0^{N(\omega)}=j\Big]$$

and statisfies the system of transport equations

$$\frac{\partial}{\partial z}\tilde{\mathcal{W}}_j^{l}(\omega,r,z)=-\Lambda_j^{c}(\omega)\tilde{\mathcal{W}}_j^{l}(\omega,r,z)+\beta_j'(\omega)\frac{\partial}{\partial r}\tilde{\mathcal{W}}_j^{l}(\omega,r,z)+\sum_{n=1}^{N(\omega)}\Gamma_{nj}^c(\omega)\big(\tilde{\mathcal{W}}_n^{l}(\omega,r,z)-\tilde{\mathcal{W}}_j^{l}(\omega,r,z)\big),$$

with initial conditions $\tilde{\mathcal{W}}_j^{l}(\omega,r,0)=\varphi(r)\delta_{jl}$. Decomposing with respect to the first jump of $(Y_t^{N(\omega)})_{t\geq 0}$, we have

$$\mathcal{W}_j^{l}(\omega,r,L)=\delta_{jl}e^{(\Gamma_{jj}^c(\omega)-\Lambda_j^c(\omega))L}\delta(r-\beta_j'(\omega)L)+\mathcal{W}_{j,c}^{l}(\omega,r,L)dr.$$

Consequently, if $j\neq l$, $\mathcal{W}_j^l(\omega,.,L)$ has a density with respect to the Lebesgue measure, and $\mathcal{W}_l^l(\omega,.,L)$ is a sum of a Dirac mass at $\beta_j'(\omega)L$ and a density with respect to the Lebesgue measure. As a result, the following proposition describes the asymptotic mean transmitted intensity.

Proposition 3.3

$$\lim_{\xi\to 0}\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\mathbb{E}\big[\big|p_{tr}^{\xi,\epsilon}(t_0,t,x,L)\big|^2\big]=\frac{1}{2\pi}\int|\widehat{f}(h)|^2dh\cdot p_{tr}^{inc}(t_0,x,L),$$

where the limits hold in $\mathcal{S}'$ with respect to t_0, and

$$\begin{aligned}p_{tr}^{inc}(t_0,x,L)=&\frac{1}{4}\sum_{j=1}^{N(\omega_0)}\phi_j^2(\omega_0,x)\phi_j^2(\omega_0,x_0)e^{(\Gamma_{jj}^c(\omega)-\Lambda_j^c(\omega))L}\delta(t_0-\beta_j'(\omega)L)\\&+\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}\frac{\beta_l(\omega_0)}{\beta_j(\omega_0)}\phi_j^2(\omega_0,x)\phi_l^2(\omega_0,x_0)\mathcal{W}_{j,c}^l(\omega_0,t_0,L).\end{aligned}$$

This result means that the transmitted wave has also an incoherent part whose typical amplitude is of order $\epsilon^{1/4}$.

Proof We have

$$\begin{aligned}\int\big|p_{tr}^{\xi,\epsilon}(t_0,t,x,L)\big|^2\varphi(t_0)dt_0=&\frac{\sqrt{\epsilon}}{8\pi}\iint\widehat{f}(h)\overline{\widehat{f}(h-\sqrt{\epsilon}s)}\widehat{\varphi}(s)\\&\times\big\langle\mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h)(\tilde{a}(\omega_0+\sqrt{\epsilon}h)),\lambda_x^\epsilon(\omega_0+\sqrt{\epsilon}h)\big\rangle_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}}\\&\times\overline{\big\langle\mathbf{T}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h-\epsilon s)(\tilde{a}(\omega_0+\sqrt{\epsilon}h-\epsilon s)),\lambda_x^\epsilon(\omega_0+\sqrt{\epsilon}h-\epsilon s)\big\rangle}_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h-\epsilon s}}dhds\\&+\mathcal{O}(\sqrt{\epsilon}).\end{aligned}$$

Using the perturbed-test-function method and after a computation, we get

$$\begin{aligned}&\iint\widehat{f}(h)\overline{\widehat{f}(h-\sqrt{\epsilon}s)}\widehat{\varphi}(s)\\&\quad\times\mathbb{E}\big[\big\langle\mathbf{V}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h,s,L)\big(\tilde{a}(\omega_0+\sqrt{\epsilon}h),\tilde{a}(\omega_0+\sqrt{\epsilon}h-\epsilon s)\big),\\&\qquad\qquad\lambda_x^\epsilon(\omega_0+\sqrt{\epsilon}h)\otimes\overline{\lambda_x^\epsilon(\omega_0+\sqrt{\epsilon}h-\epsilon s)}\big\rangle_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}\otimes\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h-\epsilon s}}\big]dhds\\&\quad=\iint\widehat{f}(h)\overline{\widehat{f}(h-\sqrt{\epsilon}s)}\widehat{\varphi}(s)\mathbb{E}\big[\big\langle\mathbf{V}^{\xi}(\omega_0,s,L)\big(\tilde{a}(\omega_0),\tilde{a}(\omega_0)\big),\tilde{\lambda}_x^\epsilon(h,s,\omega_0)\big\rangle_{\mathcal{H}_\xi^{\omega_0}\otimes\mathcal{H}_\xi^{\omega_0}}\big]dhds\\&\quad+\mathcal{O}(\sqrt{\epsilon}),\end{aligned}$$

with

$$\begin{aligned}&\mathbb{E}\Big[\big\langle\mathbf{V}^{\xi}(\omega_0,s,L)\big(\tilde{a}(\omega_0),\tilde{a}(\omega_0)\big),\tilde{\lambda}_x^\epsilon(h,s,\omega_0)\big\rangle_{\mathcal{H}_\xi^{\omega_0}\otimes\mathcal{H}_\xi^{\omega_0}}\Big]\\&=\sum_{j,m=1}^{N(\omega_0)}\mathbb{E}\Big[\mathbf{V}_{jm}^{\xi}(\omega_0,s,L)\big(\tilde{a}(\omega_0),\tilde{a}(\omega_0)\big)\Big]\overline{\tilde{\lambda}_{x,mj}^\epsilon(h,s,\omega_0)}\\&+\sum_{j=1}^{N(\omega_0)}\int_\xi^{k^2(\omega_0)}e^{\frac{1}{2}(\Gamma_{jj}^c(\omega_0)-\Gamma_{jj}^1(\omega_0)-\Lambda_j^{c,\xi}(\omega_0))L-\frac{i}{2}(\Gamma_{jj}^s(\omega_0)-\Lambda_j^{s,\xi}(\omega_0))L}\phi_j(\omega_0,x_0)\phi_j(\omega_0,x)\\&\qquad\times\phi_\gamma(\omega_0,x_0)\phi_\gamma(\omega_0,x)e^{i(\sqrt{\gamma}-\beta_j(\omega_0))\left(-L_S+\frac{L}{\epsilon}\right)}e^{-ih\beta_j'(\omega_0)\frac{L}{\sqrt{\epsilon}}}e^{-i\frac{h^2}{2}\beta_j''(\omega_0)L}d\gamma\\&+\int_\xi^{k^2(\omega_0)}\sum_{m=1}^{N(\omega_0)}e^{\frac{1}{2}(\Gamma_{mm}^c(\omega_0)-\Gamma_{mm}^1(\omega_0)-\Lambda_m^{c,\xi}(\omega_0))L+\frac{i}{2}(\Gamma_{mm}^s(\omega_0)-\Lambda_m^{s,\xi}(\omega_0))L}\phi_m(\omega_0,x_0)\phi_m(\omega_0,x)\\&\qquad\times\phi_{\gamma'}(\omega_0,x_0)\phi_{\gamma'}(\omega_0,x)e^{i(\beta_m(\omega_0)-\sqrt{\gamma'})\left(-L_S+\frac{L}{\epsilon}\right)}e^{ih\beta_m'(\omega_0)\frac{L}{\sqrt{\epsilon}}}e^{i\frac{h^2}{2}\beta_m''(\omega_0)L}d\gamma'\\&+\int_\xi^{k^2(\omega_0)}\int_\xi^{k^2(\omega_0)}\phi_{\gamma'}(\omega_0,x_0)\phi_{\gamma'}(\omega_0,x)\phi_\gamma(\omega_0,x_0)\phi_\gamma(\omega_0,x)e^{i(\sqrt{\gamma}-\sqrt{\gamma'})\left(-L_S+\frac{L}{\epsilon}\right)}d\gamma d\gamma',\end{aligned}$$

and

$$\tilde{\lambda}^{\epsilon}_{x,mj}(h,s,\omega) = \frac{\phi_j(\omega_0,x)\phi_m(\omega_0,x)}{\sqrt{\beta_j(\omega_0)\beta_m(\omega_0)}} e^{i(\beta_m(\omega_0)-\beta_j(\omega_0))\frac{L}{\epsilon}} e^{i(\beta'_m(\omega_0)-\beta'_j(\omega_0))L\frac{h}{\sqrt{\epsilon}}} \times e^{i(\beta''_m(\omega_0)-\beta''_j(\omega_0))L\frac{h^2}{2}} e^{is\beta'_m(\omega_0)L}.$$

Let us remark that the three last terms on the right side are $\mathcal{O}(\epsilon)$ because they have a term of the form $\int_{\xi}^{k^2(\omega)} \phi_\gamma(\omega,x_0)\phi_\gamma(\omega,x)e^{i\sqrt{\gamma}\frac{L}{\epsilon}} = \mathcal{O}(\epsilon)$. Here,

$$\mathbb{E}\Big[\mathbf{V}^{\xi}_{jm}(\omega_0,s,L)(\tilde{a}(\omega_0),\tilde{a}(\omega_0))\Big] = e^{Q_{jm}(\omega_0)L}\overline{\tilde{a}_j(\omega_0)}\tilde{a}_m(\omega_0) \quad \text{if } j\neq m,$$

and

$$\mathbb{E}\Big[\mathbf{V}^{\xi}_{jm}(\omega_0,s,L)(\tilde{a}(\omega_0),\tilde{a}(\omega_0))\Big] = \sum_{l=1}^{N(\omega_0)} \widehat{\mathcal{W}^{\xi,l}_j}(\omega_0,s,L)e^{-is\beta'_j(\omega_0)L}|\tilde{a}_l(\omega_0)|^2 \quad \text{if } m=j.$$

However, because of the fast phase $e^{i(\beta'_m(\omega_0)-\beta'_j(\omega_0))L\frac{h}{\sqrt{\epsilon}}}$, in the asymptotic $\epsilon\to 0$ we have only terms which correspond to the case $m=j$. Consequently,

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\int \mathbb{E}\big[|p^{\xi,\epsilon}_{tr}(t_0,t,x,L)|^2\big]\varphi(t_0)dt_0 = \frac{1}{8\pi}\int|\widehat{f}(h)|^2dh\cdot\sum_{j,l=1}^{N(\omega_0)}\frac{\beta_l(\omega_0)}{\beta_j(\omega_0)}\phi_j^2(\omega_0,x)\phi_l^2(\omega_0,x_0) \times \mathcal{W}^{\xi,l}_j(\omega_0,L)(\varphi).$$

■

Now, we study $p^{inc}_{tr}(t_0,x,L)$ in the asymptotic $L\gg 1$. In order to do that let us rescale the propagation distance using a small parameter $\tau\ll 1$, that is we consider L/τ. Then, we have

$$\lim_{\xi\to 0}\lim_{\epsilon\to 0}\frac{1}{\tau\sqrt{\epsilon}}\int\mathbb{E}\Big[\Big|p^{\xi,\epsilon}_{tr}\Big(\frac{t_0}{\tau},t,x,\frac{L}{\tau}\Big)\Big|^2\Big]\varphi(t_0)dt_0 = \frac{1}{2\pi}\int|\widehat{f}(h)|^2dh\cdot p^{inc,\tau}_{tr}(x,L)(\varphi)$$

where

$$p^{inc,\tau}_{tr}(x,L)(\varphi) = \frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}\frac{\beta_l(\omega_0)}{\beta_j(\omega_0)}\phi_j^2(\omega_0,x)\phi_l^2(\omega_0,x_0) \times\mathbb{E}\Big[e^{-\int_0^{L/\tau}\Lambda^c_{Y^{N(\omega)}_v}(\omega)dv}\varphi\Big(\tau\int_0^{L/\tau}\beta'_{Y^{N(\omega)}_v}(\omega)dv\Big)\mathbf{1}_{(Y^{N(\omega)}_L=l)}|Y^{N(\omega)}_0=j\Big].$$

Consequently, according to Theorem 2.3 page 54, $\lim_{\tau\to 0}p^{inc,\tau}_{tr}(t_0,x,L)=0$, because of the radiation losses. Then, we rescale the modal radiative damping rates, and let us consider $\tau\Lambda^c(\omega_0)$.

Proposition 3.4 *Let us assume that the radiation losses are given by $\tau\Lambda^c(\omega_0)$. Then,*

$$\lim_{\tau\to 0}p^{inc,\tau}_{tr}(t_0,x,L) = e^{-\overline{\Lambda}(\omega_0)L}H_{x_0}(\omega_0,x)\delta(t_0-\overline{\beta'(\omega_0)}L)$$

where the limit holds in $\mathcal{S}'$ with respect to t_0. Here, the transverse profile is given by

$$H_{x_0}(\omega_0,x) = \frac{1}{4N(\omega_0)}\sum_{j,l=1}^{N(\omega_0)}\beta_l(\omega_0)\phi_l^2(\omega_0,x_0)\sum_{j=1}^{N(\omega_0)}\frac{1}{\beta_j(\omega_0)}\phi_j^2(\omega_0,x).$$

This result means that the effective velocity of the incoherent wave fluctuations is the harmonic average of the modal group velocities $1/\overline{\beta'(\omega_0)}$, with

$$\overline{\beta'(\omega_0)} = \frac{1}{N(\omega_0)} \sum_{j=1}^{N(\omega_0)} \beta'_j(\omega_0),$$

and the effective radiative damping rate is the arithmetic average of the modal radiative damping rates

$$\overline{\Lambda}(\omega_0) = \frac{1}{N(\omega_0)} \sum_{j=1}^{N(\omega_0)} \Lambda^c_j(\omega_0).$$

Proposition 3.4 is a consequence of the ergodic theorem for $\big(Y_t^{N(\omega)}\big)_{t\geq 0}$. In [25, Chapter 20], the authors show, in the continuum limit $N(\omega_0) \gg 1$, that the mean intensity of the small fluctuations becomes uniform over the bounded cross-section of their waveguide model. Moreover, the spatial extent of the autocorrelation function of the small fluctuations is of order the wavelength, in the continuum limit $N(\omega_0) \gg 1$ which corresponds to the high-frequency regime $\omega_0 \nearrow +\infty$. In the following proposition, we study the transverse profile of the mean transmitted energy of the small incoherent fluctuations in a window of order the carrier wavelength $\lambda_{oc} = \frac{2\pi c}{n_1 \omega_0}$, in the ocean section $[0, d]$ and centered at any point $x \in [0, d]$.

Proposition 3.5 *In the high-frequency regime the transverse profile is given,* $\forall x \in [0, d]$ *and* $\forall \tilde{x} \in \mathbb{R}$, *by*

$$\lim_{\omega_0 \to +\infty} \lambda_{oc} H_{x_0}(\omega_0, x + \lambda_{oc}\tilde{x}) = \frac{1}{4\theta d} \arcsin(\theta) \Big[\frac{\pi}{2} - \arccos(\theta) + \frac{1}{2}\sin(2\arccos(\theta))\Big].$$

Here, $\theta = \sqrt{1 - 1/n_1^2}$, $\lambda_{oc} = \frac{2\pi c}{n_1\omega_0}$ *is the carrier wavelength in the ocean section* $[0, d]$ *of the waveguide.*

In summary, from Proposition 3.4, at time $t_0 = \overline{\beta'(\omega_0)}L$ one can observe exponentially damped small fluctuations for large propagation distance and small radiation losses. Moreover, the arrival time $\overline{\beta'(\omega_0)}L$, of the incoherent fluctuations, takes a simple form in the high-frequency regime:

$$\lim_{\omega_0 \to +\infty} \overline{\beta'(\omega_0)} = \frac{n_1}{c} \int_0^1 \frac{1}{\sqrt{1 - \theta^2 s^2}} ds = \frac{n_1}{c} \frac{\arcsin(\theta)}{\theta}.$$

From Proposition 3.5, we can see that the mean intensity of the small fluctuations becomes uniform over the ocean cross-section $[0, d]$ of the waveguide, in the high frequency regime or in the limit of a large number of propagating modes $N(\omega_0) \gg 1$.

3.4 Time Reversal in a Waveguide

Time-reversal experiments with sonar in shallow water [40, 57] were carried out by William Kuperman and his group in San Diego. This experiment is carried out in two steps. In the first step (see Figure 3.2 (a)), a source sends a pulse into the medium. The wave propagates and is recorded by a device called a time-reversal mirror. A time-reversal mirror is a device that can receive a signal, record it, and resend it time-reversed into the medium. In other words, what is recorded first is send out last. In the second step (see Figure 3.2 (b)), the wave emitted by the time-reversal mirror has the property of refocusing near the original source location, and it has been observed that random inhomogeneities enhance refocusing [19, 22].

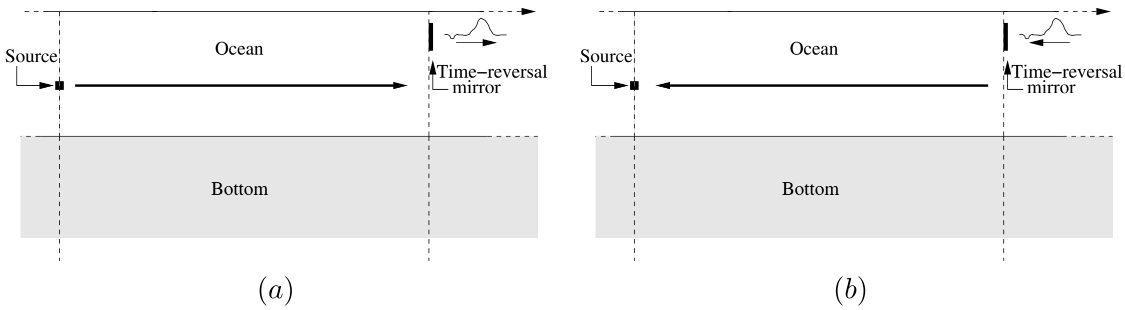


Figure 3.2: Representation of the time-reversal experiment. In (a) we represent the first step of the experiment, and in (b) we represent the second step of the experiment.

This experiment has already been analyzed in waveguides with bounded cross-section in [25, Chapter 20] and [30].

However, the properties of the fluctuations of the medium may have changed between the two steps of the experiment. That is why we distinguish, in what follows, the two steps of the experiment by using the indices 1 and 2. The influence on the time-reversal experiment of time-dependent random media is carried out in [3] for one-dimensional environments, and in [11] for three-dimensional environments with the parabolic approximation of the wave equation. In order to characterize the two realizations of the medium parameters for the two steps of the experiment, let us introduce $((V^1(x,t),V^2(x,t)), x\in[0,d], t\geq 0)$ a continuous real-valued zero-mean Gaussian field with a covariance function given by

$$\mathbb{E}\left[V^j(x,t)V^j(y,s)\right]=\gamma_0(x,y)e^{-a|t-s|}\quad\text{and}\quad\mathbb{E}\left[V^j(x,t)V^l(y,s)\right]=\tilde{\gamma}_0(x,y)e^{-a|t-s|}$$

for $(j,l)\in\{1,2\}^2$ and $j\neq l$. Here $a>0$, γ_0 and $\tilde{\gamma}_0$ are Lipschitz functions from $[0,d]\times[0,d]$ to $\mathbb{R}$, which are kernels of nonnegative operators Q_{γ_0} and $Q_{\tilde{\gamma}_0}$. As in Section 2.6.1, $(V^1(.,t),V^2(.,t))_{t\geq 0}$ can be consider as a process with values in $L^2(0,d)\times L^2(0,d)$, and we have the following results. Let

$$\mathcal{F}_t=\sigma\big((V^1(.,s),V^2(.,s)), s\leq t\big)$$

be the σ-algebra generated by $\big((V^1(.,s),V^2(.,s)), s\leq t\big)$. We have the Markov property

$$\begin{aligned}&\Big((V^1(.,t+h),V^2(.,t+h)),\Big|\mathcal{F}_t\Big)\\&\qquad=\Big((V^1(.,t+h),V^2(.,t+h)),\Big|\sigma\big(V^1(.,t),V^2(.,t)\big)\Big),\end{aligned}$$

where the equality holds in law, and this law is the one of a Gaussian field with mean

$$\mathbb{E}[V^j(.,t+h)|\mathcal{F}_t]=e^{-ah}V^j(.,t),$$

and with covariances for $(j,l)\in\{1,2\}^2$ and $j\neq l$,

$$\mathbb{E}[V^j_\varphi(t+h)V^j_\psi(t+h)-\mathbb{E}[V^j_\varphi(t+h)|\mathcal{F}_t]\mathbb{E}[V^j_\psi(t+h)|\mathcal{F}_t]|\mathcal{F}_t]=\langle Q_{\gamma_0}(\varphi),\psi\rangle_{L^2(0,d)}\left(1-e^{-2ah}\right)$$
$$\mathbb{E}[V^j_\varphi(t+h)V^l_\psi(t+h)-\mathbb{E}[V^j_\varphi(t+h)|\mathcal{F}_t]\mathbb{E}[V^l_\psi(t+h)|\mathcal{F}_t]|\mathcal{F}_t]=\langle Q_{\tilde{\gamma}_0}(\varphi),\psi\rangle_{L^2(0,d)}\left(1-e^{-2ah}\right)$$

$\forall(\varphi,\psi)\in L^2(0,d)^2$. Moreover, we also have the following two properties: $\forall T>0$, $\forall K>0$ and $\forall\mu>0$

$$\lim_{\epsilon\to 0}\mathbb{P}\Big(\epsilon^\mu\sup_{z\in[0,T]}\sup_{x\in[0,d]}\left|V^1\left(x,\frac{z}{\epsilon}\right)\right|+\left|V^2\left(x,\frac{z}{\epsilon}\right)\right|\geq K\Big)=0.$$

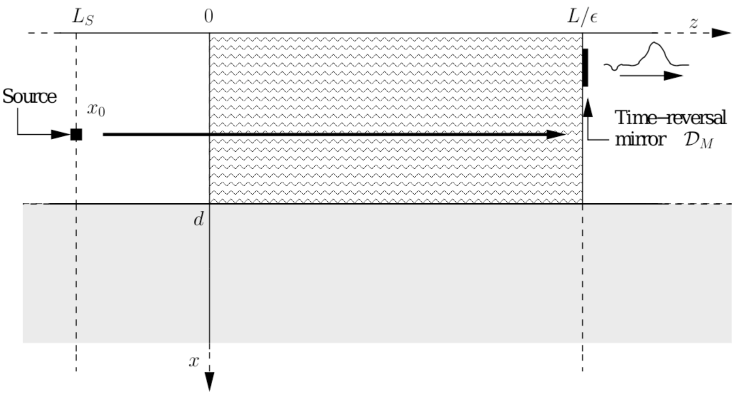


Figure 3.3: Representation of the first step of the time-reversal experiment.

$\forall n \in \mathbb{N}^*$ and $\forall z \geq 0$,

$$\mathbb{E}\Big[\Big(\int_0^d |V^1(x,z)|^2 dx\Big)^n + \Big(\int_0^d |V^2(x,z)|^2 dx\Big)^n\Big] = \mathbb{E}\Big[\Big(\int_0^d |V^1(x,0)|^2\Big)^n + \Big(\int_0^d |V^2(x,0)|^2\Big)^n\Big] < +\infty.$$

We recall that the process (V^1, V^2) is unbounded and this fact implies that the bulk modulus can take negative values. However, as in Chapter 2, this situation can be avoided since

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\exists j \in \{1,2\},\, \exists (x,z) \in [0,d]\times[0,L/\epsilon] :\, n_1 + \sqrt{\epsilon} V^j(x,z) \leq 0\Big) \leq \lim_{\epsilon\to 0} \mathbb{P}\left(\sqrt{\epsilon} \sup_{z\in[0,L]} \sup_{x\in[0,d]} \left|V^1\left(x,\frac{z}{\epsilon}\right)\right| + \left|V^2\left(x,\frac{z}{\epsilon}\right)\right| \geq n_1\right) = 0.$$

3.4.1 First Step of the Experiment

In the first step of the experiment (see Figure 3.3), a source sends a pulse into the medium, the wave propagates and is recorded by the time-reversal mirror located in the plane $z = L/\epsilon$. We assume that the time-reversal mirror occupies the transverse subdomain $\mathcal{D}_M \subset [0,d]$ and in the first step of the experiment the time-reversal mirror plays the role of a receiving array. The transmitted wave is recorded for a time interval $\left[\frac{t_0}{\epsilon}, \frac{t_1}{\epsilon}\right]$ and is re-emitted time-reversed into the waveguide toward the source. We have chosen such a time window because it is of the order of the total travel time of the section $[0, L/\epsilon]$.

According to the previous section, the wave recorded by the time-revesal mirror is given by

$$p_{tr}\left(t,x,\frac{L}{\epsilon}\right) = \frac{1}{4\pi\sqrt{\epsilon}} \int \widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right) \times \left[\sum_{j=1}^{N(\omega)} \frac{1}{\sqrt{\beta_j(\omega)}} \mathbf{T}_j^{1,\xi,\epsilon}(\omega,L)(\tilde{a}(\omega))\phi_j(\omega,x) e^{i\beta_j(\omega)\frac{L}{\epsilon}} e^{-i\omega t} + \int_\xi^{k^2(\omega)} \frac{1}{\gamma^{1/4}} \mathbf{T}_\gamma^{1,\xi,\epsilon}(\omega,L)(\tilde{a}(\omega))\phi_\gamma(\omega,x) e^{i\sqrt{\gamma}\frac{L}{\epsilon}} d\gamma e^{-i\omega t}\right] d\omega,$$

where $\mathbf{T}^{1,\xi,\epsilon}(\omega,L)$ is the transfer operator associated to V^1 during the first step of the experiment.

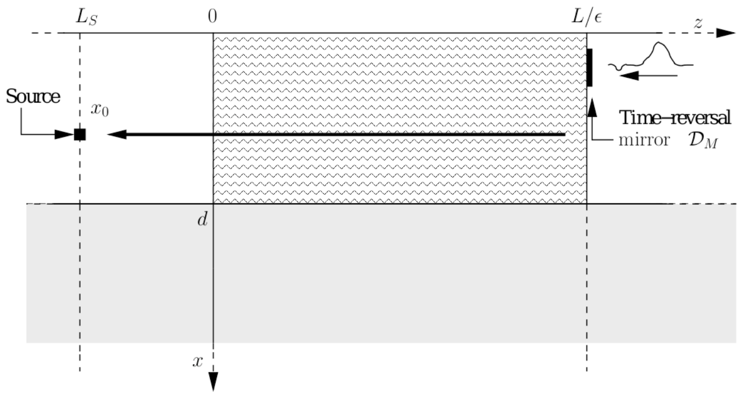


Figure 3.4: Representation of the second step of the time-reversal experiment.

3.4.2 Second Step of the Experiment

In the second step of the experiment (see Figure 3.4), the time-reversal mirror plays the role of a source array, and the time-reversed signal is transmitted back. Now, the source term is given by

$$\mathbf{F}^{\epsilon}_{TR}(t,x,z) = -f^{\epsilon}_{TR}(t,x)\delta(z-L/\epsilon)\mathbf{e}_z,$$

with

$$f^{\epsilon}_{TR}(t,x) = p^{\epsilon}_{tr}\left(\frac{t_1}{\epsilon}-t,x,\frac{L_M}{\epsilon}\right)G_1(t_1-\epsilon t)G_2(x),$$

where

$$G_1(t) = \mathbf{1}_{[t_0,t_1]}(t) \quad \text{and} \quad G_2(x) = \mathbf{1}_{\mathcal{D}_M}(x).$$

Here, G_1 represents the time window in which the transmitted wave is recorded, and G_2 represents the spatial window in which the transmitted wave is recorded. In our study of this experiment, we are interested in the spatial effects of the refocusing, so we assume that we record the field for all time at the time-reversal mirror, that is the source has the form

$$f^{\epsilon}_{TR}(t,x) = p_{tr}\left(\frac{t_1}{\epsilon}-t,x,\frac{L}{\epsilon}\right)G_2(x). \tag{3.15}$$

Now, we are interested in the propagation from $z = L/\epsilon$ to $z = 0$. The decomposition with respect to the resolution of the identity Π_ω associated to $R(\omega)$ (see Section 2.2.1) gives

$$\widehat{p}_{TR}(\omega,x,z) = \sum_{m=1}^{N(\omega)} \frac{\widehat{b}^2_m(\omega,z)}{\sqrt{\beta_m(\omega)}}e^{-i\beta_m(\omega)z}\phi_m(\omega,x) + \int_{\xi}^{k^2(\omega)} \frac{\widehat{b}^2_\gamma(\omega,z)}{\gamma^{1/4}}e^{-i\sqrt{\gamma}z}\phi_\gamma(\omega,x)d\gamma,$$

with

$$\widehat{b}^2_m(\omega,L) = \frac{\sqrt{\beta_m(\omega)}}{2}e^{i\beta_m(\omega)\frac{L}{\epsilon}}\big\langle \widehat{f}^{\epsilon}_{TR}(\omega,.),\phi_m(\omega,.)\big\rangle_H,$$
$$\widehat{b}^2_\gamma(\omega,L) = \frac{\gamma^{1/4}}{2}e^{i\sqrt{\gamma}\frac{L}{\epsilon}}\big\langle \widehat{f}^{\epsilon}_{TR}(\omega,.),\phi_\gamma(\omega,.)\big\rangle_H$$

in $\mathcal{H}^{\omega}_{\xi}$. Then, at the source location $z = L_S$, we get

$$\widehat{p}_{TR}(\omega,x,L_S) = \sum_{n=1}^{N(\omega)} \frac{\widehat{b}^2_n(\omega,0)}{\sqrt{\beta_n(\omega)}}e^{i\beta_n(\omega)L_S}\phi_n(\omega,x) + \int_{\xi}^{k^2(\omega)} \frac{\widehat{b}^2_\gamma(\omega,0)}{\gamma^{1/4}}e^{i\sqrt{\gamma}L_S}\phi_\gamma(\omega,x)d\gamma.$$

We recall that the transfer operator $\mathbf{T}^{2,\xi,\epsilon}(\omega,z)$ associated to V^2 is unitary, and therefore

$$\widehat{b}^2(\omega,0)=\overline{(\mathbf{T}^{2,\xi,\epsilon})^*(\omega,L)}\big(\widehat{b}^2(\omega,L)\big),$$

where $(\mathbf{T}^{2,\xi,\epsilon})^*(\omega,z)$ stands for the adjoint operator of $\mathbf{T}^{2,\xi,\epsilon}(\omega,z)$. Consequently,

$$\begin{aligned}\widehat{p}_{TR}(\omega,x,L_S)&=\sum_{n=1}^{N(\omega)}\frac{1}{\sqrt{\beta_n(\omega)}}\overline{(\mathbf{T}^{2,\xi,\epsilon})^*_n(\omega,L)}\big(\widehat{b}^2(\omega,L)\big)e^{-i\beta_n(\omega)L_S}\phi_n(\omega,x)\\&\quad+\int_\xi^{k^2(\omega)}\frac{1}{\gamma^{1/4}}\overline{(\mathbf{T}^{2,\xi,\epsilon})^*_\gamma(\omega,L)}\big(\widehat{b}^2(\omega,L)\big)e^{-i\sqrt{\gamma}L_S}\phi_\gamma(\omega,x)d\gamma\\&=\big\langle\tilde{b}_x(\omega),(\mathbf{T}^{2,\xi,\epsilon})^*(\omega,L)(\overline{\widehat{b}^2(\omega,L)})\big\rangle_{\mathcal{H}^\omega_\xi}\\&=\big\langle\mathbf{T}^{2,\xi,\epsilon}(\omega,L)(\tilde{b}_x(\omega)),\overline{\widehat{b}^2(\omega,L)}\big\rangle_{\mathcal{H}^\omega_\xi},\end{aligned}$$

where

$$\tilde{b}_{x,n}(\omega)=\frac{1}{\sqrt{\beta_n(\omega)}}\phi_n(\omega,x)e^{-i\beta_n(\omega)L_S}\quad\text{and}\quad\tilde{b}_{x,\gamma}(\omega)=\frac{1}{\gamma^{1/4}}\phi_\gamma(\omega,x)e^{-i\sqrt{\gamma}L_S}\tag{3.16}$$

in $\mathcal{H}^\omega_\xi$. Moreover, one can write

$$\begin{aligned}\widehat{b}^2_m(\omega,L)&=\frac{1}{4\sqrt{\epsilon}}\overline{\widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right)}e^{i\omega t_1}\big\langle\overline{\mathbf{T}^{1,\xi,\epsilon}(\omega,L)(\tilde{a}(\omega))},\lambda^\epsilon_m(\omega)\big\rangle_{\mathcal{H}^\omega_\xi},\\\widehat{b}^2_\gamma(\omega,L)&=\frac{1}{4\sqrt{\epsilon}}\overline{\widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right)}e^{i\omega t_1}\big\langle\overline{\mathbf{T}^{1,\xi,\epsilon}(\omega,L)(\tilde{a}(\omega))},\lambda^\epsilon_\gamma(\omega)\big\rangle_{\mathcal{H}^\omega_\xi},\end{aligned}$$

in $\mathcal{H}^\omega_\xi$, where $\lambda^\epsilon(\omega)$ is defined by

$$\begin{aligned}\lambda^\epsilon(\omega)_{mj}&=\sqrt{\frac{\beta_m(\omega)}{\beta_j(\omega)}}e^{-i(\beta_m(\omega)-\beta_j(\omega))\frac{L}{\epsilon}}M_{mj}(\omega),\\\lambda^\epsilon(\omega)_{m\gamma'}&=\sqrt{\frac{\beta_m(\omega)}{\sqrt{\gamma'}}}e^{-i(\beta_m(\omega)-\sqrt{\gamma'})\frac{L}{\epsilon}}M_{m\gamma'}(\omega),\\\lambda^\epsilon(\omega)_{\gamma j}&=\sqrt{\frac{\sqrt{\gamma}}{\beta_j(\omega)}}e^{-i(\sqrt{\gamma}-\beta_j(\omega))\frac{L}{\epsilon}}M_{\gamma j}(\omega),\\\lambda^\epsilon(\omega)_{\gamma\gamma'}&=\frac{\gamma^{1/4}}{\gamma'^{1/4}}e^{-i(\sqrt{\gamma}-\sqrt{\gamma'})\frac{L}{\epsilon}}M_{\gamma\gamma'}(\omega),\end{aligned}\tag{3.17}$$

with

$$M_{rs}(\omega)=\int_0^d G_2(x)\phi_r(\omega,x)\phi_s(\omega,x)dx$$

for $(r,s)\in\big(\{1,\dots,N(\omega)\}\cup(\xi,k^2(\omega))\big)^2$. $(M_{rs}(\omega))$ represents the coupling produced by the time-reversal mirror between the modes during the two steps of the time-reversal experiment. Therefore,

$$\widehat{p}_{TR}(\omega,x,L_S)=\frac{1}{4\sqrt{\epsilon}}\overline{\widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right)}e^{i\omega t_1}\big\langle\mathbf{U}^{\xi,\epsilon}(\omega,L)\big(\tilde{a}(\omega),\tilde{b}_x(\omega)\big),\lambda^\epsilon(\omega)\big\rangle_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}.$$

Here, we consider $\forall(\lambda,\mu)\in(\mathcal{H}^\omega_\xi)^2$,

$$(\lambda\otimes\mu)_{rs}=\lambda_r\mu_s$$

for $(r,s) \in \left(\{1,\dots,N(\omega)\} \cup (\xi, k^2(\omega))\right)^2$, and

$$\mathcal{H}_\xi^\omega \otimes \mathcal{H}_\xi^\omega = \left\{\lambda \otimes \mu, \quad (\lambda,\mu) \in (\mathcal{H}_\xi^\omega)^2\right\}.$$

We equip $\mathcal{H}_\xi^\omega \otimes \mathcal{H}_\xi^\omega$ with the inner product defined by

$$\begin{aligned}\langle \lambda, \mu \rangle_{\mathcal{H}_\xi^\omega \otimes \mathcal{H}_\xi^\omega} = &\sum_{j,l=1}^{N(\omega)} \lambda_{jl}\overline{\mu_{jl}} + \sum_{j=1}^{N(\omega)} \int_\xi^{k^2(\omega)} \lambda_{j\gamma'}\overline{\mu_{j\gamma'}} d\gamma' \\ &+ \int_\xi^{k^2(\omega)} \sum_{l=1}^{N(\omega)} \lambda_{\gamma l}\overline{\mu_{\gamma l}} d\gamma + \int_\xi^{k^2(\omega)} \int_\xi^{k^2(\omega)} \lambda_{\gamma\gamma'}\overline{\mu_{\gamma\gamma'}} d\gamma d\gamma'\end{aligned}$$

$\forall (\lambda,\mu) \in (\mathcal{H}_\xi^\omega \otimes \mathcal{H}_\xi^\omega)^2$, which gives a structure of Hilbert space, and the process $\mathbf{U}^{\xi,\epsilon}(\omega, L)$ is defined by

$$\mathbf{U}^{\xi,\epsilon}(\omega,L)(y^1,y^2) = \overline{\mathbf{T}^{1,\xi,\epsilon}(\omega,L)(y^1)} \otimes \mathbf{T}^{2,\xi,\epsilon}(\omega,L)(y^2)$$

$\forall (y^1,y^2) \in (\mathcal{H}_\xi^\omega)^2$.

We study the refocused wave in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_{obs}/ϵ, which is of the order the total travel time for a distance of order $1/\epsilon$. Finally, the refocused wave at the source location is given by

$$\begin{aligned}p_{TR}\Big(\frac{t_{obs}}{\epsilon} + \frac{t}{\sqrt{\epsilon}}, x, L_S\Big) &= \frac{1}{2\pi}\int \widehat{p}_{TR}(\omega, x, L_S) e^{-i\omega t} d\omega \\ &= \frac{1}{8\pi\sqrt{\epsilon}} \int \overline{\widehat{f}\left(\frac{\omega-\omega_0}{\sqrt{\epsilon}}\right)} \langle \mathbf{U}^{\xi,\epsilon}(\omega,L)(\tilde{a}(\omega), \tilde{b}_x(\omega)), \lambda^\epsilon(\omega)\rangle_{\mathcal{H}_\xi^\omega \otimes \mathcal{H}_\xi^\omega} e^{i\omega\left(\frac{t_1 - t_{obs}}{\epsilon} - \frac{t}{\sqrt{\epsilon}}\right)} d\omega,\end{aligned}$$

where $\tilde{a}(\omega)$ is defined by (3.4) and (3.5), $\tilde{b}_x(\omega)$ is defined by (3.16), and $\lambda^\epsilon(\omega)$ is defined by (3.17).

