Spectrum of random graphs

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1 Spectral measure of a fixed graph

In this section, we introduce our main definitions. We refer to Mohar and Woess [66] for an early survey on the spectrum of graphs. Related monographs include [30, 28, 29, 49, 26].

1.1 Adjacency operator

Let V be countable and $G = (V, E)$ be a non-oriented graph. Assume further that G is *locally finite*, i.e. for all $v \in V$,

$$\deg(v) = \sum_{u \in V} \mathbf{1}(\{u, v\} \in E) < \infty.$$

The *adjacency operator*, denoted by A, is defined on $\ell_c(V) \subset \ell^2(V)$, the set of vectors $\psi \in \ell^2(V)$ with finite support, by the formula

$$A\psi(u) = \sum_{v:\{u,v\} \in E} \psi(v).$$

By construction A is symmetric. Also, if $\deg(u) \le d$ for all $u \in V$ then A is a bounded operator, indeed,

$$\|A\psi\|_2^2 = \sum_u \left(\sum_{v:\{u,v\} \in E} \psi(v) \right)^2 \le \sum_u \deg(u) \sum_{v:\{u,v\} \in E} \psi(v)^2 \le \sum_v \psi(v)^2 d^2.$$

For simplicity, we will focus on the sole adjacency operator. Most claims stated here also hold for the *Laplacian operator* and the *normalized Laplacian operator* given respectively by $L = D - A$ and $D^{-1/2} A D^{-1/2}$, where D is the multiplication

$$D\psi(u) = \deg(u)\psi(u),$$

(D^{-1} is properly defined if no vertex is isolated, i.e. $\deg(v) \geq 1$ for all $v \in V$). The Laplacian is the infinitesimal generator of the continuous time simple random walk on G while the normalized Laplacian is equal to $D^{1/2} P D^{-1/2}$ where P is the transition kernel of the discrete time random walk.

1.2 Spectral measure at a vector

Being symmetric, A is closable. The von Neumanns criterion [70, Theorem X.3] implies its closure admits self-adjoint extensions. *In this paragraph, we assume further that the operator is essentially self-adjoint (i.e. it has a unique self-adjoint extension).*

For example, this assumption is fulfilled if the degrees of vertices are bounded by an integer d. Indeed, in this case, we have checked that A has norm bounded by d and A is a bounded self-adjoint operator. Note that there are examples of locally finite graphs whose adjacency operator has more than one self-adjoint extension, for references see [66, Section 3]. For a criterion of essential self-adjointness of the adjacency operator of trees, see [22] and for a characterization see Salez [71, Theorem 2.2].

For any $\psi \in \ell^2(V)$ with $\|\psi\|_2^2 = 1$, we may then define the *spectral measure with vector* ψ, denoted by μ_G^ψ, as the unique probability measure on $\mathbb{R}$, such that for all integers $k \geq 1$,

$$\int x^k d\mu_G^\psi = \langle \psi, A^k \psi \rangle.$$

For example if $|V| = n$ is finite, then A is a symmetric matrix. If $(v_1, \cdots, v_n)$ is an orthonormal basis of eigenvectors associated to eigenvalues $(\lambda_1, \cdots, \lambda_n)$, we find

$$\mu_G^\psi = \sum_{k=1}^{n} \langle v_k, \psi \rangle^2 \delta_{\lambda_k}. \tag{1}$$

If V is not finite, μ_G^ψ has a similar decomposition over the (left-continuous) resolution of the identity of A, say $\{E_{(-\infty,\lambda)}\}_{\lambda \in \mathbb{R}}$, we write $A = \int \lambda dE(\lambda)$ and we find, for any $\lambda \in \mathbb{R}$,

$$\mu_G^\psi((-\infty, \lambda)) = \langle \psi, E_{(-\infty,\lambda)} \psi \rangle. \tag{2}$$

For $v \in V$, we denote by $e_v \in \ell^2(V)$, the coordinate vector defined by $e_v(u) = \mathbf{1}(u = v)$ for all $u \in V$. Observe that for any $u, v \in V$, $\langle e_u, A^k e_v \rangle$ is the number of paths of length k from u to v in G. Consequently,

$$\int x^k \mu_G^{e_v} = |\{\text{closed paths of length } k \text{ starting from } v\}|. \tag{3}$$

The resolvent $R(z) = (A - z)^{-1}$ defined for $z \in \mathbb{C}\backslash\mathbb{R}$ is related to the walk generating function of the graph G : expanding formally, we find

$$\langle e_u, R(z)e_v\rangle = (-z)^{-1}\sum_{k\geq 0} z^{-k}\langle e_u, A^k e_v\rangle.$$

Observe also that

$$\langle e_v, R(z)e_v\rangle = \int \frac{d\mu_G^{e_v}(x)}{x - z} \tag{4}$$

is the Cauchy-Stieltjes transform of $\mu_G^{e_v}$. In these notes, we will mostly be interested by the regularity properties of the measure $\mu_G^{e_v}$. For some explicit computation of spectral measures in regular graphs, see examples below, Hora and Obata [49] and for a recent computation [9].

1.3 Operations on graphs and spectrum

There are algebraic operations on graphs for which it is possible to compute explicitly how they transform the spectral measures. In this paragraph, we consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ whose adjacency operators, A_1 and A_2 are essentially self-adjoint.

1.3.1 Cartesian product

We build a new graph $G_1 \times G_2$ on the vertex $V_1 \times V_2$ by putting the edge $\{(u_1, u_2), (v_1, v_2)\}$ if either $u_1 = v_1$ and $\{u_2, v_2\} \in E_2$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E_1$. In terms of the adjacency operator, say A, of $G_1 \times G_2$, we have

$$\langle e_{(v_1,v_2)}, Ae_{(u_1,u_2)}\rangle = \mathbf{1}(u_1 = v_1)\mathbf{1}(\{u_2, v_2\} \in E_2) + \mathbf{1}(u_2 = v_2)\mathbf{1}(\{u_1, v_1\} \in E_1).$$

For example, for integer $d \geq 1$, consider the usual graph of $\mathbb{Z}^d$ defined by putting an edge between u and v if $\|u - v\|_1 = \sum_{i=1}^d |u_i - v_i| = 1$. Then $\mathbb{Z}^d$ is equal the cartesian product of d copies of $\mathbb{Z}$.

Observe that a path of length k in $G_1 \times G_2$ can be decomposed into a path in G_1 of length ℓ and a path in G_2 of length $k - \ell$, for some $0 \leq \ell \leq k$. Conversely, a path of length ℓ starting from $u_1 \in V_1$ and a path of length $k - \ell$ starting from u_2 gives $\binom{k}{\ell}$ paths of length k in $G_1 \times G_2$ starting from (u_1, u_2). It follows easily from (3) that for any $(v_1, v_2) \in V_1 \times V_2$

$$\int x^k \mu_{G_1\times G_2}^{e_{(v_1,v_2)}} = \sum_{\ell=0}^{k}\binom{k}{\ell}\int x^\ell \mu_{G_1}^{e_{v_1}} \int x^{k-\ell}\mu_{G_2}^{e_{v_2}}.$$

So finally

$$\mu_{G_1\times G_2}^{e_{(v_1,v_2)}} = \mu_{G_1}^{e_{v_1}} * \mu_{G_2}^{e_{v_2}}, \tag{5}$$

where $*$ denotes the usual convolution.

1.3.2 Tensor product

We now build a graph $G_1 \otimes G_2$ on the vertex $V_1 \times V_2$ by putting the edge $\{(u_1, u_2), (v_1, v_2)\}$ if $\{u_1, v_1\} \in E_1$ and $\{u_2, v_2\} \in E_2$. This called the tensor or Kronecker product. The adjacency operator, say A, of $G_1 \otimes G_2$, is given by

$$\langle e_{(v_1,v_2)}, Ae_{(u_1,u_2)} \rangle = \mathbf{1}(\{u_1, v_1\} \in E_1)\mathbf{1}(\{u_2, v_2\} \in E_2),$$

For example, it is easy to check that $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z} \otimes \mathbb{Z}$.

By construction, a path of length k in $G_1 \otimes G_2$ is a path in G_1 of length k and a path in G_2 of length k. We get

$$\int x^k \mu_{G_1 \otimes G_2}^{e_{(v_1,v_2)}} = \int x^k \mu_{G_1}^{e_{v_1}} \int x^k \mu_{G_2}^{e_{v_2}},$$

and, consequently,

$$\mu_{G_1 \otimes G_2}^{e_{(v_1,v_2)}} = \mu_{G_1}^{e_{v_1}} \odot \mu_{G_2}^{e_{v_2}}, \tag{6}$$

where $\odot$ denotes the product convolution, i.e. if X_i has law μ_i for $i = 1, 2$ and X_1 and X_2 are independent then $\mu_1 \circ \mu_2$ is the law of $X_1 X_2$.

1.3.3 Free product

Assume that G_1 and G_2 are connected and let $o_i \in V_i$, $i = 1, 2$ be two distinguished vertices, called the roots. We define V as the set of finite sequences $v = (v_1, v_2, \cdots, v_k)$ such that, for any integer $i \geq 0$, $v_1 \in V_1$, $v_{2i+3} \in V_1 \backslash o_1$, $v_{2i} \in V_2 \backslash o_2$. The length of $v = (v_1, \cdots, v_k) \in V$ is set to be k. We now build a graph $G = (G_1, o_1) * (G_2, o_2)$ on the vertex V by putting the edge $\{u, v\}$, where length of u is less or equal than the length of v, if one of the four cases holds, for integer $i \geq 0$:

- $v = (v_1, \cdots, v_{2i}, v_{2i+1})$, $u = (v_1, \cdots, v_{2i}, u_{2i+1})$ and $\{u_{2i+1}, v_{2i+1}\} \in E_1$;
- $v = (v_1, \cdots, v_{2i+1}, v_{2i+2})$, $u = (v_1, \cdots, v_{2i+1}, u_{2i+2})$ and $\{u_{2i+2}, v_{2i+2}\} \in E_2$;
- $v = (v_1, \cdots, v_{2i}, v_{2i+1})$, $u = (v_1, \cdots, v_{2i})$ and $\{v_{2i+1}, o_1\} \in E_1$;
- $v = (v_1, \cdots, v_{2i+1}, v_{2i+2})$, $u = (v_1, \cdots, v_{2i+1})$ and $\{v_{2i+2}, o_2\} \in E_2$.

In words, G is obtained by gluing iteratively on each vertex of G_1 a copy of G_2 rooted at o_2 and from each vertex of G_2 a copy of G_1 rooted at o_1. If G_1 and G_2 are vertex transitive, this construction, up to isomorphisms, does not depend on the choice of the root. For example, $\mathbb{T}_d$, the infinite d-regular tree (where all vertices have degree d) is isomorphic, when d is even, to the free products of $d/2$ copies of $\mathbb{Z}$. If G_i is the Cayley graph of a group Γ_i with generating set S_i (see Subsection 1.5 for definitions), then G is the Cayley graph of the free product of the groups G_1 and G_2 with generating set the disjoint union of S_1 and S_2.

We have that

$$\mu_{(G_1,o_1)*(G_2,o_2)}^{e_{o_1}} = \mu_{(G_2,o_2)*(G_1,o_1)}^{e_{o_2}} = \mu_{G_1}^{e_{o_1}} \boxplus \mu_{G_2}^{e_{o_2}}, \tag{7}$$

where $\boxplus$ is the free convolution. For an explanation, see the monograph by Voiculescu, Dykema and Nica [77].

1.4 Finite graphs

We now look for a definition of the spectral measure of a graph. If $G = (V, E)$ is a finite graph, $|V| = n$ then we will define the *spectral measure of* G as,

$$\mu_G = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}, \tag{8}$$

where $(\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of A, counting multiplicities. In other words, μ_G is the empirical distribution of the eigenvalues of A. In the physics literature, the spectral measure is known as the *density of states*.

In terms of the spectral measure with a vector, μ_G^{ψ}, it follows from (1) that

$$\frac{1}{n} \sum_{x=1}^{n} \mu_G^{e_x} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k} \sum_{x=1}^{n} \langle v_k, e_x \rangle^2 = \mu_G. \tag{9}$$

Cycle : let C_n be a cycle of length n. The adjacency operator can be written as $A = B + B^*$, where B is the permutation matrix of a cycle of length n. Since $BB^* = B^*B = I$, the eigenvalues of B are the roots of unity and the eigenvalues of A are $\lambda_k = 2\cos(2\pi k/n)$, $1 \leq k \leq n$. We get

$$\mu_{C_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos(2\pi k/n)}.$$

As n goes to infinity, μ_{C_n} converges weakly to a arcsine distribution ν with density on $[-2, 2]$ given by

$$d\nu(x) = \frac{1}{\pi\sqrt{4 - x^2}} \mathbf{1}_{|x| \leq 2} dx, \tag{10}$$

(ν is the law of $2\cos(\pi U)$ with U uniform on $[0, 1]$).

Line segment : let $L_n = \mathbb{Z} \cap [1, n]$ be the subgraph of $\mathbb{Z}$ spanned by vertices in $\{1, \cdots, n\}$. The characteristic polynomial $P_n(x) = \det(A(L_n) - x)$ satisfies the recurrence $P_{n+2}(x) = -xP_{n+1}(x) - P_n(x)$. It follows that P_n is the Chebyshev polynomial of the second kind. The roots of P_n are $\lambda_k = 2\cos(\pi k/(n+1))$, $1 \leq k \leq n$, and we find

$$\mu_{L_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos(\pi k/(n+1))}.$$

Again, as n goes to infinity, μ_{L_n} converges weakly to a arcsine distribution ν. In view of (5), we could also compute the spectral measure of $\mathbb{Z}^d \cap [1, n]^d$ which is the cartesian product of d copies of L_n.

Complete graph : the eigenvalues of the adjacency matrix of the complete graph K_n on n vertices, are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. It follows that

$$\mu_{K_n} = \frac{1}{n}\delta_{n-1} + \frac{n-1}{n}\delta_{-1}.$$

Notice that μ_{K_n} converges weakly to δ_{-1}. It contrasts with the above situation, since the limit as $n \to \infty$ is purely atomic.

1.5 Cayley graphs

1.5.1 Definition

Let Γ be a countable group and $S \subset \Gamma$ a generating set such that for any $S^{-1} \subset S$ and the unit of Γ is not in S. The Cayley graph (on the left) $G = \mathrm{Cay}(\Gamma, S)$ associated to S is the graph with vertex set Γ and edge set $\{\{u,v\}, vu^{-1} \in S\}$. It is not hard to check that G is vertex transitive. Hence, the spectral measure at vector e_v does not depend on the choice of $v \in \Gamma$. It is then natural to define the spectral measure of G as

$$\mu_G = \mu_G^{e_v}, \tag{11}$$

(more generally, we could extend this definition to any vertex-transitive graph). In view of (9), this definition is consistent with our previous definition if G is a finite Cayley graph. This measure μ_G is usually called the *Plancherel measure* of G. Beware that this spectral measure may strongly depends on the choice of the generating set S.

It is not the scope of these notes to emphasize the connections with operator algebras. Let us recall anyway that the (left) von Neumann group algebra $\mathcal{M}$ of the discrete group Γ is the subalgebra of all bounded operators on $H = \ell^2(\Gamma)$ generated by the operators λ_v corresponding to multiplication from the left with an element $v \in \Gamma$, i.e. $\lambda_v e_u = e_{vu}$. $\mathcal{M}$ is the algebra of bounded operators on H commuting with the action of Γ on H through right multiplication. The adjacency operator is an element of $\mathcal{M}$. The canonical trace on $\mathcal{M}$ is the linear map

$$\tau(B) = \langle e_o, B e_o \rangle,$$

where o is the unit of Γ. The fact that τ is a trace follows from $\tau(\lambda_u \lambda_v) = \tau(\lambda_v \lambda_u) = \mathbf{1}(uv = o)$. With our definition of μ_G, we get that

$$\int x^k d\mu_G = \tau(A^k).$$

1.5.2 Basic examples

Let us give some example of spectral measures of Cayley graphs.

Bi-infinite path : the Cayley graph of the additive abelian group $\mathbb{Z}$ with generators $S = \{1, -1\}$.

$$d\mu_{\mathbb{Z}}(x) = \frac{1}{\pi\sqrt{4-x^2}}\mathbf{1}_{|x|\leq 2}dx = d\nu(x),$$

where ν is the arcsine distribution defined in (10).

Lattice : taking the cartesian product we find from (5) that for any integer $d \geq 1$,

$$\mu_{\mathbb{Z}^d} = \nu * \cdots * \nu,$$

where the convolution is taken d times. As already pointed, $\mathbb{Z}^2$ is also isomorphic to $\mathbb{Z} \otimes \mathbb{Z}$. It follows that (6) that $\nu \circledcirc \nu = \nu * \nu$.

Free group with d generators : let $\mathbb{T}_d$ be the infinite d-regular tree, $\mathbb{T}_d$ is isomorphic to the Cayley graph of the free group with d generators. If $d = 2k$ is even then $\mathbb{T}_d$ is isomorphic to $\mathbb{Z} * \cdots * \mathbb{Z}$ where the free product is taken k times. Kesten [56] has proved that

$$d\mu_{\mathbb{T}_d}(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}\mathbf{1}_{|x|\leq 2\sqrt{d-1}}dx.$$

It follows from (7) that if $d = 2k$, $\mu_{\mathbb{T}_d}$ is the free convolution of k times ν.

1.5.3 Lamplighter group

Spectral measures are not always absolutely continuous. Cayley graphs of lamplighter groups give examples of pure point spectral measure. In [47], Grigorchuk and Żuk have computed explicitly the spectral measure of the usual lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ and discovered that it was purely atomic, see also Lehner, Neuhauser and Woess [60]. More generally, the spectral measure of Cayley graphs on lamplighter groups are related to percolation on the walk graph, see [60]. Interestingly, lamplighter groups can also be used to build examples of spectral measures with a mass of the atom at 0 equal to any number in $(0, 1)$, see Austin [67] and Lehner and Wagner [61].

Let us explain how these Cayley graphs are built. Let Γ be a finitely generated group with unit o and set $L = (\mathbb{Z}/n\mathbb{Z}, +)$. The group Γ may be referred as the walk space and L as the lamp space. The lamplighter group $\Lambda = L \wr \Gamma$ is on the set $L^\Gamma \times \Gamma$. An element $(\eta, x) \in \Lambda$ is composed by the configuration of lamps $\eta : \Gamma \to L$ and the position of the lamplighter $x \in \Gamma$. The group operation in Λ is defined as

$$(\eta, x).(\eta', x') = (\eta + \theta_x\eta', x.x'),$$

where $(\theta_x\eta')(y) = \eta'(x^{-1}y)$, θ_x shifts the configuration by x. The unit of Λ is $\varepsilon = (\underline{0}, o)$ where $\underline{0}$ is the configuration defined by $\underline{0}(x) = 0$.

It is easier to understand the lamplighter group in terms on simple generators. For $x \in \Gamma$ and $\ell \in L$, the walk element $W_x \in \Lambda$ and switch element $S_\ell \in \Lambda$ are respectively

$$W_x = (0, x) \quad \text{and} \quad S_\ell = (\ell\delta_o, o),$$

where δ_y is the configuration defined by $\delta_y(x) = \mathbf{1}(x = y)$. In words, $(\eta, x).W_y$ moves the position of the lamplighter to $x.y$ and leaves the lamps unchanged, while $(\eta, x).S_\ell$ leaves the position of the lamplighter unchanged, it switches the light of the lamp located at x into $\eta(x) + \ell$ and leaves all other lamps unchanged, see figure 1 for an illustration of the walk and switch elements when $\Gamma = (\mathbb{Z}, +)$ and $n = 2$.

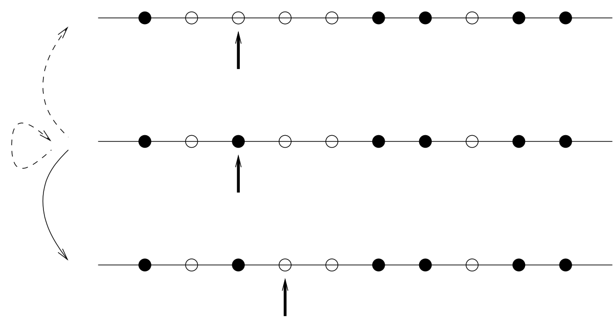

Figure 1: The position of the walker is the thick arrow, in dashed line, the multiplication by a lamp element, in plain line, by a walk element.

Consider a symmetric generating set D of Γ. It is not hard to check that $\{S_\ell, W_x : \ell \in L, x \in D\}$ is a generating set of Λ. The switch-walk generating set of Λ is given by

$$\{S_\ell.W_x : \ell \in L, x \in D\}.$$

We denote by SW the Cayley graph of Λ with this generating set. The walk-switch generating set of Λ is defined similarly with elements $W_x.S_\ell$, we denote by WS its Cayley graph. Finally, the Cayley graph SWS is associated to the usual switch-walk-switch generating set of Λ with elements $S_\ell.W_x.S_{\ell'}$.

Let $G = \mathrm{Cay}(\Gamma, D)$ be the Cayley graph associated to the generating set D. The site percolation graph $\mathrm{perc}'(G, p)$ is the random graph spanned by the open vertices of the site percolation of G with parameter p (independently each vertex is open with probability p). The next theorem due to Lehner, Neuhauser and Woess [60] relates the spectral measures of the graphs SW, WS and SWS to $\mathrm{perc}'(G, p)$.

Theorem 1.1. *For $p = 1/n$, we have*

$$\mu_{\mathrm{SW}}(\cdot/n) = \mu_{\mathrm{WS}}(\cdot/n) = \mu_{\mathrm{SWS}}(\cdot/n^2) = \mathbb{E}\mu^{e_o}_{\mathrm{perc}'(G,p)}.$$

where $\mu(\cdot/t)$ is the push-forward of the measure μ by the map $x \mapsto x/t$.

Proof. Let us sketch the argument. For ease of notation, we set $\nu = \mu_{\mathrm{WS}}(\cdot/n)$ and $\mu = \mathbb{E}\mu^{e_o}_{\mathrm{perc}'(G,p)}$. Since μ and ν have compact support it suffices to check that their moments match. For integer $k \geq 1$, let $\mathcal{W}_k$ be the set of closed walks of length k in G starting from o, that is the set $\gamma =$

$(\gamma_0, \ldots, \gamma_k) \in \Gamma^{k+1}$, such that $\gamma_0 = \gamma_k = o$ and $\{\gamma_t, \gamma_{t+1}\} \in E(G)$ for $0 \leq t \leq k-1$. The range of γ is the set $V(\gamma) = \{\gamma_t : 0 \leq t \leq k\}$. Its cardinality is denoted by $v(\gamma) = |V(\gamma)|$. We have

$$\int \lambda^k d\mu_{\mathrm{perc}'(G,p)}(\lambda) = \sum_{\gamma \in \mathcal{W}_k} \prod_{t=0}^{k} \mathbf{1}(\gamma_t \text{ is open}) = \sum_{\gamma \in \mathcal{W}_k} \prod_{x \in V(\gamma)} \mathbf{1}(x \text{ is open}).$$

Taking expectation, we get

$$\int \lambda^k d\mu(\lambda) = \sum_{\gamma \in \mathcal{W}_k} p^{|V(\gamma)|}.$$

We now compute the moments of ν. Let $d = |D|$. If A is the adjacency operator of WS, observe that $P = A/(dn)$ is the transition kernel of the simple random walk on WS. Then if $\varepsilon = (\underline{0}, o)$

$$\int \lambda^k d\nu = d^k \mathbb{P}^{\varepsilon}(S_k = \varepsilon),$$

where $S_k = (\eta_k, \gamma_k)$ is the position the the random walker on the WS-lampighter graph and $\mathbb{P}_{\varepsilon}(\cdot)$ is the law of the walk starting from $S_0 = \varepsilon$. We can decompose the random walk as $S_t = X_1 \cdots X_t = (\eta_t, \gamma_t)$ where $X_t = W_{x_t}.S_{\ell_t}$, x_t is uniform on D and independent of ℓ_t uniform on L. Then, $\gamma = (\gamma_0, \ldots, \gamma_k)$ is the trace of the walk on G, it is a simple random walk on G independent of the ℓ_t's. Moreover, we have $\eta_t(y) = \eta_{t-1}(y)$ for all $y \neq \gamma_t$ and

$$\eta_t(\gamma_t) = \eta_{t-1}(\gamma_t) + \ell_t.$$

It follows that $S_k = \varepsilon$ if and only if $\gamma_k = o$ and for each $x \in V(\gamma)$, $\eta_{\tau_x - 1}(x) + \ell_{\tau_x} = 0$, where τ_x is the last time that γ_t visits x. Since τ_x is independent of the ℓ_t's and $q + \ell_t$ is uniform on L for any $q \in L$, we deduce that

$$\begin{aligned} \mathbb{P}^{\varepsilon}(S_k = \varepsilon) &= d^{-k} \sum_{\gamma \in \mathcal{W}_k} \mathbb{P}(\forall x \in V(\gamma) : \eta_{\tau_x - 1}(x) + \ell_{\tau_x} = 0) \\ &= d^{-k} \sum_{\gamma \in \mathcal{W}_k} p^{|V(\gamma)|}. \end{aligned}$$

We thus have checked the moments of ν and μ coincide. For SW, the argument is the same, $\tau_x - 1$ is simply replaced by τ_x. The proof for SWS is the same. □

Note that if $\mathrm{perc}'(G,p)$ contains a.s. only finite connected components then the measure $\mathbb{E}\mu^{e_o}_{\mathrm{perc}'(G,p)}$ will be purely atomic (as a countable weighted sum of atomic measures is atomic). Hence, Theorem 1.1 implies for example that μ_{SW} is atomic if $G = \mathbb{Z}$ or $G = \mathbb{Z}^2$ and $n \geq 2$. In the case $G = \mathbb{Z}$, μ_{SW} can even be computed explicitly using (12) and the forthcoming (22),

$$\mathbb{E}\mu^{e_o}_{\mathrm{perc}'(\mathbb{Z},p)} = \sum_{k \geq 1} p^{k-1}(1-p)\mu_{L_k},$$

(for another method see [47, 35]). For $G = \mathbb{T}_d$, Theorem 1.1 has also been used to give an example of an atom at 0 of the spectral measures with irrational mass, see Lehner and Wagner [61] (answering a question of Atiyah), see the forthcoming Theorem 3.5).

2 Spectral measure of unimodular random graphs

We now extend our definition of spectral measures to a more general class of graphs.

We first briefly introduce the theory of local weak convergence of graph sequences and the notion of unimodularity for random rooted graphs. It was introduced by Benjamini and Schramm [15] and has then become a popular topology for studying sparse graphs. Let us briefly introduce this topology, for details we refer to Aldous and Lyons [4] and Pete [69].

A graph $G = (V, E)$ is *locally finite* if for $v \in V$, the degree of v in G (number of incident edges), $\deg_G(v)$, is finite. A *rooted graph* (G, o) is a *connected* graph $G = (V, E)$ with a distinguished vertex $o \in V$, the root. Two rooted graphs $(G_i, o_i) = (V_i, E_i, o_i)$, $i \in \{1, 2\}$, are *isomorphic* if there exists a bijection $\sigma : V_1 \to V_2$ such that $\sigma(o_1) = o_2$ and $\sigma(G_1) = G_2$, where σ acts on E_1 through $\sigma(\{u, v\}) = \{\sigma(u), \sigma(v)\}$. We will denote this equivalence relation by $(G_1, o_1) \simeq (G_2, o_2)$. In graph theory terminology, an equivalence class of rooted graph is an *unlabeled rooted graph*. We denote by $\mathcal{G}^*$ the set of unlabeled rooted locally finite graphs.

The *local topology* is the smallest topology such that for any $g \in \mathcal{G}^*$ and integer $t \geq 1$, the $\mathcal{G}^* \to \{0, 1\}$ function $f(G, o) = \mathbf{1}((G, o)_t \simeq g)$ is continuous, where $(G, o)_t$ is the induced rooted graph spanned by the vertices at graph distance at most t from o. This topology is metrizable with the metric

$$d_{\mathrm{loc}}(g, h) = \sum_{t=1}^{\infty} 2^{-t} \mathbf{1}(g_t \neq h_t). \tag{12}$$

Moreover, it is not hard to check that the space $(\mathcal{G}^*, d_{\mathrm{loc}})$ is separable and complete metric space (or Polish space).