In what follows, we consider a time-reversal mirror of the form $\mathcal{D}_M = [d_1, d_2]$ with

$$d_2 = d_M + \lambda_{oc}^{\alpha_M}\tilde{d}_2 \text{ and } d_1 = d_M - \lambda_{oc}^{\alpha_M}\tilde{d}_1,$$

where $d_M \in (0,d)$, $(\tilde{d}_2, \tilde{d}_1) \in (0,+\infty)^2$, and $\alpha_M \in [0,1]$. Here, $\lambda_{oc} = 2\pi c/(n_1\omega_0)$ is the carrier wavelength in the ocean section $[0,d]$ of the waveguide. The time-reversal coupling matrix are given by

$$\begin{aligned}M_{jl}(\omega) = &(d_2 - d_1)A_j(\omega)A_l(\omega) \\ &\times \Big[\cos\Big((\sigma_j(\omega) - \sigma_l(\omega))\Big(\frac{d_2+d_1}{2d}\Big)\pi\Big)\operatorname{sinc}\Big((\sigma_j(\omega) - \sigma_l(\omega))\Big(\frac{d_2-d_1}{2d}\Big)\pi\Big) \\ &\quad - \cos\Big((\sigma_j(\omega) + \sigma_l(\omega))\Big(\frac{d_2+d_1}{2d}\Big)\pi\Big)\operatorname{sinc}\Big((\sigma_j(\omega) + \sigma_l(\omega))\Big(\frac{d_2-d_1}{2d}\Big)\pi\Big)\Big],\end{aligned}$$

for $(j,l) \in \{1,\dots,N(\omega)\}^2$, where $A_j(\omega)$ and $\sigma_j(\omega)$ are defined in Section 2.2.1. We give only the coefficients $M_{jl}(\omega)$ for $(j,l) \in \{1,\dots,N(\omega)\}^2$, because, in what follows, that is only these terms which play a role. The parameter α_M represents the order of the magnitude of the size of the time-reversal mirror with respect to the wavelength in the ocean cross-section $[0,d]$. In fact, we shall see that the size of the mirror plays a role in the homogeneous case only when it is of the order the carrier wavelength $\lambda_{oc} = 2\pi c/(n_1\omega_0)$.

Moreover, we shall study the spatial profile of the refocused wave in the continuum limit $N(\omega_0) \gg 1$, which corresponds to the high-frequency regime $\omega_0 \nearrow +\infty$. However, we know that the main focal spot must be of order λ_{oc}, which tends to 0 in this continuum limit. Therefore, we study the spatial profile in a window of size λ_{oc} centered around x_0.

3.4.3 Refocused Field in a Homogeneous Waveguide

We study the refocused wave in a time window of order $1/\sqrt{\epsilon}$, which is comparable to the pulse width, and centered at time t_{obs}/ϵ, which is of the order the total travel time for a distance of order $1/\epsilon$. In this section we assume that the medium is homogeneous, that is $\mathbf{T}^{1,\xi,\epsilon}(\omega, L) = \mathbf{T}^{2,\xi,\epsilon}(\omega, L) = Id$. Then, the refocused wave at the original source location is given by

$$\begin{aligned} p_{TR}\Big(\frac{t_{obs}}{\epsilon} + \frac{t}{\sqrt{\epsilon}}, x, L_S\Big) = {} & e^{i\omega_0 \frac{t_1 - t_{obs}}{\epsilon}} e^{-i\omega_0 \frac{t}{\sqrt{\epsilon}}} \cdot \frac{1}{4} \sum_{j,m=1}^{N(\omega_0)} e^{i(\beta_m(\omega_0)-\beta_j(\omega_0))\left(-L_S + \frac{L}{\epsilon}\right)} M_{jm}(\omega_0) \\ & \times \phi_j(\omega_0, x_0)\phi_m(\omega_0, x) K^{\omega_0}_{j,m,L} * f\left(\frac{(\beta'_m(\omega_0) - \beta'_j(\omega_0))L + t_1 - t_{obs}}{\sqrt{\epsilon}} - t\right) \\ & + \mathcal{O}(\sqrt{\epsilon}), \end{aligned}$$

where

$$\widehat{K^{\omega_0}_{j,m,L}}(\omega) = \widehat{K^{\omega_0}_{j,L}}(\omega)\overline{\widehat{K^{\omega_0}_{m,L}}(\omega)} = e^{i(\beta''_j(\omega_0) - \beta''_m(\omega_0))L\frac{\omega^2}{2}}, \tag{3.18}$$

and $K^{\omega_0}_{j,j,L} = \delta_0$. Consequently, in the asymptotic $\epsilon \to 0$, we can observe a refocused wave only for a finite set of times given by

$$t_{jm} = t_1 + (\beta'_m(\omega_0) - \beta'_j(\omega_0))L. \tag{3.19}$$

For $m \neq j$, we have

$$\begin{aligned} p_{TR}\Big(\frac{t_{jm}}{\epsilon} + \frac{t}{\sqrt{\epsilon}}, x, L_S\Big) = {} & e^{i\omega_0 \frac{t_1 - t_{jm}}{\epsilon}} e^{-i\omega_0 \frac{t}{\sqrt{\epsilon}}} e^{i(\beta_m(\omega_0)-\beta_j(\omega_0))\left(-L_S + \frac{L}{\epsilon}\right)} M_{jm}(\omega_0) \\ & \times \frac{1}{4}\phi_j(\omega_0, x_0)\phi_m(\omega_0, x) K^{\omega_0}_{j,m,L} * f(-t) \\ & + \mathcal{O}(\sqrt{\epsilon}). \end{aligned}$$

At time t_{jm} ($j \neq m$) one can observe only one mode. In this expression we observe the mth mode, emitted by the time-reversal mirror during the second step of the experiment, which propagates toward the source location. This mode is coupled with the jth modes recorded by the time-reversal mirror during the first step of the experiment through the coefficient $M_{jm}(\omega_0)$. This coupling is produced by the time-reversal mechanism through the time-reversal mirror and is characterized by the coupling matrix $M_{jm}(\omega_0)$. Moreover, we can see that the refocused wave shape is dispersed by $K^{\omega_0}_{j,L}(t)$ during the first step of the experiment and by $K^{\omega_0}_{m,L}(-t)$ during the second step.

Now, for $t_{obs} = t_1$ we get

$$p_{TR}\Big(\frac{t_1}{\epsilon} + \frac{t}{\sqrt{\epsilon}}, x, L_S\Big) = e^{-i\omega_0 \frac{t}{\sqrt{\epsilon}}} f(-t) H^{\alpha_M}_{x_0}(\omega_0, x) + \mathcal{O}(\sqrt{\epsilon}),$$

where

$$H^{\alpha_M}_{x_0}(\omega_0, x) = \frac{1}{4}\sum_{j=1}^{N(\omega_0)} M_{jj}(\omega_0)\phi_j(\omega_0, x_0)\phi_j(\omega_0, x).$$

Here, we have a contribution of all the modes. The refocused wave is a superposition of modes where each mode is coupled with itself by the time-reversal mirror through the terms $M_{jj}(\omega_0)$. Then, we find the time-reversed pulse shape with a transverse profile that we can study in the high-frequency regime.

Proposition 3.6 *For $\alpha_M \in [0,1)$, the transverse profile of the refocused wave in the continuum limit is given by*

$$\lim_{\omega_0\to+\infty} \frac{\lambda_{oc}^{1-\alpha_M}}{\theta} H_{x_0}^{\alpha_M}\Big(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}\Big) = \frac{\tilde{d}_2+\tilde{d}_1}{d}\operatorname{sinc}(2\pi\tilde{x}).$$

The width of the focal spot is given by the diffraction limit $\lambda_{oc}/(2\theta)$.

Proof First, we have

$$\begin{aligned}\frac{d}{\tilde{d}_2+\tilde{d}_1}\frac{\lambda_{oc}^{1-\alpha_M}}{\theta}H_{x_0}^{\alpha_M}\Big(\omega_0, x_0+\frac{\lambda_{oc}}{\theta}\tilde{x}\Big) &= \frac{\lambda_{oc}}{2\theta}\left[\sum_{j=1}^{N-[N^\alpha]} + \sum_{j=N-[N^\alpha]+1}^{N}\right]\phi_j(\omega_0,x_0)\\ \phi_j(\omega_0, x_0+\lambda_{oc}\tilde{x}/\theta)\frac{A_j^2 d}{2}&\Big[1-\cos\Big(\sigma_j\frac{2d_M+\lambda_{oc}^{\alpha_M}(\tilde{d}_2-\tilde{d}_1)}{d}\Big)\operatorname{sinc}\Big(\sigma_j\frac{\lambda_{oc}^{\alpha_M}(\tilde{d}_2+\tilde{d}_1)}{d}\Big)\Big],\end{aligned}$$

and the second sum on the right of the previous equality is a $\mathcal{O}(N^{\alpha-1})$. Moreover,

$$\begin{aligned}&\Big|\frac{\lambda_{oc}}{2\theta}\sum_{j=1}^{N-[N^\alpha]}\phi_j(\omega_0,x_0)\phi_j(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\\ &\times\frac{A_j^2 d}{2}\cos\Big(\sigma_j\frac{2d_M+\lambda_{oc}^{\alpha_M}(\tilde{d}_2-\tilde{d}_1)}{d}\Big)\operatorname{sinc}\Big(\sigma_j\frac{\lambda_{oc}^{\alpha_M}(\tilde{d}_2+\tilde{d}_1)}{d}\Big)\Big| \leq K\lambda_{oc}^{1-\alpha_M}\ln(N).\end{aligned}$$

Now, for the first sum of the previous equality we have

$$\phi_j(\omega_0,x_0)\phi_j(\omega_0,x_0+\frac{\lambda_{oc}}{\theta}\tilde{x}) = \frac{A_j^2}{2}\Big[\cos\Big(\sigma_j\frac{\lambda_{oc}}{\theta d}\tilde{x}\Big)-\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\Big],$$

and

$$\begin{aligned}\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big) =& \cos\Big((\sigma_j-j\pi)\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\cos\Big(j\pi\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\\ &-\sin\Big((\sigma_j-j\pi)\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\sin\Big(j\pi\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big).\end{aligned}$$

Then using the Abel transform and Lemma 2.1, we get

$$\lambda_{oc}\Big|\sum_{j=1}^{N-[N^\alpha]}\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\Big| \leq KN^{\frac{1}{2}-\frac{3}{2}\alpha}.$$

Let us recall that

$$\sup_{j\in\{1,\dots,N-[N^\alpha]\}}\Big|A_j^2-\frac{2}{d}\Big| = \mathcal{O}(N^{\alpha-1}),$$

and then

$$\frac{\lambda_{oc}d}{8\theta}\sum_{j=1}^{N-[N^\alpha]}A_j^4\cos\Big(\sigma_j\frac{\lambda_{oc}}{\theta d}\tilde{x}\Big) = \frac{\lambda_{oc}}{2\theta d}\sum_{j=1}^{N-[N^\alpha]}\cos\Big(2\frac{j}{N}\pi\tilde{x}\Big)+\mathcal{O}(N^{\alpha-1}),$$

with

$$\lim_{\omega_0\to+\infty}\frac{\lambda_{oc}}{2\theta d}\sum_{j=1}^{N-[N^\alpha]}\cos\Big(2\frac{j}{N}\pi\tilde{x}\Big) = \int_0^1\cos(2u\pi\tilde{x})du = \operatorname{sinc}(2\pi\tilde{x}).$$

That concludes the proof of Proposition 3.6.■

3.4.4 Limit Theorem

To study the time-reversed field in the case of a random waveguide we need to know the asymptotic distribution of the process $\mathbf{U}^{\xi,\epsilon}(\omega,.)$ as ϵ goes to 0 and ξ goes to 0. Let us remark that $\forall (y^1,y^2)\in(\mathcal{H}^\omega_\xi)^2$, with $\mathcal{H}^\omega_\xi=\mathbb{C}^{N(\omega)}\times L^2(\xi,k^2(\omega))$,

$$\|\mathbf{U}^{\xi,\epsilon}(\omega,z)(y^1,y^2)\|^2_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}=\|y^1\otimes y^2\|^2_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}\quad\forall z\geq 0.$$

Let $r_y=\|y^1\otimes y^2\|_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}$,

$$\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}=\Big\{\lambda\in\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi,\|\lambda\|_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}\leq r_y\Big\}$$

the closed ball with radius r_y, and $\{g_n,n\geq 1\}$ a dense subset of $\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}$. We equip $\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}$ with the distance $d_{\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}}$ defined by

$$d_{\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}}(\lambda,\mu)=\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big|\langle\lambda-\mu,g_n\rangle_{\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}\Big|$$

$\forall(\lambda,\mu)\in\left(\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}\right)^2$, and then $(\mathcal{B}_{\mathcal{H}^\omega_\xi},d_{\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}})$ is a compact metric space.

In the two following theorems, we give only the drifts of the infinitesimal generators because, in what follows, we shall use only this part.

Theorem 3.3 *$\forall(y^1,y^2)\in(\mathcal{H}^\omega_\xi)^2$, the process $\mathbf{U}^{\xi,\epsilon}(\omega,.)(y^1,y^2)$ converge in distribution on $\mathcal{C}([0,+\infty),(\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi},d_{\mathcal{B}_{r_y,\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi}}))$ as $\epsilon\to 0$ to a limit denoted by $\mathbf{U}^{\xi}(\omega,.)(y^1,y^2)$. This limit is the unique solution of a martingale problem on $\mathcal{H}^\omega_\xi\otimes\mathcal{H}^\omega_\xi$, starting from $y^1\otimes y^2$, with drift given by*

$$\mathcal{L}^\omega_1+\mathcal{L}^\omega_{2,\xi},$$

where

$$\begin{aligned}
\mathcal{L}^\omega_1=&\sum_{\substack{j,l=1\\ j\neq l}}^{N(\omega)}\tilde{\Gamma}^c_{jl}(\omega)(U_{ll}\partial_{U_{jj}}+\overline{U_{ll}}\partial_{\overline{U_{jj}}})\\
&+\frac{1}{2}\sum_{j,l=1}^{N(\omega)}\left[\Gamma^c_{jj}(\omega)+\Gamma^c_{ll}(\omega)-(\Gamma^1_{jj}(\omega)+\Gamma^1_{ll}(\omega)-2\tilde{\Gamma}^1_{jl}(\omega))\right](U_{jl}\partial_{U_{jl}}+\overline{U_{jl}}\partial_{\overline{U_{jl}}})\\
&+\frac{1}{2}\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)}\left[\Gamma^c_{jj}(\omega)-\Gamma^1_{jj}(\omega)\right](U_{j\gamma_2}\partial_{U_{j\gamma_2}}+\overline{U_{j\gamma_2}}\partial_{\overline{U_{j\gamma_2}}})d\gamma_2\\
&+\frac{1}{2}\int_\xi^{k^2(\omega)}\sum_{l=1}^{N(\omega)}\left[\Gamma^c_{ll}(\omega)-\Gamma^1_{ll}(\omega)\right](U_{\gamma_1 l}\partial_{U_{\gamma_1 l}}+\overline{U_{\gamma_1 l}}\partial_{\overline{U_{\gamma_1 l}}})d\gamma_1\\
&+\frac{i}{2}\sum_{j,l=1}^{N(\omega)}\left[\Gamma^s_{ll}(\omega)-\Gamma^s_{jj}(\omega)\right](U_{jl}\partial_{U_{jl}}-\overline{U_{jl}}\partial_{\overline{U_{jl}}})\\
&-\frac{i}{2}\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)}\Gamma^s_{jj}(\omega)(U_{j\gamma_2}\partial_{U_{j\gamma_2}}-\overline{U_{j\gamma_2}}\partial_{\overline{U_{j\gamma_2}}})d\gamma_2\\
&+\frac{i}{2}\int_\xi^{k^2(\omega)}\sum_{l=1}^{N(\omega)}\Gamma^s_{ll}(\omega)(U_{\gamma_1 l}\partial_{U_{\gamma_1 l}}-\overline{U_{\gamma_1 l}}\partial_{\overline{U_{\gamma_1 l}}})d\gamma_1,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}^\omega_{2,\xi} =& -\frac{1}{2}\sum_{j,l=1}^{N(\omega)} \big[\Lambda^{c,\xi}_j(\omega)+\Lambda^{c,\xi}_l(\omega)\big]\big(U_{jl}\partial_{U_{jl}}+\overline{U_{jl}}\partial_{\overline{U_{jl}}}\big)\\
&-\frac{i}{2}\sum_{j,l=1}^{N(\omega)} \big[\Lambda^{s,\xi}_l(\omega)-\Lambda^{s,\xi}_j(\omega)\big]\big(U_{jl}\partial_{U_{jl}}-\overline{U_{jl}}\partial_{\overline{U_{jl}}}\big)\\
&-\frac{1}{2}\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)} [\Lambda^{c,\xi}_j(\omega)-i\Lambda^{s,\xi}_j(\omega)]U_{j\gamma_2}\partial_{U_{j\gamma_2}}+[\Lambda^{c,\xi}_j(\omega)+i\Lambda^{s,\xi}_j(\omega)]\overline{U_{j\gamma_2}}\partial_{\overline{U_{j\gamma_2}}}\\
&-\frac{1}{2}\int_\xi^{k^2(\omega)}\sum_{l=1}^{N(\omega)} [\Lambda^{c,\xi}_l(\omega)+i\Lambda^{s,\xi}_l(\omega)]U_{\gamma_1 l}\partial_{U_{\gamma_1 l}}+[\Lambda^{c,\xi}_l(\omega)-i\Lambda^{s,\xi}_l(\omega)]\overline{U_{\gamma_1 l}}\partial_{\overline{U_{\gamma_1 l}}}.
\end{aligned}$$

Here, we have considered the complex derivative with the following notations. If $U = U^1 + iU^2 \in \mathcal{H}^\omega_0\otimes\mathcal{H}^\omega_0$, we have $(U^1,U^2)\in(\mathcal{G}^\omega_0\otimes\mathcal{G}^\omega_0)^2$, where $\mathcal{G}^\omega_0=\mathbb{R}^{N(\omega)}\times L^2(0,k^2(\omega))$. Then, the operators $\partial_U=(\partial_{U_{r,s}})$ and $\partial_{\overline{U}}=(\partial_{\overline{U_{r,s}}})$ are defined by

$$\partial_U=\frac{1}{2}(\partial_{U^1}-i\partial_{U^2})\quad\text{and}\quad\partial_{\overline{U}}=\frac{1}{2}(\partial_{U^1}+i\partial_{U^2}),$$

with $\forall f\in\mathcal{C}^1((\mathcal{G}^\omega_0\otimes\mathcal{G}^\omega_0)^2,\mathbb{R})$ and $\forall\lambda=(\lambda^1,\lambda^2)\in(\mathcal{G}^\omega_0\otimes\mathcal{G}^\omega_0)^2$

$$\begin{aligned}
\sum_{n=1,2}\Big[&\sum_{j,l=1}^{N(\omega)}\lambda^n_{jl}\partial_{U^n_{jl}}f(v^1,v^2)+\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)}\lambda^n_{j\gamma_2}\partial_{U^n_{j\gamma_2}}f(v^1,v^2)d\gamma_2\\
&+\int_\xi^{k^2(\omega)}\sum_{l=1}^{N(\omega)}\lambda^n_{\gamma_1 l}\partial_{U^n_{\gamma_1 l}}f(v^1,v^2)d\gamma_1+\int_\xi^{k^2(\omega)}\int_\xi^{k^2(\omega)}\lambda^n_{\gamma_1\gamma_2}\partial_{U^n_{\gamma_1\gamma_2}}f(v^1,v^2)d\gamma_1 d\gamma_2\Big]\\
&=\sum_{n=1,2}\langle\lambda^n,\partial_{U^n}f(v^1,v^2)\rangle_{\mathcal{G}^\omega_0\otimes\mathcal{G}^\omega_0}=Df(v^1,v^2)(\lambda),
\end{aligned}$$

which is the differential of f. Moreover, $\Gamma^{c,\xi}(\omega)$, $\Gamma^{s,\xi}(\omega)$, $\Gamma^{1,\xi}(\omega)$, $\Lambda^{c,\xi}(\omega)$, and $\Lambda^{s,\xi}(\omega)$ are defined in Section 2.4.1, and

$$\begin{aligned}
\tilde{\Gamma}^c_{jl}(\omega)&=\frac{k^4(\omega)}{2\beta_j(\omega)\beta_l(\omega)}\int_0^{+\infty}\mathbb{E}[C^1_{jl}(0)C^2_{jl}(z)]\cos\big((\beta_l(\omega)-\beta_j(\omega))z\big)dz,\\
\tilde{\Gamma}^c_{jj}(\omega)&=-\sum_{\substack{l=1\\ l\neq j}}^{N(\omega)}\tilde{\Gamma}^c_{jl}(\omega),\\
\tilde{\Gamma}^1_{jl}(\omega)&=\frac{k^4(\omega)}{2\beta_j(\omega)\beta_l(\omega)}\int_0^{+\infty}\mathbb{E}[C^1_{jj}(0)C^2_{ll}(z)]dz,\quad\forall(j,l)\in\{1,\dots,N(\omega)\}^2.
\end{aligned}$$

As in Chapter 2, we can also give the asymptotic distribution of the process $\mathbf{U}^\xi(\omega,.)$ as ξ goes to 0.

Theorem 3.4 *$\forall(y^1,y^2)\in(\mathcal{H}^\omega_0)^2$, the process $\mathbf{U}^\xi(\omega,.)(y^1,y^2)$ converge in distribution on $\mathcal{C}([0,+\infty),(\mathcal{B}_{r_y,\mathcal{H}^\omega_0\otimes\mathcal{H}^\omega_0},d_{\mathcal{B}_{r_y,\mathcal{H}^\omega_0\otimes\mathcal{H}^\omega_0}}))$ as $\xi\to0$ to a limit denoted by $\mathbf{U}^0(\omega,.)(y^1,y^2)$. This limit is the unique solution of the martingale problem on $\mathcal{H}^\omega_0\otimes\mathcal{H}^\omega_0$, starting from $y^1\otimes y^2$, with drift given by*

$$\mathcal{L}^\omega_1+\mathcal{L}^\omega_2,$$

where

$$\begin{aligned}
\mathcal{L}_2^\omega = & -\frac{1}{2}\sum_{j,l=1}^{N(\omega)}\left[\Lambda_j^c(\omega)+\Lambda_l^c(\omega)\right]\left(U_{jl}\partial_{U_{jl}}+\overline{U_{jl}}\partial_{\overline{U_{jl}}}\right)\\
& -\frac{i}{2}\sum_{j,l=1}^{N(\omega)}\left[\Lambda_l^s(\omega)-\Lambda_j^s(\omega)\right]\left(U_{jl}\partial_{U_{jl}}-\overline{U_{jl}}\partial_{\overline{U_{jl}}}\right)\\
& -\frac{1}{2}\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)}\left[\Lambda_j^c(\omega)-i\Lambda_j^s(\omega)\right]U_{j\gamma_2}\partial_{U_{j\gamma_2}}+\left[\Lambda_j^c(\omega)+i\Lambda_j^s(\omega)\right]\overline{U_{j\gamma_2}}\partial_{\overline{U_{j\gamma_2}}}\\
& -\frac{1}{2}\int_\xi^{k^2(\omega)}\sum_{l=1}^{N(\omega)}\left[\Lambda_l^c(\omega)+i\Lambda_l^s(\omega)\right]U_{\gamma_1 l}\partial_{U_{\gamma_1 l}}+\left[\Lambda_l^c(\omega)-i\Lambda_l^s(\omega)\right]\overline{U_{\gamma_1 l}}\partial_{\overline{U_{\gamma_1 l}}}.
\end{aligned}$$

From Theorems 3.3 and 3.4, we have the following proposition about the autocorrelation function of the transfer operator for the two steps of the time-reversal experiment.

Proposition 3.7 *$\forall(y^1,y^2)\in\mathcal{H}_\xi^\omega\times\mathcal{H}_\xi^\omega$ and $\forall\lambda\in\mathcal{H}_\xi^\omega\times\mathcal{H}_\xi^\omega$, the autocorrelation function of the transfer operator for the two steps of the time-reversal experiment as $\epsilon\to 0$ is given by*

$$\begin{aligned}
\lim_{\epsilon\to 0}\mathbb{E}\Big[\langle\mathbf{U}^{\xi,\epsilon}(\omega,L)(y^1,y^2),\lambda\rangle_{\mathcal{H}_\xi^\omega\otimes\mathcal{H}_\xi^\omega}\Big] & = \mathbb{E}\Big[\langle\mathbf{U}^{\xi}(\omega,L)(y^1,y^2),\lambda\rangle_{\mathcal{H}_\xi^\omega\otimes\mathcal{H}_\xi^\omega}\Big]\\
= & \sum_{j,l=1}^{N(\omega)}\tilde{\mathcal{T}}_j^{\xi,l}(\omega,L)\overline{y_l^1}y_l^2\overline{\lambda_{jj}}+\sum_{\substack{j,m=1\\ j\neq m}}^{N(\omega)}e^{Q_{jm}^\xi(\omega)L}\overline{y_j^1}y_m^2\overline{\lambda_{jm}}\\
& +\sum_{j=1}^{N(\omega)}\int_\xi^{k^2(\omega)}e^{\frac{1}{2}(\Gamma_{jj}^c(\omega)-\Gamma_{jj}^1(\omega)-\Lambda_j^{c,\xi}(\omega))L-\frac{i}{2}(\Gamma_{jj}^s(\omega)-\Lambda_j^{s,\xi}(\omega))L}\overline{y_j^1}y_{\gamma'}^2\overline{\lambda_{j\gamma'}}d\gamma'\\
& +\int_\xi^{k^2(\omega)}\sum_{m=1}^{N(\omega)}e^{\frac{1}{2}(\Gamma_{mm}^c(\omega)-\Gamma_{mm}^1(\omega)-\Lambda_m^{c,\xi}(\omega))L+\frac{i}{2}(\Gamma_{mm}^s(\omega)-\Lambda_m^{s,\xi}(\omega))L}\overline{y_\gamma^1}y_m^2\overline{\lambda_{\gamma m}}d\gamma\\
& +\int_\xi^{k^2(\omega)}\int_\xi^{k^2(\omega)}\overline{y_\gamma^1}y_{\gamma'}^2\overline{\lambda_{\gamma\gamma'}}d\gamma d\gamma'.
\end{aligned}$$

Here,

$$\begin{aligned}
Q_{jm}^\xi(\omega) = & \frac{1}{2}\left[\Gamma_{jj}^c(\omega)+\Gamma_{mm}^c(\omega)-(\Gamma_{jj}^1(\omega)+\Gamma_{mm}^1(\omega)-2\tilde{\Gamma}_{jl}^1(\omega))-(\Lambda_j^{c,\xi}(\omega)+\Lambda_m^{c,\xi}(\omega))\right]\\
& +\frac{i}{2}\left[\Gamma_{mm}^s(\omega)-\Gamma_{jj}^s(\omega)-(\Lambda_m^{s,\xi}(\omega)-\Lambda_l^{s,\xi}(\omega))\right].
\end{aligned}$$

and $\tilde{\mathcal{T}}_j^{\xi,l}(\omega,z)$ is the solution of the coupled power equations

$$\begin{aligned}
\frac{d}{dz}\tilde{\mathcal{T}}_j^{\xi,l}(\omega,z) = & -\left[\Lambda_j^{c,\xi}(\omega)+\Gamma_{jj}^1(\omega)-\tilde{\Gamma}_{jj}^1(\omega)-\Gamma_{jj}^c(\omega)+\tilde{\Gamma}_{jj}^c(\omega)\right]\tilde{\mathcal{T}}_j^{\xi,l}(\omega,z)\\
& +\sum_{n=1}^{N(\omega)}\tilde{\Gamma}_{nj}^c(\omega)\left(\tilde{\mathcal{T}}_n^{\xi,l}(\omega,z)-\tilde{\mathcal{T}}_j^{\xi,l}(\omega,z)\right)
\end{aligned}$$

and $\tilde{\mathcal{T}}_j^{\xi,l}(\omega,0)=\delta_{jl}$. *Moreover,*

$$\begin{aligned}
\lim_{\xi\to 0}\mathbb{E}\Big[\langle \mathbf{U}^{\xi}(\omega,L)(y^1,y^2),\lambda\rangle_{\mathcal{H}_0^{\omega}\otimes\mathcal{H}_0^{\omega}}\Big] &= \mathbb{E}\Big[\langle \mathbf{U}^{0}(\omega,L)(y^1,y^2),\lambda\rangle_{\mathcal{H}_0^{\omega}\otimes\mathcal{H}_0^{\omega}}\Big]\\
&= \sum_{j,l=1}^{N(\omega)}\tilde{\mathcal{T}}_j^{l}(\omega,L)\overline{y_l^1}y_l^2\overline{\lambda_{jj}} + \sum_{\substack{j,m=1\\ j\neq m}}^{N(\omega)} e^{Q_{jm}(\omega)L}\overline{y_j^1}y_m^2\overline{\lambda_{jm}}\\
&+ \sum_{j=1}^{N(\omega)}\int_{\xi}^{k^2(\omega)} e^{\frac{1}{2}(\Gamma_{jj}^c(\omega)-\Gamma_{jj}^1(\omega)-\Lambda_j^c(\omega))L-\frac{i}{2}(\Gamma_{jj}^s(\omega)-\Lambda_j^s(\omega))L}\overline{y_j^1}y_{\gamma'}^2\overline{\lambda_{j\gamma'}}d\gamma'\\
&+ \int_{\xi}^{k^2(\omega)}\sum_{m=1}^{N(\omega)} e^{\frac{1}{2}(\Gamma_{mm}^c(\omega)-\Gamma_{mm}^1(\omega)-\Lambda_m^c(\omega))L+\frac{i}{2}(\Gamma_{mm}^s(\omega)-\Lambda_m^s(\omega))L}\overline{y_\gamma^1}y_m^2\overline{\lambda_{\gamma m}}d\gamma\\
&+ \int_0^{k^2(\omega)}\int_0^{k^2(\omega)}\overline{y_\gamma^1}y_{\gamma'}^2\overline{\lambda_{\gamma\gamma'}}d\gamma d\gamma',
\end{aligned}$$

where, $Q(\omega)=\lim_{\xi\to 0}Q^{\xi}(\omega)$ *and* $\tilde{\mathcal{T}}_j^{l}(\omega,z)$ *is the solution of the coupled power equations*

$$\begin{aligned}
\frac{d}{dz}\tilde{\mathcal{T}}_j^{l}(\omega_0,z) =& -\big[\Lambda_j^c(\omega_0)+\Gamma_{jj}^1(\omega_0)-\tilde{\Gamma}_{jj}^1(\omega_0)-\Gamma_{jj}^c(\omega_0)+\tilde{\Gamma}_{jj}^c(\omega_0)\big]\tilde{\mathcal{T}}_j^{l}(\omega_0,z)\\
&+\sum_{n=1}^{N(\omega_0)}\tilde{\Gamma}_{nj}^c(\omega_0)\big(\tilde{\mathcal{T}}_n^{l}(\omega_0,z)-\tilde{\mathcal{T}}_j^{l}(\omega_0,z)\big)
\end{aligned}\tag{3.20}$$

and $\tilde{\mathcal{T}}_j^{l}(\omega_0,0)=\delta_{jl}$.

Here,

$$\tilde{\mathcal{T}}_j^{l}(\omega_0,L)=\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[\overline{\mathbf{T}_j^{1,\xi,\epsilon}(\omega_0,L)(y^l)}\mathbf{T}_j^{2,\xi,\epsilon}(\omega_0,L)(y^l)\Big],\tag{3.21}$$

with $y_j^l=\delta_{jl}$, and $y_\gamma^l=0$ for $\gamma\in(0,k^2(\omega))$. $\tilde{\mathcal{T}}_j^{l}(\omega_0,L)$ is the asymptotic covariance for the jth propagating mode of the transfer operators at distance $z=L$, with respect to the two steps of the time-reversal experiment.The initial condition y^l means that an impulse equal to one charges only the lth propagating mode at $z=0$. The coupled equations (3.20) permit us to study the influence of the degree of correlation, between the two realizations of the random medium, on the amplitude of the refocused wave.

3.4.5 Refocused Field in a Changing Random Waveguide

In this section we study the refocusing of the wave when the realizations of the medium parameters are not the same between the two steps of the time-reversal experiment. With the change of variable $\omega=\omega_0+\sqrt{\epsilon}h$, the mean refocused wave is given by

$$\begin{aligned}
p_{TR}\Big(\frac{t_{obs}}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)e^{i\omega_0\big(\frac{t_{obs}-t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\big)} &= \frac{1}{8\pi}\int\overline{\widehat{f}(h)}e^{ih\big(\frac{t_1-t_{obs}}{\sqrt{\epsilon}}-t\big)}\\
&\times\langle \mathbf{U}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h,L)\big(\tilde{a}(\omega_0+\sqrt{\epsilon}h),\tilde{b}_x(\omega_0+\sqrt{\epsilon}h)\big),\lambda^{\epsilon}(\omega_0+\sqrt{\epsilon}h)\rangle_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}\otimes\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}}dh.
\end{aligned}$$

According to the previous section and using the perturbed-test-function method we get

$$\begin{aligned}
&\mathbb{E}\big[\langle \mathbf{U}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h,L)\big(\tilde{a}(\omega_0+\sqrt{\epsilon}h),\tilde{b}_x(\omega_0+\sqrt{\epsilon}h)\big),\lambda^{\epsilon}(\omega_0+\sqrt{\epsilon}h)\rangle_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}\otimes\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h}}\big]\\
&=\mathbb{E}\big[\langle \mathbf{U}^{\xi}(\omega_0,L)\big(\tilde{a}(\omega_0),\tilde{b}_x(\omega_0)\big),\tilde{\lambda}^{\epsilon}(h,\omega_0)\rangle_{\mathcal{H}_\xi^{\omega_0}\otimes\mathcal{H}_\xi^{\omega_0}}\big]+\mathcal{O}(\sqrt{\epsilon}),
\end{aligned}$$

where

$$\tilde{\lambda}^{\epsilon}_{mj}(h,\omega_0) = \sqrt{\frac{\beta_m(\omega_0)}{\beta_j(\omega_0)}} e^{-i(\beta_m(\omega_0)-\beta_j(\omega_0))\frac{L}{\epsilon}} e^{-ih(\beta'_m(\omega_0)-\beta'_j(\omega_0))\frac{L}{\sqrt{\epsilon}}} e^{-i\frac{h^2}{2}(\beta''_m(\omega_0)-\beta''_j(\omega_0))L} M_{mj}(\omega_0),$$

$$\tilde{\lambda}^{\epsilon}_{m\gamma'}(h,\omega_0) = \sqrt{\frac{\beta_m(\omega_0)}{\sqrt{\gamma'}}} e^{-i(\beta_m(\omega_0)-\sqrt{\gamma'})\frac{L}{\epsilon}} e^{-ih\beta'_m(\omega_0)\frac{L}{\sqrt{\epsilon}}} e^{-i\frac{h^2}{2}\beta''_m(\omega_0)L} M_{m\gamma'}(\omega_0),$$

$$\tilde{\lambda}^{\epsilon}_{\gamma j}(h,\omega_0) = \sqrt{\frac{\sqrt{\gamma}}{\beta_j(\omega_0)}} e^{-i(\sqrt{\gamma}-\beta_j(\omega_0))\frac{L}{\epsilon}} e^{ih\beta'_j(\omega_0)\frac{L}{\sqrt{\epsilon}}} e^{i\frac{h^2}{2}\beta''_j(\omega_0)L} M_{\gamma j}(\omega_0),$$

$$\tilde{\lambda}^{\epsilon}_{\gamma\gamma'}(h,\omega_0) = \frac{\gamma^{1/4}}{\gamma'^{1/4}} e^{-i(\sqrt{\gamma}-\sqrt{\gamma'})\frac{L}{\epsilon}} M_{\gamma\gamma'}(\omega_0),$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle \mathbf{U}^{\xi}(\omega_0,L)\big(\tilde{a}(\omega_0),\tilde{b}_x(\omega_0)\big),\tilde{\lambda}^{\epsilon}(h,\omega_0)\big\rangle_{\mathcal{H}^{\omega_0}_{\xi}\otimes\mathcal{H}^{\omega_0}_{\xi}}\Big] \\
&= \sum_{j,m=1}^{N(\omega_0)} \mathbb{E}\Big[\mathbf{U}^{\xi}_{jm}(\omega_0,L)\big(\tilde{a}(\omega_0),\tilde{b}_x(\omega_0)\big)\Big]\overline{\tilde{\lambda}^{\epsilon}_{mj}(h,\omega_0)} \\
&+ \sum_{j=1}^{N(\omega_0)} \int_{\xi}^{k^2(\omega_0)} e^{\frac{1}{2}(\Gamma^c_{jj}(\omega_0)-\Gamma^1_{jj}(\omega_0)-\Lambda^{c,\xi}_j(\omega_0))L-\frac{i}{2}(\Gamma^s_{jj}(\omega_0)-\Lambda^{s,\xi}_j(\omega_0))L} M_{\gamma,j}(\omega_0) \\
&\qquad \times \phi_j(\omega_0,x_0)\phi_{\gamma}(\omega_0,x)e^{i(\sqrt{\gamma}-\beta_j(\omega_0))\big(-L_S+\frac{L}{\epsilon}\big)} e^{-ih\beta'_j(\omega_0)\frac{L}{\sqrt{\epsilon}}} e^{-i\frac{h^2}{2}\beta''_j(\omega_0)L} d\gamma \\
&+ \int_{\xi}^{k^2(\omega_0)} \sum_{m=1}^{N(\omega_0)} e^{\frac{1}{2}(\Gamma^c_{mm}(\omega_0)-\Gamma^1_{mm}(\omega_0)-\Lambda^{c,\xi}_m(\omega_0))L+\frac{i}{2}(\Gamma^s_{mm}(\omega_0)-\Lambda^{s,\xi}_m(\omega_0))L} M_{m\gamma'}(\omega_0) \\
&\qquad \times \phi_{\gamma'}(\omega_0,x_0)\phi_m(\omega_0,x)e^{i(\beta_m(\omega_0)-\sqrt{\gamma'})\big(-L_S+\frac{L}{\epsilon}\big)} e^{ih\beta'_m(\omega_0)\frac{L}{\sqrt{\epsilon}}} e^{i\frac{h^2}{2}\beta''_m(\omega_0)L} d\gamma' \\
&+ \int_{\xi}^{k^2(\omega_0)} \int_{\xi}^{k^2(\omega_0)} M_{\gamma\gamma'}(\omega_0)\phi_{\gamma'}(\omega_0,x_0)\phi_{\gamma}(\omega_0,x)e^{i(\sqrt{\gamma}-\sqrt{\gamma'})\big(-L_S+\frac{L}{\epsilon}\big)} d\gamma d\gamma'.
\end{aligned}$$

We can see that the three last terms give a term of the form

$$\int_{\xi}^{k^2(\omega)} \phi_{\gamma}(\omega,x)\phi_{\gamma}(\omega,y)e^{i\sqrt{\gamma}\frac{L}{\epsilon}} = \mathcal{O}(\epsilon)$$

uniformly on bounded subset of $[0,+\infty)^2$. We recall that the radiating components are very small, of order $\mathcal{O}(\epsilon)$. We cannot observe the recompression of the radiating components by the time-reversal mechanism, because it holds only on a set with null Lebesgue measure.

Consequently,

$$\begin{aligned}
&\mathbb{E}\Big[p_{TR}\Big(\frac{t_{obs}}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)\Big]e^{i\omega_0\big(\frac{t_{obs}-t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\big)} = \frac{1}{4}\sum_{j,m=1}^{N(\omega_0)}\sqrt{\frac{\beta_m(\omega_0)}{\beta_j(\omega_0)}}e^{i(\beta_m(\omega_0)-\beta_j(\omega_0))\frac{L}{\epsilon}} \\
&\times M_{mj}(\omega_0)K^{\omega_0}_{j,m,L} * f\left(\frac{(\beta'_m(\omega_0)-\beta'_j(\omega_0))L+t_1-t_{obs}}{\sqrt{\epsilon}}-t\right)\mathbb{E}\Big[\mathbf{U}^{\xi}_{jm}(\omega_0,L)\big(\tilde{a}(\omega_0),\tilde{b}_x(\omega_0)\big)\Big] \\
&+\mathcal{O}(\sqrt{\epsilon}),
\end{aligned}$$

where $K^{\omega_0}_{j,m,L}$ are defined by (3.18) page 129. From Proposition 3.7, for $m \neq j$, we get

$$\begin{aligned}
\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[p_{TR}\Big(\frac{t_{jm}}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)\Big]&e^{i\omega_0\big(\frac{t_{jm}-t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}}\big)}e^{-i(\beta_m(\omega_0)-\beta_j(\omega_0))\big(-L_S+\frac{L}{\epsilon}\big)} \\
&= e^{Q_{jm}(\omega_0)L}M_{jm}(\omega_0)K^{\omega_0}_{j,m,L} * f(-t)\cdot\frac{1}{4}\phi_j(\omega_0,x_0)\phi_m(\omega_0,x),
\end{aligned}$$

where the times t_{jm} are defined by (3.19). Then, at these times we can observe the refocused waves obtained in the homogeneous case but with the damping terms $e^{Q_{jm}(\omega_0)L}$. The amplitude of the coherent refocused waves at times t_{jm} decays exponentially with respect to the propagation distance L, and therefore becomes negligible for long propagation distance.

Now, for $t_{obs} = t_1$, we have from Proposition 3.7

$$\lim_{\xi\to 0}\lim_{\epsilon\to 0}\mathbb{E}\Big[p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)\Big]e^{i\omega_0\frac{t}{\sqrt{\epsilon}}} = f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)} M_{jj}(\omega_0)\tilde{\mathcal{T}}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x),$$

where $\tilde{\mathcal{T}}_j^l(\omega_0,L)$ are the asymptotic covariance (3.21) which satisfie the coupled power equations (3.20). Here, we have a contribution of all the modes. In the case of a random waveguide we have a coupling between the modes during the propagation of the two steps of the experiment. As in the case of a homogeneous waveguide, at time $t_{obs} = t_1$, the time-reversal mechanism produces a coupling of a mode with itself through $M_{jj}(\omega_0)$, which imposes the form of the coupling produced by the random waveguide through $\tilde{\mathcal{T}}_j^l(\omega_0,z)$.

Let us remark that we can study $\tilde{\mathcal{T}}_j^l(\omega_0,z)$ as in Section 2.5.1, by using a probabilistic interpretation. Let us consider

$$\mathcal{S}_+^{N(\omega_0)} = \Big\{X\in\mathbb{R}^{N(\omega_0)},\ X_j\geq 0\quad \forall j\in\{1,\dots,N(\omega_0)\}\text{ and }\|X\|^2_{2,\mathbb{R}^{N(\omega_0)}} = \langle X,X\rangle_{\mathbb{R}^{N(\omega_0)}} = 1\Big\}$$

with $\langle X,Y\rangle_{\mathbb{R}^{N(\omega_0)}} = \sum_{j=1}^{N(\omega_0)} X_jY_j$ for $(X,Y)\in(\mathbb{R}^{N(\omega_0)})^2$, and

$$D_d(\omega_0) = diag(D_1(\omega_0),\dots,D_{N(\omega)}(\omega_0)), \tag{3.22}$$

where

$$D_j(\omega_0) = \Lambda_j^c(\omega_0) - \big[\Gamma_{jj}^c(\omega_0) - \tilde{\Gamma}_{jj}^c(\omega_0)\big] + \big[\Gamma_{jj}^1(\omega_0) - \tilde{\Gamma}_{jj}^1(\omega_0)\big].$$

Let us begin with the case where the two processes V^1 and V^2 are independent, that is the case in which $\tilde{\gamma}_0(x,y) = 0\ \forall(x,y)\in[0,d]^2$. In this case, the asymptotic covariances (3.21) become the square modulus of the asymptotic mean amplitude for the jth propagating modes (2.45),

$$\tilde{\mathcal{T}}_j^l(\omega_0,L) = e^{\big(-\Lambda_j^c(\omega_0)+\Gamma_{jj}^c(\omega_0)-\Gamma_{jj}^1(\omega_0)\big)L}\delta_{jl}.$$

Then, the mode-dependent and frequency-dependent dispersion produced during the first step of the experiment is compensated by the time-reversal mechanism. However, the mode-dependent and frequency-dependent attenuation is equal to the one of a wave which propagates over a distance $2L$. Therefore, in this particular case the time-reversal mechanism cannot recompress efficiently the recorded field during the second step of the experiment. The reason is that the two realizations of the random medium are much too different between the two steps of the time-reversal experiment.

In what follows, we shall assume that V^1 and V^2 are not independent anymore. However, even in this case, we have the following result on $\tilde{\mathcal{T}}_j^l(\omega_0,L)$.

Theorem 3.5 *Let us assume that the energy transport matrix $\tilde{\Gamma}^c(\omega_0)$ is irreducible. Then, we have*

$$\lim_{L\to+\infty}\frac{1}{L}\ln\left[\sum_{j=1}^{N(\omega)}\tilde{\mathcal{T}}_j^l(\omega_0,L)\right] = -\tilde{\Lambda}_\infty(\omega_0)$$

with

$$\tilde{\Lambda}_\infty(\omega_0) = \inf_{X\in\mathcal{S}_+^{N(\omega_0)}}\langle\big(-\tilde{\Gamma}^c(\omega_0)+D_d(\omega_0)\big)X,X\rangle_{\mathbb{R}^{N(\omega_0)}} > 0,$$

and where $D_d(\omega_0)$ is defined by (3.22).

From this result we get the following inequalities

$$0 < \min_{j\in\{1,\dots,N(\omega_0)\}} D_j(\omega_0) \quad \leq \quad \tilde{\Lambda}_\infty(\omega_0) \quad \leq \quad \overline{D}(\omega_0) = \frac{1}{N(\omega_0)} \sum_{j=1}^{N(\omega_0)} D_j(\omega_0).$$

Moreover, this result means that if the two realizations of the random medium are not fully correlated, that is $\mu \in [0,1)$ with the assumption (3.23), the amplitude of the refocused wave decays exponentially with the propagation distance, even if the radiation losses are negligible.

Let us consider the case of a strong coupling process, that is we assume that the energy transport matrix $\tilde{\Gamma}^c(\omega_0)$ can be replaced by $\frac{1}{\tau}\tilde{\Gamma}^c(\omega_0)$ with $\tau \ll 1$. Consequently, the decay rate in this regime is given by

$$\lim_{\tau\to 0} \tilde{\Lambda}^\tau_\infty(\omega_0) = \overline{D}(\omega_0),$$

and we also have

$$\lim_{\tau\to 0} \tilde{\mathcal{T}}^{\tau,l}_j(\omega_0, L) = \frac{1}{N(\omega)} \exp\Big(- \overline{D}(\omega_0)L\Big).$$

Let us remark that in the strong coupling regime the decay rate takes its largest value.

In order to investigate some particular cases relative to the changing medium, let us assume that $\forall (x,y) \in [0,d]^2$,

$$\gamma_0(x,y) = \mu\, \tilde{\gamma}_0(x,y) \quad \text{for } \mu \in (0,1]. \tag{3.23}$$

This assumption implies that

$$\tilde{\Gamma}^1(\omega_0) = \mu\Gamma^1(\omega_0) \quad \text{and} \quad \tilde{\Gamma}^c(\omega_0) = \mu\Gamma^c(\omega_0).$$

In the case of weak correlation, that is $\mu \ll 1$, the asymptotic decay rate is given by

$$\lim_{\mu\to 0} \tilde{\Lambda}^\mu_\infty(\omega_0) = \min_{j\in\{1,\dots,N(\omega_0)\}} \Lambda^c_j(\omega_0) + \Gamma^1_{jj}(\omega_0) - \Gamma^c_{jj}(\omega_0) > 0,$$

and we also have

$$\lim_{\mu\to 0} \tilde{\mathcal{T}}^{\mu,l}_j(\omega_0, L) = e^{\big(-\Lambda^c_j(\omega_0)+\Gamma^c_{jj}(\omega_0)-\Gamma^1_{jj}(\omega_0)\big)L}\delta_{jl},$$

which corresponds to the case $\mu = 0$.

More generally, for any $\mu \in [0,1)$ there exists a constant $K_{N(\omega_0)} > 0$ such that

$$\sum_{j=1}^{N(\omega_0)} \tilde{\mathcal{T}}^l_j(\omega_0, L) \leq \exp\Big(- K_{N(\omega_0)}(1-\mu)L\Big),$$

and then the amplitude of the mean refocused wave decays exponentially with respect the propagation distance L.

In the case where μ is close to 1, that is a strong correlation regime, the asymptotic decay rate in this case is given by

$$\lim_{\mu\to 1} \tilde{\Lambda}^\mu_\infty(\omega_0) = \inf_{X\in\mathcal{S}^{N(\omega_0)}_+} \big\langle\big(- \Gamma^c(\omega_0) + \Lambda^c_d(\omega_0)\big)X, X\big\rangle_{\mathbb{R}^{N(\omega_0)}},$$

and we also have

$$\lim_{\mu\to 1} \tilde{\mathcal{T}}^{\mu,l}_j(\omega_0, L) = \mathcal{T}^l_j(\omega_0, L)$$

uniformly on each bounded subset of $[0,+\infty)$, which corresponds to the case $\mu = 1$. Here, $\Lambda^c_d(\omega_0)$ is defined in Section 2.5.1, and $\mathcal{T}^l(\omega_0, L)$ is the solution of the coupled power equations (2.47) page 53. This last case will be studied more closely in Section 3.4.7 using the high-frequency approximation developed in Chapter 2.

3.4.6 Stability of the Refocused Wave

An important property of the time-reversal experiment is the stabilization of the refocused wave. This property has been shown in [18] in the one-dimensional context, and in [25, Chapter 20] in the case of a waveguide. However, when the realizations of the random medium are not the same between the two steps of the experiment, it has been shown in [3], in the one-dimensional context, that the loss of the statistical stability is related to the degree of correlation between the two realizations of the random medium. In the three-dimensional context with the parabolic approximation, it has been shown in [12], that the statistical stability is not affected by the change of the random medium between the two steps of the experiment. In this section we study, in the context of a waveguide, the effects of the change of the random medium on the statistical stability.

We recall that the mean refocused wave at the source location in the asymptotic $\epsilon \to 0$ is given by

$$\lim_{\epsilon\to 0}\mathbb{E}\Big[p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}\Big] = f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)} M_{jj}(\omega_0)\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x),$$

where the asymptotic covariances $\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,L)$, defined by (3.21), are the solution of

$$\frac{d}{dz}\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,z) = -(1-\mu)\Gamma^1_{jj}(\omega_0)\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,z) + \mathcal{L}^{\mu}\Big(\tilde{\mathcal{T}}^{\xi,l}(\omega_0,z)\Big)(j),$$

with

$$\mathcal{L}^{\xi,\mu}\phi(j) = -\Lambda_j^{c,\xi}(\omega_0)\phi(j) + (1-\mu)\Gamma^c_{jj}(\omega_0)\phi(j) + \mu\sum_{n=1}^{N(\omega_0)}\Gamma^c_{nj}(\omega_0)\big(\phi(n)-\phi(j)\big),$$

and $\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,0)=\delta_{jl}$. Now, let us introduce another probabilistic representation, in terms of the solution of a stochastic differential equation, for $\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,L)$. Let $(B^j)_{j\in\{1,\dots,N(\omega_0)\}}$ be a family of independent one-dimensional standard Brownian motions. We recall that $\Gamma^1(\omega_0)$ is a nonnegative symmetric matrix and then admits a unique symmetric square root that we denote by $\sqrt{\Gamma^1(\omega_0)}$. Let

$$Z_j(\omega_0,z) = \sum_{j=1}^{N(\omega_0)}\Big[\sqrt{\Gamma^1(\omega_0)}\Big]_{jl}B^l_z \quad \forall j\in\{1,\dots,N(\omega_0)\},$$

and $\big(X^{\xi,l}(\omega_0,.)\big)_{j\in\{1,\dots,N(\omega_0)\}}$ be the unique solution of the system of coupled Stratonovich stochastic differential equations

$$dX_j^{\xi,l}(\omega_0,z) = \mathcal{L}^{\xi,\mu}\big(X^{\xi,l}(\omega_0,z)\big)(j)dz + i\sqrt{2(1-\mu)}\,X_j^{\xi,l}(\omega_0,z)\circ dZ_j(\omega_0,z),$$

with $X_j^{\xi,l}(\omega_0,0)=\delta_{jl}$. Consequently, we have

$$\tilde{\mathcal{T}}_j^{\xi,l}(\omega_0,L) = \tilde{\mathbb{E}}\Big[X_j^{\xi,l}(\omega_0,L)\Big],$$

where the expectation $\tilde{\mathbb{E}}$ is taken with respect to the law of $(B^j)_{j\in\{1,\dots,N(\omega_0)\}}$, and

$$\begin{aligned}\lim_{\epsilon\to 0}&\mathbb{E}\Big[p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},.,L_S\Big)e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}\Big]\\ &= \tilde{\mathbb{E}}\Big[f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)} M_{jj}(\omega_0)X_j^{\xi,l}(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x)\Big].\end{aligned} \tag{3.24}$$

The following proposition extends this result.

Proposition 3.8 *The refocused wave* $p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}$ *converges in distribution as* $\epsilon\to 0$ *as a continuous process in* $(t,x)\in\mathbb{R}\times[0,+\infty)$ *to*

$$p^{\xi}_{TR}(t_1,t,x,L_S)=f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}M_{jj}(\omega_0)X^{\xi,l}_j(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x).$$

Moreover, $p^{\xi}_{TR}(t_1,t,x,L_S)$ *converges in distribution as* $\xi\to 0$ *as a continuous process in* (t,x) *to*

$$p_{TR}(t_1,t,x,L_S)=f(-t)\cdot\frac{1}{4}\sum_{j,l=1}^{N(\omega_0)}M_{jj}(\omega_0)X^{l}_j(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x),$$

where $\big(X^l(\omega_0,.)\big)_{j\in\{1,\dots,N(\omega_0)\}}$ *is the unique solution of the system of coupled Stratonovich stochastic differential equations*

$$dX^l_j(\omega_0,z)=\mathcal{L}^{\mu}\big(X^l(\omega_0,z)\big)(j)dz+i\sqrt{2(1-\mu)}\,X^l_j(\omega_0,z)\circ dZ_j(\omega_0,z),\tag{3.25}$$

with $X^l_j(\omega_0,0)=\delta_{jl}$, *and*

$$\mathcal{L}^{\mu}\phi(j)=-\Lambda^c_j(\omega_0)\phi(j)+(1-\mu)\Gamma^c_{jj}(\omega_0)\phi(j)+\mu\sum_{n=1}^{N(\omega_0)}\Gamma^c_{nj}(\omega_0)\big(\phi(n)-\phi(j)\big).$$

Consequently, the spatial profile of the refocused wave at the source location is the superposition of the $N(\omega_0)$-discrete propagating modes with random weights, which depend on the time-reversal mirror through the coefficients $M_{jj}(\omega_0)$ and on the solution of a stochastic differential equation driven by a family of $N(\omega_0)$-independent Brownian motions.

Let us remark that in the case $\mu=1$, the limit in distribution of the refocused wave is deterministic, and therefore the convergence holds in probability. The stabilization phenomenon of the refocused wave has been already observed in the context of waveguides in [25, Chapter 20] for instance.

Proof We begin by proving the tightness of the refocused wave and next we study the convergence of all finite-dimentional distributions. First of all, let us remark that

$$\sup_{\epsilon\in(0,1)}\sup_{x\in[0,+\infty)}\sup_{t\in\mathbb{R}}\Big|p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\sqrt{\epsilon}},x,L_S\Big)e^{i\omega_0\frac{t}{\sqrt{\epsilon}}}\Big|\leq K,$$

where K is a non random constant since the transfer operators $\mathbf{T}^{1,\xi,\epsilon}$ and $\mathbf{T}^{2,\xi,\epsilon}$ are unitary. Therefore, $\forall\tau>0$,

$$\begin{aligned}\sup_{\epsilon\in(0,1)}\sup_{|x_1-x_2|+|s_1-s_2|\leq\tau}&\Big|p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{s_1}{\sqrt{\epsilon}},x_1,L_S\Big)e^{i\omega_0\frac{s_1}{\sqrt{\epsilon}}}-p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{s_2}{\sqrt{\epsilon}},x_2,L_S\Big)e^{i\omega_0\frac{s_2}{\sqrt{\epsilon}}}\Big|\\&\leq K\Big[|1-e^{i\omega_0(s_2-s_1)}|\\&\quad+\int|\widehat{f}(h)|\cdot\Big[|1-e^{ih(s_1-s_2)}|+\|\tilde{b}_{x_1}(\omega_0+\sqrt{\epsilon}h)-\tilde{b}_{x_2}(\omega_0+\sqrt{\epsilon}h)\|_{\mathcal{H}^{\omega_0+\sqrt{\epsilon}h}_{\xi}}\Big]dh\Big]\\&\leq K\tau,\end{aligned}$$

and then $\forall\eta>0$

$$\lim_{\tau\to 0}\overline{\lim_{\epsilon\to 0}}\,\mathbb{P}\left(\sup_{\substack{|x_1-x_2|+\\|s_1-s_2|\leq\tau}}\Big|p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{s_1}{\sqrt{\epsilon}},x_1,L_S\Big)e^{i\omega_0\frac{s_1}{\sqrt{\epsilon}}}-p_{TR}\Big(\frac{t_1}{\epsilon}+\frac{s_2}{\sqrt{\epsilon}},x_2,L_S\Big)e^{i\omega_0\frac{s_2}{\sqrt{\epsilon}}}>\eta\Big|\right)=0.$$

We have already shown the convergence of the first moment (3.24). Now, for the high-order moments we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(e^{i\omega_0\frac{s_1}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_1}{\sqrt{\epsilon}},x_1,L_S)\Big)^{r_1}\cdots\Big(e^{i\omega_0\frac{s_q}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_q}{\sqrt{\epsilon}},x_q,L_S)\Big)^{r_q}\Big]\\
&=\frac{1}{(8\pi)^r}\int\cdots\int\prod_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}dh_{i_1i_2}\overline{\hat{f}(h_{i_1i_2})}e^{-ih_{i_1i_2}s_{i_1}}\mathbb{E}\Big[\prod_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}\\
&\big\langle\mathbf{U}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h_{i_1i_2},L)\big(\tilde{a}(\omega_0+\sqrt{\epsilon}h_{i_1i_2}),\tilde{b}(\omega_0+\sqrt{\epsilon}h_{i_1i_2})\big)\\
&\qquad\qquad,\lambda^\epsilon(\omega_0+\sqrt{\epsilon}h_{i_1i_2})\big\rangle_{\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h_{i_1i_2}}\otimes\mathcal{H}_\xi^{\omega_0+\sqrt{\epsilon}h_{i_1i_2}}}\Big],
\end{aligned}$$

with $r=\sum_{i_1=1}^q r_{i_1}$. Following, the proof of Theorem 2.1, one can show an asymptotic theorem as ϵ goes to 0 for the process

$$\big(\mathbf{U}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h_1,z),\dots,\mathbf{U}^{\xi,\epsilon}(\omega_0+\sqrt{\epsilon}h_m,z)\big)\quad\text{with } h_j\neq h_l,\forall j\neq l,$$

and which takes its values in the space $\bigotimes_{j=1}^m(\mathcal{H}_\xi^{\omega_0+h_c}\otimes\mathcal{H}_\xi^{\omega_0+h_c})$. Then, using the perturbed-test-function method we have

$$\begin{aligned}
&\lim_{\epsilon\to0}\mathbb{E}\Big[\Big(e^{i\omega_0\frac{s_1}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_1}{\sqrt{\epsilon}},x_1,L_S)\Big)^{r_1}\cdots\Big(e^{i\omega_0\frac{s_q}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_q}{\sqrt{\epsilon}},x_q,L_S)\Big)^{r_q}\Big]\\
&=\frac{1}{4^r}\prod_{1\leq i_1\leq q}f(-s_{i_1})^{r_{i_1}}\cdot\sum\prod_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}M_{j_{i_1i_2}j_{i_1i_2}}(\omega_0)\phi_{l_{i_1i_2}}(\omega_0,x_0)\phi_{l_{i_1i_2}}(\omega_0,x_{i_1})\tilde{\mathcal{T}}_{\mathbf{j}}^{\xi,\mathbf{l}}(\omega_0,L)
\end{aligned}$$

with $\mathbf{j}=(j_{i_1i_2})_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}$ and $\mathbf{l}=(l_{i_1i_2})_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}$, and where

$$\frac{d}{dz}\tilde{\mathcal{T}}_{\mathbf{j}}^{\xi,\mathbf{l}}(\omega_0,z)=-(1-\mu)\Big(\sum\Gamma^1_{j_{i_1i_2}\tilde{j}_{i_1i_2}}(\omega_0)\Big)\tilde{\mathcal{T}}_{\mathbf{j}}^{\xi,\mathbf{l}}(\omega_0,z)+\sum_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}\mathcal{L}^{\xi,\mu}(\tilde{\mathcal{T}}^{\xi,\mathbf{l}}(\omega_0,z))(j_{i_1i_2}),$$

with $\tilde{\mathcal{T}}_{\mathbf{j}}^{\xi,\mathbf{l}}(\omega_0,0)=\prod_{i_1,i_2}\delta_{j_{i_1i_2}l_{i_1i_2}}$. $\mathcal{L}^{\xi,\mu}(\tilde{\mathcal{T}}^{\xi,\mathbf{l}}(\omega_0,z))(j_{i_1i_2})$ means that the operator $\mathcal{L}^{\xi,\mu}$ acts only on the component $j_{i_1i_2}$. Therefore, we get that

$$\tilde{\mathcal{T}}_{\mathbf{j}}^{\xi,\mathbf{l}}(\omega_0,L)=\tilde{\mathbb{E}}\Big[\prod_{\substack{1\leq i_1\leq q\\1\leq i_2\leq r_{i_1}}}X_{j_{i_1i_2}}^{\xi,l_{i_1i_2}}(\omega_0,L)\Big],$$

where the expectation $\tilde{\mathbb{E}}$ is taken with respect to the law of the family $(B^j)_{j\in\{1,\dots,N(\omega_0)\}}$ of independent Brownian motion. Consequently

$$\begin{aligned}
&\lim_{\epsilon\to0}\mathbb{E}\Big[\Big(e^{i\omega_0\frac{s_1}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_1}{\sqrt{\epsilon}},x_1,L_S)\Big)^{r_1}\cdots\Big(e^{i\omega_0\frac{s_q}{\sqrt{\epsilon}}}p_{TR}(\frac{t_1}{\epsilon}+\frac{s_q}{\sqrt{\epsilon}},x_q,L_S)\Big)^{r_q}\Big]\\
&\qquad=\tilde{\mathbb{E}}\Big[\Big(p^\xi_{TR}(t_1,s_1,x_1,L_S)\Big)^{r_1}\cdots\Big(p^\xi_{TR}(t_1,s_q,x_q,L_S)\Big)^{r_q}\Big].
\end{aligned}$$

This concludes the first part of Proposition 3.8. For the second part of Proposition 3.8, using Îto's formula we get that, $\forall\xi$ and $\forall z\geq0$

$$\|X^{\xi,l}(z)\|^2_{\mathbb{C}^{N(\omega_0)}}\leq1\quad\text{a.s.},$$

and then

$$\tilde{\mathbb{E}}[\|X^{\xi,l}(L) - X^l(L)\|^2_{\mathbb{C}^{N(\omega_0)}}] \leq K \int_0^L \tilde{\mathbb{E}}[\|X^{\xi,l}(u) - X^l(u)\|^2_{\mathbb{C}^{N(\omega_0)}}]du$$
$$+ L \sup_{j\in\{1,\dots,N(\omega_0)\}} |\Lambda^c_j(\omega_0) - \Lambda^{c,\xi}_j(\omega_0)|^2.$$

Consequently, using the Gronwall's inequality, $\forall \eta > 0$

$$\lim_{\xi\to 0} \tilde{\mathbb{P}}\left(\sup_{x\in[0,+\infty)} \sup_{t\in\mathbb{R}} \left| p^{\xi}_{TR}(t_1,t,x,L_S) - p_{TR}(t_1,t,x,L_S)\right| > \eta \right) = 0.$$

This concludes the proof of Proposition 3.8. ■

Let us investigate the regime $\mu \to 0$, that is, when the two realizations of the random medium between the two steps of time-reversal experiment are weakly correlated. In this regime the system of stochastic differential equations (3.25) take a particular form, from which we can get an explicit expression of the solution. Now, let us denote by $X^{\mu,l}_j(\omega_0,.)$ the solution of (3.25) and by $X^{0,l}_j(\omega_0,.)$ the solution of the following stochastic differential equation

$$dX^{0,l}_j(\omega_0,z) = \big[-\Lambda^c_j(\omega_0) + \Gamma^c_{jj}(\omega_0) - \Gamma^1_{jj}(\omega_0)\big] X^{0,l}_j(\omega_0,z) + i\sqrt{2}X^{0,l}_j(\omega_0,L)dZ_j(\omega_0,z)$$

with $X^{0,l}_j(\omega_0,0) = \delta_{jl}$. Because there is no coupling between the $N(\omega_0)$-stochastic differential equations, we have

$$X^{0,l}_j(\omega_0,z) = \delta_{jl} e^{\left(-\Lambda^c_j(\omega_0)+\Gamma^c_{jj}(\omega_0)\right)L + i\sqrt{2}Z_j(\omega_0,L)}.$$

Proposition 3.9 $\forall \eta > 0$, *we have*

$$\lim_{\mu\to 0} \tilde{\mathbb{P}}\left(\sup_{x\in[0,+\infty)} \sup_{t\in\mathbb{R}} \left| p^{\mu}_{TR}(t_1,t,x,L_S) - p^0_{TR}(t_1,t,x,L_S)\right| > \eta \right) = 0,$$

where

$$p^0_{TR}(t_1,t,x,L_S) = f(-t)\cdot \frac{1}{4}\sum_{j=1}^{N(\omega_0)} M_{jj}(\omega_0) e^{\left(-\Lambda^c_j(\omega_0)+\Gamma^c_{jj}(\omega_0)\right)L + i\sqrt{2}Z_j(\omega_0,L)} \phi_j(\omega_0,x_0)\phi_j(\omega_0,x).$$

Consequently, in the weakly correlated regime $\mu \to 0$, the refocused wave is the superposition of the $N(\omega_0)$-propagating modes with weights depending of the time-reversal mirror through $M_{jj}(\omega_0)$, a damping term $e^{\left(-\Lambda^c_j(\omega_0)+\Gamma^c_{jj}(\omega_0)\right)L}$, and a random phase $e^{i\sqrt{2}Z_j(\omega_0,L)}$.

3.4.7 Mean Refocused Field in the Case $\mu \to 1$

This section is devoted to the study, in the strongly correlated regime $\mu \to 1$, of the transverse profile of the refocused wave in the high-frequency regime $\omega_0 \to +\infty$.

First of all, in the particular regime $\mu \to 1$, we get the stabilization of the refocused wave. In fact, using the Îto's formula we know that, $\forall \mu \in [0,1]$, $\|X^{\mu,l}(L)\|^2_{\mathbb{C}^{N(\omega_0)}} \leq 1$ a.s. , and then

$$\tilde{\mathbb{E}}[\|X^{l,\mu}(\omega_0,L) - \mathcal{T}^l(\omega_0,L)\|^2_{\mathbb{C}^{N(\omega_0)}}] \leq K_1 \int_0^L \tilde{\mathbb{E}}[\|X^{l,\mu}(\omega_0,s) - \mathcal{T}^l(\omega_0,s)\|^2_{\mathbb{C}^{N(\omega_0)}}]ds + K_2(1-\mu),$$

where $\mathcal{T}^l(\omega_0,.)$ is the solution of the coupled power equations (2.47) page 53. Consequently, we have the following result.

Proposition 3.10 *$\forall \eta > 0$, we have*

$$\lim_{\mu \to 1} \tilde{\mathbb{P}} \left(\sup_{x \in [0,+\infty)} \sup_{t \in \mathbb{R}} \left| p^{\mu}_{TR}(t_1, t, x, L_S) - p^{1}_{TR}(t_1, t, x, L_S) \right| > \eta \right) = 0,$$

where

$$p^{1}_{TR}(t_1, t, x, L_S) = f(-t) H^{\alpha_M}_{x_0}(\omega_0, x),$$

with

$$H^{\alpha_M}_{x_0}(\omega_0, x, L) = \frac{1}{4} \sum_{j,l=1}^{N(\omega_0)} M_{jj}(\omega_0) \mathcal{T}^{l}_{j}(\omega_0, L) \phi_l(\omega_0, x_0) \phi_l(\omega_0, x).$$

In what follows we consider the band-limiting idealization, introduced in Section 2.5.2, to study the transverse profile $H^{\alpha_M}_{x_0}$ in the high-frequency regime $\omega_0 \to +\infty$. With this assumption $\mathcal{T}^{l}_{j}(\omega_0, .)$ satisfies (2.51) page 59.

Mean Refocused Field with Radiation Losses

In this section, we study the transverse profile of the refocused wave in the presence of radiation losses.

Proposition 3.11 *For $\alpha_M \in [0, 1)$, the transverse profile of the refocused wave in the high-frequency $\omega_0 \to +\infty$ is given by*

$$\lim_{\omega_0 \to +\infty} \frac{\lambda_{oc}^{1-\alpha_M}}{\theta} H^{\alpha_M}_{x_0}\left(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}, L\right) = \frac{\tilde{d}_2 + \tilde{d}_1}{d} H(\tilde{x}, L),$$

where

$$H(\tilde{x}, L) = \int_0^1 \mathcal{T}_1(L, u) \cos(2\pi u \tilde{x}) du,$$

and $\mathcal{T}_1(L, u)$ is the solution of

$$\frac{\partial}{\partial z} \mathcal{T}_1(z, u) = \frac{\partial}{\partial u} \left(a_\infty(\cdot) \frac{\partial}{\partial u} \mathcal{T}_1 \right)(z, u),$$

with the boundary conditions:

$$\frac{\partial}{\partial u} \mathcal{T}_1(z, 0) = 0, \quad \mathcal{T}_1(z, 1) = 0 \quad \text{and} \quad \mathcal{T}_1(0, u) = 1,$$

$\forall z > 0$. Here,

$$a_\infty(u) = \frac{a_0}{1 - \left(1 - \frac{\pi^2}{a^2 d^2}\right)(\theta u)^2},$$

with $a_0 = \frac{\pi^2 S_0}{2 a n_1^4 d^4 \theta^2}$, $\theta = \sqrt{1 - 1/n_1^2}$, $S_0 = \int_0^d \int_0^d \gamma_0(x_1, x_2) \cos\left(\frac{\pi}{d} x_1\right) \cos\left(\frac{\pi}{d} x_2\right) dx_1 dx_2$. n_1 is the index of refraction in the ocean section $[0, d]$, $1/a = l_{z,x}$ is the correlation length of the random inhomogeneities in the longitudinal direction, and γ_0 is the covariance function of the random inhomogeneities in the transverse direction.

Consequently, the transverse profile of the refocused wave can be expressed in terms of the diffusive continuous model introduced in Section 2.5.2, with a reflecting boundary condition at $u = 0$ (the top of the waveguide) and an absorbing boundary condition at $u = 1$ (the bottom of the waveguide) which represents the radiative loss (see Figure 3.5). As it is illustrated in Section 3.4.8, the radiation losses degrade the quality of the refocusing: the amplitude of the refocused wave decays exponentially with the propagation distance (see Section 2.5.2), and the width of the focal spot increases and converges to an asymptotic value that is significantly larger than the diffraction limit $\lambda_{oc}/(2\theta)$, where λ_{oc} is the carrier wavelength in the ocean section $[0, d]$.

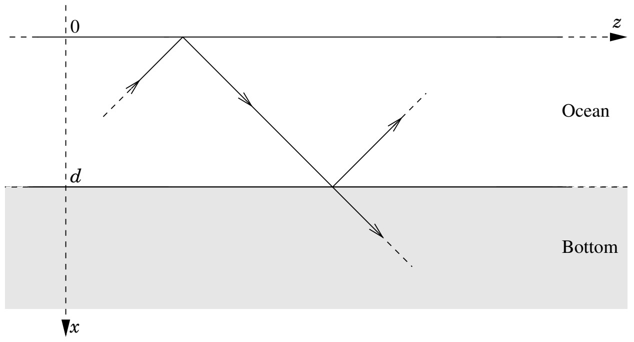


Figure 3.5: Illustration of the radiative loss in the shallow-water random waveguide model.

Proof First, we have

$$\frac{d}{\tilde{d}_2+\tilde{d}_1}\frac{\lambda_{oc}^{1-\alpha_M}}{\theta}H_{x_0}^{\alpha_M}\Big(\omega_0,x_0+\frac{\lambda_{oc}}{\theta}\tilde{x},L\Big)=\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)$$
$$\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\frac{A_j^2 d}{2}\Big[1-\cos\Big(\sigma_j\frac{2d_M+\lambda_{oc}^{\alpha_M}(\tilde{d}_2-\tilde{d}_1)}{d}\Big)\mathrm{sinc}\Big(\sigma_j\frac{\lambda_{oc}^{\alpha_M}(\tilde{d}_2+\tilde{d}_1)}{d}\Big)\Big],$$

where A_j is defined by (2.8) page 35. Moreover, using the probabilistic representation (2.49) used in the proof of Theorem 2.3 page 54,

$$\begin{aligned}\Big|\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}&\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\\&\times\frac{A_j^2 d}{2}\cos\Big(\sigma_j\frac{2d_M+\lambda_{oc}^{\alpha_M}(\tilde{d}_2-\tilde{d}_1)}{d}\Big)\mathrm{sinc}\Big(\sigma_j\frac{\lambda_{oc}^{\alpha_M}(\tilde{d}_2+\tilde{d}_1)}{d}\Big)\Big|\\&\leq\lambda_{oc}^{1-\alpha_M}N\Big[\sum_{j=2}^{N}\frac{1}{\pi(j-1)}\mathbb{P}_{\mu_N}(Y_L^N=j)+\frac{1}{\sigma_1}\mathbb{P}_{\mu_N}(Y_L^N=1)\Big]\\&\leq K\lambda_{oc}^{1-\alpha_M}\ln(N),\end{aligned}$$

where $\big(Y_t^N\big)_{t\geq 0}$ is a jump Markov process with state space $\{1,\dots,N\}$, intensity matrix $\Gamma^c(\omega_0)$, and invariant measure μ_N, which is the uniform distribution over $\{1,\dots,N\}$. Consequently, the transverse profile of the refocused wave is given by

$$\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta).$$

Let $\eta>0$ such that $\eta\ll 1$. We have

$$\begin{aligned}\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}&\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\\&=\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{[N(1-\eta)]}\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\\&\quad+\mathcal{O}(\eta)\\&=\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{[N(1-\eta)]}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)\\&\quad+\mathcal{O}(\eta),\end{aligned}$$

since we recall that

$$\lim_{N\to+\infty} \sup_{j\in\{1,\dots,N-[N^\alpha]\}} \Big| A_j^2(\omega_0) - \frac{2}{d}\Big| = 0. \tag{3.26}$$

Let $f^\eta(v) = \mathbf{1}_{[0,1-\eta]}(v)$, we have

$$\begin{aligned}\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)&\\
=\frac{\lambda_{oc}}{2\theta}\sum_{l=1}^{[N(1-\eta)]}\mathcal{T}_{f^\eta}^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)&\\
+\mathcal{O}(\eta).&\end{aligned}$$

Now, we are able to apply the high-frequency approximation given in Theorem 2.4 page 59.

$$\begin{aligned}\frac{1}{N}\sum_{l=1}^{[N(1-\eta)]}\Big|\mathcal{T}_{f^\eta}^l(\omega_0,L)-\mathcal{T}_{f^\eta}(L,l/N)\Big|&\\
\leq\sum_{l=1}^{[N(1-\eta)]-1}\int_{l/N}^{(l+1)/N}\big|\mathcal{T}_{f^\eta}^{[Nu]}(\omega_0,L)-\mathcal{T}_{f^\eta}(L,[Nu]/N)\big|du&\\
\leq\int_0^1\big|\mathcal{T}_{f^\eta}^N(\omega_0,L,u)-\mathcal{T}_{f^\eta}(L,u)\big|du&\\
+\int_0^1\Big|\mathcal{T}_{f^\eta}(L,u)-\mathcal{T}_{f^\eta}(L,[Nu]/N)\Big|du,&\end{aligned}$$

where the terms on the right side of the last inequality goes to 0 as $\omega_0\to+\infty$ by Theorem 2.4. Then

$$\begin{aligned}\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)&\\
=\frac{\lambda_{oc}}{2\theta}\sum_{l=1}^{[N(1-\eta)]}\mathcal{T}_{f^\eta}(L,l/N)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x_0+\lambda_{oc}\tilde{x}/\theta)&\\
+\mathcal{O}(\eta).&\end{aligned}$$

Moreover, we have

$$\phi_j(\omega_0,x_0)\phi_j(\omega_0,x_0+\frac{\lambda_{oc}}{\theta}\tilde{x})=\frac{A_j^2}{2}\Big[\cos\Big(\sigma_j\frac{\lambda_{oc}}{\theta d}\tilde{x}\Big)-\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\Big],$$

and

$$\begin{aligned}\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)=&\cos\Big((\sigma_j-j\pi)\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\cos\Big(j\pi\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\\
&-\sin\Big((\sigma_j-j\pi)\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\sin\Big(j\pi\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big).\end{aligned}$$

Using the Abel transform, Lemma 2.1 page 36, (3.26), and the continuity of $v\mapsto\mathcal{T}_{f^{\eta_1}}(L,v)$ on $[0,1]$, we get

$$\lim_{\omega_0\to+\infty}\lambda_{oc}\Big|\sum_{l=1}^{[N(1-\eta)]}\mathcal{T}_{f^\eta}(L,l/N)A_l^2\cos\Big(\sigma_j\frac{2x_0+\lambda_{oc}\tilde{x}/\theta}{d}\Big)\Big|=0.$$

Moreover, using (3.26) and the fact that $\lim_{\omega_0} \sup_j \lambda_{oc}|\sigma_j - j\pi| = 0$,

$$\lim_{N\to+\infty} \frac{\lambda_{oc}}{2\theta} \sum_{l=1}^{[N(1-\eta)]} \mathcal{T}_{f^\eta}(L, l/N) A_l^2 \cos\Big(\sigma_j \frac{\lambda_{oc}}{\theta d}\tilde{x}\Big)$$
$$= \lim_{N\to+\infty} \frac{\lambda_{oc}}{2\theta d^2} \sum_{l=1}^{[N(1-\eta)]} \mathcal{T}_{f^\eta}(L, l/N) \cos\Big(2\pi \frac{l}{N}\tilde{x}\Big)$$
$$= (1-\eta) \int_0^{1-\eta} \mathcal{T}_{f^\eta}(L, u) \cos(2\pi u\tilde{x}) du.$$

Consequently, from the decomposition used in the proof of Theorem 2.5 page 60, we have

$$\|\mathcal{T}_{f^\eta}(L, .) - \mathcal{T}_1(L, .)\|_{L^2([0,1])} \leq \|f^\eta - 1\|_{L^2([0,1])},$$

and then

$$\overline{\lim_{\omega_0\to+\infty}} \Big| \frac{\lambda_{oc}}{2\theta} \sum_{j,l=1}^{N} \frac{A_j^2 d}{2} \mathcal{T}_j^l(\omega_0, L)\phi_l(\omega_0, x_0)\phi_l(\omega_0, x_0 + \lambda_{oc}\tilde{x}/\theta)$$
$$- \int_0^1 \mathcal{T}_1(L, v) \cos(2\pi u\tilde{x}) du\Big| \leq K\eta.$$

This concludes the proof of Proposition 3.11.■

In order to study the case $\alpha_M = 1$, let us introduce some notations. Let $\mathcal{E} = \bigcup_{M\geq 1} \mathcal{E}_M$, where

$$\mathcal{E}_M = \left\{ \sum_{j=1}^{M} \alpha_j \phi_j, \quad (\alpha_j)_j \in \mathbb{R}^M \right\},$$

and

$$\phi_j(x) = \sqrt{\frac{2}{d}} \sin\Big(j\frac{\pi}{d}x\Big) \quad \forall x \in [0, d], \forall j \geq 1.$$

Let us remark that $(\phi_j)_j$ is a basis of $L^2(0, d)$.