We now consider $\mathcal{P}(\mathcal{G}^*)$ the set of probability measures on $\mathcal{G}^*$. An element $\rho \in \mathcal{P}(\mathcal{G}^*)$ is the law of (G, o), a random rooted graph. Since $\mathcal{G}^*$ is a Polish space, we may safely consider the *local weak topology* on $\mathcal{P}(\mathcal{G}^*)$. Recall that it is the smallest topology such that for any continuous bounded function $f : \mathcal{G}^* \to \mathbb{R}$, the function $\rho \mapsto \mathbb{E}_\rho f(G, o)$ is continuous, where under $\mathbb{P}_\rho$, (G, o) has law ρ. It is well known that this weak convergence is metrizable by the Lévy-Prohorov distance which we will denote by d_{wloc} (the actual definition of the Lévy-Prohorov distance will not be used here). Then $(\mathcal{P}(\mathcal{G}^*), d_{\mathrm{wloc}})$ is also a separable and complete metric space.

For a finite graph $G = (V, E)$ and $v \in V$, one writes $G(v)$ for the connected component of G at v. One defines the probability measure $U(G) \in \mathcal{P}(\mathcal{G}^*)$ as the law of the equivalence class of the rooted graph $(G(o), o)$ where the root o is sampled uniformly on V :

$$U(G) = \frac{1}{|V|} \sum_{v \in V} \delta_{g(v)},$$

where $g(v)$ is the equivalence class of $(G(v), v)$. See Figure 2 for a concrete example. In the passage from G to $U(G)$ we have lost some information on the graph G, notably the labels of the vertices.

If $(G_n)_{n \geq 1}$, is a sequence of finite graphs, we shall say that G_n has *local weak limit* (or Benjamini-Schramm limit) $\rho \in \mathcal{P}(\mathcal{G}^*)$ if $U(G_n) \to \rho$ weakly in $\mathcal{P}(\mathcal{G}^*)$. A measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *sofic* if

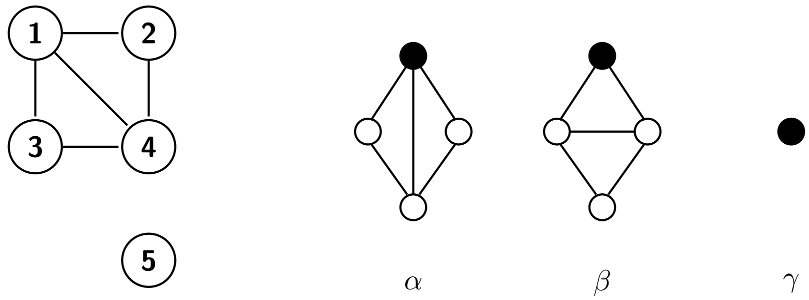

Figure 2: Example of a graph G and its empirical neighborhood distribution. Here $U(G) = \frac{1}{5}(2\delta_\alpha + 2\delta_\beta + \delta_\gamma)$, where $\alpha, \beta, \gamma \in \mathcal{G}^*$ are the unlabeled rooted graphs depicted above (the black vertex is the root), with $g(1) = g(4) = \alpha$, $g(2) = g(3) = \beta$, $g(5) = \gamma$.

there exists a sequence of finite graphs $(G_n)_{n \geq 1}$, whose local weak limit is ρ. In other words, the set of sofic measures is the closure of the set $\{U(G) : G \text{ finite}\}$. The set of sofic measures will be denoted by $\mathcal{P}_{\text{sof}}(\mathcal{G}^*)$.

We may define similarly locally finite connected graphs with two roots (G, o, o') and extend the notion of isomorphisms to such structures. We define $\mathcal{G}^{**}$ as the set of equivalence classes of graphs (G, o, o') with two roots and associate its natural local topology. A function f on $\mathcal{G}^{**}$ can be extended to a function on connected graphs with two roots (G, o, o') through the isomorphism classes. Then, a measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *unimodular* if for any measurable function $f : \mathcal{G}^{**} \to \mathbb{R}_+$, we have

$$\mathbb{E}_\rho \sum_{v \in V} f(G, o, v) = \mathbb{E}_\rho \sum_{v \in V} f(G, v, o), \tag{13}$$

where under $\mathbb{P}_\rho$, (G, o) has law ρ. It is immediate to check that if G is finite then $U(G)$ is unimodular : indeed, if u and v are in the same connected component then by definition, $G(u) = G(v)$. It follows that

$$\begin{aligned}
\mathbb{E}_{U(G)} \sum_{v \in V} f(G, o, v) &= \frac{1}{|V|} \sum_{u \in V} \sum_{v \in V(G(u))} f(G(u), u, v) \\
&= \frac{1}{|V|} \sum_{v \in V} \sum_{u \in V(G(v))} f(G(v), u, v) \\
&= \mathbb{E}_{U(G)} \sum_{v \in V} f(G, v, o).
\end{aligned}$$

We will denote by $\mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ the set of unimodular measures.

Lemma 2.1. *The set $\mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ is closed for the local weak topology.*

Proof. We follow [15]. Let $\rho_n \to \rho$ and $f : \mathcal{G}^{**} \to \mathbb{R}_+$. Let $t > 0$ and $g \in \mathcal{G}_*$ with radius from the root at most t, observe that by dominated convergence (13) holds for $f_{t,g}(G, u, v) =$

$t \wedge f(G,u,v)\mathbf{1}(d_G(u,v) \le t)\mathbf{1}((G,u)_t \simeq g)$. Then, summing over all countably many g, it holds for $f_t(G,u,v) = t \wedge f(G,u,v)\mathbf{1}(d_G(u,v) \le t)$. By monotone convergence, it also holds for f. $\square$

In particular, the above lemma implies that all sofic measures are unimodular, the converse is open, for a discussion see [4]. It is however known that all unimodular probability measures supported on rooted trees are sofic, see Elek [39], Bowen [24], and for alternative proofs [14, 19]. In this last reference, the asymptotics number of graphs G with n vertices and m edges such that $U(G)$ is close to a given $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$ is computed when ρ is supported on rooted trees.

Let $G = (\Gamma, E)$ be a Cayley graph of a discrete group Γ with generating set $S = S^{-1}$, $E = \{\{u,v\}, vu^{-1} \in S\}$. Let o be the unit of Γ. Then the counting measure on Γ, $\nu = \sum_{v \in \Gamma} \delta_v$ is unimodular in the group theoretic sense (invariant by left and right multiplication). In particular, any function $f : \Gamma \times \Gamma \to \mathbb{R}_+$ invariant by right multiplication (i.e. such that $f(u,v) = f(u\gamma, v\gamma)$ for all $\gamma \in \Gamma$) will satisfy

$$\sum_{v \in \Gamma} f(o,v) = \sum_{v \in \Gamma} f(o,v^{-1}) = \sum_{v \in \Gamma} f(v,o).$$

It implies that if we define the measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ which puts a Dirac mass at the equivalence class of (G,o), then ρ is unimodular.

With a slight abuse of language, we shall say that a random rooted graph (G,o) is unimodular if the law of its equivalence class in $\mathcal{G}^*$ is unimodular.

2.1 Extension to weighted graphs

A weighted graph (G,ω) is a graph $G = (V,E)$ equipped with a weight function $\omega : V^2 \to \mathbb{Z}$ such that $\omega(u,v) = 0$ if $u \neq v$ and $\{u,v\} \notin E$. The weight function is edge-symmetric if $\omega(u,v) = \omega(v,u)$ and $\omega(u,u) = 0$. Note that, for edge-symmetric weight functions, the set of edges such that $\omega(e) = k$ spans a subgraph of G. It is straightforward to extend the local weak topology to weighted graphs. The definition of unimodularity carries over naturally to the weighted graphs (see the definition of unimodular network in [4]).

2.2 Examples of unimodular graphs and local weak limits

Finite window approximation of a lattice : consider an integer $d \ge 1$, the graph of $\mathbb{Z}^d$ and $L_n = \mathbb{Z}^d \cap [1,n]^d$. Then, the local weak limit of L_n is the Dirac mass of the equivalence class of $(\mathbb{Z}^d, o)$. Indeed, if t is an integer $(L_n, v)_t \simeq (\mathbb{Z}^d, o)_t$ for all $v \in V(L_n)$ which are distance at least t from $\mathbb{Z}^d \backslash [1,n]^d$. It follows that $(L_n,v)_t \simeq (\mathbb{Z}^d,o)_t$ for all but $O(tn^{d-1}) = o(|V(L_n)|)$ vertices.

The same argument will work for any amenable graph. As an exercise, what is the local weak limit of a complete binary tree T_n of height n ?

Percolation on a lattice : Consider an integer $d \ge 1$ and the usual bond percolation on the graph of $\mathbb{Z}^d$ where each edge is kept with probability $p \in [0,1]$, we obtain a random subgraph G of $\mathbb{Z}^d$.

Then, a.s. the local weak limit of $G_n = G \cap [1,n]^d$ is $\mathrm{perc}(\mathbb{Z}^d, p)$, the law of the equivalence class of $(G(o), o)$.

Unimodular Galton-Watson trees : Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with positive and finite mean. The unimodular Galton-Watson tree with degree distribution P (commonly known as size-biased Galton-Watson tree) is the law of the random rooted tree obtained as follows. The root has a number d of children sampled according to P, and, given d, the subtrees of the children of the root are independent Galton-Watson trees with offspring distribution

$$\widehat{P}(k) = \frac{(k+1)P(k+1)}{\sum_\ell \ell P(\ell)}. \tag{14}$$

These unimodular trees appear naturally as a.s. local weak limits of uniform random graphs with a given degree distribution, see e.g. [38, 33, 17]. It is also well known that the Erdős-Rényi $G(n, c/n)$ has a.s. local weak limit the Galton-Watson tree with offspring distribution $\mathrm{Poi}(c)$. Note that if P is $\mathrm{Poi}(c)$ then $\widehat{P} = P$. The percolation on the hypercube $\{0,1\}^n$ with parameter c/n has the same a.s. local weak limit.

Skeleton tree : The infinite skeleton tree which consists of a semi-infinite line $\mathbb{Z}_+$ rooted at 0 with i.i.d. critical Poisson Galton-Watson trees $\mathrm{Poi}(1)$ attached to each of the vertices of $\mathbb{Z}_+$. It is the a.s. local weak limit of the uniformly sampled spanning tree on n labeled vertices.

2.3 Spectral measure

Remark that if two rooted graphs (G_1, o_1) and (G_2, o_2) are isomorphic then the spectral measures $\mu_{G_i}^{e_{o_i}}$, $i = 1, 2$ are equal. It thus makes sense to define $\mu_G^{e_o}$ for elements $(G, o) \in \mathcal{G}^*$. Then, if $\rho \in \mathcal{P}(\mathcal{G}^*)$ is supported on graphs with bounded degrees, we may consider the expected spectral measure at the root vector :

$$\mu_\rho = \mathbb{E}_\rho \mu_G^{e_o}. \tag{15}$$

In particular, if $|V| = n$ is finite, we find from (1)

$$\mu_{U(G)} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}.$$

It is consistent with our previous definition (8). Similarly, if G is a Cayley graph and $\rho = \delta_{(G,o)}$ we find $\mu_\rho = \mu_G$ which is consistent with (11).

It is not clear a priori how to extend this construction to random graphs without bounded degrees. It can be difficult to check that adjacency operators are essentially self-adjoint. It turns out however that for unimodular measures, A is always ρ-a.s. essentially self-adjoint and $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$ is thus well-defined without any bounded degree assumption.

Proposition 2.2. *For any $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$,*

(i) the adjacency operator A is ρ-a.s. essentially self adjoint,

(ii) if $\rho_n \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ and $\rho_n \to \rho$, then μ_{ρ_n} converges weakly to μ_ρ.

In particular, if a sequence of finite graphs $(G_n)_{n\geq 1}$ has local weak limit ρ then the empirical distribution of the eigenvalues of their adjacency matrices converges weakly to μ_ρ. In the next subsection, we will reinforce this convergence. Restricted to sofic measures, the proof of this proposition is contained in [22], [21]. To bypass this limitation, we introduce some concepts of operator algebras. The idea being that to any unimodular measure we can associate a Von Neumann algebra which is analog to the group algebra considered above.

Consider a Von Neumann algebra $\mathcal{M}$ of bounded linear operators on a Hilbert space H with a normalized trace τ. If $A \in \mathcal{M}$ is self-adjoint, we denote by ν_A its spectral measure, i.e. the probability measure such that

$$\tau(A^k) = \int x^k d\nu_A(x).$$

The rank of A is defined as

$$\text{rank}(A) = 1 - \nu_A(\{0\}).$$

This is the natural notion of rank. Indeed, consider a closed vector space S of H such that, P_S, the orthogonal projection to S, is an element of $\mathcal{M}$. The von Neumann dimension of such vector space S is

$$\dim(S) := \tau(P_S) = \text{rank}(P_S). \tag{16}$$

We refer for example to Kadison and Ringrose [52].

There is a natural Von Neumann algebra associated to unimodular measures. More precisely, let $\mathcal{G}^*$ denote the set of equivalence classes of locally finite connected (possibly weighted) graphs endowed with the local weak topology. There is a canonical way to represent an element $(G, o) \in \mathcal{G}^*$ as a rooted graph on the vertex set $V(G) = \{o, 1, 2, \cdots, N\}$ with $N \in \mathbb{N} \cup \{\infty\}$, see Aldous and Lyons [4]. We set $V = \{o, 1, 2, \cdots\}$, $H = \ell^2(V)$ and define $\mathcal{B}(H)$ as the set of bounded linear operators on H. Then, for a fixed $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$, we consider the Hilbert space $\mathcal{H}$ of ρ-measurable functions $\psi : \mathcal{G}^* \to H$, such that $\mathbb{E}_\rho \|\psi\|_2^2 < \infty$ with inner product $\mathbb{E}_\rho \langle \psi, \varphi \rangle$. Let us denote by $L^\infty(\mathcal{G}^*, \mathcal{B}(H), \rho)$ the ρ-measurable maps $B : \mathcal{G}^* \to \mathcal{B}(H)$ with $\|B\| \in L^\infty(\mathcal{G}^*, \rho)$.

For any bijection $\sigma : V \to V$, we consider the orthogonal operator λ_σ defined for all $v \in V$, $\lambda_\sigma(e_v) = e_{\sigma(u)}$. We introduce the algebra $\mathcal{M}$ of operators in $L^\infty(\mathcal{G}^*, \mathcal{B}(H), \rho)$ which commutes with the operators λ_σ, i.e. for any bijection σ, ρ-a.s. $B(G, o) = \lambda_\sigma^{-1} B(\sigma(G), o) \lambda_\sigma$. In particular, $B(G, o)$ does not depend on the root. It can be checked that $\mathcal{M}$ is a von Neumann algebra of operators on the Hilbert space $\mathcal{H}$ (see [4, §5] and Lyons [64] for details). Moreover, the linear map $\mathcal{M} \to \mathbb{C}$ defined by

$$\tau(B) = \mathbb{E}_\rho \langle e_o, B e_o \rangle,$$

where $B = B(G,o) \in \mathcal{M}$ and under, $\mathbb{E}_\rho$, (G,o) has distribution ρ, is a normalized faithful trace. Observe finally that any $(G,o) = (V(G), E, o) \in \mathcal{G}^*$ can be extended to a graph on V (all vertices in $V \backslash V(G)$ are isolated). Then, the adjacency operator $A : (G,o) \mapsto A(G)$ defines a densely defined operator affiliated to $\mathcal{M}$ (see again [64] for details). We may now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. Statement (i) is a consequence of Nelson [68]. First, since $A : (G,o) \mapsto A(G)$ is affiliated to $\mathcal{M}$, from [51, Remark 5.6.3], $\bar{A}$, the closure of A, is also affiliated to $\mathcal{M}$. Moreover, from [68, Theorem 1], A^* is affiliated to $\mathcal{M}$ (see discussion below [73, Theorem 2.2]). To prove statement (i) it is sufficient to check that $\bar{A} = A^*$ (indeed, denoting by $\mathcal{R}$ the range of an operator, if $\mathcal{R}(\bar{A} + iI) = \mathcal{H}$ then ρ-a.s. $\mathcal{R}(A(G) + iI) = H$). Now, we introduce $V_n(G) = \{v \in V : \deg_G(v) \leq n \text{ and for all } \{u,v\} \in E(G), \deg_G(u) \leq n\}$ and let $P_n \in \mathcal{M}$ be the projection onto $\mathcal{H}_n = \{\psi \in \mathcal{H} : \rho\text{-a.s. } \mathrm{supp}(\psi(G,o)) \subset V_n(G)\}$. Observe that for $\psi \in \mathcal{H}_n$, then ρ-a.s.

$$\|AP_n\psi\|^2 = \sum_{v \in V} \mathbf{1}(\deg(v) \leq n) \left(\sum_{u:\{u,v\} \in E} \psi(u) \right)^2 \leq n \sum_{v \in V} \sum_{u:\{u,v\} \in E} \psi^2(u) \leq n^2 \|\psi\|^2.$$

Hence, AP_n is bounded and it follows that $\mathcal{H}_n$ is both in the domain of $\bar{A}$ and A^*. We deduce that $\bar{A}$ and A^* coincide on $\mathcal{H}_n$. Moreover, since ρ is a probability measure on locally finite graphs,

$$\mathbb{P}_\rho \left(\deg(o) > n \text{ or } \exists v : \{v,o\} \in E, \deg(v) > n \right) = \varepsilon(n) \to 0. \tag{17}$$

Finally, since for any $B \in \mathcal{M}$, $\dim(\ker(B)) \geq \mathbb{P}_\rho(e_o \in \ker(B(G,o)))$, we deduce from (16) that $\dim(\mathcal{H}_n) \geq \mathbb{P}_\rho(e_o \in V_n(G)) \geq 1 - \varepsilon(n)$. From [68, Theorem 3], $\bar{A}$ and A^* are equal.

Let us prove statement (ii). Consider a sequence (ρ_n) converging to ρ in the local weak topology. From the Skorokhod's representation theorem one can define a common probability space such that the rooted graphs (G_n, o) converge for the local topology to (G,o) where (G_n,o) has distribution ρ_n and (G,o) has distribution ρ. Then, the following two facts hold true: (a) for any compactly supported $\psi \in \ell^2(V)$, for n large enough, $A_n\psi = A\psi$, where A_n and A are the adjacency operators of G_n and G. And, (b) if $\mathbb{P}$ denotes the probability measure of the joint laws of (G_n,o) and (G,o), from statement (i), $\mathbb{P}$-a.s. A and A_n are essentially self-adjoint with common core, the compactly supported $\psi \in \ell^2(V)$. These last two facts imply the strong resolvent convergence, see e.g. [70, Theorem VIII.25(a)]. From (4), the Cauchy-Stieltjes transform of $\mu_A^{e_o}$ is a diagonal coefficient of the resolvent,

$$\langle e_o, (A - zI)^{-1} e_o \rangle = \int \frac{d\mu_A^{e_o}}{x - z}.$$

It implies that ρ-a.s. $\mu_{A_n}^{e_o}$ converges weakly to $\mu_A^{e_o}$ (recall that the pointwise convergence of Cauchy-Stieltjes transform on $\mathbb{C}_+$ is equivalent to weak convergence). Taking expectation, we get $\mu_{\rho_n} = \mathbb{E}_\rho \mu_{A_n}^{e_o}$ converges weakly to $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$. $\square$

We conclude this paragraph with a perturbation inequality on the average spectral measures. Recall that the Kolmogorov-Smirnov distance between two probability measures on $\mathbb{R}$ is the L^∞

norm of their cumulative distribution functions :

$$d_{KS}(\mu,\nu)=\sup_{t\in\mathbb{R}}|\mu(-\infty,t]-\nu(-\infty,t]|.$$

We have that $d_{KS}(\mu,\nu)\geq d_L(\mu,\nu)$ where d_L is the Lévy distance,

$$d_L(\mu,\nu)=\inf\{\varepsilon>0:\forall t\in\mathbb{R},\mu(-\infty,t-\varepsilon]-\varepsilon\leq\nu(-\infty,t]\leq\mu(-\infty,t+\varepsilon]+\varepsilon\}$$

The following simple lemma is the operator algebra analog of a well known rank inequality for matrices (see e.g. Bai and Silverstein [10, Theorem A.43]).

Lemma 2.3. *If $A,B\in\mathcal{M}$ are self-adjoint,*

$$d_{KS}(\nu_A,\nu_B)\leq\operatorname{rank}(A-B).$$

Proof. We should prove that for any $J=(-\infty,t]$ we have $|\nu_A(J)-\nu_B(J)|\leq\operatorname{rank}(A-B)$. There is a convenient variational expression for $\nu_A(J)$:

$$\nu_A(J)=\max\{\tau(P):PAP\leq tP,P\in\mathcal{P}\},\tag{18}$$

where $\mathcal{P}\subset\mathcal{M}$ is the set of projection operators ($P=P^*=P^2$) and $S\leq T$ means that $T-S$ is a non-negative operator. This maximum is reached for P equal to the spectral projection on the interval J, (see e.g. Bercovici and Voiculescu [16, Lemma 3.2]).

Now let $Q\in\mathcal{P}$ such that $\nu_B(J)=\tau(Q)$ and $QBQ\leq tQ$. We denote H the range of Q and we consider the projection operator R on $H\cap\ker(A-B)$. Observe that $RAR=RBR\leq tR$. In particular, from (18), we get

$$\tau(R)=\dim(H(Q)\cap\ker(A-B))\leq\nu_A(J).\tag{19}$$

Then, the formula for closed linear subspaces, U,V,

$$\dim(U+V)+\dim(U\cap V)=\dim(U)+\dim(V),$$

(see [50, exercice 8.7.31]) yields

$$\begin{aligned}\dim(H\cap\ker(A-B)) &\geq \dim(H)+\dim(\ker(A-B))-1\\ &\geq \dim(H)-\operatorname{rank}(A-B).\end{aligned}$$

By definition $\dim(H)=\nu_B(J)$ and Equation (19) imply that

$$\nu_B(I)-\operatorname{rank}(A-B)\leq\nu_A(I).$$

Reversing the role of A and B allows to conclude. □

For integer $n\geq 1$, if $G=(V,E)$ is locally finite, denote by $G_n=(V,E_n)$ the subgraph spanned by edges adjacent to vertices of degree at most n : $E_n=\{\{u,v\}\in E:\deg(u)\vee\deg(v)\leq n\}$. If $\rho\in\mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$, let ρ_n be the law of $(G_n(o),o)$ where (G,o) has distribution ρ and $G_n(o)$ is the connected component of o in G_n. It is easy to check that $\rho_n\in\mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$. By construction, the operator $A(G_n)$ has norm at most n. The next corollary will be useful.

Corollary 2.4. *If $\rho, \rho_n \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ are as above,*

$$d_{KS}(\mu_\rho, \mu_{\rho_n}) \le \mathbb{P}_\rho(\deg(o) > n \text{ or } \exists v : v, o \in E, \deg(v) > n).$$

Proof. Consider the von Neumann algebra $\mathcal{M}$ associated to ρ. We define $A_n : \mathcal{G}^* \mapsto A_n(G)$ as the adjacency operator spanned by edges adjacent to vertices of degree at most n. Since $\|A_n(G)\| \le n$, we have $A_n \in \mathcal{M}$ and $\nu_{An} = \mu_{\rho_n}$. Now, with $\varepsilon(n)$ as in (17), we deduce that, for any $n, m \in \mathbb{N}$,

$$\operatorname{rank}(A_n - A_{n+m}) \le 1 - \mathbb{P}_\rho(A_n e_o = A_{n+m} e_o) \le \varepsilon(n).$$

Using Lemma 2.3, we find that μ_{ρ_n} is a Cauchy sequence for the Kolmogorov-Smirnov distance. The space $(\mathcal{P}(\mathbb{R}), d_{KS})$ is a complete metric space. It follows that μ_{ρ_n} converges weakly to some probability measure denoted by μ and $d_{KS}(\mu, \mu_{\rho_n}) \le \varepsilon(n)$. However, from Proposition 2.2(ii), as ρ_n converges weakly to ρ, $\mu = \mu_\rho$. ☐

2.4 Pointwise continuity of the spectral measure

In this last case, if moreover for some $\theta > 0$ and for all $v \in V(G_n)$, $\deg_{G_n}(v) \le \theta$, then using Lück's approximation, the convergence can even be reinforced to the pointwise convergence of all atoms. The next result is proved in Àbert, Thom and Viràg [1], see also [63, 75] for nearly equivalent statements.

Theorem 2.5. *Let $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$. If $(G_n)_{n\ge1}$ is a sequence of finite graphs such that $U(G_n) \to \rho$ then*

$$\lim_{n\to\infty} d_{KS}(\mu_\rho, \mu_{G_n}) = 0.$$

Consequently for any $\lambda \in \mathbb{R}$, $\mu_{G_n}(\{\lambda\}) \to \mu_\rho(\{\lambda\})$.

Notice that Proposition 2.2 and the Portemanteau theorem implies that for any λ, $\limsup_n \mu_{G_n}(\lambda) \le \mu_\rho(\lambda)$, $\limsup_n \mu_{G_n}(-\infty, \lambda] \le \mu_\rho(-\infty, \lambda]$ and $\liminf_n \mu_{G_n}(-\infty, \lambda) \ge \mu_\rho(-\infty, \lambda)$. Hence the convergence for the Kolmogorov-Smirnov distance is equivalent to weak convergence together with convergence of all atoms. Since $\mu_G(\lambda) = 0$ for all finite graphs and all non-algebraic integers, a striking consequence of Theorem 2.5 is the next result (first proved in the context of sofic groups by Thom [75]).

Corollary 2.6. *If $\rho \in \mathcal{P}(\mathcal{G}^*)$ is sofic then all atoms of μ_ρ are algebraic integers.*

Even for Cayley graphs, it is an open problem to prove whether the statement of Corollary 2.6 holds for all $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$. A negative answer would disprove the conjecture that all unimodular graphs are sofic.

We now turn to the proof of Theorem 2.5. We will follow the proof of [1]. It is essentially a consequence of the next result :

Proposition 2.7. *Let $\lambda \in \mathbb{R}$, $\theta, \varepsilon \in \mathbb{R}_+$. There exists a continuous function $\delta : \mathbb{R} \to [0,1]$ with $\delta(0) = 0$ depending on λ and θ such that, for any finite graph G where all degrees are bounded θ, we have*

$$\mu_G(\lambda) \leq \mu_G((\lambda - \varepsilon, \lambda + \varepsilon)) \leq \mu_G(\lambda) + \delta(\varepsilon).$$

The strength of Proposition 2.7 is a uniform control with respect to size of the graph on the mass around a small interval. It will be a consequence of the repulsion of the distinct eigenvalues coming from the fact that the adjacency matrix has integer coefficients.

Proof of Theorem 2.5. From Corollary 2.4, we may restrict to the case where G_n and G have degrees bounded by θ for some $\theta > 0$. Also, as already pointed, it is enough to prove that

$$\liminf \mu_{G_n}(\lambda) \geq \mu_\rho(\lambda).$$

From Portemanteau theorem, for any $\varepsilon > 0$, we have

$$\liminf \mu_{G_n}(\lambda - \varepsilon/2, \lambda + \varepsilon/2) \geq \mu_\rho(\lambda - \varepsilon/2, \lambda + \varepsilon/2) \geq \mu_\rho(\lambda).$$

We get from Proposition 2.7 that

$$\mu_\rho(\lambda) \leq \liminf \mu_{G_n}(\lambda) + \delta(\varepsilon).$$

It remains to take $\varepsilon \to 0$. □

Proof of Proposition 2.7. We start with two simple remarks. The set $\mathcal{A}(k,\theta)$ of algebraic integers of degree at most k such that all roots of its minimal polynomial (the Galois conjugates) have modulus at most θ is finite. Indeed, the coefficients of the minimal polynomial are integers with absolute value bounded by $|\theta|^\ell \binom{k}{\ell}$ for some $1 \leq \ell \leq k$. Also, if $x \in \mathbb{R}$ is an algebraic integer of degree k then

$$\mu_G(x) \leq \frac{1}{k}.$$

(if x is an eigenvalue of A then all its Galois conjugates are also eigenvalues of A with the same multiplicity).