Proposition 3.12 *For $\alpha_M = 1$, in the continuum limit $N(\omega_0) \gg 1$, we have*

$$\lim_{\omega_0\to+\infty} H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L) = 0$$

in $\mathcal{E}'$, which is the topological dual of $\mathcal{E}$ equipped with the weak topology, and where

$$\lim_{\omega_0\to+\infty} \tilde{H}^1_{x_0}\Big(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}, L\Big) = \theta \frac{\tilde{d}_2 + \tilde{d}_1}{d} H(\tilde{x}, L).$$

Here, $H(\tilde{x}, L)$ is defined in Proposition 3.11.

Proof Let $M \geq 1$ and $f^M = \sum_{j=1}^M \alpha_j \phi_j \in \mathcal{E}_M$. Moreover, let

$$\forall x \in [0, d], \quad \tilde{f}^M(x) = \sum_{j=1}^{M} \alpha_j \phi_j(\omega_0, x).$$

Using (3.26) and because we have

$$\sup_{j\in\{1,\dots,M\}} |\sigma_j - j\pi| = \mathcal{O}\left(\frac{1}{N}\right),$$

then,

$$\sup_{x\in[0,d]} |f^M(x) - \tilde{f}^M(x)| = \mathcal{O}\left(\frac{1}{N}\right).$$

Finally, by letting

$$\tilde{H}^1_{x_0}(\omega_0, x, L) = \frac{\tilde{d}_2 + \tilde{d}_1}{d}\frac{\lambda_{oc}}{2\theta}\sum_{j,l=1}^{N}\frac{A_j^2 d}{2}\mathcal{T}_j^l(\omega_0, L)\phi_l(\omega_0, x_0)\phi_l(\omega_0, x),$$

we have

$$\begin{aligned}\langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L), f^M\rangle_{L^2(0,d)} =&\langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L), f^M - \tilde{f}^M\rangle_{L^2(0,d)}\\ &+ \langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L), \tilde{f}^M\rangle_{L^2(0,d)},\end{aligned}$$

with

$$\begin{aligned}&\left|\langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1(\omega_0, ., L), f^M - \tilde{f}^M\rangle_{L^2(0,d)}\right|\\ &\qquad\leq \frac{K}{N}N\Big[\sum_{j=2}^{N}\frac{1}{\pi(j-1)}\mathbb{P}_{\mu_N}(Y_L^N = j) + \frac{1}{\sigma_1}\mathbb{P}_{\mu_N}(Y_L^N = 1)\Big]\\ &\qquad\leq K\frac{\ln(N)}{N},\end{aligned}$$

and

$$\begin{aligned}&\left|\langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L), \tilde{f}^M\rangle_{L^2(0,d)}\right|\\ &\qquad\leq K\sum_{l=1}^{M}\Big[\mathcal{T}^l_{f^\eta}(\omega_0, L) + \frac{1}{[N\eta]+1}\mathbb{P}(Y_L^N \in \{[N\eta]+1, \dots, N\} | Y_0^N = l)\Big]\end{aligned}$$

for $\eta > 0$, and $f^\eta(v) = \mathbf{1}_{[0,\eta]}$. Therefore, it suffices to study $\sum_{l=1}^M \mathcal{T}^l_{f^\eta}(\omega_0, L)$. To do this, let g^η be a smooth function with compact support included in $[0, 2\eta]$ and such that $0 \leq f^\eta \leq g^\eta \leq f^{2\eta}$. Using the second part of Theorem 2.5 page 60,

$$\overline{\lim_{N\to+\infty}}\sum_{l=1}^{M}\mathcal{T}^l_{g^\eta}(\omega_0, L) = M\mathcal{T}_{g^\eta}(L, 0) \leq M\overline{\mathbb{P}}_0(x(t) \in [0, 2\eta]).$$

Here, we have used the probabilistic representation of $\mathcal{T}^l_{g^\eta}(\omega_0, L)$ introduced in the proof of Theorem 2.5, where $\overline{\mathbb{P}}_0$ is the unique solution of a martingale problem starting from 0. However, the probabilistic representation can be chosen such that the associated diffusion process has transition probabilities absolutely continuous with respect to the Lebesgue measure [27]. Therefore,

$$\lim_{\omega_0\to+\infty}\langle H^1_{x_0}(\omega_0, ., L) - \tilde{H}^1_{x_0}(\omega_0, ., L), \tilde{f}^M\rangle_{L^2(0,d)} = 0,$$

and the rest of the proof is the same as that of Proposition 3.11. ■

Consequently, in the case of a random waveguide, the order of magnitude α_M of the time-reversal mirror plays no role in the transverse profile compared to the homogeneous case (see Section 3.4.7).

Mean Refocused Field with Negligible Radiation Losses

In this section, we study the transverse profile of the refocused wave in the case where the radiation losses are negligible, that is, $\Lambda^c(\omega)$ can be replaced by $\tau\tilde{\Lambda}^c(\omega)$ with $\tau \ll 1$. Therefore, we have

$$\forall L > 0, \qquad \sup_{z\in[0,L]} \|\mathcal{T}_j^{\tau,l}(\omega,z) - \mathcal{T}_j^{0,l}(\omega,z)\|_{2,\mathbb{R}^{N(\omega)}} = \mathcal{O}(\tau),$$

where $\mathcal{T}^{0,l}(\omega,.)$ satisfies

$$\frac{d}{dz}\mathcal{T}_N^{0,l}(z) = \Gamma^c_{N-1\,N}\left(\mathcal{T}_{N-1}^{0,l}(z) - \mathcal{T}_N^{0,l}(z)\right),$$
$$\frac{d}{dz}\mathcal{T}_j^{0,l}(z) = \Gamma^c_{j-1\,j}\left(\mathcal{T}_{j-1}^{0,l}(z) - \mathcal{T}_j^{0,l}(z)\right) + \Gamma^c_{j+1\,j}\left(\mathcal{T}_{j+1}^{0,l}(z) - \mathcal{T}_j^{0,l}(z)\right) \text{ for } j \in \{2,\dots,N-1\},$$
$$\frac{d}{dz}\mathcal{T}_1^{0,l}(z) = \Gamma^c_{2\,1}\left(\mathcal{T}_2^{0,l}(z) - \mathcal{T}_1^{0,l}(z)\right),$$

with $\mathcal{T}_j^{0,l}(0) = \delta_{jl}$. Consequently,

$$\lim_{\tau\to 0}\sup_{x\in[0,+\infty)}\left|H_{x_0}^{\tau,\alpha_M}(\omega_0,x,L) - H_{x_0}^{0,\alpha_M}(\omega_0,x,L)\right| = 0,$$

where

$$H_{x_0}^{0,\alpha_M}(\omega_0,x,L) = \frac{1}{4}\sum_{j,l=1}^{N(\omega_0)} M_{jj}(\omega_0)\mathcal{T}_j^{0,l}(\omega_0,L)\phi_l(\omega_0,x_0)\phi_l(\omega_0,x).$$

Proposition 3.13 *For $\alpha_M \in [0,1)$, with negligible radiation losses, the transverse profile of the refocused wave in the continuum limit $N(\omega_0) \gg 1$ is given by*

$$\lim_{\omega_0\to+\infty}\frac{\lambda_{oc}^{1-\alpha_M}}{\theta}H_{x_0}^{0,\alpha_M}\left(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}, L\right) = \frac{\tilde{d}_2+\tilde{d}_1}{d}\operatorname{sinc}(2\pi\tilde{x}).$$

Proof Following the proof of Propostion 3.11 and using Theorem 2.6 page 63, we get

$$\lim_{\omega_0\to+\infty}\frac{\lambda_{oc}^{1-\alpha_M}}{\theta}H_{x_0}^{0,\alpha_M}\left(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}, L\right) = \frac{\tilde{d}_2+\tilde{d}_1}{d}\int_0^1 \mathcal{T}_1(L,u)\cos(2\pi u\tilde{x})du,$$

where $\mathcal{T}_1(z,v)$ is a solution of

$$\frac{\partial}{\partial z}\mathcal{T}_1(z,u) = \frac{\partial}{\partial u}\left(a_\infty(\cdot)\frac{\partial}{\partial u}\mathcal{T}_1\right)(z,u),$$

with the boundary conditions

$$\frac{\partial}{\partial u}\mathcal{T}_1(z,0) = 0, \quad \frac{\partial}{\partial u}\mathcal{T}_1(z,1) = 0, \quad \text{and} \quad \mathcal{T}_1(0,u) = 1.$$

However, this problem admits only one solution, which is $\mathcal{T}_1(z,u) = 1$. ■

The transverse profile of the refocused wave is studied using the diffusive continuous model introduced in Section 2.5.3, with two reflecting boundary conditions at $u = 0$ (the top of the waveguide) and $u = 1$ (the bottom of the waveguide). Here, the two reflecting boundary conditions mean that there is no radiative loss anymore (see Figure 3.6), and then the energy is conserved. This is for this reason that $\mathcal{T}_1(z,u) = 1$. Consequently, the sinc profile obtained in Proposition 3.13 is the best transverse profile that we can obtain.

In the same way, we have the following result for $\alpha_M = 1$.

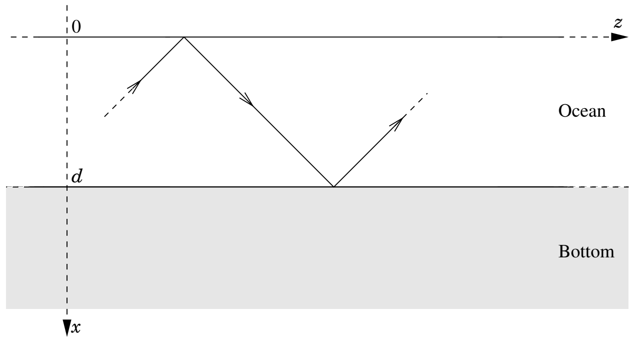


Figure 3.6: Illustration of negligible radiation losses in the shallow-water random waveguide model.

Proposition 3.14 *For $\alpha_M = 1$, in the continuum limit $N(\omega_0) \gg 1$, we have*

$$\lim_{\omega_0 \to +\infty} H^{0,1}_{x_0}(\omega_0, ., L) - \tilde{H}^{0,1}_{x_0}(\omega_0, ., L) = 0$$

in $\mathcal{E}'$, and where

$$\lim_{\omega_0 \to +\infty} \tilde{H}^{0,1}_{x_0}\Big(\omega_0, x_0 + \frac{\lambda_{oc}}{\theta}\tilde{x}, L\Big) = \theta \frac{\tilde{d}_2 + \tilde{d}_1}{d} \operatorname{sinc}(2\pi\tilde{x}).$$

These results are consistent with the ones obtained in [25, Chapter 20] where the authors have obtained the sinc function for transverse profile, and which does not depend on the size of the time-reversal mirror.

3.4.8 Numerical Illustration

In this section we illustrate the spatial focusing of the refocused wave around the source location. First, we represent the evolution of $\mathcal{T}_1(L, u)$, in presence of radiation losses, with respect to L. Here, $\mathcal{T}_1(L, u)$ is the mean mode power for the $[N(\omega_0)u]$th propagating mode in the continuum limit $N(\omega_0) \gg 1$, which is the solution of the partial differential equation in Proposition 3.11.

Second, we represent the spatial profile $H(\tilde{x}, L)$ of the refocused wave, and finally we illustrate the resolution of the refocused wave as the propagation distance L becomes large.

In this section, we consider the following values of the parameters. For the sake of simplicity, we take $a_0 = 1$ and the inverse of the correlation length of the random inhomogeneities in the longitudinal direction is $a = 1$. Moreover, we take $n_1 = 2$ for index of refraction in the ocean section $[0, d]$, and depth $d = 20$.

We saw in Proposition 3.11 and Proposition 3.12 that $\mathcal{T}_1(L, u)$, in the presence of radiation losses, plays an important role in the transverse profile of the refocused wave. In Figure 3.7, we illustrate the influence of the radiation losses on $\mathcal{T}_1(L, u)$ as the propagation distance L increase. As we can see in Figure 3.8 and Figure 3.9, the radiation losses degrade the quality of the refocusing. Moreover, for $L \gg 1$, one can see a threshold of the quality of the resolution since

$$H_{x_0}(\tilde{x}, L) \underset{L \gg 1}{\simeq} e^{\lambda_1 L} \int_0^1 \phi_{\infty,1}(v) dv \int_0^1 \phi_{\infty,1}(u) \cos(2\pi\tilde{x}u) du,$$

where $\lambda_1 < 0$ and $\phi_{\infty,1}$ are defined in the proof of Theorem 2.5 page 60.

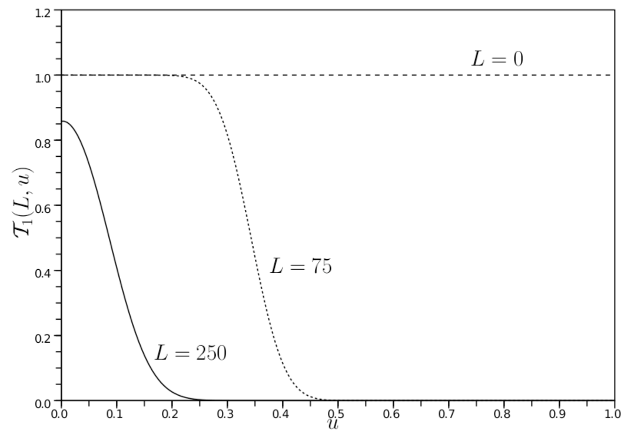
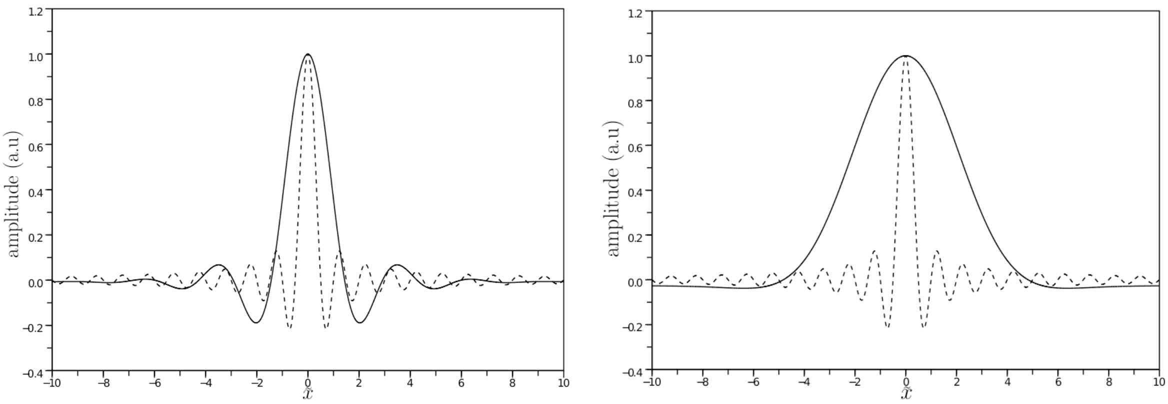


Figure 3.7: Representation of $\mathcal{T}_1(L, u)$, in the presence of radiation losses, with respect to the propagation distance L.



(a) (b)

Figure 3.8: Normalized transverse profile. In (a) and (b) the dashed curves are the transverse profiles in the case where the radiation losses are negligible, and the solid curves represent the transverse profile $H(\tilde{x}, L)$. In (a) we represent $H(\tilde{x}, L)$ with $L = 75$, and in (b) we represent $H(\tilde{x}, L)$ with $L = 250$.

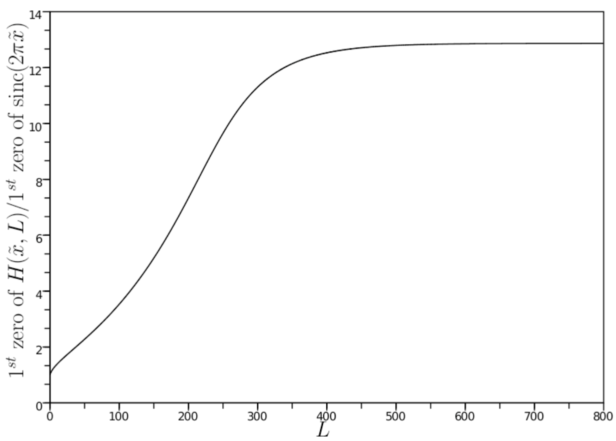


Figure 3.9: Representation of the evolution of the resolution with respect to the propagation distance L.

Conclusion

In this chapter we have analyzed the pulse propagation and the time reversal of waves in a shallow-water acoustic waveguide with random perturbations.

We have shown that a broadband pulse can be decomposed into a superposition of modal waves with different arrival times and different modal speeds. As in [17], the statistics of the transmitted wave can be described by a front stabilization theory. We have studied the incoherent wave fluctuations, which requires the analysis of the distribution of the transfer operator at two nearby frequencies, and we have derived an effective system of transport equations which takes into account the effect of the radiation losses. The intensity of the wave fluctuations is exponentially damped and becomes uniform across the waveguide section $[0, d]$ as long as the propagation distance is large.

We have studied the time-reversal experiment of a broadband pulse in the case where the medium may have changed between the two steps of the experiment. We have shown that the loss of the statistical stability of the refocused wave is related to the degree of correlation between the two realizations of the random medium. In the case where the two realizations are not sufficiently correlated, the amplitude of the refocused wave decreases exponentially with the propagation distance. In the case where the two realizations are sufficiently correlated, we obtain the statistical stability of the refocused wave. Moreover, using the continuous diffusive models developed in Section 2.5.2 and 2.5.3, we have seen that radiation losses degrade the quality of the refocused transverse profile as the propagation distance increases.

In this chapter, we have shown that the size of the focal spot in the time-reversal experiment is, at least in the most favorable case, limited by the diffraction limit. However, we shall see in Chapter 4 that the focal spot can be smaller than the diffraction limit by inserting a strongly heterogeneous random section in the vicinity of the source.

Chapter 4

Time Reversal SuperResolution in Random Waveguides

Introduction

Time-reversal refocusing has been studied in different contexts: in one-dimensional media [18, 25], in three-dimensional randomly layered media [26], in the paraxial approximation [15, 10, 49], and in random waveguides [30, 25]. In all these contexts it has been shown that the focal spot can be smaller than the Rayleigh resolution formula $\lambda L/D$ (where λ is the carrier wavelength, L is the propagation distance, and D is the mirror diameter). However, the focal spot is still larger than the diffraction limit $\lambda/2$.

Mathias Fink and his group at ESPCI have proposed an approach to obtain a *superresolution* effect, that is to refocus beyond the diffraction limit, with a far-field time-reversal mirror [42]. This approach consists in adding a random distribution of scatterers in the vicinity of the source. The proposed physical explanation is as follows. The small-scale features (position and shape) of the source are carried by high evanescent modes, and these modes decay exponentially fast with the propagation distance, so that this information is usually not transmitted up to the time-reversal mirror, which is located in the far field. The random medium located around the source location permits to convert high modes into low propagating modes. In other words, the inhomogeneities of the random slab induce mode coupling, so that the information on small scales of the source is transferred to the propagating modes and reaches the time-reversal mirror. During the time-reversal experiment these modes are regenerated in the vicinity of the source from the backpropagated propagating modes, and therefore they can participate in the refocusing process. An application of this result to wireless communication is presented in [42].

In this Chapter, even though the work of Fink and his group was on time reversal of electromagnetic waves, we consider a two-dimensional acoustic waveguide model. The main goal of this chapter is to present a mathematical proof that the focal spot can indeed be smaller than the diffraction limit. Before the mathematical analysis, we give some physical explanations to describe the important phenomena induced by the insertion of a section in the vicinity of the source for a long waveguide. First, the case of a waveguide with homogeneous speed of propagation c_0 (see Figure 4.1 (a)) is well known; see, for instance, [25], where the authors obtain the classical diffraction limit. Namely, the focal spot has radius equal to the carrier wavelength over two. In this case, the small-scale features (position and shape) of the source are carried by high evanescent modes that decay exponentially fast with the propagation distance. Consequently, these modes do not reach the time-reversal mirror, which is located in the far field. Only low modes are recorded by the time-reversal mirror. In the

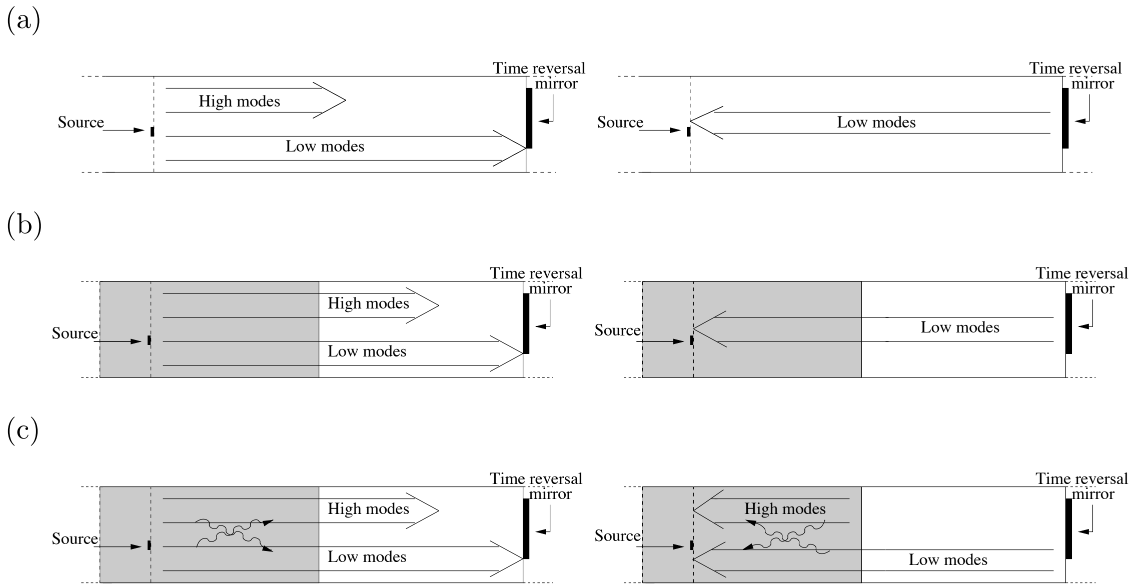


Figure 4.1: Representation of mode propagation in the time reversal experiment. In (a) we represent a homogeneous waveguide, in (b) we add a homogeneous section with low propagation speed, and in (c) we add a randomly heterogeneous section with low background propagation speed.

second step of the time-reversal experiment, the mirror sends back the recorded low modes that carry only the large-scale features of the original source. This loss of information is responsible for the diffraction-limited transverse profile computed in Proposition 4.3. In what follows, we refer to high or low modes relatively to a waveguide with homogeneous speed of propagation c_0. Experiments have shown that the situation changes dramatically when a section of medium with low speed of propagation $c_1 \ll c_0$ is inserted in the vicinity of the source. In this chapter, we will compare the two following cases with the homogeneous case.

First, we assume that a homogeneous section with low speed of propagation is inserted in the vicinity of the source, as illustrated in Figure 4.1 (b), such that some high modes of the previous case are propagating modes in this first section. However, we assume that the major part of the waveguide has speed of propagation c_0 so the high modes and the small-scale features of the source do not reach the time-reversal mirror. Therefore, as in the homogeneous case, only low modes are recorded by the time-reversal mirror and the small-scale features of the source are lost. The transverse profile obtained in this case is described in Proposition 4.2.

Second, if the additional section with low speed of propagation is randomly perturbed, then coupling mechanisms, between propagating modes of the first section, allow small-scale features of the source, which are carried by the high modes, to be transferred to low modes. Even if the high modes do not propagate over large distances in the second part of the waveguide and are not recorded by the time-reversal mirror, a part of the small-scale features of the source reaches the time-reversal mirror since they are carried by the low modes which are recorded by the time-reversal mirror. This fact is illustrated in Figure 4.1 (c). These low modes, time-reversed, will come back to the randomly perturbed section in the second step of the time-reversal experiment, and by coupling mechanisms they will regenerate high modes with the small-scale features of the source. This regeneration of small-scale features of the source is responsible for the superresolution described in Proposition 4.4.

The organization of Chapter 4 is as follows: In the first section, we describe the waveguide model that we consider in this chapter for the experiment. In Section 4.2, we reduce the

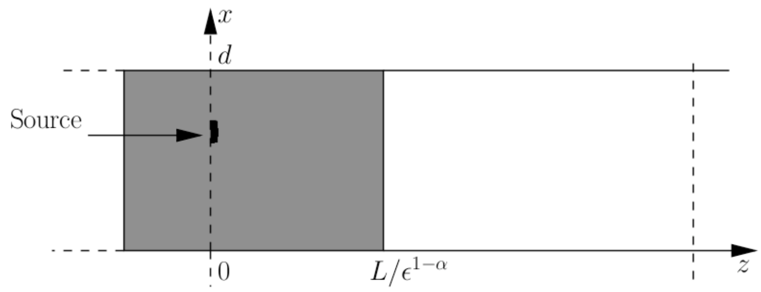


Figure 4.2: Representation of the waveguide model with bounded cross-section $[0,d]$, and two sections $(-\infty, L/\epsilon^{1-\alpha})$ and $(L/\epsilon^{1-\alpha}, +\infty)$.

study of the wave propagation in the random section to the study of a system of differential equations with random coefficients by using a modal decomposition. Moreover, we introduce some assumptions needed for the study of the time-reversal process. In Section 4.3, we state the asymptotic results that we will use in the following section. In Section 4.4, we consider the time-reversal experiment in the random waveguide presented in Section 4.1. We analyze the refocused field to emphasize the superresolution effect and show the statistical stability. Finally, the appendix is devoted to the proofs of the theorems stated in Section 4.3.

4.1 Waveguide Model

For the sake of simplicity, we do not consider in this chapter the same waveguide model as in Chapters 2 and 3. The waveguide model that we consider in this chapter is the same as in [25, Chapter 20] and [30], that is, with a bounded-cross section. As a result, in this Chapter we shall not consider the influence of radiative losses on the time-reversal experiment.

We consider a two-dimensional linear acoustic wave model. The conservation equations of mass and linear momentum are given by

$$\begin{aligned} \rho^\epsilon(x,z)\frac{\partial \mathbf{u}}{\partial t} + \nabla p &= \mathbf{F}^\epsilon, \\ \frac{1}{K^\epsilon(x,z)}\frac{\partial p}{\partial t} + \nabla . \mathbf{u} &= 0, \end{aligned} \tag{4.1}$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity, ρ^ϵ is the density of the medium, K^ϵ is the bulk modulus, and the source is modeled by the forcing term $\mathbf{F}^\epsilon(t,x,z)$. The third coordinate z represents the propagation axis along the waveguide. The transverse section of the waveguide is a bounded interval denoted by $[0,d]$, with $d>0$ and $x\in[0,d]$ representing the transverse coordinate. We assume that the medium parameters are given by (see Figure 4.2)

$$\frac{1}{K^\epsilon(x,z)} = \begin{cases} \epsilon^{2\alpha_K}\frac{1}{K}\left(1+\sqrt{\epsilon}V\left(x,\frac{z}{\epsilon^\alpha}\right)\right) & \text{if} \quad x\in(0,d), \quad z\in[0,L/\epsilon^{1-\alpha}] \\ \epsilon^{2\alpha_K}\frac{1}{K} & \text{if} \quad x\in(0,d), \quad z\in(-\infty,0) \\ \frac{1}{K} & \text{if} \quad x\in(0,d), \quad z\in(L/\epsilon^{1-\alpha},+\infty), \end{cases}$$

$$\rho^\epsilon(x,z) = \begin{cases} \epsilon^{-2\alpha_\rho}\bar{\rho} & \text{if} \quad x\in(0,d), \quad z\in(-\infty,L/\epsilon^{1-\alpha}] \\ \bar{\rho} & \text{if} \quad x\in(0,d), \quad z\in(L/\epsilon^{1-\alpha},+\infty), \end{cases}$$

where α_ρ and α_K are such that $\alpha_\rho - \alpha_K = \alpha \in (0,1]$ and where V, which models the spatial inhomogeneities, is described in Section 2.6.1. In what follows, we will see that the important parameter is α, because it determines the order of the sound speed of the first section. This configuration means that the order of the sound speed of the section $(-\infty, L/\epsilon^{1-\alpha})$ is small compared to that of the section $(L/\epsilon^{1-\alpha}, +\infty)$. The first section can represent a solid with random inhomogeneities, and the second can represent a homogeneous gas or liquid. The case $\alpha = 0$ is equivalent to that studied in [30] and [25, Chapter 20], in which no superresolution effect can be detected. The parameter α represents a possible configuration of the waveguide model, but we will see in Theorem 4.1 that the set of possible configurations for which we will apply an asymptotic analysis is more restricted.

We consider a source that emits a signal in the z-direction with carrier frequency ω_0. The source is localized in the plane $z = 0$.

$$\mathbf{F}^\epsilon(t,x,z) = f^\epsilon(t)\Psi(x)\delta(z)\mathbf{e}_z, \text{ where } f^\epsilon(t) = \frac{1}{2\epsilon^\alpha} f(\epsilon^p t)e^{-i\omega_0 t} \text{ with } p \in (0,1), \tag{4.2}$$

$\Psi(x)$ is the transverse profile of the source and $\mathbf{e}_z$ is the unit vector pointing in the z-direction. The source amplitude is large, of order $1/\epsilon^\alpha$, because transmission coefficients at the interface $z = L/\epsilon^{1-\alpha}$ are small, of order $\epsilon^{\alpha/2}$. However, we shall see in Section 4.4.6 that the transmission coefficients can be made of order one by inserting a quarter wavelength plate. Note that the condition $p > 0$ simplifies the algebra, and the condition $p < 1$ corresponds to the broadband case and ensures the statistical stability property discussed in Section 4.4.5. In the configuration (4.2), the relative bandwidth is of order ϵ^p, and the carrier wavelength is of order ϵ^α in the $(-\infty, L/\epsilon^{1-\alpha})$ section and of order one in $(L/\epsilon^{1-\alpha}, +\infty)$.

Let us recall that the process V is unbounded and this fact implies that the bulk modulus can take negative values. However, this situation can be avoided by working on the event

$$\left(\forall (x,z) \in [0,d] \times [0, L/\epsilon^{1-\alpha}],\ 1 + \sqrt{\epsilon} V\left(x, \frac{z}{\epsilon^\alpha}\right) > 0\right),$$

since by the property (2.55) page 66

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\exists (x,z) \in [0,d] \times [0, L/\epsilon^{1-\alpha}] : \ 1 + \sqrt{\epsilon} V\left(x, \frac{z}{\epsilon^\alpha}\right) \leq 0\right)$$
$$\leq \lim_{\epsilon \to 0} \mathbb{P}\left(\sqrt{\epsilon} \sup_{z \in [0,L]} \sup_{x \in [0,d]} \left| V\left(x, \frac{z}{\epsilon}\right)\right| \geq 1\right) = 0.$$

4.2 Waveguide Propagation

4.2.1 Propagation in Homogeneous Waveguides

In this section, we assume that the medium parameters are given by

$$\rho^\epsilon(x,z) = \frac{\bar{\rho}}{\epsilon^{2\alpha_\rho}} \text{ and } K^\epsilon(x,z) = \frac{\bar{K}}{\epsilon^{2\alpha_K}}, \quad \forall (x,z) \in (0,d) \times \mathbb{R}.$$

From the conservation equations (4.1), we can derive the wave equation for the pressure field,

$$\Delta p - \frac{1}{c^{\epsilon 2}} \frac{\partial^2 p}{\partial t^2} = \nabla . \mathbf{F}^\epsilon, \tag{4.3}$$

where $c^\epsilon = \epsilon^\alpha \sqrt{\frac{\bar{K}}{\bar{\rho}}} = \epsilon^\alpha c$ and $\Delta = \partial_x^2 + \partial_z^2$. We consider Dirichlet boundary conditions

$$p(t,0,z) = p(t,d,z) = 0, \quad \forall (t,z) \in [0,+\infty) \times \mathbb{R}.$$

We recall that the Fourier transform and the inverse Fourier transform, with respect to time, are defined by

$$\widehat{f}(\omega) = \int f(t) e^{i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int \widehat{f}(\omega) e^{-i\omega t} d\omega.$$

In the half-space $z > 0$ (resp., $z < 0$), taking the Fourier transform in (4.3), we get that $\widehat{p}(\omega, x, z)$ satisfies the time harmonic wave equation without source term

$$\partial_z^2 \widehat{p}(\omega, x, z) + \partial_x^2 \widehat{p}(\omega, x, z) + \frac{k^2(\omega)}{\epsilon^{2\alpha}} \widehat{p}(\omega, x, z) = 0,$$

with Dirichlet boundary conditions $\widehat{p}(\omega, 0, z) = \widehat{p}(\omega, d, z) = 0$, $\forall (t, z) \in [0, +\infty) \times \mathbb{R}$. Here $k(\omega) = \frac{\omega}{c}$. The source term implies the following jump conditions for the pressure field across the plane $z = 0$

$$\begin{aligned} \widehat{p}(\omega, x, 0^+) - \widehat{p}(\omega, x, 0^-) &= \widehat{f^\epsilon}(\omega) \Psi(x), \\ \partial_z \widehat{p}(\omega, x, 0^+) - \partial_z \widehat{p}(\omega, x, 0^-) &= 0. \end{aligned}$$

We can decompose this solution in a spectral basis of $L^2(0, d)$, which can be chosen as the set of eigenfunctions $(\phi_j(x))_{j \geq 1}$ of $-\partial_x^2$

$$-\partial_x^2 \phi_j(x) = \lambda_j \phi_j(x) \quad \text{and} \quad \int_0^d \phi_j(x) \phi_l(x) dx = \delta_{jl} \quad \forall j, l \geq 1,$$

where δ_{jl} denotes the Kronecker symbol. This family is given by

$$\phi_j(x) = \sqrt{\frac{2}{d}} \sin\left(\frac{j\pi}{d} x\right) \text{ with } \lambda_j = \frac{j^2 \pi^2}{d^2} \text{ for } j \geq 1,$$

and corresponds to the basis of the unperturbed waveguide. Let us remark that in this waveguide model the spectral decomposition does not depend on the frequency. Thus, we can write

$$\widehat{p}(\omega, x, z) = \sum_{j \geq 1} \widehat{p}_j(\omega, z) \phi_j(x). \tag{4.4}$$

This implies that $\forall j \geq 1$, $\widehat{p}_j(\omega, z)$ satisfies the differential equation

$$\frac{d^2}{dz^2} \widehat{p}_j(\omega, z) + \left(\frac{k^2(\omega)}{\epsilon^{2\alpha}} - \lambda_j\right) \widehat{p}_j(\omega, z) = 0. \tag{4.5}$$

For each frequency ω,

$$\epsilon^{2\alpha} \lambda_{N_\epsilon(\omega)} \leq k^2(\omega) < \epsilon^{2\alpha} \lambda_{N_\epsilon(\omega)+1}$$

with $N_\epsilon(\omega) = [\frac{k(\omega) d}{\pi \epsilon^\alpha}]$. There are two cases. First, for $j \leq N_\epsilon(\omega)$, these modes represent the propagating modes, and we define the associated modal wavenumbers by

$$\beta_j^\epsilon(\omega) = \sqrt{\frac{k^2(\omega)}{\epsilon^{2\alpha}} - \lambda_j}.$$

Second, for $j > N_\epsilon(\omega)$, these modes represent evanescent modes, and in this case we define the modal wavenumbers by

$$\beta_j^\epsilon(\omega) = \sqrt{\lambda_j - \frac{k^2(\omega)}{\epsilon^{2\alpha}}}.$$

Finally, using (4.5) and (4.4), the pressure field can be written as an expansion over the complete set of modes

$$\begin{aligned} \widehat{p}(\omega, x, z) = & \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega) z} \phi_j(x) + \sum_{j \geq N_\epsilon(\omega)+1} \frac{\widehat{c}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-\beta_j^\epsilon(\omega) z} \phi_j(x)\right] \mathbf{1}_{(0,+\infty)}(z) \\ & + \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{b}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-i\beta_j^\epsilon(\omega) z} \phi_j(x) + \sum_{j \geq N_\epsilon(\omega)+1} \frac{\widehat{d}_{j,0}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{\beta_j^\epsilon(\omega) z} \phi_j(x)\right] \mathbf{1}_{(-\infty,0)}(z), \end{aligned} \tag{4.6}$$

where $\widehat{a}^{\epsilon}_{j,0}(\omega)$ (resp., $\widehat{b}^{\epsilon}_{j,0}(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating in the right half-space $z > 0$ (resp., left half-space $z < 0$), and $\widehat{c}^{\epsilon}_{j,0}(\omega)$ (resp., $\widehat{d}^{\epsilon}_{j,0}(\omega)$) is the amplitude of the jth right-going (resp., left-going) evanescent mode in the right half-space $z > 0$ (resp., left half-space $z < 0$). We recall that the source is located in the plane $z = 0$ with the transverse profile $\Psi(x)$.