Now, we set $n = |V(G)|$ and we denote by $\lambda_1, \cdots, \lambda_n$ the eigenvalues of the adjacency matrix A of G. They are the roots of the characteristic polynomial $P \in \mathbb{Z}[x]$ of A. Moreover for all i, $|\lambda_i| \leq \theta$. From what precedes, we deduce that for all there exists an integer $k(\varepsilon)$ such that $k(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the open set $I = (\lambda - \varepsilon, \lambda + \varepsilon) \backslash \{\lambda\}$ does not intersect $\mathcal{A}(k(\varepsilon), \theta)$. Consequently, for any $x \in I$, $\mu_G(x) \leq 1/k(\varepsilon)$. We introduce the scalars

$$\alpha = n^{-2}|\{(i,j) : \lambda_i = \lambda_j, \lambda_i \in I\}|$$
$$\beta = n^{-2}|\{(i,j) : \lambda_i \neq \lambda_j, \lambda_i \in I, \lambda_j \in I\}|.$$

From what precedes,

$$\alpha = \sum_{x\in\sigma(A)\cap I} \mu(x)^2 \leq \frac{1}{k} \sum_{x\in\sigma(A)} \mu(x) = \frac{1}{k(\varepsilon)}.$$

Hence,

$$\beta = \mu(I)^2 - \alpha \geq \mu(I)^2 - \frac{1}{k(\varepsilon)}. \tag{20}$$

We introduce

$$D = D(\lambda_1, \cdots, \lambda_n) = \prod_{(i,j):\lambda_i\neq\lambda_j} (\lambda_i - \lambda_j).$$

Observe that D is invariant by permutation, hence it can be written in terms of the elementary symmetric polynomials. Since $P \in \mathbb{Z}[x]$, we get that $D \in \mathbb{Z}$. In particular $|D| \geq 1$ and we find

$$1 \leq (2\varepsilon)^{n^2\beta}(2\theta)^{n^2},$$

the above inequality is the key relation which allows to quantify the repulsion of the distinct eigenvalues. Taking logarithm, we find

$$0 \leq \beta \log(2\varepsilon) + \log(2\theta).$$

Using (20) yields to, for any $0 < \varepsilon < 1/2$,

$$\mu(I)^2 \leq \frac{1}{k(\varepsilon)} + \frac{\log(2\theta)}{|\log(2\varepsilon)|}.$$

The conclusion follows. □

3 Atoms and eigenvectors

In this section, we will give criteria for existence of a continuous part in μ_ρ where $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$. To motivate the sequel, let us give some comments on the atomic part of μ_ρ.

3.1 Finite graphs

First, it is important to keep in mind that atoms are related to eigenspaces : if A is ρ-a.s. a bounded operator, then the spectral resolution of A gives that (see (2))

$$\mu_\rho(\lambda) = \mathbb{E}\mu_G^{e_o}(\lambda) = \mathbb{E}\langle e_o, E_{\{\lambda\}}e_o\rangle = \dim(S_{\{\lambda\}}), \tag{21}$$

where $S_{\{\lambda\}}$ is the vector space spanned by vectors $\psi \in \ell^2(V)$ such that $A\psi = \lambda\psi$ and $\dim(\cdot)$ is the von Neumann dimension defined by (16).

Also, for any Borel $B \subset \mathbb{R}$, we apply unimodularity to the function $f(G, u, v) = \mu_G^{e_u}(B)/|V|$ if G is finite and f equal to 0 otherwise. We find

$$\mathbb{E}_\rho\mu_G^{e_o}(B)\mathbf{1}_{|V|<\infty} = \mathbb{E}_\rho\frac{1}{|V|}\sum_{v\in V}\mu_G^{e_v}(B)\mathbf{1}_{|V|<\infty} = \mathbb{E}_\rho\mu_G(B)\mathbf{1}_{|V|<\infty}, \tag{22}$$

where we have used (9). It follows that

$$\mu_\rho \succeq \mathbb{E}_\rho \mu_G \mathbf{1}_{|V|<\infty}, \tag{23}$$

where $\mu \preceq \nu$ means that that $\mu(B) \leq \nu(B)$ for any Borel B. A countable sum of atomic measures is atomic. We deduce the following simple lemma.

Lemma 3.1. *If $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$ is supported on finite graphs then μ_ρ is purely atomic.*

We denote by $\mathcal{T}^* \subset \mathcal{G}^*$, the set of unlabeled rooted trees and by $\mathcal{T}^*_f \subset \mathcal{T}^*$ the subset of finite trees. Salez [72] has proved any algebraic integer λ is an eigenvalue of the adjacency matrix of a finite tree. Hence, a corollary of his result, (23) and Theorem 2.5 is

Lemma 3.2. *Let $\rho = \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$ whose support contains $\mathcal{T}^*_f$. If $\lambda \in \mathbb{R}$ is an algebraic integer then $\mu_\rho(\lambda) > 0$ otherwise $\mu_\rho(\lambda) = 0$. In particular, the pure point part of μ_ρ is dense in $\mathbb{R}$.*

For example, let us consider the case where $\rho = \mathrm{UGW}(\mathrm{Poi}(c))$ is the distribution of a Poisson Galton-Watson tree with mean offspring $c > 0$. Recall that if $0 < c \leq 1$ then ρ-a.s. T is finite. We deduce from Lemma 3.1 that if $0 < c \leq 1$, μ_ρ is purely atomic. Moreover, for any $c > 0$, the support of ρ contains $\mathcal{T}^*_f$ and we may apply lemma 3.2 to $\mathrm{UGW}(\mathrm{Poi}(c))$ for any $c > 0$.

3.2 Finite pending subgraphs

The atomic part of μ_ρ does not only come from finite graphs. It may also come from the existence of finite subgraphs. If $g, g' \in \mathcal{G}^*$, we denote by $g \cup g'$ the rooted graph obtained by identifying the two roots of g and g' and taking the disjoint union of the edge and vertex sets. We write that $g \subset g'$ if there exists $\gamma \in \mathcal{G}^*$ such that $g \cup \gamma = g'$. We also define g_+ as the graph obtained by adding a new neighboring vertex to the root and defining the new root as being this new vertex. The next result generalizes Lemma 3.2.

Lemma 3.3. *Let $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$ such that for any $\tau \in \mathcal{T}^*_f$, $\mathbb{P}_\rho(\tau \subset (G, o)) > 0$. If $\lambda \in \mathbb{R}$ is an algebraic integer then $\mu_\rho(\lambda) > 0$ otherwise $\mu_\rho(\lambda) = 0$. In particular, the pure point part of μ_ρ is dense in $\mathbb{R}$.*

As an application, take $\rho = \mathrm{UGW}(\mathrm{Poi}(c))$ and $c > 1$. Then, T is infinite with positive probability, $p = 1 - e^{-cp}$. The measure ρ_∞ (resp. ρ_f) is defined as the law of (T, o) conditioned on T infinite (resp. finite) is a unimodular measure. We have $\mu_\rho = p\mu_{\rho_\infty} + (1-p)\mu_{\rho_f}$. It not hard to check that the assumption of Lemma 3.3 holds for ρ_∞. We deduce that the total mass of atoms in μ_ρ is larger than $1 - p$ and it does not come solely from the contribution of finite trees.

Proof of Lemma 3.3. We are going to build finitely supported eigenvectors. Let $\lambda \in \mathbb{R}$ be an algebraic integer. From [72] there exists a finite tree $t \in \mathcal{T}^*_f$ such that $A(t)$ has eigenvalue λ with associated eigenvector ψ, $\|\psi\|_2 = 1$. We may also require that t isλ- irreducible in the sense that that λ is simple and for all $v \in V(t)$, $\psi(v) \neq 0$. Indeed, if it is the case that $\psi(v) = 0$, the two trees obtained by removing the vertex v have also eigenvalue λ (with eigenvector the restriction of ψ to their vertex set). For any $\gamma \in \mathcal{G}^*$, we consider $g(\gamma) = t \cup (\gamma \cup t_+)_+$, see Figure 3.

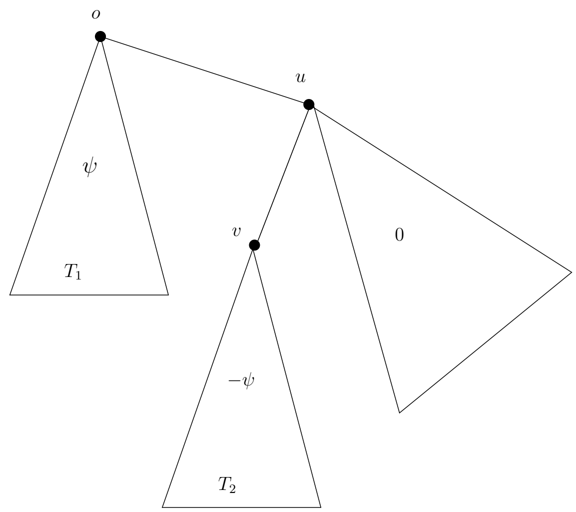

Figure 3: Construction of the eigenvector

As in Corollary 2.4, we introduce the truncated version ρ_n of ρ. By assumption on an event with probability (under ρ) at least p, there exists γ such that $(G, o) \simeq g(\gamma)$. If the truncation of the degrees, n, is high enough, this will also holds under ρ_n with the same probability. We write, with obvious notation, $(G, o) = (T_1, o) \cup (G_2, o)$, with $(T_1, o) \simeq t$ and $(G_2, o) \simeq (\gamma \cup t_+)_+$ (see Figure 3). Let us call u the neighbor of o with subgraph $(G_2, u) \simeq (\gamma \cup t_+)$ and v the neighboring vertex of u in $V(G_2)$ with subgraph $(T_2, v) \simeq t$. We consider the vector φ equal to 0 on $V(G_2 \backslash T_2)$, equal (up to isomorphism) to $-\psi/\sqrt{2}$ on T_2 and $\psi/\sqrt{2}$ on T_1. If A is the adjacency operator of G, we find that $A\varphi = \lambda\varphi$, indeed, we have $A\varphi(u) = \varphi(o) + \varphi(v) = 0 = \lambda\varphi(u)$ and all other vertices satisfy the eigenvalue equation. Moreover, by hypothesis $\varphi(o) = \psi(o) \neq 0$.

With the notation of (21), $\varphi \in E_{\{\lambda\}}$ and it yields to

$$\mu_G^{e_o}(\lambda) \geq \langle \varphi, e_o \rangle^2 = \psi(o)^2/2.$$

Finally, taking expectation, under ρ_n, we get that $\mu_{\rho_n}(\lambda) > p\psi(o)^2/2$. Letting n go to infinity, the conclusion follows from Corollary 2.4. □

At least for finite graphs, it is possible to extend the idea of the proof of Lemma 3.3 to compute the mass of an atom. Let us consider an algebraic integer λ and a finite rooted graph L which is λ-irreducible, in the sense that λ is a simple eigenvalue of the adjacency matrix of L and its eigenvector has no zero entries.

Lemma 3.4. *Let $\lambda \in \mathbb{R}$ and L be a λ-irreducible rooted graph. Assume that G is a finite graph* *and $o \in V(G)$ is a vertex such that $L_+ \subset (G, o)$ then*

$$\dim \ker(A(G) - \lambda) = \dim \ker(A(G \backslash L_+) - \lambda).$$

Proof. Denote by u the root of L and $G' = G \backslash L_+$. We have that $Ae_o = e_u + \alpha$ where $\alpha \in \ell^2(V(G'))$. Let $\varphi \in \ker(A(G) - \lambda)$. We denote by φ_L and $\varphi_{G'}$ the restrictions of φ to L and G'. We find $A(L)\varphi_L = \lambda\varphi_L + \varphi(o)e_o$. In particular $\varphi(o)e_o \in \mathrm{im}(A(L) - \lambda) = \ker(A(L) - \lambda)^\perp$. By assumption, there is a unique vector ψ in $\ker(A(L) - \lambda)$ such that $\psi(u) = 1$. We get that $\varphi(o) = 0$, $\varphi_{G'} \in \ker(A(G') - \lambda)$ and $\varphi_L = c\psi$ for some $c \in \mathbb{R}$. We also have $0 = (A(G)\varphi)(o) = c + \alpha^* \varphi_{G'}$. Hence, c is uniquely determined by $\varphi_{G'}$ and there is an isomorphism between $\ker(A(G) - \lambda)$ and $\ker(A(G') - \lambda)$. □

For $\lambda = 0$, observe that a single vertex graph is 0-irreducible. Hence, the above lemma gives an algorithm, the *leaf removal algorithm*, to compute recursively the rank of the adjacency matrix of a finite tree. Interestingly, it also the size of the maximal number of vertices covered by a finite tree. In this context, it was notably studied by Bauer and Golinelli [12, 11].

3.3 Computation of the atom at 0

We have seen so far two ways to generate some masses at a given $\lambda \in \mathbb{R}$: either by a finite graph or by a finite pending subgraph (both are associated with finitely supported eigenvectors). It may not cover all cases, as Theorem 1.1 illustrates for the lamplighter group. Also, for $\lambda = 0$, it is a consequence of [22] that for any $d \geq 0$, there exist D and $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{T}^*)$ such that ρ-a.s. all degrees of vertices are in $[d, D]$ and $\mu_\rho(0) > 0$. With $d \geq 2$, it implies that unimodular trees without any finite pending subtrees can have a spectral measure with a pure point part.

The exact value of $\mu_\rho(0)$ can also be computed in non-trivial examples. The main result of [22] is the following.

Theorem 3.5. *Let ρ be the distribution of a UGW tree T whose degree distribution π has a finite second moment, and let ϕ be the generating function of π. Then, $A(T)$ is ρ-a.s. essentially self adjoint and*

$$\mu_\rho(\{0\}) = \max_{x \in [0,1]} M(x),$$

where

$$M(x) = \phi'(1)x\overline{x} + \phi(1 - x) + \phi(1 - \overline{x}) - 1, \qquad \textit{with} \qquad \overline{x} = \phi'(1 - x)/\phi'(1).$$

3.4 Quantum Percolation

For simplicity, the above discussion was focused on unimodular trees. We may also study $\rho = \mathrm{perc}(\mathbb{Z}^d, p)$, the law of the connected component of the origin in bond percolation in $\mathbb{Z}^d$ where each edge is present independently with probability $0 < p < 1$. In this case, the measure μ_ρ has support $[-2d, 2d]$. Lemma 3.1 implies that for $0 < p < p_c(d)$, μ_ρ is purely atomic. For $p_c(d) < p < 1$,

μ_ρ has a dense pure point part, even when we condition the law of ρ on the event that connected component of the origin is infinite. This fact was first observed by Kirkpatrick and Eggarter [58] and Chayes et al. [27].

Physicists are mainly interested in eigenvectors and existence of continuous spectrum. Shortly after the seminal work of Anderson [7], the study of random Hamiltonians generated by percolation on the Euclidean lattice was initiated in [31, 32] under the name of *quantum percolation*. For clarity, let us define the Anderson model. Consider a transitive graph $G = (V, E)$ on a countable vertex set V (for example G is the graph of $\mathbb{Z}^d$ or G is the infinite d-regular tree). The Anderson tight-binding model on G is formally defined by the operator on $\ell^2(V)$,

$$H = A + \lambda V;$$

where A is the adjacency operator of G, $\lambda > 0$ is the strength of disorder and V is a diagonal operator with, for $x \in V$, $Vf(x) = V_x f(x)$ and $(V_x)_{x\in V}$ independent and identically distributed real random variables. If $G = \mathbb{T}_d$ is the infinite d-regular tree, then there is a intrinsic equation satisfied by the law of diagonal terms of the resolvent of H. This is the starting point of all rigorous statements on the fact that B has absolutely continuous spectrum at small disorder λ (under mild assumptions on the law of V, see [59, 54, 2]). Proving the existence of Anderson delocalization for random Schrödinger operators on the Euclidean lattice at small noise remains the main open challenge in the area. On the other end, if λ is large enough, H has purely atomic spectrum, this phenomenon is called Anderson localization, we refer to [36] for a recent account on the topic.

Quantum percolation is even harder to study, the randomness is now on the graph geometry itself. One of the issue of quantum percolation models is that the lack of regularity of percolation graphs does not allow to use Wegner estimates, that is the regularity of the density of states is already difficult to study. In the simplified setting of random trees, robust criteria for the existence of continuous spectrum are still to be found. The only know results are on small perturbations of infinite regular trees [53, 18]. They are based on ideas first developed for the study of random Schrödinger operators on the infinite regular tree [59, 44, 2, 54]. In [53, 18] it is notably shown that Galton-Watson trees whose offspring distribution is sufficiently close to a Dirac mass at $d \geq 2$ have an absolutely continuous spectrum (it applies for example to percolation on the infinite d-regular tree $\mathbb{T}_d$). In section 5 we will come back to the study of the eigenvectors of finite graphs.

4 Existence of continuous spectral measure

This section is based on a joint work with Sen and Viràg [23].

4.1 A few answers and many questions

Percolation on $\mathbb{Z}^2$ As above, we consider an integer $d \geq 2$ and the edge percolation on $\mathbb{Z}^d$ where each edge of the graph of $\mathbb{Z}^d$ is removed independently with probability $1 - p \in [0, 1]$. Let

$\mathrm{perc}(\mathbb{Z}^d, p)$ is the law of (G, o), the connected component containing the origin rooted at the origin. As already pointed for $p < p_c$, $\mu_{\mathrm{perc}(\mathbb{Z}^d,p)}$ is purely atomic, and for $p = 1$, $\mathrm{perc}(\mathbb{Z}^d, 1)$ is simply $\mathbb{Z}^d$ rooted at the origin and its spectral measure is absolutely continuous (it is the convolution of d arcsine distributions).

Theorem 4.1. *Assume $d = 2$ and let $\rho = \mathrm{perc}(\mathbb{Z}^2, p)$. For any $p > p_c = 1/2$, μ_ρ has a non-trivial continuous part.*

Unimodular trees A weighted graph (G, ω) is a graph $G = (V, E)$ equipped with a weight function $\omega : V^2 \to \mathbb{Z}$ such that $\omega(u, v) = 0$ if $u \neq v$ and $\{u, v\} \notin E$. The weight function is edge-symmetric if $\omega(u, v) = \omega(v, u)$ and $\omega(u, u) = 0$. Note that, for edge-symmetric weight functions, the set of edges such that $\omega(e) = k$ spans a subgraph of G. A *line ensemble* of G is a edge-symmetric weight function $L : V^2 \to \{0, 1\}$ such that for all $v \in V$,

$$\sum_u L(u, v) \in \{0, 2\}.$$

Now, consider a unimodular graph (G, o). If, on an enlarged probability space, the weighted graph (G, L, o) is unimodular and L is a.s. a line ensemble then we shall say that L is *an invariant line ensemble* of (G, o). We shall say that a vertex $v \in V$ is in L if $\sum_u L(u, v) = 2$ and outside L otherwise.

Theorem 4.2. *Let (T, o) be a unimodular tree with law ρ. If L is an invariant line ensemble of (T, o) then for each real λ,*

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \notin L)\mu_{\rho'}(\lambda)$$

where, if $\mathbb{P}(o \notin L) > 0$, ρ' is the law of the rooted tree $(T \backslash L(o), o)$ conditioned on the root $o \notin L$. In particular, the total mass of atoms of μ_ρ is bounded above by $\mathbb{P}(o \notin L)$.

We will check in §4.4.1 below that the measure ρ' is indeed unimodular. As a consequence, if (T, o) has an invariant line ensemble such that $\mathbb{P}(o \in L) = 1$ then μ_ρ is continuous. Our next result gives the existence of invariant line ensemble for a large class of unimodular trees. We recall that for a rooted tree (T, o), a topological end is just an infinite simple path in T starting from o.

Proposition 4.3. *Let (T, o) be a unimodular tree. If T has at least two topological ends with positive probability, then (T, o) has an invariant line ensemble L with positive density: $\mathbb{P}(o \in L) > 0$. Moreover, we have the following lower bounds.*

(i) $\mathbb{P}(o \in L) \geq \frac{1}{6} \frac{(\mathbb{E} \deg(o) - 2)^2}{\mathbb{E} \deg(o)^2}$ *as long as the denominator is finite.*

(ii) *Let q be the probability that $T \setminus \{o\}$ has at most one infinite component. If $\deg(o) \leq d$ a.s., then $\mathbb{P}(o \in L) \geq \frac{1}{3} (\mathbb{E} \deg(o) - 2q)/d$.*

One of the natural examples where the conditions of Proposition 4.3 are not satisfied is the infinite skeleton tree which consists of a semi-infinite line $\mathbb{Z}_+$ with i.i.d. critical Poisson Galton-Watson trees attached to each of the vertices of $\mathbb{Z}_+$. It is the local weak limit of the uniform trees on n labeled vertices.

Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with positive and finite mean. The unimodular Galton-Watson tree with degree distribution P (commonly known as size-biased Galton-Watson tree) is the law of the random rooted tree obtained as follows. The root has a number d of children sampled according to P, and, given d, the subtrees of the children of the root are independent Galton-Watson trees with offspring distribution

$$\widehat{P}(k) = \frac{(k+1)P(k+1)}{\sum_\ell \ell P(\ell)}. \tag{24}$$

These unimodular trees appear naturally as a.s. local weak limits of random graphs with a given degree distribution, see e.g. [38, 33, 17]. It is also well known that the Erdős-Rényi $G(n, c/n)$ has a.s. local weak limit the Galton-Watson tree with offspring distribution $\mathrm{Poi}(c)$. Note that if P is $\mathrm{Poi}(c)$ then $\widehat{P} = P$. The percolation on the hypercube $\{0,1\}^n$ with parameter c/n has the same a.s. local weak limit.

If P has first moment μ_1 and second moment μ_2, then the first moment of $\widehat{P}$ is $\widehat{\mu} = (\mu_2 - \mu_1)/\mu_1$. If $P \neq \delta_2$ and $\widehat{\mu} \leq 1$, then the unimodular Galton-Watson tree is a.s. finite. If $\widehat{\mu} > 1$ ($\widehat{\mu} = \infty$ is allowed), the tree is infinite with positive probability. Proposition 4.3 now implies the following phase transition exists for the existence of a continuous part in the spectral measure.

Corollary 4.4. *Let ρ be a unimodular Galton-Watson tree with degree distribution $P \neq \delta_2$. The first moment of $\widehat{P}$ is denoted by $\widehat{\mu}$. Then μ_ρ contains a non-trivial continuous part if and only if $\widehat{\mu} > 1$.*

Note that for some choices of P, it is false that the total mass of the atomic part of μ_ρ is equal to the probability of extinction of the tree, it is only a lower bound (see [22]).

Let us conclude the intoduction with a few open questions.

Open questions

Question 4.5. *Consider a unimodular Galton-Watson tree with degree distribution P with finite support and $P(0) = P(1) = 0$. Does the expected spectral measure have only finitely many atoms?*

Theorem 4.1 naturally inspires the following question. We strongly believe that the answer is yes.

Question 4.6. *Does supercritical bond percolation on $\mathbb{Z}^d$ have a continuous part in its expected spectral measure for every $d \geq 2$?*

In view of the result of Grigorchuk and Żuk [47] on the lamplighter group, the next problem has some subtlety

Question 4.7. *Is there some monotonicity in the weights of the atoms of the spectral measure (for some non-trivial partial order on unimodular measures)?*

Our main results concern percolation on lattices and trees. It motivates the following question.

Question 4.8. *What can be said about the regularity of the spectral measure for other nonamenable/hyperbolic graphs and for other planar graphs (such as the uniform infinite planar triangulation in Angel and Schramm [8])?*

We have seen that regular trees with degree at least 2 contain invariant line ensembles with density 1. A quantitative version of this would be that if the degree is concentrated, then the density is close to 1. Based on the last part of Proposition 4.3. the following formulation is natural.

Question 4.9. *Is there a function f with $f(x) \to 1$ as $x \to 1$ so that every unimodular tree of maximal degree $d \geq 2$ contains an invariant line ensemble with density at least $f(\mathbb{E}\deg(o)/d)$?*

Two open questions (Questions 4.23 and 4.24) can be found in subsection 4.4.4.

4.2 Two Tools for bounding eigenvalues multiplicities

We will develop two simple tools to prove the existence of a continuous part of the spectral measure of unimodular graphs. We will give many examples where those two tools can be applied. Let us state two results.

4.2.1 Monotone labeling

In this paragraph, we will use a carefully chosen labeling of the vertices of a graph to prove regularity of its spectrum, the intuition being that a labeling gives an order to solve the eigenvalue equation at each vertex.

Definition 4.10. *Let $G = (V, E)$ be a graph. A map $\eta : V \to \mathbb{Z}$ is a labeling of the vertices of G with integers. We shall call a vertex v*

(i) **prodigy** *if it has a neighbor w with $\eta(w) < \eta(v)$ so that all other neighbors of w also have label less than $\eta(v)$,*

(ii) **level** *if not prodigy and if all of its neighbors have the same or lower labels,*

(iii) **bad** *if none of the above holds.*

See Figure 4 for an illustration of these definitions.

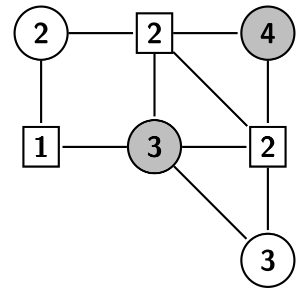

Figure 4: A labeling of a graph. The prodigy, level and bad vertices are marked with $\bullet$, $\circ$ and $\square$ respectively.

Finite graphs. We start with the simpler case of finite graphs.

Theorem 4.11. *Let G be a finite graph, and consider a labeling η of its vertices with integers. Let ℓ, b denote the number of level and bad vertices, respectively. For any eigenvalue λ with multiplicity m we have, if ℓ_j is the multiplicity of the eigenvalue λ in the subgraph induced by level vertices with label j,*

$$m \leq b + \sum_j \ell_j.$$

Consequently, for any multiplicities $m_1, \ldots, m_k$ of distinct eigenvalues we have

$$m_1 + \ldots + m_k \leq kb + \ell.$$

Proof. Let S be the eigenspace for the eigenvalue λ of multiplicity m. Consider the set of bad vertices, and let B be the space of vectors which vanish on that set. For every integer j, let L_j denote the set of level vertices with label j and let A_j denote the eigenspace of λ in the induced subgraph of L_j. With the notation of the theorem, $\dim(A_j) = \ell_j$. We extend the vectors in A_j to the whole graph by setting them to zero outside L_j. Let $A_j^\perp$ be the orthocomplement of A_j. Recall that for any vector spaces A, B we have $\dim(A \cap B) \geq \dim A - \operatorname{codim} B$. Using this, let $S' = S \cap B \cap \bigcap_j A_j^\perp$, and note that

$$\dim S' \geq \dim S - \operatorname{codim} B - \sum_j \operatorname{codim} A_j^\perp = m - b - \sum_j \dim A_j. \tag{25}$$

However, we claim that the subspace S' is trivial. Let $f \in S'$. We now prove, by induction on the label j of the vertices, low to high, that f vanishes on vertices with label j. Suppose that f vanishes on all vertices with label strictly below j. Clearly, f vanishes on all bad vertices since $f \in B$. Consider a prodigy v with label j. Then, by induction hypothesis, v has a neighbor w so that f vanishes on all of the neighbors of w except perhaps at v. But the eigenvalue equation

$$\lambda f(w) = \sum_{u \sim w} f(u)$$

implies that f also vanishes at v. Now, observe that the outer vertex boundary of L_j (all vertices that have a neighbor in L_j but are not themselves in L_j) is contained in the union of the set of bad vertices, the set of level vertices with label strictly below j and the set of prodigy with label j. Hence, we know that f vanishes on the outer vertex boundary of L_j. This means that the restriction of f to L_j has to satisfy the eigenvector equation. But since $f \in A_j^\perp$, we get that $f(v) = 0$ for $v \in L_j$, and the induction is complete.

We thus have proved that S' is trivial. Thus Equation (25) implies that $m \leq b + \sum_j \mathrm{dim} A_j$. It gives the first statement of Theorem 4.11.