Substituting (4.6) into

$$\partial_z^2 \widehat{p}(\omega, x, z) + \partial_x^2 \widehat{p}(\omega, x, z) + \frac{k^2(\omega)}{\epsilon^{2\alpha}} \widehat{p}(\omega, x, z) = \widehat{f^{\epsilon}}(\omega)\Psi(x)\delta_0'(z), \tag{4.7}$$

multiplying by $\phi_j(x)$, and integrating over $(0, d)$ permit us to express the mode amplitudes

$$\widehat{a}^{\epsilon}_{j,0}(\omega) = -\widehat{b}^{\epsilon}_{j,0}(\omega) = \frac{\sqrt{\beta^{\epsilon}_j(\omega)}}{4\epsilon^{\alpha+p}} \widehat{f}\left(\frac{\omega - \omega_0}{\epsilon^p}\right)\theta_j,$$

$$\widehat{c}_{j,0}(\omega) = -\widehat{d}_{j,0}(\omega) = -\frac{\sqrt{\beta^{\epsilon}_j(\omega)}}{4\epsilon^{\alpha+p}} \widehat{f}\left(\frac{\omega - \omega_0}{\epsilon^p}\right)\theta_j,$$

where $\forall j \geq 1$,

$$\theta_j = \langle \Psi, \phi_j \rangle_{L^2(0,d)} = \int_0^d \Psi(x)\phi_j(x)dx.$$

4.2.2 Mode Coupling in Random Waveguides

In this section, we study the expansion of $\widehat{p}(\omega, x, z)$ when a random section $z \in [0, L/\epsilon^{1-\alpha}]$ is inserted between two homogeneous waveguides:

$$\frac{1}{K^{\epsilon}(x,z)} = \begin{cases} \epsilon^{2\alpha_K} \frac{1}{K}\left(1 + \sqrt{\epsilon} V\left(x, \frac{z}{\epsilon^{\alpha}}\right)\right) & \text{if} \quad x \in (0,d), \quad z \in [0, L/\epsilon^{1-\alpha}] \\ \epsilon^{2\alpha_K} \frac{1}{K} & \text{if} \quad x \in (0,d), \quad z \in (-\infty, 0) \\ \frac{1}{K} & \text{if} \quad x \in (0,d), \quad z \in (L/\epsilon^{1-\alpha}, +\infty), \end{cases}$$

$$\rho^{\epsilon}(x,z) = \begin{cases} \epsilon^{-2\alpha_{\rho}}\bar{\rho} & \text{if} \quad x \in (0,d), \quad z \in (-\infty, L/\epsilon^{1-\alpha}] \\ \bar{\rho} & \text{if} \quad x \in (0,d), \quad z \in (L/\epsilon^{1-\alpha}, +\infty). \end{cases}$$

In this region, the pressure field can be decomposed on the basis of eigenmodes of the unperturbed waveguide

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N_{\epsilon}(\omega)} \widehat{p}_j(\omega, z)\phi_j(x) + \sum_{j > N_{\epsilon}(\omega)} \widehat{q}_j(\omega, z)\phi_j(x).$$

Evanescent modes correspond to $j > N_{\epsilon}(\omega)$, and $N_{\epsilon}(\omega)$ goes to $+\infty$ as ϵ goes to 0. Therefore, we will neglect the modes $j > N_{\epsilon}(\omega)$. Note that it could be possible to incorporate the modes $j > N_{\epsilon}(\omega)$ using the method described in Chapter 2 or in [25, Chapter 20], but this would lead to complicated algebra without modifying the overall result. Indeed, we shall check a posteriori that the mode decomposition of the wave is supported by a number of modes of order one as ϵ goes to 0. Consequently, we shall consider in what follows the decomposition

$$\widehat{p}(\omega, x, z) = \sum_{j=1}^{N_{\epsilon}(\omega)} \widehat{p}_j(\omega, z)\phi_j(x),$$

where $\widehat{p}_j(\omega, z)$ satisfies

$$\frac{d^2}{dz^2}\widehat{p}_j(\omega, z) + \beta^{\epsilon}_j(\omega)^2 \widehat{p}_j(\omega, z) + \epsilon^{\frac{1}{2} - 2\alpha} k^2(\omega) \sum_{l=1}^{N_{\epsilon}(\omega)} C_{jl}\left(\frac{z}{\epsilon^{\alpha}}\right)\widehat{p}_l(\omega, z) = 0 \tag{4.8}$$

with $\forall (j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$

$$C_{jl}(z) = \langle \phi_j, \phi_l V(.,z)\rangle_{L^2(0,d)} = \int_0^d \phi_j(x)\phi_l(x)V(x,z)dx.$$

Let us recall that $\forall (j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$, the coefficient C_{jl} represents the coupling between the jth propagating mode with the lth propagating mode. Next, we introduce the amplitudes of the generalized right- and left-going modes $\widehat{a}_j(\omega,z)$ and $\widehat{b}_j(\omega,z)$ for $j \in \{1,\dots,N_\epsilon(\omega)\}$. They are given by

$$\widehat{p}_j(\omega,z) = \frac{1}{\sqrt{\beta_j^\epsilon(\omega)}}\left(\widehat{a}_j(\omega,z)e^{i\beta_j^\epsilon(\omega)z} + \widehat{b}_j(\omega,z)e^{-i\beta_j^\epsilon(\omega)z}\right),$$
$$\frac{d}{dz}\widehat{p}_j(\omega,z) = i\sqrt{\beta_j^\epsilon(\omega)}\left(\widehat{a}_j(\omega,z)e^{i\beta_j^\epsilon(\omega)z} - \widehat{b}_j(\omega,z)e^{-i\beta_j^\epsilon(\omega)z}\right).$$

In the absence of random perturbation, these amplitudes are constant. In the presence of random perturbations, we obtain from (4.8) the coupled mode equation

$$\frac{d}{dz}\widehat{a}_j(\omega,z) = \epsilon^{\frac{1}{2}-\alpha}\frac{ik^2}{2}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon^\alpha}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l(\omega,z)e^{i\left(\beta_l^\epsilon-\beta_j^\epsilon\right)z} + \widehat{b}_l(\omega,z)e^{-i\left(\beta_l^\epsilon+\beta_j^\epsilon\right)z}\right),$$
$$\frac{d}{dz}\widehat{b}_j(\omega,z) = -\epsilon^{\frac{1}{2}-\alpha}\frac{ik^2}{2}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon^\alpha}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l(\omega,z)e^{i\left(\beta_l^\epsilon+\beta_j^\epsilon\right)z} + \widehat{b}_l(\omega,z)e^{-i\left(\beta_l^\epsilon-\beta_j^\epsilon\right)z}\right),$$

$\forall j \in \{1,\dots,N_\epsilon(\omega)\}$.

Let us define the rescaled processes

$$\widehat{a}_j^\epsilon(\omega,z) = \widehat{a}_j\left(\omega,\frac{z}{\epsilon^{1-\alpha}}\right) \text{ and } \widehat{b}_j^\epsilon(\omega,z) = \widehat{b}_j\left(\omega,\frac{z}{\epsilon^{1-\alpha}}\right) \text{ for } z \in (0,L),$$

$\forall j \in \{1,\dots,N_\epsilon(\omega)\}$. These scalings correspond to the size of the random section $(0, L/\epsilon^{1-\alpha})$. They satisfy the rescaled coupled mode equation

$$\begin{aligned}\frac{d}{dz}\widehat{a}_j^\epsilon(\omega,z) &= \frac{ik^2}{2\sqrt{\epsilon}}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l^\epsilon(\omega,z)e^{i\epsilon^\alpha\left(\beta_l^\epsilon-\beta_j^\epsilon\right)\frac{z}{\epsilon}} + \widehat{b}_l^\epsilon(\omega,z)e^{-i\epsilon^\alpha\left(\beta_l^\epsilon+\beta_j^\epsilon\right)\frac{z}{\epsilon}}\right),\\ \frac{d}{dz}\widehat{b}_j^\epsilon(\omega,z) &= -\frac{ik^2}{2\sqrt{\epsilon}}\sum_{l=1}^{N_\epsilon(\omega)}\frac{C_{jl}\left(\frac{z}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta_j^\epsilon\beta_l^\epsilon}}\left(\widehat{a}_l^\epsilon(\omega,z)e^{i\epsilon^\alpha\left(\beta_l^\epsilon+\beta_j^\epsilon\right)\frac{z}{\epsilon}} + \widehat{b}_l^\epsilon(\omega,z)e^{-i\epsilon^\alpha\left(\beta_l^\epsilon-\beta_j^\epsilon\right)\frac{z}{\epsilon}}\right),\end{aligned} \tag{4.9}$$

This system is endowed with the boundary conditions $\forall j \in \{1,\dots,N_\epsilon(\omega)\}$,

$$\widehat{a}_j^\epsilon(\omega,0) = \widehat{a}_{j,0}^\epsilon(\omega) \text{ and } \widehat{b}_j^\epsilon(\omega,L) = 0.$$

Note that $\forall j \in \{1,\dots,N_\epsilon(\omega)\}$, $\widehat{a}_{j,0}^\epsilon(\omega)$ represents the initial amplitude of the jth propagating mode generated by the source at $z = 0^+$. The second condition means that no wave comes from the right. We can rewrite (4.9) in a vector-matrix form as

$$\frac{d}{dz}X^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^\epsilon\left(\omega,\frac{z}{\epsilon}\right)X^\epsilon(\omega,z),$$

where

$$X^\epsilon(\omega,z) = \begin{bmatrix}\widehat{a}^\epsilon(\omega,z)\\ \widehat{b}^\epsilon(\omega,z)\end{bmatrix}, \quad \mathbf{H}^\epsilon(\omega,z) = \begin{bmatrix}\mathbf{H}^{a,\epsilon}(\omega,z) & \mathbf{H}^{b,\epsilon}(\omega,z)\\ \overline{\mathbf{H}^{b,\epsilon}}(\omega,z) & \overline{\mathbf{H}^{a,\epsilon}}(\omega,z)\end{bmatrix},$$

and $\forall(j,l) \in \{1,\dots,N_\epsilon(\omega)\}^2$,

$$\begin{aligned}\mathbf{H}^{a,\epsilon}_{jl}(\omega,z) &= \frac{ik^2(\omega)}{2}\frac{C_{jl}(z)}{\epsilon^\alpha\sqrt{\beta^\epsilon_j(\omega)\beta^\epsilon_l(\omega)}}e^{i\epsilon^\alpha\left(\beta^\epsilon_l(\omega)-\beta^\epsilon_j(\omega)\right)z},\\ \mathbf{H}^{b,\epsilon}_{jl}(\omega,z) &= \frac{ik^2(\omega)}{2}\frac{C_{jl}(z)}{\epsilon^\alpha\sqrt{\beta^\epsilon_j(\omega)\beta^\epsilon_l(\omega)}}e^{-i\epsilon^\alpha\left(\beta^\epsilon_l(\omega)+\beta^\epsilon_j(\omega)\right)z}.\end{aligned} \tag{4.10}$$

Now, we introduce the propagator matrix $\mathbf{P}^\epsilon(\omega,z)$, that is, the $2N_\epsilon(\omega)\times 2N_\epsilon(\omega)$ matrix solution of the differential equation

$$\frac{d}{dz}\mathbf{P}^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^\epsilon\left(\omega,\frac{z}{\epsilon}\right)\mathbf{P}^\epsilon(\omega,z) \quad \text{with} \quad \mathbf{P}^\epsilon(\omega,0)=\mathbf{I}.$$

This relation implies

$$\begin{bmatrix}\widehat{a}^\epsilon(\omega,z)\\ \widehat{b}^\epsilon(\omega,z)\end{bmatrix} = \mathbf{P}^\epsilon(\omega,z)\begin{bmatrix}\widehat{a}^\epsilon(\omega,0)\\ \widehat{b}^\epsilon(\omega,0)\end{bmatrix},$$

and the symmetry of $\mathbf{H}^\epsilon(\omega,z)$ gives a particular form of the propagator:

$$\mathbf{P}^\epsilon(\omega,z) = \begin{bmatrix}\mathbf{P}^a_\epsilon(\omega,z) & \mathbf{P}^b_\epsilon(\omega,z)\\ \overline{\mathbf{P}^b_\epsilon}(\omega,z) & \overline{\mathbf{P}^a_\epsilon}(\omega,z)\end{bmatrix},$$

where $\mathbf{P}^a_\epsilon(\omega,z)$ and $\mathbf{P}^b_\epsilon(\omega,z)$ are $N_\epsilon(\omega)\times N_\epsilon(\omega)$ matrices which represent, respectively, the coupling between right-going modes and the coupling between right-going and left-going modes.

4.2.3 Band-Limiting Idealization and Forward Scattering Approximation

In this section, we introduce a band-limiting idealization hypothesis in which the power spectral density of the random fluctuations is assumed to be limited in both the transverse and the longitudinal directions. We already have introduced this assumption in Section 2.5.2 for the study of the coupled power equations in the high-frequency regime and also for the study of the time-reversal experiment in Section 3.4.7. In the same way, this hypothesis simplifies in this chapter the study of the time-reversal experiment. Note that $\forall j\geq 1$ and $z\in[0,+\infty)$, we have

$$\begin{aligned}\mathbb{E}[C_{jl}(z)^2] &= \int_0^d\int_0^d \gamma(x,y)\phi_j(x)\phi_l(x)\phi_j(y)\phi_l(y)dxdy\\ &= S(j-l,j-l)+S(j+l,j+l)-S(j-l,j+l)-S(j+l,j-l),\end{aligned}$$

where

$$S(a,b) = \frac{4}{d^2}\int_0^d\int_0^d\gamma(x,y)\cos\left(a\frac{\pi}{d}x\right)\cos\left(b\frac{\pi}{d}y\right)dxdy.$$

We assume that the support of S lies in the square $\left[-\frac{3}{2},\frac{3}{2}\right]\times\left[-\frac{3}{2},\frac{3}{2}\right]$. Our compact support hypothesis implies

$$C_{jl}(z)=0 \quad \text{if } |j-l|>1,$$

which is tantamount to a nearest neighbor coupling. More precisely, this assumption implies that $\forall(j,l)\in\{1,\dots,N_\epsilon(\omega)\}^2$ the jth mode amplitude can exchange information with the lth amplitude mode if they are direct neighbors, that is, if they satisfy $|j-l|\leq 1$.

Now, we consider the forward scattering approximation already discussed in Section 2.3.4 for the waveguide model studied in Chapter 2 and Chapter 3. Considering the first exit times of a closed ball related to the weak topology and by considering the process $\mathbf{P}^\epsilon$ in an appropriate finite dimensional dual space, the same proof as the one in Section 4.5.1 shows that $\mathbf{P}^\epsilon$ converges in law. The limit processes of $\mathbf{P}^a_\epsilon$ and $\mathbf{P}^b_\epsilon$, as $\epsilon \to 0$, are coupled through the coefficients

$$\int_0^{+\infty} \mathbb{E}[C_{jl}(0)C_{jl}(z)]\cos(2k(\omega)z)dz,$$

because of the factor $e^{\pm i\epsilon\left(\beta^\epsilon_l(\omega)+\beta^\epsilon_j(\omega)\right)z}$ in $\mathbf{H}^{b,\epsilon}_{jl}(\omega,z)$ and the fact that $\forall j \geq 1$,

$$\lim_{\epsilon\to 0} \epsilon^\alpha \beta^\epsilon_j(\omega) = k(\omega). \tag{4.11}$$

We assume that the power spectral density of the process V, i.e. the Fourier transform of its z-autocorrelation function, possesses a cut-off wavenumber strictly less than $2k(\omega)$. In other words, we consider the case where

$$\int_0^{+\infty} \mathbb{E}[C_{jl}(0)C_{jl}(z)]\cos(2k(\omega)z)dz = 0 \quad \forall j,l \geq 1.$$

Consequently, the limit coupling between $\mathbf{P}^a_\epsilon(\omega,z)$ and $\mathbf{P}^b_\epsilon(\omega,z)$ becomes zero. Moreover, the initial condition $\mathbf{P}^b_\epsilon(\omega,0) = \mathbf{0}$ implies that $\mathbf{P}^b_\epsilon$ converges to $\mathbf{0}$. In this forward scattering approximation, we can neglect the left-going propagating modes in the asymptotic $\epsilon \to 0$. With this assumption, one can consider the simplified coupled amplitude equation given by

$$\frac{d}{dz}\widehat{a}^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{a,\epsilon}\left(\omega\frac{z}{\epsilon}\right)\widehat{a}^\epsilon(\omega,z) \quad \text{with} \quad \widehat{a}^\epsilon(\omega,0) = \widehat{a}_0(\omega).$$

Finally, we introduce the transfer matrix $\mathbf{T}^\epsilon(\omega,z)$, which is the $N_\epsilon(\omega)\times N_\epsilon(\omega)$ matrix solution of

$$\frac{d}{dz}\mathbf{T}^\epsilon(\omega,z) = \frac{1}{\sqrt{\epsilon}}\mathbf{H}^{a,\epsilon}\left(\omega,\frac{z}{\epsilon}\right)\mathbf{T}^\epsilon(\omega,z) \quad \text{with} \quad \mathbf{T}^\epsilon(\omega,0) = \mathbf{I}. \tag{4.12}$$

From this equation, one can check that the transfer matrix $\mathbf{T}^\epsilon(\omega,z)$ is unitary since $\mathbf{H}^{a,\epsilon}(\omega,z)$ is skew-Hermitian.

4.3 The Coupled Mode Process

This section presents the theoretical results needed in this chapter. In our configuration the number of propagating modes is not fixed. Then, we must extend the limit theorem stated in [48], where the number of propagating modes is fixed. The first result concerns the diffusion-approximation for a solution of an ordinary differential equation with random coefficients. This result is a version of that stated in [48], where the dimension of the system is fixed, adapted to the case where the dimension of the system goes to infinity in the asymptotic ϵ goes to 0. The second result, which follows from Theorem 4.1, is about the asymptotic behavior of the expectation of the product of two transfer coefficients. These two results will be used in the following section to compute the refocused wave in the asymptotic regime ϵ goes to 0. The third result concerns the high-frequency approximation to the coupled power equations obtained in Proposition 4.1. Using a probabilistic representation of solutions of this equation, we establish a convergence in law to a continuous diffusion process. From Theorem 4.2, we give the high-frequency approximation to the coupled power equations that will allow us to compute the transverse profile of the refocused wave and show that randomness enhances spatial refocusing beyond the diffraction limit.

Let $\mathcal{H} = l^2(E,\mathbb{C})$, with $E = (\mathbb{N}^*)^2$, equipped with the inner product be defined by

$$\forall(\lambda,\mu)\in\mathcal{H}\times\mathcal{H}, \quad \langle\lambda,\mu\rangle_{\mathcal{H}} = \sum_{j,m\geq 1}\lambda_{jm}\overline{\mu_{jm}}.$$

Let us fix $(l, n) \in (\mathbb{N}^*)^2$ and consider

$$\mathbf{U}^\epsilon_{jm}(\omega, z) = \overline{\mathbf{T}^\epsilon_{jl}(\omega, z)}\mathbf{T}^\epsilon_{mn}(\omega, z),$$

which is an $\mathcal{H}$-valued process such that $\forall z \geq 0$

$$\|\mathbf{U}^\epsilon(\omega, z)\|_{\mathcal{H}} = 1.$$

Note that we have dropped the indexes l and n in the previous definition because they do not play any role in (4.12). Moreover, let

$$\mathcal{B}_{\mathcal{H}} = \left\{\lambda \in \mathcal{H}, \|\lambda\|_{\mathcal{H}} = \sqrt{\langle \lambda, \lambda \rangle_{\mathcal{H}}} \leq 1\right\}$$

the unit ball of $\mathcal{H}$, and $\{g_n, n \geq 1\}$ a dense subset of $\mathcal{B}_{\mathcal{H}}$. We equip $\mathcal{B}_{\mathcal{H}}$ with the distance $d_{\mathcal{B}_{\mathcal{H}}}$ defined by

$$d_{\mathcal{B}_{\mathcal{H}}}(\lambda, \mu) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \left|\langle \lambda - \mu, g_n \rangle_{\mathcal{H}}\right|,$$

$\forall(\lambda, \mu) \in \mathcal{B}_{\mathcal{H}}{}^2$, and then $(\mathcal{B}_{\mathcal{H}}, d_{\mathcal{B}_{\mathcal{H}}})$ is a compact metric space.

Theorem 4.1 *For $\alpha \in (0, 1/4)$, the family of processes $(\mathbf{U}^\epsilon(\omega, .))_{\epsilon\in(0,1)}$ converges in distribution on $\mathcal{C}([0, +\infty), (\mathcal{B}_{\mathcal{H}}, d_{\mathcal{B}_{\mathcal{H}}}))$ as $\epsilon \to 0$ to a limit denoted by $\mathbf{U}(\omega, .)$. This limit is the unique solution of the infinite-dimensional stochastic differential equation*

$$d\mathbf{U}(\omega, z) = J^\omega(\mathbf{U}(\omega, z))dz + \psi^\omega_1(\mathbf{U}(\omega, z))(dB^1_z) + \psi^\omega_2(\mathbf{U}(\omega, z))(dB^2_z),$$

with $\mathbf{U}_{jm}(\omega, 0) = \delta_{jl}\delta_{mn}$. $(B^\eta_{jm})_{\substack{\eta=1,2\\ j,m\geq 1}}$ is a family of independent one-dimensional standard Brownian motions and

$$\begin{aligned}
J^\omega(U)_{jm} &= \Lambda(\omega)\left[(U_{j+1j+1}\delta_{jm} - U_{jm}) + (U_{j-1j-1}\delta_{jm} - U_{jm})\right],\\
\psi^\omega_1(U)(\lambda)_{jm} &= \sqrt{\frac{\Lambda(\omega)}{2}}(U_{j+1m}\lambda_{jj+1} - U_{j-1m}\lambda_{j-1j} + U_{jm+1}\lambda_{mm+1} - U_{jm-1}\lambda_{m-1m}),\\
\psi^\omega_2(U)(\lambda)_{jm} &= i\sqrt{\frac{\Lambda(\omega)}{2}}(-U_{j+1m}\lambda_{jj+1} - U_{j-1m}\lambda_{j-1j} + U_{jm+1}\lambda_{mm+1} + U_{jm-1}\lambda_{m-1m})
\end{aligned}$$

$\forall(U, \lambda) \in \mathcal{H} \times l^2(E, \mathbb{R})$, with $\Lambda = \frac{k^2(\omega)}{2a}S(1, 1)$. We use the convention $(y_{0,m})_{m\geq1} = (y_{j,0})_{j\geq1} = 0$ for $y \in \mathcal{H}$.

This theorem gives the asymptotic behavior of the statistical properties of the matrix $\mathbf{U}^\epsilon$ in terms of the diffusion model given by the infinite-dimensional stochastic differential equation.

The proof of this theorem, given in the appendix, is based on a martingale approach using the perturbed-test-function method already used in the proof of Theorem 2.1.

Proposition 4.1

$$\begin{aligned}
\lim_{\epsilon\to0} \mathbb{E}\left[\overline{\mathbf{T}^\epsilon_{jj}(\omega, L)}\mathbf{T}^\epsilon_{mm}(\omega, L)\right] &= \mathbb{E}\left[\mathbf{U}_{jm}(\omega, L)\right]\\
&= \begin{cases} e^{-\Lambda(\omega)L} & \text{if } j \neq m \text{ and } j = 1 \text{ or } m = 1,\\ e^{-2\Lambda(\omega)L} & \text{if } j \neq m \neq 1,\end{cases}\\
\lim_{\epsilon\to0} \mathbb{E}\left[\overline{\mathbf{T}^\epsilon_{jl}(\omega, L)}\mathbf{T}^\epsilon_{jl}(\omega, L)\right] &= \mathbb{E}\left[\mathbf{U}_{jj}(\omega, L)\right] = \mathcal{T}^l_j(\omega, L),\\
\lim_{\epsilon\to0} \mathbb{E}\left[\overline{\mathbf{T}^\epsilon_{jl}(\omega, L)}\mathbf{T}^\epsilon_{mn}(\omega, L)\right] &= \mathbb{E}\left[\mathbf{U}_{jm}(\omega, L)\right] = 0 \text{ in the other cases,}
\end{aligned}$$

where $\left(\mathcal{T}_j^l(\omega,z)\right)_{j\geq 1}$ *is the solution of the coupled power equations,*

$$\frac{d}{dz}\mathcal{T}_j^l(\omega,z) = \Lambda(\omega)\left[\mathcal{T}_{j+1}^l(\omega,z)+\mathcal{T}_{j-1}^l(\omega,z)-2\mathcal{T}_j^l(\omega,z)\right], \quad j\geq 1,$$
$$\frac{d}{dz}\mathcal{T}_1^l(\omega,z) = \Lambda(\omega)\left[\mathcal{T}_2^l(\omega,z)-\mathcal{T}_1^l(\omega,z)\right],$$

with $\mathcal{T}_j^l(\omega,0)=\delta_{jl}$.

$\mathcal{T}_j^l(\omega,L)$ is the expected power of the jth propagating mode at the propagation distance $z=L$, when at $z=0$ the energy is concentrated on the lth propagating mode. These equations represent the transfer of energy between propagating modes, and Λ is the energy transport coefficient. As in Section 2.5, we are interested in studying this equation in the high-frequency regime, that is, when $\omega \gg 1$. To this end we take a probabilistic representation of this equation. We introduce the jump Markov process $(X_t)_{t\geq 0}$ whose state space is $\mathbb{N}^*$ and whose infinitesimal generator is

$$\mathcal{L}_X\varphi(j) = \Lambda(\omega)(\varphi(j+1)+\varphi(j-1)-2\varphi(j)), \quad j\geq 2,$$
$$\mathcal{L}_X\varphi(1) = \Lambda(\omega)(\varphi(2)-\varphi(1)).$$

We get

$$\mathcal{T}_j^l(\omega,L) = \mathbb{P}(X_L=j|X_0=l) = \mathbb{P}\left(\frac{X_L}{N}=\frac{j}{N}\middle|\frac{X_0}{N}=\frac{l}{N}\right),$$

where $N(\omega)=\left[\frac{\omega d}{\pi\bar{c}}\right]$ is the number of propagating modes in the homogeneous part of the waveguide model $(L/\epsilon^{1-\alpha},+\infty)$. The normalization of the last equality is the same as the one used in the proof of Theorems 2.4 and 2.6. As in Chapter 3, the continuous diffusive regime that we get in Theorem 4.2 will be used in the next section to compute the transverse profile of the refocused wave.

We can consider $(\mathcal{T}^l(\omega,L))_{l\geq 1}$ as a family of probability measures on $\mathbb{R}_+$. Let $\forall\varphi\in\mathcal{C}_b^0([0,+\infty))$, $\forall u\in[0,+\infty)$, and $z\geq 0$,

$$\mathcal{T}_\varphi^N(z,u) = \mathcal{T}_\varphi^{[Nu]}(\omega,z) = \sum_{j\geq 1}\varphi\Big(\frac{j}{N}\Big)\mathcal{T}_j^{[Nu]}(z).$$

Theorem 4.2 *Let* $u\geq 0$. $\forall\varphi\in\mathcal{C}_b^0([0,+\infty))$ *and* $\forall z\geq 0$, *we have*

$$\lim_{\omega\to+\infty}\mathcal{T}_\varphi^{N(\omega)}(z,u) = \mathcal{T}_\varphi(z,u) = \int_{\mathbb{R}_+}\varphi(v)\mathcal{W}(z,u,v)dv,$$

where $\forall z>0$ *and* $\forall(u,v)\in[0,+\infty)^2$,

$$\frac{\partial}{\partial z}\mathcal{W}(z,u,v) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial u^2}\mathcal{W}(z,u,v),$$

with

$$\frac{\partial}{\partial u}\mathcal{W}(z,0,v) = 0 \text{ and } \mathcal{W}(0,u,v)=\delta(u-v).$$

and $\sigma^2 = \frac{\pi^2}{d^2 a}S(1,1)$.

This theorem is a continuum approximation in the limit of a large number of propagating modes. This approximation gives us, in the high-frequency regime, a diffusion model for the transfer of energy between propagating modes. In our case, the diffusion model of the coupled power equations takes a particularly simple form; it is the heat equation with a reflecting barrier. Let us note that $\mathcal{W}(z,u,v)$ can be computed. We have, $\forall z>0$ and $\forall(u,v)\in[0,+\infty)^2$,

$$\mathcal{W}(z,u,v) = \frac{1}{\sqrt{2\pi\sigma^2 z}}\left(e^{-\frac{(v-u)^2}{2\sigma^2 z}}+e^{-\frac{(v+u)^2}{2\sigma^2 z}}\right).$$

4.4 Time Reversal in a Waveguide

4.4.1 First Step of the Time Reversal Experiment

In the first step of the experiment, a source sends a pulse into the medium, and the wave propagates and is recorded by the time-reversal mirror. In this section we obtain the integral representation of the wave recorded by the time-reversal mirror.

A source is located in the plane $z = 0$ and emits a pulse $f^\epsilon(t)$ of the form (4.2),

$$f^\epsilon(t) = \frac{1}{2\epsilon^\alpha} f(\epsilon^p t) e^{-i\omega_0 t} \text{ with } p \in (0,1).$$

A time-reversal mirror is located in the plane $z = L_M/\epsilon^{1-\alpha}$, it occupies the transverse subdomain $\mathcal{D}_M \subset [0, d]$ and in the first step of the experiment the time-reversal mirror plays the role of a receiving array. The transmitted wave is recorded for a time interval $\left[\frac{t_0}{\epsilon}, \frac{t_1}{\epsilon}\right]$ at the time-reversal mirror and is re-emitted time-reversed into the waveguide toward the source. We have chosen such a time window because it is of the order of the total travel time of the two sections. We recall that the propagation distance is of order $1/\epsilon^{1-\alpha}$ and the sound speed is of order ϵ^α in $(-\infty, L/\epsilon^{1-\alpha})$, and the propagation distance is of order $1/\epsilon^{1-\alpha}$ and the sound speed is of order 1 in $(L/\epsilon^{1-\alpha}, L_M/\epsilon^{1-\alpha})$.

The Fourier transform of the pressure field at the end of the random section $[0, L/\epsilon^{1-\alpha}]$ is given by

$$\widehat{p}_{tr}^{0,\epsilon}\left(\omega, x, \frac{L^-}{\epsilon^{1-\alpha}}\right) = \sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_j^\epsilon(\omega, L)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega)\frac{L}{\epsilon^{1-\alpha}}} \phi_j(x).$$

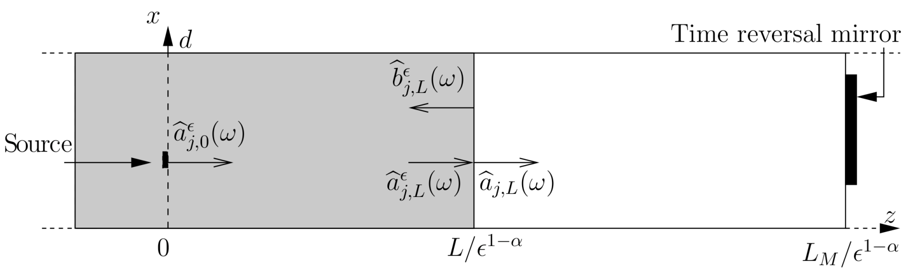
Jumps of the medium parameters at $z = L/\epsilon^{1-\alpha}$ imply that the incoming pulse produces a reflected and a transmitted field. The modal decomposition obtained in Section 4.2.1 for the first part of the waveguide can be obtained in the same way for the second part with $\epsilon = 1$. The decomposition over the eigenmodes gives

$$\begin{aligned}
\widehat{p}_{tr}^{L,\epsilon}(\omega, x, z) = &\left[\sum_{j=1}^{N(\omega)} \frac{\widehat{a}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{b}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x)\right. \\
&+ \left.\sum_{j=N(\omega)+1}^{N_\epsilon(\omega)} \frac{\widehat{c}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{-\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{d}_{j,L}(\omega)}{\sqrt{\beta_j(\omega)}} e^{\beta_j(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x)\right] \mathbf{1}_{(L/\epsilon^{1-\alpha}, +\infty)}(z) \\
&+ \left[\sum_{j=1}^{N_\epsilon(\omega)} \frac{\widehat{a}_{j,L}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{i\beta_j^\epsilon(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) + \frac{\widehat{b}_{j,L}^\epsilon(\omega)}{\sqrt{\beta_j^\epsilon(\omega)}} e^{-i\beta_j^\epsilon(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_j(x)\right] \mathbf{1}_{(0, L/\epsilon^{1-\alpha})}(z),
\end{aligned} \tag{4.13}$$

where $\widehat{a}_{j,L}(\omega)$ (resp., $\widehat{b}_{j,L}(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating, and $\widehat{c}_{j,L}(\omega)$ (resp., $\widehat{d}_{j,L}(\omega)$) is the amplitude of the jth right-going (resp., left-going) evanescent mode in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$. Moreover, $\widehat{a}_{j,L}^\epsilon(\omega)$ (resp., $\widehat{b}_{j,L}^\epsilon(\omega)$) is the amplitude of the jth right-going (resp., left-going) mode propagating in the section $(0, L/\epsilon^{1-\alpha})$. Note that we have kept the evanescent modes $j > N(\omega)$, in the waveguide section $(L/\epsilon^{1-\alpha}, +\infty)$, in the expression (4.13) because $N(\omega)$ is of order one.

From the continuity of the pressure and velocity fields, we get $\forall j \in \{1, \dots, N(\omega)\}$

$$\begin{bmatrix} \widehat{a}_{j,L}(\omega) \\ \widehat{b}_{j,L}(\omega) \end{bmatrix} = \begin{bmatrix} r_j^{\epsilon,+} & r_j^{\epsilon,-} \\ r_j^{\epsilon,-} & r_j^{\epsilon,+} \end{bmatrix} \begin{bmatrix} \widehat{a}_{j,L}^\epsilon(\omega) \\ \widehat{b}_{j,L}^\epsilon(\omega) \end{bmatrix}, \text{ where } r_j^{\epsilon,\pm} = \frac{1}{2}\left[\sqrt{\frac{\beta_j(\omega)}{\beta_j^\epsilon(\omega)}} \pm \sqrt{\frac{\beta_j^\epsilon(\omega)}{\beta_j(\omega)}}\right],$$



and $\forall j \in \{N(\omega)+1,\dots,N_\epsilon(\omega)\}$

$$\begin{bmatrix}\widehat{c}_{j,L}(\omega)\\ \widehat{d}_{j,L}(\omega)\end{bmatrix} = \begin{bmatrix} r^{\epsilon,i}_j & \overline{r^{\epsilon,i}_j} \\ \overline{r^{\epsilon,i}_j} & r^{\epsilon,i}_j\end{bmatrix}\begin{bmatrix}\widehat{a}^{\epsilon}_{j,L}(\omega)\\ \widehat{b}^{\epsilon}_{j,L}(\omega)\end{bmatrix}, \text{where } r^{\epsilon,i}_j = \frac{1}{2}\left[\sqrt{\frac{\beta_j(\omega)}{\beta^\epsilon_j(\omega)}} - i\sqrt{\frac{\beta^\epsilon_j(\omega)}{\beta_j(\omega)}}\right]$$

with

$$\widehat{a}^{\epsilon}_{j,L}(\omega) = \widehat{a}^{\epsilon}_j(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}},\ \widehat{b}_{j,L}(\omega) = 0,\ \text{and } \widehat{d}_{j,L}(\omega) = 0.$$

The two last conditions mean that no wave comes from the right. In fact, in the first part of the experiment the time-reversal mirror records the signal and does not produce reflected waves. Solving these equations allows us to express the transmitted and the reflected coefficients. Consequently, $\forall j \in \{1,\dots,N(\omega)\}$, we have

$$\widehat{a}_{j,L}(\omega) = \tau^{\epsilon,+}_j(\omega)\widehat{a}^{\epsilon}_j(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} \text{ and } \widehat{b}^{\epsilon}_{j,L}(\omega) = -\frac{r^{\epsilon,-}_j}{r^{\epsilon,+}_j}\widehat{a}^{\epsilon}_j(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}},$$

where

$$\tau^{\epsilon,+}_j(\omega) = \frac{1}{r^{\epsilon,+}_j(\omega)} \tag{4.14}$$

is the transmission coefficient of the interface $z = L/\epsilon^{1-\alpha}$, and $\forall j \in \{N+1,\dots,N_\epsilon(\omega)\}$

$$\widehat{c}_{j,L}(\omega) = -\frac{i}{r^{\epsilon,i}_j}\widehat{a}^{\epsilon}_j(\omega, L)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} \text{ and } \widehat{b}^{\epsilon}_{j,L}(\omega) = -\frac{\overline{r^{\epsilon,i}_j}}{r^{\epsilon,i}_j}e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}}.$$

We can remark that $\forall j \in \{1,\dots,N(\omega)\}$, the transmission coefficients $\tau^{\epsilon,+}_j(\omega)$, which are defined by (4.14), are of order $\epsilon^{\alpha/2}$. We recall that we have taken a source amplitude of order $1/\epsilon^\alpha$ in (4.2). This fact will allow us to have, after the second step of the time-reversal experiment, a refocused wave of order one. However, we recall that we shall see, in section 4.4.6, that the transmission coefficients can be made of order one by inserting a quarter wavelength plate.

The reflected wave produced at the interface $z = L/\epsilon^{1-\alpha}$ does not reach the time-reversal mirror. Moreover, $L_M/\epsilon^{1-\alpha}$ is sufficiently large so that one can assume that the evanescent modes, that is, the jth right-going modes for $j \in \{N(\omega)+1,\dots,N_\epsilon(\omega)\}$ in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$ which decrease exponentially fast, do not reach the time-reversal mirror either. Therefore, only the transmitted propagating wave

$$p^\epsilon_{tr}\left(\frac{t}{\epsilon}, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) = \frac{1}{2\pi}\int \sum_{j=1}^{N(\omega)} \frac{\widehat{a}^{\epsilon}_j(\omega, L)}{\sqrt{\beta_j(\omega)}}\tau^{\epsilon,+}_j(\omega)e^{i\beta^\epsilon_j(\omega)\frac{L}{\epsilon^{1-\alpha}}}e^{i\beta_j(\omega)\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}\phi_j(x)e^{-i\omega\frac{t}{\epsilon}}d\omega \tag{4.15}$$

is recorded by the time-reversal mirror.

4.4.2 Second Step of the Time-Reversal Experiment

In the second step of the time-reversal experiment, the time-reversal mirror plays the role of a source array, and the time-reversed signal is transmitted back. This source term is given by

$$\mathbf{F}^{\epsilon}_{TR}(t,x,z) = -f^{\epsilon}_{TR}(t,x)\delta(z - L_M/\epsilon^{1-\alpha})\mathbf{e}_z,$$

with

$$f^{\epsilon}_{TR}(t,x) = p^{\epsilon}_{tr}\left(\frac{t_1}{\epsilon} - t, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) G_1(t_1 - \epsilon t) G_2(x),$$

and

$$G_1(t) = \mathbf{1}_{[t_0,t_1]}(t) \quad \text{and} \quad G_2(x) = \mathbf{1}_{\mathcal{D}_M}(x).$$

Here, G_1 represents the time window in which the transmitted wave is recorded, and G_2 represents the spatial window in which the transmitted wave is recorded. As in Chapter 3, we are interested in the spatial effects of the refocusing, so we assume that we record the field for all time at the time-reversal mirror, i.e.,

$$f^{\epsilon}_{TR}(t,x) = p^{\epsilon}_{tr}\left(\frac{t_1}{\epsilon} - t, x, \frac{L_M}{\epsilon^{1-\alpha}}\right) G_2(x). \tag{4.16}$$

However, in this chapter, we assume that the two realizations of the random medium are the same during the two steps of the time-reversal experiment. Let us remark that the same work as in Section 3.4.6 can be done in the configuration of this chapter.

We study the propagation from $z = L_M/\epsilon^{1-\alpha}$ to $z = 0$. The decomposition on the eigenmodes gives

$$\widehat{p}^{L_M,\epsilon}_{TR}\left(\omega, x, \frac{z}{\epsilon^{1-\alpha}}\right) = \sum_{m=1}^{N(\omega)} \frac{\widehat{b}_{m,L_M}(\omega)}{\sqrt{\beta_m(\omega)}} e^{-i\beta_m(\omega)\left(\frac{z-L_M}{\epsilon^{1-\alpha}}\right)} \phi_m(x)$$

in the homogeneous part $(L/\epsilon^{1-\alpha}, L_M/\epsilon^{1-\alpha})$ of the waveguide, with

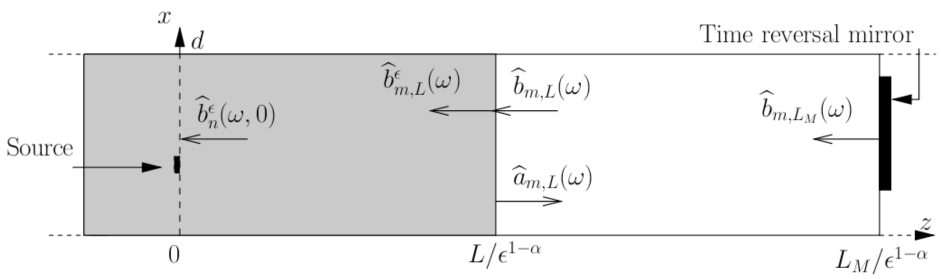
$$\widehat{b}_{m,L_M}(\omega) = \frac{\sqrt{\beta_m(\omega)}}{2} \int_0^d \widehat{f}^{\epsilon}_{TR}(\omega,x)\phi_m(x)dx, \tag{4.17}$$

where

$$\widehat{f}^{\epsilon}_{TR}(\omega,x) = \sum_{j=1}^{N(\omega)} \frac{\overline{\widehat{a}^{\epsilon}_j(\omega,L)}}{\sqrt{\beta_j(\omega)}} \overline{\tau^{\epsilon,+}_j(\omega)} e^{-i\beta^{\epsilon}_j(\omega)\frac{L}{\epsilon^{1-\alpha}}} e^{-i\beta_j(\omega)\left(\frac{L_M - L}{\epsilon^{1-\alpha}}\right)} \phi_j(x) G_2(x) e^{i\omega\frac{t_1}{\epsilon}}, \tag{4.18}$$

and $\widehat{b}_{m,L_M}(\omega) = 0$ for $m > N$. We are now interested in the refocused wave near the source location. The transmission through the interface $z = L/\epsilon^{1-\alpha}$ and the back propagation in the random section are treated in the same way as the first step of the time-reversal experiment. The eigenmode decomposition at the interface $z = L/\epsilon^{1-\alpha}$ is given by

$$\begin{aligned}
\widehat{p}^{L,\epsilon}_{TR}(\omega,x,z) = &\left[\sum_{m=1}^{N(\omega)} \frac{\widehat{a}_{m,L}(\omega)}{\sqrt{\beta_m(\omega)}} e^{i\beta_m(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_m(x) + \frac{\widehat{b}_{m,L}(\omega)}{\sqrt{\beta_m(\omega)}} e^{-i\beta_m(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_m(x)\right] \mathbf{1}_{(L/\epsilon^{1-\alpha},+\infty)}(z) \\
&+ \left[\sum_{m=1}^{N_\epsilon(\omega)} \frac{\widehat{a}^{\epsilon}_{m,L}(\omega)}{\sqrt{\beta^{\epsilon}_m(\omega)}} e^{i\beta^{\epsilon}_m(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_m(x) + \frac{\widehat{b}^{\epsilon}_{m,L}(\omega)}{\sqrt{\beta^{\epsilon}_m(\omega)}} e^{-i\beta^{\epsilon}_m(\omega)\left(z - \frac{L}{\epsilon^{1-\alpha}}\right)} \phi_m(x)\right] \mathbf{1}_{(0,L/\epsilon^{1-\alpha})}(z),
\end{aligned} \tag{4.19}$$



where $\widehat{a}_{m,L}(\omega)$ (resp., $\widehat{b}_{m,L}(\omega)$) is the amplitude of the mth right-going (resp., left-going) mode propagating in the homogeneous section $(L/\epsilon^{1-\alpha}, +\infty)$, and $\widehat{a}^{\epsilon}_{m,L}(\omega)$ (resp., $\widehat{b}^{\epsilon}_{m,L}(\omega)$) is the amplitude of the mth right-going (resp. left-going) mode propagating in the section $(0, L/\epsilon^{1-\alpha})$.

From the continuity of the pressure and velocity fields, we get $\forall m \in \{1,\dots,N_\epsilon(\omega)\}$

$$\begin{bmatrix}\widehat{a}_{m,L}(\omega)\\ \widehat{b}_{m,L}(\omega)\end{bmatrix} = \begin{bmatrix} r^{\epsilon,+}_m & r^{\epsilon,-}_m \\ r^{\epsilon,-}_m & r^{\epsilon,+}_m\end{bmatrix}\begin{bmatrix}\widehat{a}^{\epsilon}_{m,L}(\omega)\\ \widehat{b}^{\epsilon}_{m,L}(\omega)\end{bmatrix}.$$

However, the source emits only $N(\omega)$ propagating modes; therefore, $\widehat{a}_{m,L}(\omega) = \widehat{b}_{m,L}(\omega) = 0$ for $m > N(\omega)$ and for $m \le N(\omega)$

$$\widehat{a}^{\epsilon}_{m,L}(\omega) = 0 \text{ and } \widehat{b}_{m,L}(\omega) = \widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)}.$$

The first condition means that no wave comes from the left in this forward approximation that we are considering. Solving this equation permits us to express the transmitted and the reflected coefficients. $\forall m \in \{1,\dots,N(\omega)\}$,

$$\widehat{a}_{m,L}(\omega) = \frac{r^{\epsilon,-}_m}{r^{\epsilon,+}_m}\widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)}, \quad \widehat{b}^{\epsilon}_{m,L}(\omega) = \tau^{\epsilon,+}_m(\omega)\widehat{b}_{m,L_M} e^{-i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)},$$

where $\tau^{\epsilon,+}_m(\omega) = \frac{1}{r^{\epsilon,+}_m(\omega)}$ and $\widehat{b}^{\epsilon}_{m,L}(\omega) = 0$ $\forall m \in \{N(\omega)+1,\dots,N_\epsilon(\omega)\}$. Thus, we have obtained the expression of the boundary conditions at the plane $z = L/\epsilon^{1-\alpha}$. Now, we are interested in the back propagation through the random section from $z = L/\epsilon^{1-\alpha}$ to $z = 0$;

$$\widehat{p}^{\epsilon}_{TR}(\omega, x, 0) = \sum_{n=1}^{N_\epsilon(\omega)} \frac{\widehat{b}^{\epsilon}_n(\omega,0)}{\sqrt{\beta^{\epsilon}_n(\omega)}}\phi_n(x).$$

Since the transfer matrix $\mathbf{T}^{\epsilon}(\omega, z)$ is unitary,

$$\begin{aligned}\widehat{b}^{\epsilon}_n(\omega,0) &= \sum_{m=1}^{N(\omega)} \mathbf{T}^{\epsilon}_{mn}(\omega,L)\widehat{b}^{\epsilon}_m(\omega,L)e^{i\beta^{\epsilon}_m(\omega)\frac{L}{\epsilon^{1-\alpha}}}\\ &= \sum_{m=1}^{N(\omega)} \mathbf{T}^{\epsilon}_{mn}(\omega,L)\tau^{\epsilon,+}_m(\omega)\widehat{b}_{m,L_M} e^{i\beta_m(\omega)\left(\frac{L-L_M}{\epsilon^{1-\alpha}}\right)} e^{i\beta^{\epsilon}_m(\omega)\frac{L}{\epsilon^{1-\alpha}}},\end{aligned}$$

and using (4.15), (4.16), (4.17), and (4.18) we get

$$\begin{aligned}\widehat{b}_{m,L_M}(\omega) = \frac{1}{8\epsilon^p}\sum_{j=1}^{N(\omega)}\sum_{l=1}^{N_\epsilon(\omega)} &\sqrt{\frac{\beta^{\epsilon}_l(\omega)\beta_m(\omega)}{\beta_j(\omega)}} M_{mj}\theta_l \overline{\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)\mathbf{T}^{\epsilon}_{jl}(\omega,L)}\\ &\times \frac{1}{\epsilon^\alpha}\overline{\tau^{\epsilon,+}_j(\omega)}\tau^{\epsilon,+}_m(\omega)e^{-i\beta^{\epsilon}_j(\omega)\frac{L}{\epsilon^{1-\alpha}}}e^{-i\beta_j(\omega)\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}e^{i\omega\frac{t_1}{\epsilon}},\end{aligned}$$

where

$$M_{jl} = \int_0^d G_2(x)\phi_j(x)\phi_l(x)dx.$$

The matrix (M_{jl}) represents the coupling produced by the time-reversal mirror between the propagating modes during the two steps of the time-reversal experiment. We recall that $\widehat{b}_{m,L_M}$ is the projection over the mth propagating mode for the Fourier transform of the time-reversed signal recorded by the time-reversal mirror. Therefore, the refocused wave is

$$\begin{aligned} p^\epsilon_{TR}\left(\frac{t}{\epsilon}, x, 0\right) = & \frac{1}{16\pi\epsilon^p} \int \sum_{j,m=1}^{N(\omega)} \sum_{l,n=1}^{N_\epsilon(\omega)} \sqrt{\frac{\beta^\epsilon_l(\omega)\beta_m(\omega)}{\beta_j(\omega)\beta^\epsilon_n(\omega)}} M_{mj}\theta_l\phi_n(x) \\ & \times \overline{\mathbf{T}^\epsilon_{jl}(\omega, L)}\mathbf{T}^\epsilon_{mn}(\omega, L)\frac{1}{\epsilon^\alpha}\overline{\tau^{\epsilon,+}_j(\omega)}\tau^{\epsilon,+}_m(\omega)\overline{\widehat{f}\left(\frac{\omega-\omega_0}{\epsilon^p}\right)} \\ & \times e^{i(\beta_m(\omega)-\beta_j(\omega))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)} e^{i(\beta^\epsilon_m(\omega)-\beta^\epsilon_j(\omega))\frac{L}{\epsilon^{1-\alpha}}} e^{i\omega\frac{t_1-t}{\epsilon}} d\omega. \end{aligned} \tag{4.20}$$

Now, we make the change of variable $\omega = \omega_0 + \epsilon^p h$. Consequently, (4.20) becomes

$$\begin{aligned} e^{-i\omega_0\frac{t_1-t}{\epsilon}} p^\epsilon_{TR}\left(\frac{t}{\epsilon}, x, 0\right) = & \frac{1}{16\pi} \int \sum_{j,m=1}^{N(\omega_0+\epsilon^p h)} \sum_{l,n=1}^{N_\epsilon(\omega_0+\epsilon^p h)} \sqrt{\frac{\beta^\epsilon_l(\omega_0+\epsilon^p h)\beta_m(\omega_0+\epsilon^p h)}{\beta_j(\omega_0+\epsilon^p h)\beta^\epsilon_n(\omega_0+\epsilon^p h)}} \\ & \times \overline{\mathbf{T}^\epsilon_{jl}(\omega_0+\epsilon^p h, L)}\mathbf{T}^\epsilon_{mn}(\omega_0+\epsilon^p h, L)\frac{1}{\epsilon^\alpha}\overline{\tau^{\epsilon,+}_j(\omega_0+\epsilon^p h)}\tau^{\epsilon,+}_m(\omega_0+\epsilon^p h) \\ & \times M_{mj}\theta_l\phi_n(x)\overline{\widehat{f}(h)} e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)} \\ & \times e^{i(\beta^\epsilon_m(\omega_0+\epsilon^p h)-\beta^\epsilon_j(\omega_0+\epsilon^p h))\frac{L}{\epsilon^{1-\alpha}}} e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)} dh. \end{aligned} \tag{4.21}$$

In what follows: we consider the following

1. A source with transverse profile of the form

$$\forall x \in [0, d], \quad \Psi(x) = \sum_{l=1}^{\zeta} \phi_l(x_0)\phi_l(x),$$

   where we assume that $\zeta \gg N(\omega_0)$. Then, $\theta_l = \phi_l(x_0)$ for $l \in \{1, \dots, \zeta\}$ and $\theta_l = 0$ for $l \geq \zeta + 1$. This profile is an approximation of a Dirac distribution at x_0, which models a point source at x_0.

2. A time-reversal mirror of the form $\mathcal{D}_M = [d_1, d_2]$ with

$$d_2 = d_M + \lambda_0^{\alpha_M}\tilde{d}_2 \text{ and } d_1 = d_M - \lambda_0^{\alpha_M}\tilde{d}_1,$$

   where $d_M \in (0, d)$, $(\tilde{d}_2, \tilde{d}_1) \in (0, +\infty)^2$, and $\alpha_M \in [0, 1]$. The time-reversal coupling matrix is given by

$$\begin{aligned} M_{jl} = \frac{d_2 - d_1}{d} \Bigg[& \cos\left((j-l)\left(\frac{d_2+d_1}{2d}\right)\pi\right) \operatorname{sinc}\left((j-l)\left(\frac{d_2-d_1}{2d}\right)\pi\right) \\ & - \cos\left((j+l)\left(\frac{d_2+d_1}{2d}\right)\pi\right) \operatorname{sinc}\left((j+l)\left(\frac{d_2-d_1}{2d}\right)\pi\right)\Bigg]. \end{aligned}$$

   The parameter α_M represents the order of the magnitude of the size of the mirror with respect to the carrier wavelength $\lambda_0 = 2\pi c/\omega_0$. In fact, we shall see that the size of the mirror plays a role in the homogeneous case only when it is of the order the carrier wavelength λ_0.

Moreover, we shall study the spatial profile of the refocused wave in the high-frequency regime $\omega_0 \nearrow +\infty$. However, we know that the main focal spot must be of order λ_0, which tends to 0 in this continuum limit. Therefore, we shall study the spatial profile in a window of size λ_0 centered around the source location x_0.

4.4.3 Homogeneous Waveguide

Here we examine the homogeneous case, that is, the case in which the section $[0, L/\epsilon^{1-\alpha}]$ has homogeneous parameters $\bar{K}/\epsilon^{2\alpha_K}$ and $\bar{\rho}/\epsilon^{2\alpha_\rho}$. In these conditions we have $\mathbf{T}^\epsilon_{jl}(\omega, z) = \delta_{jl}$. We recall that the continuum limit $N(\omega_0) \gg 1$ is achieved in the high frequency regime $\omega_0 \nearrow +\infty$ and the carrier wavelength is given by $\lambda_0 = 2\pi c/\omega_0$.

Proposition 4.2 *The refocused field is given by*

$$\lim_{\epsilon \to 0} e^{i\omega_0 \frac{t}{\epsilon^p}} p^\epsilon_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, x, 0\right) = H^{\alpha_M}_{x_0}(\omega_0, x) f(-t),$$

where

$$H^{\alpha_M}_{x_0}(\omega_0, x) = \frac{1}{2} \sum_{j=1}^{N(\omega_0)} \frac{\beta_j(\omega_0)}{k(\omega_0)} M_{jj} \phi_j(x_0)\phi_j(x)$$

For $\alpha_M \in [0, 1)$, the transverse profile of the refocused wave in the continuum limit is given by

$$\lim_{\omega_0 \to +\infty} \lambda_0^{1-\alpha_M} H^{\alpha_M}_{x_0}(\omega_0, x_0 + \lambda_0 \tilde{x}) = \frac{\tilde{d}_2 + \tilde{d}_1}{d} H^{(1)}(\tilde{x}),$$

where

$$H^{(1)}(\tilde{x}) = \int_0^1 \sqrt{1 - u^2} \cos(2\pi \tilde{x} u)\, du. \tag{4.22}$$

Proof First, we have $\forall p \in (0, 1)$ and $\forall \alpha \in (0, 1]$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^\alpha} \overline{\tau^{\epsilon,+}_j(\omega_0 + \epsilon^p h)} \tau^{\epsilon,+}_m(\omega_0 + \epsilon^p h) = 4\frac{\sqrt{\beta_j(\omega_0)\beta_m(\omega_0)}}{k(\omega_0)}. \tag{4.23}$$

Second, we will fix the parameters p and α in order to give, for illustration, a simpler proof. Let $p = 1/2$ and $\alpha = 1/6$. These two values allow us to have a not too long truncated expansion (4.24); then the refocused field is given by the deterministic expression, for $\epsilon \ll 1$,

$$\begin{aligned} p^\epsilon_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^{1/2}}, x, 0\right) &\simeq \frac{1}{2} \sum_{j,m=1}^{N(\omega_0)} \frac{\beta_m(\omega_0)}{k(\omega_0)} M_{mj} \theta_j \phi_m(x) e^{i(\beta_m(\omega_0) - \beta_j(\omega_0))\left(\frac{L_M - L}{\epsilon^{5/6}}\right)} \\ &\times e^{i(\beta^\epsilon_m(\omega_0) - \beta^\epsilon_j(\omega_0))\frac{L}{\epsilon^{5/6}}} e^{-i\omega_0 \frac{t}{\epsilon^{1/2}}} \\ &\times f\left(\left[\epsilon^{1/6}(\beta'_m(\omega_0) - \beta'_j(\omega_0))(L_M - L) + \epsilon^{1/3}(m^2 - j^2)\frac{\pi^2 c^2}{\omega_0^2 d^2} \frac{L}{2c}\right] \frac{1}{\epsilon^{1/2}} - t\right), \end{aligned}$$

since

$$\begin{aligned} \left(\beta^\epsilon_m(\omega_0 + \epsilon^{1/2} h) - \beta^\epsilon_j(\omega_0 + \epsilon^{1/2} h)\right) \frac{L}{\epsilon^{1-\alpha}} &= \left(\beta^\epsilon_m(\omega_0) - \beta^\epsilon_j(\omega_0)\right) \frac{L}{\epsilon^{5/6}} \\ &+ \frac{h}{2c}(m^2 - j^2)\frac{\pi^2 c^2}{\omega_0^2 d^2} \frac{L}{\epsilon^{1/6}} + o(1). \end{aligned} \tag{4.24}$$

Finally, the transverse profile is given by

$$\frac{d}{\tilde{d}_2+\tilde{d}_1}\lambda_0^{1-\alpha_M}H_{x_0}^{\alpha_M}(\omega_0,x_0+\lambda_0\tilde{x}) = \frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\frac{j^2}{N^2}}\cos\left(2\pi\tilde{x}\frac{j}{N}\right)$$
$$+\frac{\lambda_0}{2d}\sum_{j=1}^{N}\frac{\beta_j(\omega_0)}{k(\omega_0)}\cos\left(j\frac{\pi}{d}(2x_0+\lambda_0\tilde{x})\right)$$
$$-\frac{d}{\tilde{d}_2+\tilde{d}_1}\frac{\lambda_0^{1-\alpha_M}}{2}\sum_{j=1}^{N}\frac{\beta_j(\omega_0)}{j\pi k(\omega_0)}\phi_j(x_0)\phi_j(x_0+\lambda_0\tilde{x})$$
$$\times\cos\left(j\pi\left(\frac{d_2+d_1}{d}\right)\right)\sin\left(j\pi\left(\frac{d_2-d_1}{d}\right)\right)$$
$$+o(1).$$

Using the Abel transform, the second and the third sums on the right are $\mathcal{O}(1)$. This completes the proof of the Proposition. ■

To finish this section, we consider the difference between the previous profile (obtained in the case where the homogeneous section $[0, L/\epsilon^{1-\alpha}]$, with the parameters $\bar{K}/\epsilon^{2\alpha_K}$ and $\bar{\rho}/\epsilon^{2\alpha_\rho}$, is present) and the one in which this homogeneous section is missing (that is, the waveguide is homogeneous with parameters $\bar{K}$ and $\bar{\rho}$). The second profile is given, in [25, Chapter 20], by

$$H_{x_0,\text{no section}}^{\alpha_M}(\omega_0,x) = \frac{1}{2}\sum_{j=1}^{N(\omega_0)}M_{jj}\phi_j(x_0)\phi_j(x),$$

which we can rewrite in the continuum limit $N(\omega_0)\gg 1$.

Proposition 4.3 *For $\alpha_M\in[0,1)$, the spatial profile in the continuum limit is given by*

$$\lim_{\omega_0\to+\infty}\lambda_0^{1-\alpha_M}H_{x_0,\text{no section}}^{\alpha_M}(\omega_0,x_0+\lambda_0\tilde{x}) = \frac{\tilde{d}_2+\tilde{d}_1}{d}\text{sinc}(2\pi\tilde{x}); \tag{4.25}$$

where the sinc *function is defined by* $\text{sinc}(v)=\sin(v)/v$.

The formula (4.25) corresponds to the classical diffraction limit with a focal spot of radius $\lambda_0/2$. In Figure 4.3, we compare, in the homogeneous case, the spatial profile (4.22) in the case where the homogeneous section $[0, L/\epsilon^{1-\alpha}]$ is present with the profile (4.25), where this section is missing. We can see that the main focal spot, in the case where a section is inserted, is larger than the focal spot produced when this section is missing (see Figure 4.5). The use of this section does not improve the refocusing in the homogeneous case. It is necessary to use an inhomogeneous section to induce mode coupling in order to enhance refocusing, as we shall see in the next section.

4.4.4 Mean Refocused Field in the Random Case

Taking the expectation of (4.21), we obtain the mean refocused wave

$$\mathbb{E}\left[e^{-i\omega_0\frac{t_1-t}{\epsilon}}p_{TR}^{\epsilon}\left(\frac{t}{\epsilon},x,0\right)\right] = \frac{1}{16\pi}\int\sum_{j,m=1}^{N(\omega_0+\epsilon^p h)}\sum_{n=1}^{N_\epsilon(\omega_0+\epsilon^p h)}\sum_{l=1}^{\zeta}\sqrt{\frac{\beta_l^{\epsilon}(\omega_0+\epsilon^p h)\beta_m(\omega_0+\epsilon^p h)}{\beta_j(\omega_0+\epsilon^p h)\beta_n^{\epsilon}(\omega_0+\epsilon^p h)}}$$
$$\times\mathbb{E}\left[\overline{\mathbf{T}_{jl}^{\epsilon}(\omega_0+\epsilon^p h,L)}\mathbf{T}_{mn}^{\epsilon}(\omega_0+\epsilon^p h,L)\right]M_{mj}\phi_l(x_0)\phi_n(x)\overline{\hat{f}(h)}$$
$$\times\frac{1}{\epsilon^{\alpha}}\overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)}\tau_m^{\epsilon,+}(\omega_0+\epsilon^p h)e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M-L}{\epsilon^{1-\alpha}}\right)}$$
$$\times e^{i(\beta_m^{\epsilon}(\omega_0+\epsilon^p h)-\beta_j^{\epsilon}(\omega_0+\epsilon^p h))\frac{L}{\epsilon^{1-\alpha}}}e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)}dh.$$

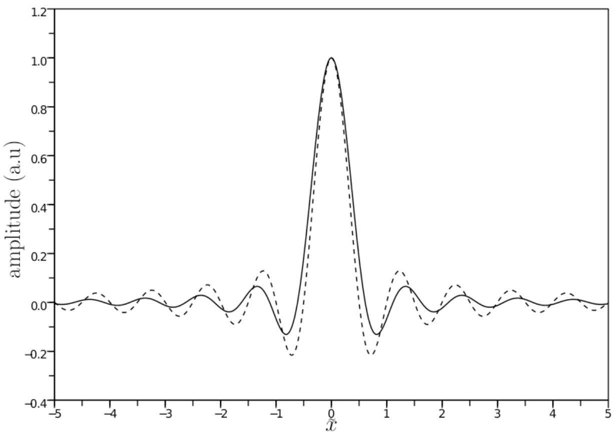


Figure 4.3: Normalized transverse profiles in the homogeneous waveguide in the case where $\alpha_M \in [0,1)$. The dashed curve is the transverse profile in the case where the section is missing $\mathrm{sinc}(2\pi\tilde{x})$ and the solid curve is the refocusing profile $H^{(1)}(\tilde{x})$ in the case where we add a homogeneous section.

We shall establish the convergence of the mean refocused wave in the topological dual space $\mathcal{E}'$ equipped with the weak topology, with $\mathcal{E} = \bigcup_{M\geq 1} \mathcal{E}_M$ and where

$$\mathcal{E}_M = \left\{ \sum_{j=1}^{M} \mu_j \phi_j, \quad (\mu_j)_j \in \mathbb{R}^M \right\}.$$

$\mathcal{E}_M$ is equipped with the topology induced by $\langle .,. \rangle_{L^2(0,d)}$ and $\mathcal{E}$ with the inductive limit topology. This topology is a the same as the one used in Proposition 3.12 and Proposition 3.14. Consequently, It suffices to study $\langle \mathbb{E}[e^{i\omega_0 \frac{t}{\epsilon^p}} p^\epsilon_{TR}(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, ., 0)], \phi_n \rangle_{L^2(0,d)}$ for $n \in \mathbb{N}^*$. Using Proposition 4.1, we get

$$\begin{aligned}
\lim_{\zeta\to+\infty} \lim_{\epsilon\to 0} \left\langle \mathbb{E}\left[e^{i\omega_0 \frac{t}{\epsilon^p}} p^\epsilon_{TR}\left(\frac{t_1}{\epsilon} + \frac{t}{\epsilon^p}, ., 0\right)\right], \phi_n \right\rangle_{L^2(0,d)} \\
= \frac{1}{2} \sum_{j=1}^{N(\omega_0)} \sum_{l\geq 1} \frac{\beta_j(\omega_0)}{k(\omega_0)} \mathcal{T}^l_j(\omega_0, L) M_{jj} \phi_l(x_0) f(-t) \delta_{ln} + \mathcal{O}(N^2 e^{-\Lambda L}) \\
= \langle f(-t) H^{\alpha_M}_{x_0}(\omega_0, .), \phi_n \rangle_{L^2(0,d)} + \mathcal{O}(N^2 e^{-\Lambda L}),
\end{aligned}$$

where the transverse profile is given by

$$H^{\alpha_M}_{x_0}(\omega_0, x) = \frac{1}{2} \sum_{l\geq 1} \sum_{j=1}^{N(\omega_0)} \frac{\beta_j(\omega_0)}{k(\omega_0)} \mathcal{T}^l_j(\omega_0, L) M_{jj} \phi_l(x) \phi_l(x_0).$$

In the continuum limit, the terms which correspond to $j \neq m$ decay exponentially because of the damping term $e^{-\Lambda L}$ since $\Lambda \simeq N^2\sigma^2/2$.

Proposition 4.4 *For $\alpha_M \in [0,1]$, in the continuum limit $N(\omega_0) \gg 1$, we have*

$$\lim_{\omega_0\to+\infty} \left(H^{\alpha_M}_{x_0}(\omega_0, .) - \tilde{H}^{\alpha_M}_{x_0}(\omega_0, .) \right) = 0$$

in $\mathcal{E}'$, where

$$\lim_{\omega_0\to+\infty} \lambda_0^{1-\alpha_M} \tilde{H}^{\alpha_M}_{x_0}(\omega_0, x_0 + \lambda_0 \tilde{x}) = \frac{\tilde{d}_2 + \tilde{d}_1}{d} H^{(2)}(\tilde{x}, L),$$

with

$$H^{(2)}(\tilde{x}, L) = e^{-\tilde{x}^2/r_c^2} H^{(1)}(\tilde{x}) = e^{-\tilde{x}^2/r_c^2} \int_0^1 \sqrt{1-u^2} \cos(2\pi \tilde{x} u)\, du, \tag{4.26}$$

and

$$r_c = \frac{1}{\pi\sigma\sqrt{2L}} = \frac{d}{\pi^2}\sqrt{\frac{a}{2LS(1,1)}}. \tag{4.27}$$

From this proposition, in contrast with Proposition 4.2 which considers a homogeneous waveguide, the time-reversal coupling matrix does not play any role in the transverse profile of the mean refocused wave. This result is consistent with those obtained in [25, 30] and those of Section 3.4.7.

Proof Let $n \in \mathbb{N}^*$; we have

$$\lim_{\zeta\to+\infty} \langle H_{x_0}^{\alpha_M}(\omega_0, .), \phi_n \rangle_{L^2(0,d)} = \frac{1}{2}\phi_n(x_0) \sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{k(\omega_0)} M_{jj} \mathcal{T}_j^n(\omega_0, L).$$

Using the probabilistic interpretation of $\mathcal{T}_j^n(\omega_0, L)$ in Section 4.3, we get

$$\begin{aligned}\sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{k(\omega_0)} M_{jj} \mathcal{T}_j^n(\omega_0, L) = {} & \frac{d_2 - d_1}{2d}\mathbb{E}\left[\sqrt{1-\left(\frac{X_L}{N}\right)^2}\mathbf{1}_{\left(\frac{X_L}{N}\in\left\{\frac{1}{N},\cdots,1\right\}\right)}\Big|\frac{X_0}{N} = \frac{n}{N}\right] \\ & - \frac{1}{2}\sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \cos\left(j\pi\frac{d_2+d_1}{d}\right) \sin\left(j\pi\frac{d_2-d_1}{d}\right) \\ & + o(1).\end{aligned}$$

Moreover, using Theorem 4.2

$$\begin{aligned}\mathbb{E}\left[\sqrt{1-\left(\frac{X_L}{N}\right)^2}\mathbf{1}_{\left(\frac{X_L}{N}\in\left\{\frac{1}{N},\cdots,1\right\}\right)}\Big|\frac{X_0}{N} = \frac{n}{N}\right] = {} & \mathbb{E}\left[\sqrt{1-\left(\sigma B_L + \frac{n}{N}\right)^2}\mathbf{1}_{\left(\sigma B_L + \frac{n}{N}\in[-1,1]\right)}\right] \\ & + o(1),\end{aligned}$$

and we have the following result.

Lemma 4.1

$$\lim_{N\to+\infty} \sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \cos\left(j\pi\frac{d_2+d_1}{d}\right) \sin\left(j\pi\frac{d_2-d_1}{d}\right) = 0.$$

Proof It suffices to show that

$$\lim_{N\to+\infty} \sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) = 0.$$

Let $\eta \in (0,1)$; we have

$$\begin{aligned}\sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \leq {} & \mathbb{P}\left(\frac{X_L}{N} \in \left\{\frac{1}{N}, \cdots, \frac{[N\eta]}{N}\right\}\Big|\frac{X_0}{N} = \frac{n}{N}\right) \\ & + \frac{1}{[N\eta]+1} \sum_{j=[N\eta]+1}^{N} \mathbb{P}(X_L = j | X_0 = n).\end{aligned}$$

Therefore,

$$\overline{\lim_{N}} \sum_{j=1}^{N} \frac{\beta_j(\omega_0)}{j\pi k(\omega_0)} \mathcal{T}_j^n(\omega_0, L) \leq \mathbb{P}(\sigma B_L \in [0, \eta]),$$

and we get the result by letting $\eta \searrow 0$.□

This lemma shows that the time-reversal coupling matrix does not play any role in the transverse profile of the mean refocused wave. Consequently,

$$\lim_{\omega_0\to+\infty}\left\langle H^{\alpha_M}_{x_0}(\omega_0,.)-\tilde{H}^{\alpha_M}_{x_0}(\omega_0,.),\phi_n\right\rangle_{L^2(0,d)}=0,$$

where

$$\tilde{H}^{\alpha_M}_{x_0}(\omega_0,x)=\frac{d_2-d_1}{2d}\sum_{l\geq1}\mathbb{E}\left[\sqrt{1-\left(\sigma B_L+\frac{l}{N}\right)^2}\mathbf{1}_{(\sigma B_L+\frac{l}{N}\in[-1,1])}\right]\phi_l(x_0)\phi_l(x).$$

Lemma 4.2 *In the continuum limit $N(\omega_0)\gg1$, we have*

$$\lambda_0^{1-\alpha_M}\tilde{H}^{\alpha_M}_{x_0}(\omega_0,x_0+\lambda_0\tilde{x})=\frac{\tilde{d}_2+\tilde{d}_1}{d}e^{-2L\sigma^2\pi^2\tilde{x}^2}\int_0^1\sqrt{1-u^2}\cos\left(2\pi\tilde{x}u\right)du.$$

Proof The proof is an application of the Poisson formula,

$$\sum_{m\in\mathbb{Z}}\widehat{F}_u(m)e^{imv}=2\pi\sum_{m\in\mathbb{Z}}F_u(v+2m\pi),$$

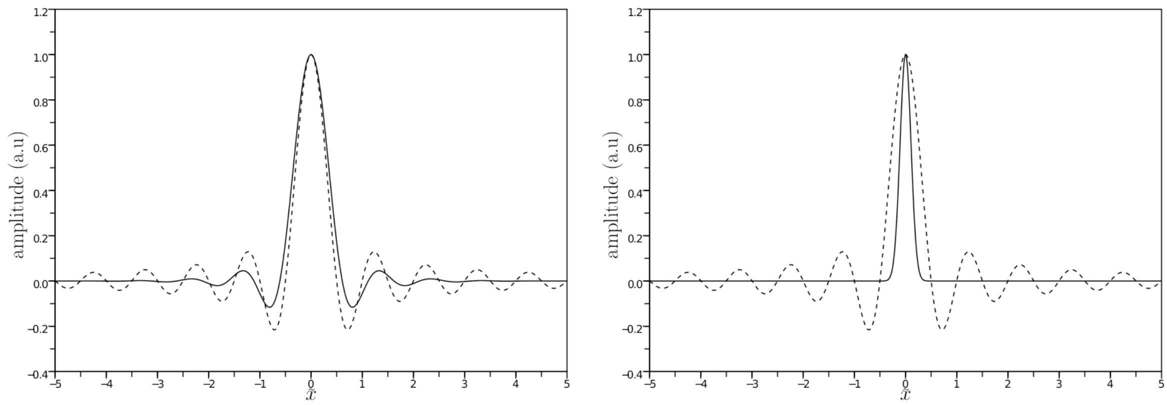
with $\widehat{F}_u(m)=e^{-\frac{(m-Nu)^2}{2N^2\sigma^2L}}$ and $F_u(t)=\frac{\sqrt{2\pi N^2\sigma^2L}}{2\pi}e^{-t^2N^2\frac{\sigma^2}{2}L+itNu}$. Thus, we obtain

$$\begin{aligned}\frac{d}{d_2-d_1}&\tilde{H}^{\alpha_M}_{x_0}(\omega_0,x_0+\lambda_0\tilde{x})\\&=\frac{N}{d}\sum_{l\in\mathbb{Z}}e^{-N^2\frac{\sigma^2}{2}L\left(\frac{\pi}{d}\lambda_0\tilde{x}+2l\pi\right)^2}\int_0^1\sqrt{1-u^2}\cos\left[\left(\frac{\pi}{d}\lambda_0\tilde{x}+2l\pi\right)Nu\right]du\\&-\frac{N}{d}\sum_{l\in\mathbb{Z}}e^{-N^2\frac{\sigma^2}{2}L\left(\frac{\pi}{d}(\lambda_0\tilde{x}+2x_0)+2l\pi\right)^2}\int_0^1\sqrt{1-u^2}\cos\left[\left(\frac{\pi}{d}(\lambda_0\tilde{x}+2x_0)+2l\pi\right)Nu\right]du.\end{aligned}$$

Finally, we take only the term $l=0$ in the first sum on the right because the rest of the first sum and the second sum are of order $\mathcal{O}(e^{-CN^2})$ uniformly in $\tilde{x}$. Moreover, we have $\lim_{\omega_0}\lambda_0N/(2d)=1$. □

This last result completes the proof of Proposition 4.4. ■

In Figure 4.4, we illustrate the differences between the transverse profiles of the refocused wave in the homogeneous case and when a random section is inserted. In order to show that random inhomogeneities enhance refocusing of the time-reversed waves, we consider two configurations. (a) illustrates the case where $\sigma\ll1$ (weak fluctuations). We can see that the focal spot in the case where we add a section can be larger than in the case where this section is missing. In contrast, (b) illustrates the case where σ is large enough to have side-lobe suppression and a focal spot which is narrower than in the case where the random section is missing. In figure 4.5, we illustrate the improvement of resolution with respect to σ by using the FWHM, that is the *full width at half maximum*, which is a useful tool for studying the width of peaks. In the case where the random perturbed section is missing, the FWHM of the transverse profile given in Proposition 4.3, is of order $\lambda_0/2$. However, when this section is inserted the FWHM of the transverse profile, given in Proposition 4.4 is narrower than in the previous case for σ large enough. Consequently, if σ is large enough, the resolution is $<\lambda_0/2$.



(a) (b)

Figure 4.4: Normalized transverse profiles in a random waveguide. Here $L = 1$. In (a) and (b) we illustrate the case where $\alpha_M \in [0, 1)$. The dashed curves are the transverse profiles in the case where the section is missing $\mathrm{sinc}(2\pi\tilde{x})$, and the solid curves are the transverse profiles $H^{(2)}(\tilde{x}, L)$ in the case where we add a random section, with $\sigma = 0.5$ in (a), and $\sigma = 7$ in (b).

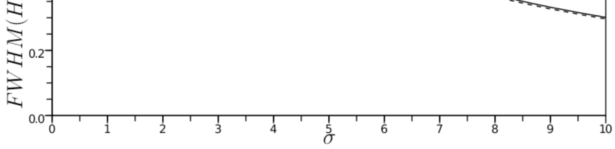


Figure 4.5: Ratio between the FWHM of the profile $H^{(2)}(\tilde{x}, L)$ obtained when we add a random section and that of the profile obtained when this section is missing $\mathrm{sinc}(2\pi\tilde{x})$, in terms of the standard deviation σ. Here $L = 1$. The solid curve represents the case where $\alpha_M \in [0, 1)$, and the dashed curve represents the case where $\alpha_M = 1$.