For the second statement, let $A_{i,j}$ denote the eigenspace of λ_i in the induced subgraph of L_j. Summing over i the above inequality, we get

$$m_1 + \ldots + m_k \leq bk + \sum_j \sum_i \mathrm{dim} A_{i,j} \leq bk + \sum_j |L_j| = bk + \ell.$$

$\square$

Unimodular graphs. We now prove the same theorem for unimodular random graphs which may possibly be infinite. To make the above proof strategy work, we need a suitable notion of normalized dimension for infinite dimensional subspaces of $\ell^2(V)$. This requires some basic concepts of operator algebras. First, as usual, if (G, o) is a unimodular random graph, we shall say that a labeling $\eta : V(G) \to \mathbb{Z}$ is invariant if on an enlarged probability space, the vertex-weighted rooted graph (G, η, o) is unimodular.

We have seen in the proof of Proposition 2.2 that there is a natural Von Neumann algebra associated to unimodular measures. For a fixed $\rho \in \mathcal{P}_{\mathrm{uni}}(\mathcal{G}^*)$, we introduce the algebra $\mathcal{M}$ of operators in $L^\infty(\mathcal{G}^*, \mathcal{B}(H), \rho)$ which commutes with the operators λ_σ, i.e. for any bijection σ, ρ-a.s. $B(G, o) = \lambda_\sigma^{-1} B(\sigma(G), o) \lambda_\sigma$. In particular, $B(G, o)$ does not depend on the root. It is a Von Neumann algebra and the linear map $\mathcal{M} \to \mathbb{C}$ defined by

$$\tau(B) = \mathbb{E}_\rho \langle e_o, B e_o \rangle,$$

where $B = B(G, o) \in \mathcal{M}$ and under, $\mathbb{E}_\rho$, G has distribution ρ, is a normalized faithful trace (see [4, §5] and Lyons [64]).

A closed vector space S of H such that, P_S, the orthogonal projection to S, is an element of $\mathcal{M}$ will be called an invariant subspace. Recall that the von Neumann dimension of such vector space S is just

$$\mathrm{dim}(S) := \tau(P_S) = \mathbb{E}_\rho \langle e_o, P_S e_o \rangle.$$

We refer e.g. to Kadison and Ringrose [52].

Theorem 4.12. *Let (G, o) be unimodular random graph with distribution ρ, and consider an invariant labeling η of its vertices with integers. Let ℓ, b denote the probability that the root is level or* *bad, respectively. For integer j and real λ, let ℓ_j be the von Neumann dimension of the eigenspace of λ in the subgraph spanned by level vertices with label j. The spectral measure μ_ρ satisfies*

$$\mu_\rho(\lambda) \le b + \sum_j \ell_j.$$

Consequently, for any distinct real numbers $\lambda_1, \ldots, \lambda_k$, we have

$$\mu_\rho(\lambda_1) + \ldots + \mu_\rho(\lambda_k) \le kb + \ell.$$

In particular, if $b = 0$, then the atomic part of μ_ρ has total weight at most ℓ.

Proof. We first assume that there are only finitely many labels. Let S be the eigenspace of λ : that is the subspace of $f \in \ell^2(V)$ satisfying, for all $w \in V$,

$$\lambda f(w) = \sum_{u \sim w} f(u). \tag{26}$$

Consider the set of bad vertices, and let B be the space of vectors which vanish on that set. For every integer j let L_j denote the set of level vertices with label j. Let A_j denote the eigenspace of λ in the induced subgraph of L_j; extend the vectors in A_j to the whole graph by setting them to zero outside L_j. Let $A_j^\perp$ be the orthocomplement of A_j.

For any two invariant vector spaces R,Q we have

$$\dim(R \cap Q) \ge \dim(R) + \dim(Q) - 1,$$

(see e.g. [50, exercice 8.7.31]). Setting $S' = S \cap B \cap \bigcap_j A_j^\perp$, it yields to

$$\dim(S') \ge \dim(S) + \dim(B) - 1 + \sum_j (\dim(A_j^\perp) - 1) = \mu_\rho(\lambda_i) - b - \sum_j \dim(A_j).$$

However, we claim that the subspace V_i' is trivial. Let $f \in V_i'$. We now prove, by induction on the label j of the vertices, low to high, that f vanishes on vertices with label j. The argument is exactly similar to the case of finite graphs presented before. Suppose that f vanishes on all vertices with label strictly below j. Clearly, f vanishes on all bad vertices since $f \in B$. Consider a prodigy v with label j. Then v has a neighbor w so that f vanishes on all of the neighbors of w except perhaps at v. But the eigenvalue equation (26) implies that f also vanishes at v. By now, we know that f vanishes on the outer vertex boundary of L_j. This means that the restriction of f to L_j has to satisfy the eigenvector equation. But since $f \in A_j^\perp$, we get that $f(v) = 0$ for $v \in L_j$, and the induction is complete.

We have proved that $\mu_\rho(\lambda_i) \le b + \sum_j \dim(A_j)$: it is the first statement of the theorem in the case of finitely many labels. When there are infinitely many labels, for every ε, we can find n so that $\mathbb{P}(|\eta(o)| > n) \le \varepsilon$. We can relabel all vertices with $|\eta(v)| > n$ by $-n-1$; this may make them bad vertices, but will not make designation of vertices with other labels worse. The argument for finitely many labels gives

$$\mu_\rho(\lambda) \leq b + \varepsilon + \sum_{j=-n-1}^{n} \dim(A_j) \leq b + 2\varepsilon + \sum_{j=-n}^{n} \dim(A_j) \leq b + 2\varepsilon + \sum_j \ell_j,$$

and letting $\varepsilon \to 0$ completes the proof of the first statement.

For the second statement, let $A_{i,j}$ denote the eigenspace of λ_i in the induced subgraph of L_j. Summing over i the above inequality, we get

$$\mu_\rho(\lambda_1) + \ldots + \mu_\rho(\lambda_k) \leq bk + \sum_j \sum_i \dim(A_{i,j}) \leq bk + \sum_j \mathbb{P}(o \in L_j) = bk + \ell.$$

□

Vertical percolation. There are simple examples where we can apply Theorems 4.11-4.12. Consider the graph of $\mathbb{Z}^2$. We perform a vertical percolation by removing some vertical edge $\{(x,y),(x,y+1)\}$. We restrict to the $n \times n$ box $[0,n-1]^2 \cap \mathbb{Z}^2$. We obtain this way a finite graph Λ_n on n^2 vertices. We consider the labeling $\eta((x,y)) = x$. It appears that all vertices with label different from 0 are prodigy. The vertices on the y-axis are bad and there are no level vertices. By Theorem 4.11, the multiplicity of any eigenvalue of the adjacency matrix of Λ_n is bounded by $n = o(n^2)$.

Similarly, let $p \in [0,1]$. We remove each vertical edge $\{(x,y),(x,y+1)\}$ independently with probability $1-p$. We obtain a random graph $\Lambda(p)$ with vertex set $\mathbb{Z}^2$. Now, we root this graph $\Lambda(p)$ at the origin and obtain a unimodular random graph. We claim that its spectral measure is continuous for any $p \in [0,1]$. Indeed, let $k \geq 1$ be an integer and U be a random variable sampled uniformly on $\{0,\cdots,k-1\}$. We consider the labeling $\eta((x,y)) = x + U \bmod(n)$. It is not hard to check that this labeling is invariant. Moreover, all vertices such that $\eta(x,y) \neq 0$ are prodigy while vertices such that $\eta(x,y) = 0$ are bad. It follows from Theorem 4.12 that the mass of any atom of the spectral measure is bounded by $1/k$. Since k is arbitrary, we deduce that the spectral measure is continuous.

The same holds on $\mathbb{Z}^d$, $d \geq 2$, in the percolation model where we remove edges of the form $\{u, u+e_k\}$, with $u \in \mathbb{Z}^d$, $k \in \{2,\cdots,d\}$.

4.2.2 Minimal path matchings

In this subsection, we give a new tool to upper bound the multiplicities of eigenvalues.

Definition 4.13. *Let $G = (V,E)$ be a finite graph, $I = \{i_1,\cdots,i_b\}$ and $J = \{j_1,\cdots,j_b\}$ be two disjoint subsets of V of equal cardinal. A* **path matching** *$\Pi = \{\pi_\ell\}_{1\leq\ell\leq b}$ from I to J is a collection of self-avoiding paths $\pi_\ell = (u_{\ell,1},\cdots,u_{\ell,p_\ell})$ in G such that for some permutation σ on $\{1,\cdots,b\}$ and all $1 \leq \ell \neq \ell' \leq b$,*

- $\pi_{\ell'} \cap \pi_\ell = \emptyset$,
- $u_{\ell,1} = i_\ell$ *and* $u_{\ell,p_\ell} = j_{\sigma(i_\ell)}$.

We will call σ *the* **matching map** *of* Π. *The length of* Π *is defined as the sum of the lengths of the paths*

$$|\Pi| = \sum_{\ell=1}^{b} |\pi_\ell| = \sum_{\ell=1}^{b} |p_\ell|.$$

Finally, Π *is a* **minimal path matching** *from* I *to* J *if its length is minimal among all possible paths matchings.*

Connections between multiplicities of eigenvalues and paths have already been noticed for a long time, see e.g. Godsil [46]. The following theorem and its proof are a generalization of Kim and Shader [57, Theorem 8] (which is restricted to trees).

Theorem 4.14. *Let* $G = (V, E)$ *be a finite graph and* $I, J \subset V$ *be two subsets of cardinal* b. *Assume that the sets of path matchings from* I *to* J *is not empty and that all minimal path matchings from* I *to* J *have the same matching map. Then if* $|V| - \ell$ *is the length of a minimal path matching and if* $m_1, \cdots, m_r$ *are the multiplicities of the distinct eigenvalues* $\lambda_1, \cdots, \lambda_r$ *of the adjacency matrix of* G*, we have*

$$\sum_{i=1}^{r} (m_i - b)_+ \leq \ell.$$

Consequently, for any $1 \leq k \leq r$,

$$m_1 + \cdots + m_k \leq kb + \ell.$$

We will aim at applying Theorem 4.14 with b small and $|V| - \ell$ proportional to $|V|$. Observe that ℓ is the number of vertices not covered by the union of paths involved in a minimal path matching. Theorem 4.14 and Theorem 4.12 have the same flavor but they are not equivalent. We note that, contrary to Theorem 4.11-Theorem 4.12, we do not have a version of Theorem 4.14 which holds for possibly infinite unimodular graphs. Unlike Theorem 4.11, we do not have either a version which bounds the multiplicity of an eigenvalue in terms of its multiplicities in subgraphs. On the other hand, Theorem 4.14 will be used to show the existence of non-trivial continuous part for the expected spectral measure of two dimensional supercritical bond percolation. It is not clear how to apply Theorem 4.11 or Theorem 4.12 to get this result.

Following [57], the proof of Theorem 4.14 is based on the divisibility properties of characteristic polynomials of subgraphs. For $I, J \subset V$, we define $(A - x)_{I,J}$ has the matrix $(A - x)$ where the rows with indices in I and columns with indices in J have been removed. We define the polynomial associated to the (I, J)-minor as :

$$P_{I,J}(A) : x \mapsto \det(A - x)_{I,J}.$$

We introduce the polynomial

$$\Delta_b(A) = \mathrm{GCD}\left(P_{I,J}(A) : |I| = |J| = b\right),$$

where GCD is the greatest common divisor in the ring of polynomials $\mathbb{R}[x]$: by convention, GCD is a monic polynomial. Recall also that any polynomial divides 0. Observe that if $|I| = b$ then $P_{I,I}(A)$ is a polynomial of degree $|V| - b$. It follows that the degree of Δ_b is at most $|V| - b$.

The next lemma is the key to relate multiplicities of eigenvalues and characteristic polynomial of subgraphs.

Lemma 4.15. *If A is a symmetric matrix and $m_1, \cdots, m_r$ are the multiplicities of its distinct eigenvalues $\lambda_1, \cdots, \lambda_r$, we have*

$$\Delta_b(A) = \prod_{i=1}^{r} (x - \lambda_i)^{(m_i - b)_+}.$$

Consequently,

$$\sum_{i=1}^{r} (m_i - b)_+ = \deg(\Delta_b(A)).$$

Proof. We set $|V| = n$. If $B(x) \in \mathcal{M}_n(\mathbb{R}[x])$ is an $n \times n$ matrix with polynomial entries, we may define analogously $P_{I,J}(B(x)) = \det B(x)_{I,J}$ and $\Delta_b(B(x))$ (we retrieve our previous definition with $B(x) = A - x$). Let $B_1(x), \cdots, B_n(x)$ be the columns of $B(x)$. The multi-linearity of the determinant implies that

$$\begin{aligned} &\det(w_{11}B_1(x) + w_{21}B_2(x) + \cdots + w_{n1}B_n(x), B_2(x), \cdots, B_n(x))_{I,J} \\ &\quad = \sum_{j=1}^{n} w_{j1} \det(B_i(x), \cdots, B_n(x))_{I,J^{(j)}} \end{aligned}$$

is a weighted sum of determinants of the minors of the form $(I, J^{(j)})$, where $J^{(j)} = (J \setminus \{1\}) \cup \{j\}$ if $1 \in J$ and $J^{(j)} = J$ if $1 \notin J$. It is thus divisible by $\Delta_b(B(x))$. The same holds for the rows of $B(x)$. We deduce that if $U, W \in \mathcal{M}_n(\mathbb{R})$, $\Delta_b(B(x))$ divides $\Delta_b(UB(x)W)$. It follows that if U and W are invertible

$$\Delta_b(UB(x)W) = \Delta_b(B(x)).$$

We may now come back to our matrix A. Since A is symmetric, the spectral theorem gives $A = UDU^*$ with U orthogonal matrix and D diagonal matrix with m_i entries equal to λ_i. We have $U(D - x)U^* = A - x$. Hence, from what precedes

$$\Delta_b(A - x) = \Delta_b(D - x).$$

It is immediate to check that if $I \neq J$, $P_{I,J}(D - x) = 0$ and

$$P_{I,I}(D - x) = \prod_{k \notin I} (D_{kk} - x) = \prod_{i=1}^{r} (\lambda_i - x)^{m_i - m_i(I)},$$

where $m_i(I) = \sum_{k\in I} \mathbf{1}(D_{kk} = \lambda_i)$. The lemma follows easily. $\square$