4.4.5 Statistical Stability

Pulse stabilization is proved by a frequency decoherence argument, see [18] in the context of a one-dimensional medium and [25, Chapter 20] in the context of waveguides. In our case, to prove the self-averaging property, we study the second order moment of the refocused wave $\epsilon^{i\omega_0 \frac{t}{\epsilon^p}} p^\epsilon_{TR}\left(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^p},x,0\right)$. As in section 4.5.1, we prove a limit theorem for

$$\left(\overline{\mathbf{T}^\epsilon_{.l}(\omega+\epsilon^p h,.)}\mathbf{T}^\epsilon_{.n}(\omega+\epsilon^p h,.)\mathbf{T}^\epsilon_{.l'}(\omega+\epsilon^p h',.)\overline{\mathbf{T}^\epsilon_{.n'}(\omega+\epsilon^p h',.)}\right)_\epsilon$$

and show that, $\forall p\in(0,1)$ and $\alpha\in\left(0,\frac{1}{4}\wedge\frac{1-p}{2}\right)$,

$$\begin{aligned}&\mathbb{E}\Big[\overline{\mathbf{T}^\epsilon_{jl}(\omega+\epsilon^p h,.)}\mathbf{T}^\epsilon_{mn}(\omega+\epsilon^p h,.)\mathbf{T}^\epsilon_{j'l'}(\omega+\epsilon^p h',.)\overline{\mathbf{T}^\epsilon_{m'n'}(\omega+\epsilon^p h',.)}\Big]\\&\quad=\mathbb{E}\left[\overline{\mathbf{T}^\epsilon_{jl}(\omega+\epsilon^p h,L)}\mathbf{T}^\epsilon_{mn}(\omega+\epsilon^p h,L)\right]\mathbb{E}\left[\mathbf{T}^\epsilon_{j'l'}(\omega+\epsilon^p h',L)\overline{\mathbf{T}^\epsilon_{m'n'}(\omega+\epsilon^p h',L)}\right]\\&\quad+\mathcal{O}\left(\epsilon^{(1/2)\wedge(1-2\alpha-p)}\right)\end{aligned}$$

$\forall K\geq 1$ and $\forall(j,l,m,n,j',l',m',n')\in\{1,\dots,K\}^4$. Consequently, we have $\forall\varphi\in\mathcal{E}$

$$\begin{aligned}&\lim_{\epsilon\to0}\mathbb{E}\Big[\Big|\Big\langle e^{i\omega_0\frac{t}{\epsilon^p}}p^\epsilon_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^p},.,0\Big),\varphi\Big\rangle_{L^2(0,d)}\Big|^2\Big]\\&\qquad\qquad=\lim_{\epsilon\to0}\Big|\mathbb{E}\Big[\Big\langle e^{i\omega_0\frac{t}{\epsilon^p}}p^\epsilon_{TR}\Big(\frac{t_1}{\epsilon}+\frac{t}{\epsilon^p},x,0\Big),\varphi\Big\rangle_{L^2(0,d)}\Big]\Big|^2.\end{aligned}$$

4.4.6 Quarter Wavelength Plate

In this section, we explain how the transmission coefficients through the interface $z=L/\epsilon^{1-\alpha}$ can be made of order one. We have seen that the previous transmission coefficients, defined by (4.14), are particularly small, of order $\epsilon^{\alpha/2}$. This poor transmission can be corrected by inserting a quarter wavelength plate. A description of this antireflective process can be found in [25, Chapter 3]. This method is often used in echographic imaging; it consists in adding a thin layer to enhance the transmission through an interface with the minimum loss of energy. In our situation, we will obtain a transmission of order one when it was of order $\epsilon^{\alpha/2}$ without this method. Here, we consider a source that emits a pulse of the form

$$f^\epsilon(t)=\frac{1}{2}f(\epsilon^p t)e^{-i\omega_0 t}.$$

Note that we no longer need the factor $1/\epsilon^\alpha$ as in (4.2) in order to get a refocused signal of order one. The medium parameters of this thin homogeneous layer located in the region $(L/\epsilon^{1-\alpha},L^\epsilon_c)$ are given by

$$\rho^\epsilon(x,z)=\frac{\bar\rho}{\epsilon^{\alpha_\rho}}\text{ and }K^\epsilon(x,z)=\frac{\bar K}{\epsilon^{\alpha_K}}\quad\forall(x,z)\in(0,d)\times(L/\epsilon^{1-\alpha},L^\epsilon_c)\text{ with }L^\epsilon_c=\frac{L}{\epsilon^{1-\alpha}}+\epsilon^{\alpha/2}\frac{\lambda_0}{4}.$$

In the section $(L/\epsilon^{1-\alpha},L^\epsilon_c)$, the modal wavenumbers are

$$\tilde\beta^\epsilon_j(\omega)=\sqrt{\frac{k^2(\omega)}{\epsilon^\alpha}-j^2\frac{\pi^2}{d^2}},\quad j=1,\dots,\left[\frac{k(\omega)d}{\epsilon^{\alpha/2}\pi}\right].$$

From the continuity of the pressure and velocity fields, the transmission coefficients of the layer become

$$\tau^{\epsilon,+}_j(\omega)=\frac{T^{0,\epsilon}_j(\omega)T^{1,\epsilon}_j(\omega)e^{i\tilde\beta^\epsilon_j(\omega)(L^\epsilon_c-L/\epsilon^{1-\alpha})}}{1+R^{0,\epsilon}_j(\omega)R^{1,\epsilon}_j(\omega)e^{2i\tilde\beta^\epsilon_j(\omega)(L^\epsilon_c-L/\epsilon^{1-\alpha})}},$$

with

$$T_j^{0,\epsilon}(\omega) = \frac{2\sqrt{\beta_j^\epsilon(\omega)\tilde{\beta}_j^\epsilon(\omega)}}{\beta_j^\epsilon(\omega) + \tilde{\beta}_j^\epsilon(\omega)}, \qquad T_j^{1,\epsilon}(\omega) = \frac{2\sqrt{\beta_j(\omega)\tilde{\beta}_j^\epsilon(\omega)}}{\beta_j(\omega) + \tilde{\beta}_j^\epsilon(\omega)},$$

$$R_j^{0,\epsilon}(\omega) = \frac{\beta_j^\epsilon(\omega) - \tilde{\beta}_j^\epsilon(\omega)}{\beta_j^\epsilon(\omega) + \tilde{\beta}_j^\epsilon(\omega)}, \qquad R_j^{1,\epsilon}(\omega) = \frac{\tilde{\beta}_j^\epsilon(\omega) - \beta_j(\omega)}{\tilde{\beta}_j^\epsilon(\omega) + \beta_j(\omega)},$$

where $T_j^{0,\epsilon}$ and $R_j^{0,\epsilon}$ (resp., $T_j^{1,\epsilon}$ and $R_j^{1,\epsilon}$) are the transmission and reflection coefficients of the interface between the sections $(0, L/\epsilon^{1-\alpha})$ and $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ (resp., $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ and $(L_c^\epsilon, L_M/\epsilon^{1-\alpha})$). Consequently, the refocused wave is given by

$$\begin{aligned} e^{-i\omega_0 \frac{t_1-t}{\epsilon}} p_{TR}^\epsilon\left(\frac{t}{\epsilon}, x, 0\right) = \frac{1}{16\pi} \int \sum_{j,m=1}^{N(\omega_0+\epsilon^p h)} \sum_{n=1}^{N_\epsilon(\omega_0+\epsilon^p h)} \sum_{l=1}^{\zeta} \sqrt{\frac{\beta_l^\epsilon(\omega_0+\epsilon^p h)\beta_m(\omega_0+\epsilon^p h)}{\beta_j(\omega_0+\epsilon^p h)\beta_n^\epsilon(\omega_0+\epsilon^p h)}} \\ \times \overline{\mathbf{T}_{jl}^\epsilon(\omega+\epsilon^p h, L)}\mathbf{T}_{mn}^\epsilon(\omega+\epsilon^p h, L)\overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)}\tau_m^{\epsilon,+}(\omega_0+\epsilon^p h) \\ \times M_{mj}\phi_l(x_0)\phi_n(x)\overline{\widehat{f}(h)}e^{i(\beta_m(\omega_0+\epsilon^p h)-\beta_j(\omega_0+\epsilon^p h))\left(\frac{L_M}{\epsilon^{1-\alpha}}-L_c^\epsilon\right)} \\ \times e^{i(\beta_m^\epsilon(\omega_0+\epsilon h)-\beta_j^\epsilon(\omega_0+\epsilon h'))\frac{L}{\epsilon^{1-\alpha}}} e^{ih\left(\frac{t_1-t}{\epsilon^{1-p}}\right)} dh' dh. \end{aligned} \tag{4.28}$$

Note that the only difference between (4.21) and (4.28) is the expression for the product of transmission coefficients $\overline{\tau_j^{\epsilon,+}(\omega)}\tau_m^{\epsilon,+}(\omega)$. The limit as $\epsilon \to 0$ of this product is (4.23) in the absence of quarter wavelength plate. In the presence of the quarter wavelength plate, it is given by

$$\lim_{\epsilon\to 0} \overline{\tau_j^{\epsilon,+}(\omega_0+\epsilon^p h)}\tau_m^{\epsilon,+}(\omega_0+\epsilon^p h) = 4\frac{\sqrt{\beta_j(\omega_0)\beta_m(\omega_0)}}{k(\omega_0)} \frac{1}{\left(\frac{\beta_j(\omega_0)}{k(\omega_0)}+1\right)\left(\frac{\beta_m(\omega_0)}{k(\omega_0)}+1\right)}.$$

From this result, we can analyze the mean refocused wave and see that the statistical stability is not affected. The homogeneous spatial profile, with $\alpha_M = 1$, becomes

$$\frac{1}{2}\sum_{j=1}^{N(\omega_0)} \frac{\beta_j(\omega_0)}{k(\omega_0)} \frac{1}{\left(\frac{\beta_j(\omega_0)}{k(\omega_0)}+1\right)^2} M_{mj}\phi_j(x_0)\phi_j(x_0+\lambda_0\tilde{x}),$$

and in the case where $\alpha_M \in [0,1)$, we have in the continuum limit $N(\omega_0) \gg 1$

$$\int_0^1 \frac{\sqrt{1-u^2}}{\left(1+\sqrt{1-u^2}\right)^2} \cos(2\pi\tilde{x}u)\, du.$$

In the random case, the expression of the mean refocused field (4.26) becomes

$$e^{-\tilde{x}^2/r_c^2} \int_0^1 \frac{\sqrt{1-u^2}}{\left(1+\sqrt{1-u^2}\right)^2} \cos(2\pi\tilde{x}u)\, du$$

in the continuum limit $N(\omega_0) \gg 1$, where r_c is defined by (4.27).

To summarize, random inhomogeneities in the section $(0, L/\epsilon^{1-\alpha})$ ensure a conversion between low and high modes, and the quarter wavelength plate $(L/\epsilon^{1-\alpha}, L_c^\epsilon)$ ensures an efficient transmission from the perturbed section $(0, L/\epsilon^{1-\alpha})$ to the homogeneous medium $(L_c^\epsilon, L_M/\epsilon^{1-\alpha})$.

Conclusion

In this chapter we have analyzed a time-reversal experiment in a homogeneous waveguide in which a heterogeneous section is inserted in the vicinity of the source. The role played by these inhomogeneities is quite different from the regime studied in [30] or in Chapter 3, in which the random fluctuations are weak and distributed throughout the waveguide. In the latter case, randomness can enhance spatial refocusing up to the usual diffraction limit. But in our configuration, the random section permits us to refocus beyond this diffraction limit, and this effect is statistically stable in that it does not depend on the particular realization of the random section. The role of this random section is to ensure a strong conversion between low modes (that can propagate over large distances) and high modes (that carry the information about the small-scale features of the source). The insertion of a quarter wavelength plate completes the experimental set-up. It ensures an efficient transmission from the random section to the homogeneous one. It could be possible to build other experimental configurations (with a rough surface, for instance) in order to achieve super-resolution. The important ingredient is that a time-reversible mechanism should convert high modes to low (propagating) modes in the vicinity of the source.

4.5 Appendix

4.5.1 Proof of Theorem 4.1

The proof of this theorem follows the ideas of the proof of Theorem 2.1, which is based on a martingale approach using the perturbed-test-function method. First, using a particular tightness criteria, we shall prove the tightness of the family $(\mathbf{U}^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ on $\mathcal{C}([0,+\infty),(\mathcal{B}_\mathcal{H},d_{\mathcal{B}_\mathcal{H}}))$. In a second part, we shall characterize all subsequence limits as being solutions of a martingale problem in a Hilbert space, and using the stochastic calculus in infinite-dimensional Hilbert spaces we will see that this martingale problem is well posed.

From the definition of the metric $d_{\mathcal{B}_\mathcal{H}}$, we can use the tightness criteria of Theorem 2.10 page 67, which was already used in the proof of Theorem 2.1.

For any $\lambda\in\mathcal{H}$, we set $\mathbf{U}^\epsilon_\lambda(\omega,z)=\langle\mathbf{U}^\epsilon(\omega,z),\lambda\rangle_\mathcal{H}$. According to Theorem 2.10, the family $(\mathbf{U}^\epsilon(\omega,.))_\epsilon$ is tight on $\mathcal{C}([0,+\infty),(\mathcal{B}_\mathcal{H},d_{\mathcal{B}_\mathcal{H}}))$ if and only if the family $(\mathbf{U}^\epsilon_\lambda(\omega,.))_\epsilon$ is tight on $\mathcal{C}([0,+\infty),\mathbb{C})$ $\forall\lambda\in\mathcal{H}$. Furthermore, $\|\mathbf{U}^\epsilon(\omega,z)\|_\mathcal{H}=1$ $\forall z\geq 0,\forall\epsilon\in(0,1)$, and $(\mathbf{U}^\epsilon(\omega,.))_\epsilon$ is a family of continuous processes. Then, it is sufficient to prove that $(\mathbf{U}^\epsilon_\lambda(\omega,.))_\epsilon$ is tight in $\mathcal{D}([0,+\infty),\mathbb{C})$ $\forall\lambda$ in a dense subset of $\mathcal{H}$. Let $\mathcal{E}_\mathcal{H}$ be the subspace of sequences with finite support equipped with the induced inner product. We have chosen $\mathcal{E}_\mathcal{H}$ for two reasons. First, $\mathcal{E}_\mathcal{H}$ is a dense subset of $\mathcal{H}$. Second, thanks to the band-limiting idealization, it allows one to avoid in (4.12) the unboundedness of $N_\epsilon(\omega)$ and the fact that $\epsilon\beta^\epsilon_j(\omega)$ goes to 0 for j of order $N_\epsilon(\omega)$ when ϵ goes to 0.

As in the proof of Theorem 2.1, we consider the complex case for more convenient manipulations. Letting $\lambda\in\mathcal{E}_\mathcal{H}$, we consider the equation

$$\frac{d}{dt}\mathbf{U}^\epsilon_\lambda(\omega,t)=\frac{1}{\sqrt{\epsilon}}F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega,t),C\left(\frac{t}{\epsilon}\right),\frac{t}{\epsilon}\right),$$

where

$$\begin{aligned}F^\epsilon_{jm}(\mathbf{U},C,s)=&\frac{-ik^2(\omega)}{2}\sum_{q=1}^{N_\epsilon(\omega)}\frac{C_{jq}}{\epsilon^\alpha\sqrt{\beta^\epsilon_j\beta^\epsilon_q}}e^{i\epsilon^\alpha\left(\beta^\epsilon_j-\beta^\epsilon_q\right)s}\mathbf{U}_{qm}\\&+\frac{ik^2(\omega)}{2}\sum_{q=1}^{N_\epsilon(\omega)}\frac{C_{mq}}{\epsilon^\alpha\sqrt{\beta^\epsilon_m\beta^\epsilon_q}}e^{i\epsilon\left(\beta^\epsilon_q-\beta^\epsilon_m\right)s}\mathbf{U}_{jq}.\end{aligned}$$

The proof of this theorem is based on the perturbed-test-function approach. Using the notion of a pseudogenerator recalled in Section 2.6.2, we prove tightness and characterize all subsequence limits.

Tightness

We shall consider the classical complex derivative with the following notation: If $v = \alpha + i\beta$, then $\partial_v = \frac{1}{2}(\partial_\alpha - i\partial_\beta)$ and $\partial_{\overline{v}} = \frac{1}{2}(\partial_\alpha + i\partial_\beta)$.

Proposition 4.5 $\forall\lambda \in \mathcal{E}_\mathcal{H}$, *the family* $(\mathbf{U}^\epsilon_\lambda(\omega, .))_{\epsilon\in(0,1)}$ *is tight in* $\mathcal{D}([0, +\infty), \mathbb{C})$.

Proof (of Proposition 4.5) According to Theorem 4 in [41], we need to show the following three lemmas. Let $\lambda \in \mathcal{E}_\mathcal{H}$, f be a smooth function, and $f^\epsilon_0(t) = f(\mathbf{U}^\epsilon_\lambda(\omega, t))$. Thus,

$$\begin{aligned}\mathcal{A}^\epsilon f^\epsilon_0(t) &= \frac{1}{\sqrt{\epsilon}}\partial_v f(\mathbf{U}^\epsilon_\lambda(\omega, t))\, F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega, t), C\left(\frac{t}{\epsilon}\right), \frac{t}{\epsilon}\right)\\ &+ \frac{1}{\sqrt{\epsilon}}\partial_{\overline{v}} f(\mathbf{U}^\epsilon_\lambda(\omega, t))\, \overline{F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega, t), C\left(\frac{t}{\epsilon}\right), \frac{t}{\epsilon}\right)}.\end{aligned}$$

Let

$$\begin{aligned}f^\epsilon_1(t) &= \frac{1}{\sqrt{\epsilon}}\int_t^{+\infty}\mathbb{E}^\epsilon_t\left[F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega, t), C\left(\frac{u}{\epsilon}\right), \frac{u}{\epsilon}\right)\right]\partial_v f(\mathbf{U}^\epsilon_\lambda(\omega, t))\, du\\ &+ \frac{1}{\sqrt{\epsilon}}\int_t^{+\infty}\mathbb{E}^\epsilon_t\left[\overline{F^\epsilon_\lambda\left(\mathbf{U}^\epsilon(\omega, t), C\left(\frac{u}{\epsilon}\right), \frac{u}{\epsilon}\right)}\right]\partial_{\overline{v}} f(\mathbf{U}^\epsilon_\lambda(\omega, t))\, du.\end{aligned}$$

Lemma 4.3 $\forall T > 0$, $\lim_\epsilon \sup_{0\le t\le T}|f^\epsilon_1(t)| = 0$ *almost surely, and* $\sup_{t\ge 0}\mathbb{E}[|f^\epsilon_1(t)|] = \mathcal{O}(\sqrt{\epsilon})$.

Proof (of Lemma 4.3) Using the Markov property of the Gaussian field, we get

$$\begin{aligned}f^\epsilon_1(t) &= \sqrt{\epsilon}\partial_v f(\mathbf{U}^\epsilon_\lambda(\omega, t))\Bigg[\sum_{j,m} -\frac{ik^2}{2}\sum_{|q-j|\le 1}\frac{C_{jq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta^\epsilon_j\beta^\epsilon_q}}e^{i\epsilon^\alpha(\beta^\epsilon_j-\beta^\epsilon_q)\frac{t}{\epsilon}}\\ &\times \mathbf{U}^\epsilon_{qm}(\omega, t)\frac{a + i\epsilon^\alpha(\beta^\epsilon_j - \beta^\epsilon_q)}{a^2 + \epsilon^{2\alpha}(\beta^\epsilon_j - \beta^\epsilon_q)^2}\overline{\lambda_{jm}}\\ &+ \frac{ik^2}{2}\sum_{|q-m|\le 1}\frac{C_{mq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta^\epsilon_m\beta^\epsilon_q}}e^{i\epsilon^\alpha(\beta^\epsilon_q-\beta^\epsilon_m)\frac{t}{\epsilon}}\mathbf{U}^\epsilon_{jq}(\omega, t)\frac{a + i\epsilon^\alpha(\beta^\epsilon_q - \beta^\epsilon_m)}{a^2 + \epsilon^{2\alpha}(\beta^\epsilon_q - \beta^\epsilon_m)^2}\overline{\lambda_{jm}}\Bigg]\\ &+ \sqrt{\epsilon}\partial_{\overline{v}} f(\mathbf{U}^\epsilon_\lambda(\omega, t))\Bigg[\sum_{j,m}\frac{ik^2}{2}\sum_{|q-j|\le 1}\frac{C_{jq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta^\epsilon_j\beta^\epsilon_q}}e^{i\epsilon(\beta^\epsilon_q-\beta^\epsilon_j)\frac{t}{\epsilon}}\\ &\times \overline{\mathbf{U}^\epsilon_{qm}(\omega, t)}\frac{a + i\epsilon^\alpha(\beta^\epsilon_q - \beta^\epsilon_j)}{a^2 + \epsilon^{2\alpha}(\beta^\epsilon_q - \beta^\epsilon_j)^2}\lambda_{jm}\\ &- \frac{ik^2}{2}\sum_{|q-m|\le 1}\frac{C_{mq}\left(\frac{t}{\epsilon}\right)}{\epsilon^\alpha\sqrt{\beta^\epsilon_m\beta^\epsilon_q}}e^{i\epsilon^\alpha(\beta^\epsilon_m-\beta^\epsilon_q)\frac{t}{\epsilon}}\overline{\mathbf{U}^\epsilon_{jq}(\omega, t)}\frac{a + i\epsilon^\alpha(\beta^\epsilon_m - \beta^\epsilon_q)}{a^2 + \epsilon^{2\alpha}(\beta^\epsilon_m - \beta^\epsilon_q)^2}\lambda_{jm}\Bigg].\end{aligned}$$

Using (2.54) page 65, we obtain

$$\mathbb{E}[|f^\epsilon_1(t)|] \le \sqrt{\epsilon}K(f, \lambda).$$

For the first part, we get

$$|f^\epsilon_1(t)| \le K(\lambda, f)\sqrt{\epsilon}\sup_{0\le t\le T}\sup_{x\in[0,d]}\left|V\left(x, \frac{t}{\epsilon}\right)\right|,$$

and we conclude with (2.55) page 66.□

Lemma 4.4 $\{\mathcal{A}^\epsilon\left(f_0^\epsilon + f_1^\epsilon\right)(t), \epsilon \in (0,1), 0 \le t \le T\}$ *is uniformly integrable.*

Proof (of Lemma 4.4) After a computation, we get

$$\mathcal{A}^\epsilon\left(f_0^\epsilon + f_1^\epsilon\right)(t) = \tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t), \left(C_{jl}\left(\frac{t}{\epsilon}\right)C_{mn}\left(\frac{t}{\epsilon}\right)\right)_{j,l,m,n}, \frac{t}{\epsilon}\right),$$

where

$$\begin{aligned}\tilde{F}_\lambda^\epsilon(\mathbf{U},\mathbf{C},s) &= \partial_v f(\mathbf{U})\tilde{F}_\lambda^{1,\epsilon}(\mathbf{U},\mathbf{C},s) + \partial_{\overline{v}} f(\mathbf{U})\overline{\tilde{F}_\lambda^{1,\epsilon}(\mathbf{U},\mathbf{C},s)} \\ &+ \partial_v^2 f(\mathbf{U})\tilde{F}_\lambda^{2,\epsilon}(\mathbf{U},\mathbf{C},s) + \partial_{\overline{v}}^2 f(\mathbf{U})\overline{\tilde{F}_\lambda^{2,\epsilon}(\mathbf{U},\mathbf{C},s)} \\ &+ \partial_{\overline{v}}\partial_v f(\mathbf{U})\tilde{F}_\lambda^{3,\epsilon}(\mathbf{U},\mathbf{C},s) + \partial_v\partial_{\overline{v}} f(\mathbf{U})\overline{\tilde{F}_\lambda^{3,\epsilon}(\mathbf{U},\mathbf{C},s)},\end{aligned}$$

with

$$\begin{aligned}&\tilde{F}_\lambda^{1,\epsilon}(\mathbf{U},\mathbf{C},s) \\ &= \frac{k^4}{4}\sum_{j,m}\left[\sum_{q,q'=1}^{N_\epsilon} -\frac{\mathbf{C}_{jqqq'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_q^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_{q'}^\epsilon)s}\mathbf{U}_{q'm}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2}\right. \\ &+ \frac{\mathbf{C}_{jqmq'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_m^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon+\beta_{q'}^\epsilon-\beta_m^\epsilon)s}\mathbf{U}_{qq'}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2} \\ &+ \frac{\mathbf{C}_{jq'mq}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_{q'}^\epsilon\beta_m^\epsilon\beta_q^\epsilon}} e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_{q'}^\epsilon+\beta_q^\epsilon-\beta_m^\epsilon)s}\mathbf{U}_{q'q}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2} \\ &\left.-\frac{\mathbf{C}_{mqqq'}}{\epsilon^{2\alpha}\sqrt{\beta_m^\epsilon\beta_q^\epsilon\beta_q^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_{q'}^\epsilon-\beta_m^\epsilon)s}\mathbf{U}_{jq'}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2}\right]\overline{\lambda_{jm}},\end{aligned}$$

$$\begin{aligned}&\tilde{F}_\lambda^{2,\epsilon}(\mathbf{U},\mathbf{C},s) \\ &= \frac{k^4}{4}\sum_{\substack{j,m\\j',m'}}\left[\sum_{q,q'=1}^{N_\epsilon} -\frac{\mathbf{C}_{jqj'q'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_{j'}^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon+\beta_{j'}^\epsilon-\beta_{q'}^\epsilon)s}\mathbf{U}_{qm}\mathbf{U}_{q'm'}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2}\right. \\ &+ \frac{\mathbf{C}_{jqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_{m'}^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon+\beta_{q'}^\epsilon-\beta_{m'}^\epsilon)s}\mathbf{U}_{qm}\mathbf{U}_{j'q'}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2} \\ &+ \frac{\mathbf{C}_{j'q'mq}}{\epsilon^{2\alpha}\sqrt{\beta_{j'}^\epsilon\beta_{q'}^\epsilon\beta_m^\epsilon\beta_q^\epsilon}} e^{i\epsilon^\alpha(\beta_{j'}^\epsilon-\beta_{q'}^\epsilon+\beta_q^\epsilon-\beta_m^\epsilon)s}\mathbf{U}_{jq}\mathbf{U}_{q'm'}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2} \\ &\left.-\frac{\mathbf{C}_{mqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta_m^\epsilon\beta_q^\epsilon\beta_{m'}^\epsilon\beta_{q'}^\epsilon}} e^{i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon+\beta_{q'}^\epsilon-\beta_{m'}^\epsilon)s}\mathbf{U}_{jq}\mathbf{U}_{j'q'}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2}\right]\overline{\lambda_{jm}\lambda_{j'm'}},\end{aligned}$$

$$\begin{aligned}
&\tilde{F}_\lambda^{3,\epsilon}(\mathbf{U},\mathbf{C},s)\\
&= \frac{k^4}{4}\sum_{\substack{j,m\\ j',m'}}\Bigg[\sum_{q,q'=1}^{N_\epsilon}\frac{\mathbf{C}_{jqj'q'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_{j'}^\epsilon\beta_{q'}^\epsilon}}e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon-\beta_{j'}^\epsilon+\beta_{q'}^\epsilon)s}\mathbf{U}_{qm}\overline{\mathbf{U}_{q'm'}}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2}\\
&-\frac{\mathbf{C}_{jqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta_j^\epsilon\beta_q^\epsilon\beta_{m'}^\epsilon\beta_{q'}^\epsilon}}e^{i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon-\beta_{q'}^\epsilon+\beta_{m'}^\epsilon)s}\mathbf{U}_{qm}\overline{\mathbf{U}_{j'q'}}\frac{a+i\epsilon^\alpha(\beta_j^\epsilon-\beta_q^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_j^\epsilon-\beta_q^\epsilon)^2}\\
&-\frac{\mathbf{C}_{j'q'mq}}{\epsilon^{2\alpha}\sqrt{\beta_{j'}^\epsilon\beta_{q'}^\epsilon\beta_{m}^\epsilon\beta_{q}^\epsilon}}e^{i\epsilon^\alpha(-\beta_{j'}^\epsilon+\beta_{q'}^\epsilon+\beta_{q}^\epsilon-\beta_{m}^\epsilon)s}\mathbf{U}_{jq}\overline{\mathbf{U}_{q'm'}}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2}\\
&+\frac{\mathbf{C}_{mqm'q'}}{\epsilon^{2\alpha}\sqrt{\beta_{m}^\epsilon\beta_{q}^\epsilon\beta_{m'}^\epsilon\beta_{q'}^\epsilon}}e^{i\epsilon^\alpha(\beta_{q}^\epsilon-\beta_{m}^\epsilon-\beta_{q'}^\epsilon+\beta_{m'}^\epsilon)s}\mathbf{U}_{jq}\overline{\mathbf{U}_{j'q'}}\frac{a+i\epsilon^\alpha(\beta_q^\epsilon-\beta_m^\epsilon)}{a^2+\epsilon^{2\alpha}(\beta_q^\epsilon-\beta_m^\epsilon)^2}\Bigg]\overline{\lambda_{jm}}\lambda_{j'm'}.
\end{aligned}$$

From this expression, using (2.54) page 65, we can check that $\sup_{\epsilon,t}\mathbb{E}\left[\,|\mathcal{A}^\epsilon\left(f_0^\epsilon+f_1^\epsilon\right)(t)|^2\,\right]<+\infty$. □

Lemma 4.5

$$\lim_{K\to+\infty}\overline{\lim_{\epsilon\to0}}\,\mathbb{P}\left(\sup_{0\le t\le T}|\mathbf{U}_\lambda^\epsilon(\omega,t)|\ge K\right)=0.$$

Proof (of Lemma 4.5) We have

$$|\mathbf{U}_\lambda^\epsilon(\omega,t)|=\left|\sum_{j,m\ge1}\mathbf{U}_{jm}^\epsilon(\omega,t)\overline{\lambda_{jm}}\right|\le\|\lambda\|_{\mathcal{H}}.$$

□

This last lemma completes the proof Proposition 4.5. ■

Martingale problem

In this section, using a well-posed martingale problem, we characterize all subsequence limits. In what follows, we consider a converging subsequence of $(\mathbf{U}^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ which converges to a limit $\mathbf{U}(\omega,.)$. For the sake of simplicity we denote by $(\mathbf{U}^\epsilon(\omega,.))_{\epsilon\in(0,1)}$ the subsequence.

Convergence Result

Proposition 4.6 *$\forall\lambda\in\mathcal{E}_{\mathcal{H}}$ and $\forall f$ smooth test function,*

$$\begin{aligned}
&f(\mathbf{U}_\lambda(\omega,t))-\int_0^t\partial_v f\left(\mathbf{U}_\lambda(\omega,s)\right)\langle J(\mathbf{U}(\omega,s)),\lambda\rangle_{\mathcal{H}}+\partial_{\overline{v}}f\left(\mathbf{U}_\lambda(\omega,s)\right)\overline{\langle J(\mathbf{U}(\omega,s)),\lambda\rangle_{\mathcal{H}}}\\
&+\partial_v^2 f\left(\mathbf{U}_\lambda(\omega,s)\right)\langle K\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}+\partial_{\overline{v}}^2 f\left(\mathbf{U}_\lambda(\omega,s)\right)\overline{\langle K\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}\\
&+\partial_{\overline{v}}\partial_v f\left(\mathbf{U}_\lambda(\omega,s)\right)\langle L\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}+\partial_v\partial_{\overline{v}} f\left(\mathbf{U}_\lambda(\omega,s)\right)\overline{\langle L\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}ds
\end{aligned}$$

is a martingale, where

$$\begin{aligned}
J(\mathbf{U})_{jm} &= \Lambda\big[(\mathbf{U}_{j+1j+1}\delta_{jm} - \mathbf{U}_{jm}) + (\mathbf{U}_{j-1j-1}\delta_{jm} - \mathbf{U}_{jm})\big],\\
K(\mathbf{U})(\lambda)_{jm} &= \frac{\Lambda}{2}\big[\mathbf{U}_{j-1m}\big(\langle \mathbf{U}_{j-1},\lambda_j\rangle_2 - \langle \mathbf{U}_j,\lambda_{j-1}\rangle_1\big)\\
&\qquad + \mathbf{U}_{j+1m}\big(\langle \mathbf{U}_{j+1},\lambda_j\rangle_2 - \langle \mathbf{U}_j,\lambda_{j+1}\rangle_1\big)\big]\\
&+ \frac{\Lambda}{2}\big[\mathbf{U}_{jm-1}\big(\langle \mathbf{U}_{m-1},\lambda_m\rangle_1 - \langle \mathbf{U}_m,\lambda_{m-1}\rangle_2\big)\\
&\qquad + \mathbf{U}_{jm+1}\big(\langle \mathbf{U}_{m+1},\lambda_m\rangle_1 - \langle \mathbf{U}_m,\lambda_{m+1}\rangle_2\big)\big],\\
L(\mathbf{U})(\lambda)_{jm} &= \frac{\Lambda}{2}\big[\mathbf{U}_{j-1m}\big(\overline{\langle \mathbf{U}_{j-1},\lambda_j\rangle_1} - \overline{\langle \mathbf{U}_j,\lambda_{j-1}\rangle_2}\big)\\
&\qquad + \mathbf{U}_{j+1m}\big(\overline{\langle \mathbf{U}_{j+1},\lambda_j\rangle_1} - \overline{\langle \mathbf{U}_j,\lambda_{j+1}\rangle_2}\big)\big]\\
&+ \frac{\Lambda}{2}\big[\mathbf{U}_{jm-1}\big(\overline{\langle \mathbf{U}_{m-1},\lambda_m\rangle_2} - \overline{\langle \mathbf{U}_m,\lambda_{m-1}\rangle_1}\big)\\
&\qquad + \mathbf{U}_{jm+1}\big(\overline{\langle \mathbf{U}_{m+1},\lambda_m\rangle_2} - \overline{\langle \mathbf{U}_m,\lambda_{m+1}\rangle_1}\big)\big],
\end{aligned}$$

with

$$\langle \lambda_j,\mu_j\rangle_1 = \sum_{m\geq 1}\lambda_{jm}\overline{\mu_{jm}},\quad \langle \lambda_m,\mu_m\rangle_2 = \sum_{j\geq 1}\lambda_{jm}\overline{\mu_{jm}}$$

$\forall j,m\geq 1$, and for $(\mathbf{U},\lambda,\mu)\in\mathcal{H}\times\mathcal{E}_{\mathcal{H}}\times\mathcal{E}_{\mathcal{H}}$.

Proof (of Proposition 4.6) Let

$$\begin{aligned}
f_2^\epsilon(t) = \int_t^{+\infty}\mathbb{E}_t^\epsilon\Bigg[&\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(C_{jl}\left(\frac{u}{\epsilon}\right)C_{mn}\left(\frac{u}{\epsilon}\right)\right)_{j,l,m,n},\frac{u}{\epsilon}\right)\Bigg]\\
&-\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(\mathbb{E}[C_{jl}(0)C_{mn}(0)]\right)_{j,l,m,n},\frac{u}{\epsilon}\right)du.
\end{aligned}$$

Lemma 4.6

$$\sup_{t\geq 0}\mathbb{E}\left[|f_2^\epsilon(t)|\right] = \mathcal{O}(\epsilon)$$

and

$$\mathcal{A}^\epsilon\left(f_0^\epsilon + f_1^\epsilon + f_2^\epsilon\right)(t) = \tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(S(j-l,m-n)\right)_{j,l,m,n},\frac{t}{\epsilon}\right) + A(\epsilon,t),$$

where $\sup_{t\geq 0}\mathbb{E}\left[|A(\epsilon,t)|\right] = \mathcal{O}(\sqrt{\epsilon})$.

Proof (of Lemma 4.6) A change of variable gives

$$\begin{aligned}
f_2^\epsilon(t) &= \epsilon\int_0^{+\infty}\mathbb{E}_t^\epsilon\Bigg[\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(C_{jl}\left(u+\frac{t}{\epsilon}\right)C_{mn}\left(u+\frac{t}{\epsilon}\right)\right)_{j,l,m,n},u+\frac{t}{\epsilon}\right)\Bigg]\\
&-\tilde{F}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),\left(\mathbb{E}[C_{jl}(0)C_{mn}(0)]\right)_{j,l,m,n},u+\frac{t}{\epsilon}\right)du\\
&= \epsilon B(\epsilon,t).
\end{aligned}$$

By a computation, we can check that $\sup_{\epsilon,t\geq 0}\mathbb{E}\left[|B(\epsilon,t)|\right] < +\infty$. The second part of this lemma follows a long but straightforward computation. □

We consider $\tilde{G}_\lambda^\epsilon(\mathbf{U}^\epsilon(\omega,t),\frac{t}{\epsilon}) = \tilde{F}_\lambda^\epsilon(\mathbf{U}^\epsilon(\omega,t),\left(S(j-l,m-n)\right)_{j,l,m,n},\frac{t}{\epsilon})$ and let

$$f_3^\epsilon(t) = -\int_0^t\left[\tilde{G}_\lambda^\epsilon(\mathbf{U}^\epsilon(\omega,t),\frac{u}{\epsilon}) - \lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon(\mathbf{U}^\epsilon(\omega,t),s)ds\right]du.$$

Lemma 4.7 *We have*

$$\sup_{t\geq 0}\mathbb{E}\left[|f_3^\epsilon(t)|\right]=\mathcal{O}(\epsilon^{1-2\alpha}).$$

Then, we need to have $\alpha\in(0,1/2)$.

Proof (of Lemma 4.7) After a change of variable, we get

$$f_3^\epsilon(t)=-\epsilon\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),u\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds\right]du,$$

and

$$\sup_{t,\epsilon}\mathbb{E}\left[\left|\int_0^{\frac{t}{\epsilon}}\left[\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),u\right)-\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds\right]du\right|\right]\leq\frac{K}{\epsilon^{2\alpha}}.$$

□

Let $f^\epsilon(t)=f_0^\epsilon(t)+f_1^\epsilon(t)+f_2^\epsilon(t)+f_3^\epsilon(t)$. With the boundness condition (2.54) page 65, a computation gives

$$\mathcal{A}^\epsilon f^\epsilon(t)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\tilde{G}_\lambda^\epsilon\left(\mathbf{U}^\epsilon(\omega,t),s\right)ds+C(\epsilon,t).$$

We assume that the following nondegeneracy condition holds. $\forall\epsilon\in(0,1)$, the wavenumbers $\beta_j^\epsilon(\omega)=\beta_j(\omega/\epsilon^\alpha)$ are distinct along with their sums and differences. Consequently, we get

$$\begin{aligned}\mathcal{A}^\epsilon f^\epsilon(t)=&\,\partial_v f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\langle J(\mathbf{U}^\epsilon(\omega,t)),\lambda\rangle_{\mathcal{H}}\\&+\partial_{\overline{v}} f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\overline{\langle J(\mathbf{U}^\epsilon(\omega,t)),\lambda\rangle_{\mathcal{H}}}\\&+\partial_v^2 f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\langle K\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\\&+\partial_{\overline{v}}^2 f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\overline{\langle K\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}\\&+\partial_{\overline{v}}\partial_v f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\langle L\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\\&+\partial_v\partial_{\overline{v}} f\left(\mathbf{U}_\lambda^\epsilon(\omega,t)\right)\overline{\langle L\left(\mathbf{U}^\epsilon(\omega,t)\right)(\lambda),\lambda\rangle_{\mathcal{H}}}\\&+C(\epsilon,t),\end{aligned}\tag{4.29}$$

where $\sup_{t\geq 0}\mathbb{E}\left[|C(\epsilon,t)|\right]=\mathcal{O}(\epsilon^{\frac{1}{2}-2\alpha})$. Then, we need to have $\alpha\in(0,1/4)$. By Theorem 2.11 page 73, $\left(M_{f^\epsilon}^\epsilon(t)\right)_{t\geq 0}$ is an $(\mathcal{F}_t^\epsilon)$-martingale; this implies that for every bounded continuous function h, every sequence $0<s_1<\cdots<s_n\leq s<t$, and every family $(\lambda_j)_{j\in\{1,\dots,n\}}$ we have

$$\mathbb{E}\left[h\left(\mathbf{U}_{\lambda_j}^\epsilon(\omega,s_j),1\leq j\leq n\right)\left(f^\epsilon(t)-f^\epsilon(s)-\int_s^t\mathcal{A}^\epsilon f^\epsilon(u)du\right)\right]=0.$$

Finally, using (4.29) and (4.11) with lemmas 4.3, 4.6, and 4.7, we get the announced result of Proposition 4.6. ■

Uniqueness To show uniqueness, we decompose $\mathbf{U}(\omega,.)$ into real and imaginary parts. Then let us consider the new process

$$\mathbf{Y}(\omega,t)=\begin{bmatrix}\mathbf{Y}^1(\omega,t)\\\mathbf{Y}^2(\omega,t)\end{bmatrix},\text{ where }\mathbf{Y}^1(\omega,t)=Re\left(\mathbf{U}(\omega,t)\right)\text{ and }\mathbf{Y}^2(\omega,t)=Im\left(\mathbf{U}(\omega,t)\right).$$

Let $\mathcal{G}=l^2\left(E,\mathbb{R}\right)$. $\mathcal{G}\times\mathcal{G}$ is endowed with the inner product defined by

$$\langle\mathbf{T},\mathbf{S}\rangle_{\mathcal{G}\times\mathcal{G}}=\sum_{j,m\geq 1}\mathbf{T}_{jm}^1\mathbf{S}_{jm}^1+\mathbf{T}_{jm}^2\mathbf{S}_{jm}^2$$

$\forall(\mathbf{T},\mathbf{S}) \in \mathcal{G}\times\mathcal{G}$. We also use the notation $\mathbf{Y}_\lambda(\omega,t) = \langle \mathbf{Y}(\omega,t),\lambda\rangle$ with $\lambda \in \mathcal{G}\times\mathcal{G}$. We introduce the operator φ on $\mathcal{G}\times\mathcal{G}$ given by

$$\begin{aligned}\varphi : \mathcal{G}\times\mathcal{G} &\longrightarrow \mathcal{G}\times\mathcal{G},\\ \begin{bmatrix}\mathbf{T}^1\\ \mathbf{T}^2\end{bmatrix} &\longmapsto \begin{bmatrix}\mathbf{T}^2\\ -\mathbf{T}^1\end{bmatrix}.\end{aligned}$$

Let f be a smooth function on $\mathbb{R}$. By Proposition 4.6, we get the following result.

Proposition 4.7 *$\forall\lambda \in \mathcal{E}_{\mathcal{G}\times\mathcal{G}}$,*

$$f\left(\mathbf{Y}_\lambda(\omega,t)\right) - \int_0^t \langle J(\mathbf{Y}(\omega,s)),\lambda\rangle_{\mathcal{G}\times\mathcal{G}}\, f'\left(\mathbf{Y}_\lambda(\omega,s)\right) + \frac{1}{2}\left\langle A\left(\mathbf{Y}(\omega,s)\right)(\lambda),\lambda\right\rangle_{\mathcal{G}\times\mathcal{G}} f''\left(\mathbf{Y}_\lambda(\omega,s)\right) ds$$

is a martingale, where

$$\begin{aligned}A(\mathbf{Y})(\lambda)_{jm} = \frac{\Lambda}{2}\Big[&\mathbf{Y}_{j+1m}\big[\langle \mathbf{Y}_{j+1},\lambda_j\rangle_1 - \langle \mathbf{Y}_j,\lambda_{j+1}\rangle_2 + \langle \mathbf{Y}_{j+1},\lambda_j\rangle_2 - \langle \mathbf{Y}_j,\lambda_{j+1}\rangle_1\big]\\ &+ \mathbf{Y}_{j-1m}\big[\langle \mathbf{Y}_{j-1},\lambda_j\rangle_1 - \langle \mathbf{Y}_j,\lambda_{j-1}\rangle_2 + \langle \mathbf{Y}_{j-1},\lambda_j\rangle_2 - \langle \mathbf{Y}_j,\lambda_{j-1}\rangle_1\big]\\ &+ \mathbf{Y}_{jm+1}\big[\langle \mathbf{Y}_{m+1},\lambda_m\rangle_1 - \langle \mathbf{Y}_m,\lambda_{m+1}\rangle_2 + \langle \mathbf{Y}_{m+1},\lambda_m\rangle_2 - \langle \mathbf{Y}_m,\lambda_{m+1}\rangle_1\big]\\ &+ \mathbf{Y}_{jm-1}\big[\langle \mathbf{Y}_{m-1},\lambda_m\rangle_1 - \langle \mathbf{Y}_m,\lambda_{m-1}\rangle_2 + \langle \mathbf{Y}_{m-1},\lambda_m\rangle_2 - \langle \mathbf{Y}_m,\lambda_{m-1}\rangle_1\big]\\ &+ \varphi(\mathbf{Y})_{j+1m}\big[\langle \varphi(\mathbf{Y})_{j+1},\lambda_j\rangle_1 - \langle \varphi(\mathbf{Y})_j,\lambda_{j+1}\rangle_2 - \langle \varphi(\mathbf{Y})_{j+1},\lambda_j\rangle_2 + \langle \varphi(\mathbf{Y})_j,\lambda_{j+1}\rangle_1\big]\\ &+ \varphi(\mathbf{Y})_{j-1m}\big[\langle \varphi(\mathbf{Y})_{j-1},\lambda_j\rangle_1 - \langle \varphi(\mathbf{Y})_j,\lambda_{j-1}\rangle_2 - \langle \varphi(\mathbf{Y})_{j-1},\lambda_j\rangle_2 + \langle \varphi(\mathbf{Y})_j,\lambda_{j-1}\rangle_1\big]\\ &+ \varphi(\mathbf{Y})_{jm+1}\big[\langle \varphi(\mathbf{Y})_{m+1},\lambda_m\rangle_2 - \langle \varphi(\mathbf{Y})_m,\lambda_{m+1}\rangle_1 - \langle \varphi(\mathbf{Y})_{m+1},\lambda_m\rangle_1 + \langle \varphi(\mathbf{Y})_m,\lambda_{m+1}\rangle_2\big]\\ &+ \varphi(\mathbf{Y})_{jm-1}\big[\langle \varphi(\mathbf{Y})_{m-1},\lambda_m\rangle_2 - \langle \varphi(\mathbf{Y})_m,\lambda_{m-1}\rangle_1 - \langle \varphi(\mathbf{Y})_{m-1},\lambda_m\rangle_1 - \langle \varphi(\mathbf{Y})_m,\lambda_{m-1}\rangle_2\big]\Big]\end{aligned}$$

for $(\mathbf{Y},\lambda) \in (\mathcal{G}\times\mathcal{G})^2$.

Proof (of Proposition 4.7) By Proposition 4.6,

$$f\left(\mathbf{Y}_\lambda(\omega,t)\right) - \int_0^t \langle J(\mathbf{Y}(\omega,s)),\lambda\rangle_{\mathcal{G}\times\mathcal{G}}\, f'\left(\mathbf{Y}_\lambda(\omega,s)\right) + \frac{1}{2} Re\left(\langle (L+K)\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\right) f''\left(\mathbf{Y}_\lambda(\omega,s)\right) ds$$

is a martingale, where we also have denoted by λ the sequence $\lambda^1 + i\lambda^2$. In addition,

$$Re\Big(\langle \mathbf{U}(\omega,t)_j,\lambda_j\rangle\Big) = \langle \mathbf{Y}(\omega,t)_j,\lambda_j\rangle \text{ and } Im\Big(\langle \mathbf{U}(\omega,t)_j,\lambda_j\rangle\Big) = \langle \varphi\left(\mathbf{Y}(\omega,t)\right)_j,\lambda_j\rangle,$$

and we get $Re\left(\langle (L+K)\left(\mathbf{U}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{H}}\right) = \langle A\left(\mathbf{Y}(\omega,s)\right)(\lambda),\lambda\rangle_{\mathcal{G}\times\mathcal{G}}$. ■

From this last proposition, for $f(x) = x$ and $f(x) = x^2$, we get that

$$\langle M(t),\lambda\rangle_{\mathcal{G}\times\mathcal{G}} = M_\lambda(t) = \left\langle \mathbf{Y}(\omega,t) - \int_0^t J(\mathbf{Y}(\omega,s))ds, \lambda\right\rangle_{\mathcal{G}\times\mathcal{G}}$$

is a continuous martingale with quadratic variation given by

$$< M_\lambda > (t) = \int_0^t \langle A(\mathbf{Y}(\omega,s))(\lambda),\lambda\rangle_{\mathcal{G}\times\mathcal{G}}\, ds.$$

Proposition 4.8 $\forall f \in \mathcal{C}_b^2(\mathcal{G}\times\mathcal{G})$,

$$f(\mathbf{Y}(\omega,t)) - \int_0^t Lf(\mathbf{Y}(\omega,s))ds \tag{4.30}$$

is a continuous martingale, where $\forall \mathbf{Y} \in \mathcal{G}\times\mathcal{G}$

$$Lf(\mathbf{Y}) = \frac{1}{2} trace\left(A(\mathbf{Y})D^2 f(\mathbf{Y})\right) + \langle J(\mathbf{Y}), Df(\mathbf{Y})\rangle_{\mathcal{G}\times\mathcal{G}}\,.$$

Moreover, the martingale problem associated to the generator L is well posed.

Proof (of Proposition 4.8) We begin with the following lemma.

Lemma 4.8

$$\begin{aligned} A &: \mathcal{G}\times\mathcal{G} \longrightarrow L_1^+(\mathcal{G}\times\mathcal{G})\,,\\ J &: \mathcal{G}\times\mathcal{G} \longrightarrow \mathcal{G}\times\mathcal{G}, \end{aligned}$$

where $L_1^+(\mathcal{G}\times\mathcal{G})$ is a set of nonnegative operators with finite trace. We have, $\forall \mathbf{Y}\in\mathcal{G}\times\mathcal{G}$, $A(\mathbf{Y}) = \sigma^*(\mathbf{Y})\circ\sigma(\mathbf{Y})$ with

$$\sigma : \mathcal{G}\times\mathcal{G} \longrightarrow L_2(\mathcal{G}\times\mathcal{G}),$$

where $L_2(\mathcal{G}\times\mathcal{G})$ is the set of Hilbert-Schmidt operators on $\mathcal{G}\times\mathcal{G}$, σ^* is the adjoint operator of σ, and

$$\begin{aligned} &\sigma^1(\mathbf{Y})(\lambda)_{jm} = \sqrt{\frac{\Lambda}{2}}\Big(\langle \mathbf{Y}_{j+1}, \lambda_j\rangle_1 + \langle \mathbf{Y}_{j+1}, \lambda_j\rangle_2 - \langle \mathbf{Y}_j, \lambda_{j+1}\rangle_1 - \langle \mathbf{Y}_j, \lambda_{j+1}\rangle_2\Big)\delta_{j+1m},\\ &\sigma^2(\mathbf{Y})(\lambda)_{jm} =\\ &\qquad\sqrt{\frac{\Lambda}{2}}\Big(\langle \varphi(\mathbf{Y})_{j+1}, \lambda_j\rangle_1 - \langle \varphi(\mathbf{Y})_{j+1}, \lambda_j\rangle_2 + \langle \varphi(\mathbf{Y})_j, \lambda_{j+1}\rangle_1 - \langle \varphi(\mathbf{Y})_j, \lambda_{j+1}\rangle_2\Big)\delta_{j+1m}. \end{aligned}$$

Proof $\forall(\mathbf{Y},\lambda,\mu)\in(\mathcal{G}\times\mathcal{G})^3$, we have

$$\begin{aligned} \langle A(\mathbf{Y})(\lambda),\mu\rangle_{\mathcal{G}\times\mathcal{G}} &= \frac{\Lambda}{2}\sum_{j\geq 1}\left(\langle \mathbf{Y}_{j+1}, \lambda_j\rangle_1 + \langle \mathbf{Y}_{j+1}, \lambda_j\rangle_2 - \langle \mathbf{Y}_j, \lambda_{j+1}\rangle_1 - \langle \mathbf{Y}_j, \lambda_{j+1}\rangle_2\right)\\ &\times\left(\langle \mathbf{Y}_{j+1}, \mu_j\rangle_1 + \langle \mathbf{Y}_{j+1}, \mu_j\rangle_2 - \langle \mathbf{Y}_j, \mu_{j+1}\rangle_1 - \langle \mathbf{Y}_j, \mu_{j+1}\rangle_2\right)\\ &+\left(\langle \varphi(\mathbf{Y})_{j+1}, \lambda_j\rangle_1 - \langle \varphi(\mathbf{Y})_{j+1}, \lambda_j\rangle_2 + \langle \varphi(\mathbf{Y})_j, \lambda_{j+1}\rangle_1 - \langle \varphi(\mathbf{Y})_j, \lambda_{j+1}\rangle_2\right)\\ &\times\left(\langle \varphi(\mathbf{Y})_{j+1}, \mu_j\rangle_1 - \langle \varphi(\mathbf{Y})_{j+1}, \mu_j\rangle_2 + \langle \varphi(\mathbf{Y})_j, \mu_{j+1}\rangle_1 - \langle \varphi(\mathbf{Y})_j, \mu_{j+1}\rangle_2\right)\\ &= \langle \sigma(\mathbf{Y})(\lambda), \sigma(\mathbf{Y})(\mu)\rangle_{\mathcal{G}\times\mathcal{G}}\,. \end{aligned}$$

Let $(e_{jl}^{\eta})_{\substack{\eta=1,2\\ j,l\geq 1}}$ be the family of elements in $\mathcal{G}\times\mathcal{G}$ defined by

$$e_{jl}^1 = \begin{bmatrix}\delta_{jl}\\ \mathbf{0}\end{bmatrix} \text{ and } e_{jl}^2 = \begin{bmatrix}\mathbf{0}\\ \delta_{jl}\end{bmatrix}.$$

This family defines a basis of $\mathcal{G}\times\mathcal{G}$ and $\forall \mathbf{Y}\in\mathcal{G}\times\mathcal{G}$,

$$trace(A(\mathbf{Y})) = \sum_{\substack{\eta=1,2\\ j,l\geq 1}} \langle A(\mathbf{Y})(e_{jl}^{\eta}), e_{jl}^{\eta}\rangle_{\mathcal{G}\times\mathcal{G}} = \sum_{\substack{\eta=1,2\\ j,l\geq 1}} \|\sigma(\mathbf{Y})(e_{jl}^{\eta})\|_{\mathcal{G}\times\mathcal{G}}^2 \leq 16\|\mathbf{Y}\|_{\mathcal{G}\times\mathcal{G}}^2.$$

□

From this lemma and following the proof of Theorem 4.1.4 in [63], (4.30) is a martingale. In fact, the key point is that the process $\mathbf{Y}(\omega, .)$ takes its values in $\mathcal{B}_{\mathcal{G}\times\mathcal{G}}$. Moreover, by Theorem 3.2.2 and 4.4.1 in [63], the martingale problem is well posed since σ is linear in x and $\forall \mathbf{Y} \in \mathcal{G}\times\mathcal{G}$

$$\|\sigma(\mathbf{Y})\| = \|\sigma^*(\mathbf{Y})\| \leq 4\|\mathbf{Y}\|_{\mathcal{G}\times\mathcal{G}}.$$

That concludes the proof of Proposition 4.7.■

At this point, we cannot assert that $\mathbf{Y}(\omega, .)$ is uniquely determined. In fact, we need to know if its law is supported by $\mathcal{C}([0,+\infty),(\mathcal{G}\times\mathcal{G},\|.\|_{\mathcal{G}\times\mathcal{G}}))$. Let

$$\mathcal{S}_{\mathcal{G}\times\mathcal{G}} = \left\{\lambda \in \mathcal{G}\times\mathcal{G}, \|\lambda\|_{\mathcal{G}\times\mathcal{G}} = \sqrt{\langle\lambda,\lambda\rangle_{\mathcal{G}\times\mathcal{G}}} = 1\right\}.$$

Proposition 4.9 *The law of the continuous process $(\mathbf{Y}(\omega,t))_{t\geq 0}$ is support by the space $\mathcal{C}([0,+\infty),(\mathcal{S}_{\mathcal{G}\times\mathcal{G}},\|.\|_{\mathcal{G}\times\mathcal{G}}))$, and more generally by $\mathcal{C}([0,+\infty),(\mathcal{G}\times\mathcal{G},\|.\|_{\mathcal{G}\times\mathcal{G}}))$. Consequently, $(\mathbf{Y}(\omega,t))_{t\geq 0}$ is uniquely characterised as being the unique solution of the martingale problem associated to the generator L.*

Proof $(M(t))_{t\geq 0}$ is a bounded weakly-continuous martingale with values in $\mathcal{G}\times\mathcal{G}$. Furthermore, from the following proposition, $(M(t))_{t\geq 0}$ is also a bounded strongly-continuous martingale with values in $\mathcal{G}\times\mathcal{G}$.

Proposition 4.10 *Let $(M_t^n)_{t\geq 0}$ be a sequence of continuous $(\boldsymbol{F}_t)$-martingale with values in a separable banach space $\boldsymbol{B}$. Assuming that*

$$\lim_{n\to+\infty} \mathbb{E}\Big[\|M_t^n - M_t\|_{\boldsymbol{B}}^2\Big] = 0.$$

Then, $(M_t)_{t\geq 0}$ is an $\boldsymbol{F}_t$-martingale, almost surely continuous.

Consequently, we can use the representation theorem, Theorem 4.3.5 in [63]. Then, there exists a cylindrical Brownian motion $(B_t)_{t\geq 0}$ defined on $\mathcal{G}\times\mathcal{G}$ such that

$$\mathbf{Y}(\omega,t) = \mathbf{Y}(\omega,0) + \int_0^t J(\mathbf{Y}(\omega,s))ds + \int_0^t \sigma^*(\mathbf{Y}(\omega,s))dB_s. \tag{4.31}$$

Using the Ito's formula given by Theorem 3.1.3 in [63], we get that

$$\|\mathbf{Y}(\omega,t)\|_{\mathcal{G}\times\mathcal{G}} = 1, \quad \forall t \geq 0.$$

Then, the process $(\mathbf{Y}(\omega,t))_{t\geq 0}$ is strongly-continunous with values in $\mathcal{B}_{\mathcal{G}\times\mathcal{G}}$, and more generally in $\mathcal{G}\times\mathcal{G}$. ■

Using (4.31) and the definition of the last integral, we have

$$\begin{aligned}\left\langle e_{jl}^{\eta}, \int_0^t \sigma^*(\mathbf{Y}(\omega,s))dB_s\right\rangle_{\mathcal{G}\times\mathcal{G}} &= \int_0^t \left\langle \sigma(\mathbf{Y}(\omega,s))(e_{jl}^{\eta}), dB_s\right\rangle_{\mathcal{G}\times\mathcal{G}} \\ &= \sum_{\substack{\theta=1,2\\ r,s\geq 1}} \int_0^t \left\langle \sigma(\mathbf{Y}(\omega,s))(e_{jl}^{\eta}), e_{rs}^{\theta}\right\rangle_{\mathcal{G}\times\mathcal{G}} dB_s(e_{rs}^{\theta}).\end{aligned}$$

By Theorem 3.2.2 in [35], $(B_t)_{t\geq 0}$ can be decomposed as follows:

$$B_t(h) = \sum_{\substack{\eta=1,2\\ j,l\geq 1}} \langle e_{jl}^{\eta}, h\rangle_{\mathcal{G}\times\mathcal{G}} B_{jl}^{\eta}(t), \quad \forall h \in \mathcal{G}\times\mathcal{G}$$

with $(B^{\eta}_{jl})_{\substack{\eta=1,2\\ j,l\geq 1}}$ a family of independent one-dimensional Brownian motions. Finally, a computation gives

$$\begin{aligned}d\mathbf{U}(\omega,t) &= d\mathbf{Y}^1(\omega,t) + i\, d\mathbf{Y}^2(\omega,t)\\ &= J(\mathbf{U}(\omega,t))dt + \psi_1(\mathbf{U}(\omega,t))(dB^1_t) + \psi_2(\mathbf{U}(\omega,t))(dB^2_t),\end{aligned}$$

From this equation, we can get the conservation relation:

$$\|\mathbf{U}(\omega,t)\|_{\mathcal{H}} = 1, \quad \forall t \geq 0.$$

4.5.2 Proof of Theorem 4.2

The proof of this theorem follows closely the ideas developed in Theorem 2.4. In a first step, we introduce a new process, it is an adapted version of the first one, which has a symmetric state space about 0 and which is more convenient for manipulations. In a second step we shall prove the tightness using Theorem 3 in [41]. Moreover, the size of the jumps are equal to $1/N$. Then, all accumulation points are supported by the set of continuous functions. Consequently, the last step consists of adapting Lemma 11.1.1 and 11.1.3 in [59] to the Skorokhod topology.

We begin by introducing a new process. Let $(Y_t)_{t\geq 0}$ be a jump Markov process on $\mathbb{Z}$ with generator $\tilde{\mathcal{L}}$ given by

$$\begin{aligned}\tilde{\mathcal{L}}\phi(j) &= \Lambda(\omega)(\phi(j+1) + \phi(j-1) - 2\phi(j)), \quad j \neq 0,\\ \tilde{\mathcal{L}}\phi(0) &= \frac{\Lambda(\omega)}{2}(\phi(1) + \phi(-1) - 2\phi(0)), \quad j = 0.\end{aligned}$$

One can check that, starting from the same point and $\forall t \geq 0$, X_t and $1 + |Y_t|$ have the same law. In what follows, we will denote by $\mathbb{Q}^N_{d(N)}$ the law of the normalized process $(Y_t/N)_{t\geq 0}$ starting from $d(N) = (l(N) - 1)/N$. According to Theorem 3 in [41], we will not directly prove the tightness of the normalized process, but truncations of this process, and we will be able to conclude thanks to an adapted version of Lemma 11.1.1 in [59] to the Skorokhod topology on $\mathcal{D}([0,+\infty),\mathbb{R})$. We also introduce some notation. Let $\mathcal{M} = \sigma(x(u), u \geq 0)$, $\mathcal{M}_t = \sigma(x(u), u \leq t)$, and

$$M^N_f(t) = f(x(u)) - f(x(0)) - \int_0^t \tilde{\mathcal{L}}^N f(x(s))ds,$$

which is an $(\mathcal{M}_t)$-martingale under $\mathbb{Q}^N_{d(N)}$ and where

$$\begin{aligned}\tilde{\mathcal{L}}^N\phi(j) &= \Lambda(\omega)\left[\phi\left(\frac{j+1}{N}\right) + \phi\left(\frac{j-1}{N}\right) - 2\phi\left(\frac{j}{N}\right)\right], \quad j \neq 0,\\ \tilde{\mathcal{L}}^N\phi(0) &= \frac{\Lambda(\omega)}{2}\left[\phi\left(\frac{1}{N}\right) + \phi\left(\frac{-1}{N}\right) - 2\phi(0)\right], \quad j = 0.\end{aligned}$$

Tightness of $\left(\mathbb{Q}^{N,M}_{d(N)}\right)_N$

Let $M \geq 1$, large enough to have $\sup_N d(N) \leq M$, and $\tau_M = \inf(u \geq 0, |x(u)| \geq M)$. We denote by $\mathbb{Q}^{N,M}_{d(N)}$ the law of $(Y_{t\wedge\tau_M}/N)_{t\geq 0}$ starting from $d(N)$. We remark that $\mathbb{Q}^{N,M}_{d(N)} = \mathbb{Q}^N_{d(N)}$ on $\mathcal{M}_{\tau_M}$. It becomes easy to see that

$$\lim_{K\to+\infty} \mathbb{Q}^{N,M}_{d(N)}\left(\sup_{u\geq 0}|x(u)| \geq K\right) = 0.$$

Moreover, $(M_f^N(t\wedge\tau_M))_{t\geq 0}$ is an $(\mathcal{M}_t)$-martingale under $\mathbb{Q}^N_{d(N)}$. Consequently, $\forall\ 0\leq s\leq t$,

$$\begin{aligned}\mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}\left[(x(t)-x(s))^2\right] &= \mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}\left[(M^N_{Id}(t)-M^N_{Id}(s))^2\right]\\ &= \mathbb{E}_s^{\mathbb{Q}^{N}_{d(N)}}\left[< M^N_{Id} >_{t\wedge\tau_M} - < M^N_{Id} >_{s\wedge\tau_M}\right]\\ &\leq 2\frac{\Lambda}{N^2}(t-s),\end{aligned}$$

where $\mathbb{E}_s^{\mathbb{Q}^{N,M}_{d(N)}}$ is the conditional expectation under $\mathbb{Q}^{N,M}_{d(N)}$ given $\mathcal{M}_s$. Thus, by Theorem 3 in [41], $(\mathbb{Q}^{N,M}_{d(N)})_N$ is tight in $\mathcal{D}([0,+\infty),\mathbb{R})$.

Convergence

We consider f a smooth function and $(\mathbb{Q}^{N',M}_{d(N')})_{N'}$ a converging subsequence to $\mathbb{Q}^M_y$. Let $0\leq s\leq t$ and Φ be a bounded continuous $\mathcal{M}_s$-measurable function. We have

$$\mathbb{E}^{\mathbb{Q}^{N',M}_{d(N')}}\left[M_f^{N'}(t\wedge\tau_M)\Phi\right]=\mathbb{E}^{\mathbb{Q}^{N',M}_{d(N')}}\left[M_f^{N'}(s\wedge\tau_M)\Phi\right]. \tag{4.32}$$

However, $\Lambda(\omega)=\frac{k^2(\omega)S(1,1)}{2a}\underset{\omega\gg 1}{\sim}N^2\frac{\sigma^2}{2}$,

$$\lim_{N\to+\infty}\sup_{v\in\left[-M,-\frac{1}{N}\right]\cup\left[\frac{1}{N},M\right]}\left|\tilde{\mathcal{L}}^N f\left(\frac{[Nv]}{N}\right)-\frac{\sigma^2}{2}f''(v)\right|=0 \text{ and } \lim_{N\to+\infty}\tilde{\mathcal{L}}^N f(0)=\frac{1}{2}\frac{\sigma^2}{2}f''(0).$$

To correct the problem in $v=0$, we have the following lemma.

Lemma 4.9

$$\mathbb{E}^{\mathbb{Q}^N_{d(N)}}\left[\int_0^t \mathbf{1}_{(x(u)=0)}du\right]=\mathcal{O}\left(\frac{1}{N^2}\right).$$

Proof $\mathbb{E}^{\mathbb{Q}^N_{d(N)}}\left[\int_0^t \mathbf{1}_{(x(u)=0)}du\right]$ is the mean time spent by $(\frac{Y_t}{N})_{t\geq 0}$ in the state 0. We denote by $(\tilde{X}_t)_{t\geq 0}$ the traffic of the *M/M/1* queue with traffic rate $\rho=1$. In addition to the Markov property,

$$\mathbb{E}^{\mathbb{Q}^N_{d(N)}}\left[\int_0^t \mathbf{1}_{(x(u)=0)}du\right]=\mathbb{E}_{d(N)}\left[\int_0^t \mathbf{1}_{(\tilde{X}_u=0)}du\right]\leq\mathbb{E}_0\left[\int_0^t \mathbf{1}_{(\tilde{X}_u=0)}du\right]\leq\int_0^t\mathbb{P}_0(\tilde{X}_u=0)du.$$

However, explicit expressions of the transition probabilities for this queue can be found in [6, Theorem 8.5] . In our case,

$$\mathbb{P}_0(\tilde{X}_t=0)=e^{-2\Lambda t}\left(I_0(2\Lambda t)+I_1(2\Lambda t)\right),$$

where I_n is the modified Bessel function of order n, given by $I_n(t)=\sum_{k\geq 0}\frac{t^{n+2k}}{2^{n+2k}k!(n+k)!}$. Then, we get

$$\int_0^t\mathbb{P}_0(\tilde{X}_u=0)du\leq\frac{1}{2\Lambda}e^{-2\Lambda t}\sum_{k\geq 0}\frac{(2\Lambda)^{2k+1}}{(2k+1)!}+\frac{(2\Lambda)^{2k+2}}{(2k+2)!}\leq\frac{1}{2\Lambda}.$$

□

Consequently, letting $N'\to+\infty$ in (4.32), we obtain that under $\mathbb{Q}^M_y$,

$$f(x(t\wedge\tau_M))-f(x(0))-\frac{\sigma^2}{2}\int_0^{t\wedge\tau_M}f''(x(u))du$$

is an $(\mathcal{M}_t)$-martingale. If we denote by $\mathbb{W}^\sigma_y$ the law of the process $(\sigma B_t+y)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a standard Brownian motion, we have $\mathbb{Q}^M_y=\mathbb{W}^\sigma_y$ on $\mathcal{M}_{\tau_M}$. Finally, we conclude that $\mathbb{Q}^N_{d(N)}$ converges to $\mathbb{W}^\sigma_y$ thanks to an adapted version of Lemma 11.1.1 in [59] to the Skorokhod topology on $\mathcal{D}([0,+\infty),\mathbb{R})$.■

Bibliography

[1] R. Adler, *The Geometry of Random Fields*, Wiley, London, 1981.

[2] R.J Adler and J. Taylor, *Random fields and geometry*, Springer, New York, 2007.

[3] D.G. Alfaro Vigo, J.-P. Fouque, J. Garnier, and A. Nachbin, *Robustness of time reversal for waves in time-dependent random media*, Stochastic Process. Appl., 111 (2004), pp. 289–313.

[4] P.W. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev., 109 (1958) pp. 1492–1505.

[5] M. Asch, W. Kolher, G. Papanicolaou, M. Postel and B. White, *Frequency content of randomly scattered signals*, SIAM Review, 33 (1991), pp. 519–625.

[6] S. Asmussen, *Applied probability and queues*, Springer, New York, 2003.

[7] F. Bailly, J.-F. Clouet, and J.-P. Fouque, *Parabolic and white noise approximation for waves in random media*, SIAM J. Appl. Math, 56 (1996), pp. 1445–1470.

[8] G. Bal, A. Fannjiang, G. Papanicolaou, and L. Ryzhik, *Radiative transport in a periodic structure*, Jour. Stat. Phys., 95 (1999), pp. 479–494.

[9] G. Bal, G. Papanicolaou, and L. Ryzhik, *Radiative transport limit for random Schrödinger equation*, Nonlinearity, 15 (2002), pp. 513-529.

[10] G. Bal, G. Papanicolaou, and L. Ryzhik, *Self-averaging in time reversal for the parabolic wave equation*, Stochast. Dyn., 2 (2002), pp. 507–531.

[11] G. Bal and L. Ryzhik, *Stability of time reversed waves in changing media*, Disc. Cont. Dyn. Syst. A , 12 (2005), pp. 793–815.

[12] G. Bal and R. Verastegui, *Time Reversal in Changing Environment*, SIAM Multiscale Model. Simul., 2 (2004), pp. 639–661.

[13] A. Bamberger, B. Engquist, L. Halpern, and P. Joly, *Parabolic wave equation approximations in heterogeneous media*, SIAM J. Appl. Math, 48 (1988), pp. 99–128.

[14] P. Billingsley, *Convergence of probability measure*, 2^{nd} ed., Wiley InterScience, 1999.

[15] P. Blomgren, G. Papanicolaou, and H. Zhao, *Super-resolution in time-reversal acoustics*, J. Acous. Soc. Am.,111 (2002), pp. 230–248.

[16] R. Carmona and J.-P. Fouque, *Diffusion-approximation for the advection-diffusion of a passive scalar by a space-time gaussian velocity field*, Seminar on Stochastic Analysis, Random Fields and Applications. (Edited by E. Bolthausen, M. Dozzi and F. Russo). Birkhauser, Basel (1994) 37-50.

[17] J.-F. Clouet and J.-P. Fouque, *Spreading of a pulse traveling in random media*, Ann. Appl. Henri Probab., 4 (1994), pp. 1083–1097.

[18] J.-F. Clouet and J.-P. Fouque, *A time-reversal method for an acoustical pulse propagating in randomly layered media*, Wave Motion, 25 (1997), pp. 361–368.

[19] A. Derode, P. Roux, and M. Fink, *Robust acoustic time reversal with high-order multiple scattering*, Phys. Rev. Lett., 75 (1995), pp. 4206–4209.

[20] M.D. Donsker and S.R.S. Varadhan, *Asymptotic evaluation of certain Markov process expectations for large time, I-IV*, Comm. Pure Appl. Math, 28 (1975), pp. 1–47, 279–301, 29 (1979), pp. 368–461, 36 (1983) pp. 183–212.

[21] M. Fink, *Time-reversed acoustic*, Scientific American, 281 (1999), pp. 91–97.

[22] M. Fink, D. Casserau, A. Derode, C. Prada, P. Roux, M. Tanter, J.-L. Thomas, F. Wu, *Time-reversed acoustic*, Reports on progress in Physics, 63 (2000), pp. 1933–1995.

[23] J.-P. Fouque, *La convergence en loi pour les processus à valeur dans un espace nucléaire*, Ann. Inst. Henri Poincaré, 20 (1984), pp. 225–245.

[24] J.-P. Fouque, J. Garnier, A. Nachbin, and K. Sølna, *Time reversal refocusing for point source in randomly layered media*, Wave Motion, 42 (2005), pp. 238–260.

[25] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Springer, New York, 2007.

[26] J.-P. Fouque, J. Garnier, and K. Sølna, *Time reversal super resolution in randomly layered media*, Wave Motion, 43 (2006), pp. 646–666.

[27] A. Friedman, *Partial differential equations of parabolic type*, Englewood Cliffs, NJ, Prentice Hall, 1964.

[28] J. Garnier, *A multi-scaled diffusion-approximation theorem. Applications to wave propagation in random media*, ESAIM Probab. Statist., 1 (1997), pp. 183–206 .

[29] J. Garnier, *The role of evanescent modes in randomly perturbed single-mode waveguides* , Discrete and Continuous Dynamical Systems-Series B, 8 (2007), pp. 455–472.

[30] J. Garnier and G. Papanicolaou, *Pulse propagation and time reversal in random waveguides*, SIAM J. Appl. Math, 67 (2007), pp. 1718–1739.

[31] J. Garnier and K. Sølna, *Effective transport equations and enhanced backscattering in random waveguides* , SIAM J. Appl. Math, 68 (2008), pp. 1574–1599.

[32] I.J. Goldsheid, S.A. Molchanov, and L.A. Pastur, *A random one-dimensional Schroedinger operator has pure point spectrum*, Functional Anal. Appl., 11 (1977), pp. 1–10.

[33] C. Gomez, *Time-reversal superresolution in random waveguides*, SIAM Multiscale Model. Simul. 7 (2009), pp. 1348–1386.

[34] A. Ishimaru, *Wave propagation and scattering in random media*, Academic press, San Diego, 1978.

[35] G. Kallianpur and J. Xiong, *Stochastic differential equations in infinite dimensional spaces*, IMS Lecture notes-monograph series, 1995.

[36] R.Z. Khasminskii, *A limit theorem for solutions of differential equations with random right hand side*, Theory Probab. Appl. 11 (1966), pp. 390–406.

[37] V.I. Klyatskin, *Stochastic equations and waves in randomly inhomogeneous media*, Nauka, Moscow, 1980. (In Russian.)

[38] W. Kohler, *Power reflexion at the input of a randomly perturbed rectangular waveguide*, SIAM J. Appl. Math, 32 (1977), pp. 521–533.

[39] W. Kohler and G. Papanicolaou, *Wave propagation in randomly inhomogeneous ocean*, Lecture Notes in Physics, Vol. 70, J. B. Keller and J. S. Papadakis, eds., Wave Propagation and Underwater acoustics, Springer-Verlag, Berlin, 1977.

[40] W.A. Kuperman, W.S. Hodgkiss, H.C. Song, T. Hakal, C. Ferla, D.R. Jackson, *Phase conjugation in the ocean, experimental demonstration of an acoustic time-reversal mirror*, J. Acoust. Soc. Am. 103 (1998), pp. 25–40.

[41] H.-J. Kushner, *Approximation and weak convergence methods for random processes*, MIT press, Cambridge, 1984.

[42] G. Lerosey, J. De Rosny, A. Tourin, and M. Fink, *Focusing beyound the diffraction limit with far-field time reversal*, Science, 315 (2007), pp. 1120–1122.

[43] R. Magnanini and F. Santosa, *Wave propagation in a 2-D optical waveguide*, SIAM J. Appl. Math, 61 (2000), pp. 1237–1252.

[44] D. Marcuse, *Theory of dielectric optical waveguides*, 2^{nd} ed., Academic press, New York, 1991.

[45] R.H. Mellen, D.G. Browning, and J.M. Ross, *Attenuation in randomly inhomogeneous sound chanels*, J. Acoust. Soc. Am., 56 (1974), no. 3, pp. 80–82.

[46] M. Metivier, *Stochastic partial differential equations in infinite dimensional spaces*, Scuola normale superiore, Pisa, 1988.

[47] I. Mitoma, *On the sample continuity of $\mathcal{S}'$-processes*, J. Math. Soc. Japan, 35 (1983), pp. 629–636.

[48] G. Papanicolaou and W. Kohler, *Asymptotic theory of mixing stochastic ordinary differential equations*, Comm. Pure Appl. Math, 27 (1974), pp. 641–668.

[49] G. Papanicolaou, L. Ryzhik and K. Sølna, *Statistical stability in time reversal*, SIAM J. Appl. Math, 64 (2004), pp. 1133–1155.

[50] G. Papanicolaou and S. Weinryb, *A functional limit theorem for waves reflected by a random medium*, Appl. Math Optim., 30 (1994), pp. 307–334.

[51] C.I. Pekeris, *Theory of propagation of explosive sound in shallow water*, propagation of sound in the ocean, geological society of America, memoir 27, pp. 1–117.

[52] E. Perrey-Debain and I.D. Abrahams, *A diffusion analysis approach to TE mode propagation in randomly perturbed optical waveguides*, SIAM J. Appl. Math, 68 (2007), pp. 523–543.

[53] P. Roux and M. Fink, *Time reversal in a waveguide: Study of the temporal and spacial focusing*, J. Acoust. Soc. Am., 107 (2000), pp. 2418–2429.

[54] H.E. Rowe, *Electromagnetic propagation in multi-mode random media*, Wiley, New York, 1999.

[55] H. Schaefer, *Topological vector spaces*, Springer-Verlag, New York, 1971.

[56] P. Sheng, Ed., *Scattering and localization of classical waves in random media*, World Scientific, Singapore, 1989.

[57] C. Song, W.A. Kuperman, and W.S. Hodgkiss, *Iterative time reversal in the ocean*, J. Acoust. Soc. Amer., 105 (1999), pp. 3176–3184.

[58] C.M. Soucoulis, E.N. Economou, G.S. Grest, and M.H. Cohen, *Existence of Anderson localization of classical waves in random two-component medium*, Phys. Rev. Lett., 62 (1989), pp. 575–578.

[59] D.W. Stroock and S.R.S. Varadhan, *Multidimensional diffusion processes*, Springer-Verlag, Berlin, 1979.

[60] B. White, P. Sheng, Z.-Q. Zhang, and G. Papanicolaou, *Wave localization characteristics in the time domaine*, Phys. Rev. Lett., 59 (1987), no. 3, pp. 1918–1921.

[61] C. Wilcox, *Spectral analysis of the Pekeris operator in the theory of acoustic wave propagation in shallow water*, Arch. Rational Mech. Anal, 60 (1975/76), no. 3, pp. 259–300.

[62] C. Wilcox, *Transient electromagnetic wave propagation in a dielectric waveguide*, in Symposia Mathematica, vol. XVIII (Convegno sulla Teoria Matematica dell'Elettromagnetismo, INDAM, Rome, 1974), Academic Press, London, 1976, pp. 239–277.

[63] M. Yor, *Existence et unicité de diffusion à valeurs dans un espace de Hilbert*, Ann. Inst. Henri Poincaré, 10 (1974), pp. 55–88.

Résumé : Cette thèse porte sur la propagation et le retournement temporel des ondes dans des guides d'ondes aléatoirement perturbés. L'étude de la propagation dans les guides d'ondes aléatoires est devenue indispensable face au grand nombre de situations pouvant se modéliser de cette manière : comme par exemple en télécommunication, en acoustique sous-marine ou en géophysique.

Le travail présenté dans cette thèse se décompose en trois chapitres. Dans un premier chapitre, on s'intéresse à la propagation des ondes dans un guide d'onde océanique inhomogène. On propose des équations effectives permettant de modéliser la propagation des ondes dans ce milieu. Ces équations décrivent le rôle des modes propagatifs, évanescents et radiatifs sur la propagation, et permettent de quantifier la perte radiative d'énergie dans le fond océanique. Dans un second chapitre, on s'intéresse à la propagation et à la refocalisation par retournement temporel d'une impulsion dans le modèle de guide d'onde océanique du premier chapitre. On obtient une description de l'onde refocalisée prenant en compte la perte radiative dans le fond océanique et l'évolution des fluctuations du milieu entre les deux étapes de l'expérience de retournement temporel. Dans le dernier chapitre, on s'intéresse à la refocalisation par retournement temporel dans un modèle de guide d'onde simple. On obtient un phénomène de *super-résolution* par l'insertion, devant la source, d'une section inhomogène à faible vitesse de propagation, c'est à dire qu'on obtient des tailles de taches focales plus concentrées qu'en milieu homogène.

Mots-clés : Propagation d'ondes, Retournement temporel des ondes, Guides d'ondes acoustiques, Milieux aléatoires, Analyse asymptotiques, Théorèmes limites.

WAVE PROPAGATION AND TIME REVERSAL IN RANDOM WAVEGUIDES

Abstract : This thesis concerns wave propagation and time reversal of waves in randomly perturbed waveguides. The study of wave propagation phenomena in random waveguides is an interesting subject with numerous domains of applications: for instance in telecommunication, underwater acoustics and geophysics.

This thesis is composed of three chapters. In a first Chapter, we are interested in wave propagation in inhomogeneous oceanic waveguides, and we derive effective equations which model wave propagation in such media. These equations describe the role of the propagating, radiating, and evanescent modes, and allow us to quantify the radiative loss of energy in the ocean bottom during the propagation. In a second chapter we study pulse propagation and time-reversal refocusing in the perturbed waveguide model introduced in the first chapter. We get a description of the refocused wave which takes into account the radiative loss in the ocean bottom, and the evolution of the random fluctuations of the medium between the two steps of the time-reversal experiment. In a last chapter, we study time-reversal refocusing in a simple waveguide model. In this model we get a *superresolution* phenomena by inserting a random section with low speed of propagation in the vicinity of the source, that is, we get more concentrated focal spots than in the homogenous waveguides.

Keywords : Wave propagation, Time reversal of waves, Acoustic waveguides, Random media, Asymptotic analysis, Limit theorems.

Laboratoire de Probabilités et Modèles Aléatoires
CNRS UMR 7599,
Université Paris Diderot - Paris 7, UFR de mathématiques (Site Chevaleret)
Case 7012 75205 Paris Cedex, France.