Proof of Theorem 4.14. We set $|V| = n$. We can assume without loss of generality that $I\cap J = \emptyset$ and the matching map of minimal length matchings is the identity. We consider the matrix $B \in \mathcal{M}_n(\mathbb{R})$ obtained from A by setting

$$\text{for } 1 \leq \ell \leq b,\ Be_{j_\ell} = e_{i_\ell} \ \text{ and for } j \notin J,\ Be_j = \sum_{i\notin I} A_{ij} e_i.$$

In graphical terms, B is the adjacency matrix of the oriented graph $\bar{G}$ obtained from G as follows : (1) all edges adjacent to a vertex in J are oriented inwards, (2) all edges adjacent to a vertex in I are oriented outwards, and (3) for all $1 \leq \ell \leq b$, an oriented edge from j_ℓ to i_ℓ is added. We define

$$B(x) = B - xD,$$

where D is the diagonal matrix with entry $D_{ii} = 1 - \mathbf{1}(i \in I \cup J)$. Expanding the determinant along the columns J, it is immediate to check that

$$\det B(x) = \det(A - x)_{I,J}.$$

We find

$$P_{I,J}(A) = \sum_\tau (-1)^\tau \prod_{v\in V} B(x)_{v,\tau(v)} = \sum_\tau (-1)^\tau Q_\tau(x),$$

where the sum is over all permutations of V. Consider a permutation such that $Q_\tau \neq 0$. We decompose τ into disjoint cycles. Observe that $Q_\tau \neq 0$ implies that any cycle of length at least 2 coincides with a cycle in the oriented graph $\bar{G}$. Hence, $Q_\tau = 0$ unless $\tau(j_\ell) = i_\ell$ and $(\tau^k(i_\ell), k \geq 0)$ is a path in $\bar{G}$. We define $\sigma(i_\ell) = \tau^{p_\ell}(i_\ell)$ as the first element in J which is met in the path. We may decompose these paths into disjoints path $\pi_\ell = (\tau^k(i_\ell), 0 \leq k \leq p_\ell)$ in G from i_ℓ to $j_{\sigma(\ell)}$. It defines a path matching $\Pi = \{\pi_1, \cdots, \pi_b\}$. The contribution to Q_τ of any cycle of length at least 2 is 1 (since off-diagonal entries of A and B are 0 or 1). Also, the signature of disjoint cycles is the product of their signatures. So finally, it follows that

$$P_{I,J}(A) = \sum_\Pi \varepsilon(\Pi) \det(B(x)_{\Pi,\Pi}) = \sum_\Pi \varepsilon(\Pi) \det((A - x)_{\Pi,\Pi}), \tag{27}$$

where the sum is over all path matchings from I to J and $\varepsilon(\Pi)$ is the signature of the permutation τ on Π defined by, if $\Pi = \{\pi_1, \cdots, \pi_b\}$, $\pi_\ell = (i_{\ell,1}, \cdots, i_{\ell,p_\ell})$ and σ is the matching map of Π : for $1 \leq k \leq p_\ell - 1$, $\tau(i_{\ell,k}) = i_{\ell,k+1}$ and $\tau(i_{\ell,p_\ell}) = \tau(j_{\sigma(\ell)}) = i_{\sigma(\ell)}$.

Observe that $\det((A-x)_{\Pi,\Pi})$ is a polynomial of degree $n-|\Pi|$ and leading coefficient $(-1)^{n-|\Pi|}$. Recall also that the signature of a cycle of length k is $(-1)^{k+1}$. By assumption, if Π is a minimal path matching then its matching map is the identity : it follows that

$$\varepsilon(\Pi) = (-1)^{n-\ell+b}.$$

Hence, from (27), $P_{I,J}(A)$ is a polynomial of degree ℓ and leading coefficient $m(-1)^b$ where m is the number of minimal path matchings. By assumption $\Delta_b(A)$ divides $P_{I,J}(A)$ in particular $\deg(\Delta_b(A)) \leq \ell$. It remains to apply Lemma 4.15. $\square$

Vertical percolation (revisited). Let us revisit the example of vertical percolation on $\mathbb{Z}^2$ introduced in the previous paragraph. We consider the graph Λ_n on the vertex set $[0, n-1]^2 \cap \mathbb{Z}^2$ where some vertical edges $\{(x, y), (x, y+1)\}$ have been removed. We set $I = \{(0, 0), (0, 1), \cdots, (0, n-1)\}$ and $J = \{(n-1, 0), (n-1, 1), \cdots, (n-1, n-1)\}$. Consider the path matchings from I to J. Since none of the horizontal edges of the graph of $\mathbb{Z}^2$ have been removed, the minimal path matching is unique, it matches $(0, k)$ to $(n-1, k)$ along the path $((0, k), (1, k), \cdots, (n-1, k))$. In particular, the length of the minimal path matching is n^2. We may thus apply Theorem 4.14 : we find that the multiplicity of any eigenvalue is bounded by $n = o(n^2)$. On this example, Theorems 4.11 and 4.14 give the same bound on the multiplicities.

Lamplighter group. The assumption that all minimal path matchings have the same matching map is important in the proof of Theorem 4.14. It is used to guarantee that the polynomial in (27) is not identically zero. Consider a Følner sequence B_n in the Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ [47] where B_n consists of the vertices of the form $(v, k) \in \mathbb{Z}_2^{\mathbb{Z}} \times \mathbb{Z}$ with $v(i) = 0$ for $|i| > n$ and $|k| \leq n$. There is an obvious minimal matching in B_n covering all the vertices where each path is obtained by shifting the marker from $-n$ to n keeping the configurations of the lamps unaltered along the way. But the condition on the unicity of the matching map is not fulfilled. In this case, it is not hard to check that there is a perfect cancellation on the right hand side of (27). It is consistent with the fact that spectral measure of this lamplighter group is purely atomic.

4.3 Supercritical edge percolation on $\mathbb{Z}^2$

In this section, we will prove Theorem 4.1 by finding an explicit lower bound on the total mass of the continuous part of μ_ρ in terms of the speed of the point-to-point first passage percolation on $\mathbb{Z}^2$. We fix $p > p_c(\mathbb{Z}^2) = 1/2$.

We will use a finite approximation of $\mathbb{Z}^2$. Let $\Lambda_n(p)$ be the (random) subgraph of the lattice $\mathbb{Z}^2$ obtained by restricting the p-percolation on $\mathbb{Z}^2$ onto the $(n+1) \times (n+1)$ box $[0, n]^2 \cap \mathbb{Z}^2$. We simply write Λ_n for $\Lambda_n(1)$. As mentioned in the introduction, $\mathrm{perc}(\mathbb{Z}^2, p)$ is the local weak limit of $U(\Lambda_n(p))$ and hence by Proposition 2.2, we have that $\mathbb{E}\mu_n^p$ converges weakly to μ_ρ as $n \to \infty$, where μ_n^p is the empirical eigenvalue distribution of $\Lambda_n(p)$ and the average $\mathbb{E}$ is taken w.r.t. the randomness of $\Lambda_n(p)$.

Now, assume that, given a realization of the random graph $\Lambda_n(p)$, we can find two disjoint subsets of vertices U and V of $\Lambda_n(p)$ with $|U| = |V|$ and a minimal vertex-disjoint path matching M_n of $\Lambda_n(p)$ between U and V such that

(i) The vertices of U and V are uniquely paired up in any such minimal matching of $\Lambda_n(p)$ between U and V.

(ii) $|U| = o(n^2)$.

(iii) There exists a constant $c > 0$ such that the size of M_n is at least cn^2, with probability converging to one.

If such a matching exists satisfying property (i), (ii) and (iii) as above, then Theorem 4.14 says that for any finite subset $S \subset \mathbb{R}$,

$$\mathbb{P}(\mu_n^p(S) \leq 1 - c) = 1 - o(1),$$

and consequently, $\mathbb{E}\mu_n^p(S) \leq (1 - c) + o(1)$. Then by Lück approximation (see [76, Corollary 2.5], [75, Theorem 3.5] or [1]) $\mu_\rho(S) = \lim_{n\to\infty} \mathbb{E}\mu_n^p(S) \leq 1 - c$ for any finite subset S, which implies that the total mass of the continuous part of μ_ρ is at least c. Hence, in order to prove Theorem 4.1, it is sufficient to prove the existence with high probability of such pair of disjoint vertices.

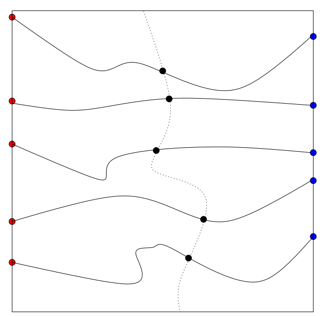

Figure 5: Path matchings.

A natural way to construct this is to find a linear number of vertex-disjoint paths in $\Lambda_n(p)$ between its left and right boundary (see Figure 5). Suppose that there exists a collection of m disjoint left-to-right crossings of $\Lambda_n(p)$ that matches the vertex $(0, u_i)$ on the left boundary to the vertex (n, v_i) on the right boundary for $1 \leq i \leq m$. Without loss of generality, we can assume $0 \leq u_1 < u_2 < \cdots < u_m \leq n$. Since two vertex-disjoint left-to-right crossings in $\mathbb{Z}^2$ can never cross each other, we always have $0 \leq v_1 < v_2 < \cdots < v_m \leq n$. Now we take $U = \{(0, u_i) : 1 \leq i \leq m\}$ and $V = \{(n, v_i) : 1 \leq i \leq m\}$. We consider all vertex-disjoint path matchings between U and V in $\Lambda_n(p)$ (there exists at least one such matching by our hypothesis) and take M_n to be a minimal matching between U and V. Clearly, the property (i) and (ii) above are satisfied. Since any left-to-right crossing contains at least $(n + 1)$ vertices, the size of M_n is at least $(n + 1)m$. Thus to satisfy the property (iii) we need to show that with high probability we can find at least cn many vertex-disjoint left-to-right crossings in $\Lambda_n(p)$.

Towards this end, let ℓ_n denote the maximum number of vertex-disjoint paths in $\Lambda_n(p)$ between its left and right boundary. By Menger's theorem, ℓ_n is also equal to the size of a minimum vertex cut of $\Lambda_n(p)$, that is, a set of vertices of smallest size that must be removed to disconnect the left and right boundary of $\Lambda_n(p)$ (see Figure 5). Note that to bound ℓ_n from below, it suffices to find a lower bound on the size of a minimum edge cut of $\Lambda_n(p)$, since the size of a minimum edge cut is always bounded above by 4 times the size of a minimum vertex cut. This is because deleting all the edges incident to the vertices in a minimum vertex cover gives an edge cut. The reason behind considering minimum edge cut instead of minimum vertex cut is that the size of the former can be related to certain line-to-line first passage time in the dual graph of Λ_n, whose edges are weighted by i.i.d. $\mathrm{Ber}(p)$. We describe this connection below.

Let Λ_n^* (called the dual of Λ_n) be a graph with vertices $\{(x+\frac{1}{2}, y+\frac{1}{2}) : 0 \le x \le n-1, -1 \le y \le n\}$, with all edges of connecting the pair of vertices with ℓ_1-distance exactly 1, except for those in top and bottom sides. To each edge e of Λ_n^*, we assign a random weight of value 1 or 0 depending on whether the unique edge of Λ_n, which e crosses, is present or absent in the graph $\Lambda_n(p)$. Hence, the edge weights of Λ_n^* are i.i.d. $\mathrm{Ber}(p)$. Now here is the crucial observation. The size of minimum edge cut of $\Lambda_n(p)$, by duality, is same as the minimum weight of a path from the top to bottom boundary of Λ_n^*. Moreover, since the dual lattice of $\mathbb{Z}^2$ is isomorphic to $\mathbb{Z}^2$, the minimum weight of a top-to-bottom crossing in Λ_n^* is equal in distribution to the line-to-line passage time $t_{n+1,n-1}(\mathrm{Ber}(p))$ in $\mathbb{Z}^2$, where

$$t_{n,m}(F) := \inf\Big\{ \sum_{e\in\gamma} t(e) : \gamma \text{ is a path in } \mathbb{Z}^2 \text{ joining } (0,a),(n,b) \text{ for some } 0 \le a,b \le m \\ \text{and } \gamma \text{ is contained in } [0,n]\times[0,m]\Big\},$$

and $t(e)$, the weight of edge e of $\mathbb{Z}^2$, are i.i.d. with nonnegative distribution F. By Theorem 2.1(a) of [48], for any nonnegative distribution F, we have

$$\liminf_{n\to\infty} \frac{1}{n} t_{n,n}(F) \ge \nu(F) \quad a.s., \tag{28}$$

where $\nu(F) < \infty$ is called the speed (or time-constant) of the first passage percolation on $\mathbb{Z}^2$ with i.i.d. F edge weights, that is,

$$\frac{1}{n} a_{0,n}(F) \to \nu(F) \quad \text{in probability},$$

where

$$a_{0,n}(F) := \inf\Big\{ \sum_{e\in\gamma} t(e) : \gamma \text{ is a path in } \mathbb{Z}^2 \text{ joining } (0,0),(n,0)\Big\}.$$

It is a classical fact due to Kesten [56] that the speed is strictly positive or $\nu(F) > 0$ if and only if $F(0) < p_c(\mathbb{Z}^2) = \frac{1}{2}$. This ensures that $\nu(\mathrm{Ber}(p)) > 0$ in the supercritical regime $p > \frac{1}{2}$. Therefore, for any $\varepsilon > 0$, with probability tending to one,

$$t_{n+1,n-1}(\mathrm{Ber}(p)) \ge t_{n+1,n+1}(\mathrm{Ber}(p)) \ge \big(\nu(\mathrm{Ber}(p)) - \varepsilon\big)(n+1),$$

which implies that

$$\lim_{n\to\infty} \mathbb{P}\left(\ell_n \geq \frac{1}{4}\big(\nu(\mathrm{Ber}(p)) - \varepsilon\big)n\right) = 1.$$

Hence the property (3) is satisfied with $c = \frac{1}{4}\big(\nu(\mathrm{Ber}(p)) - \varepsilon\big)$ for any $\varepsilon > 0$. Therefore, the total mass of the continuous part of μ_ρ is bounded below by $\frac{1}{4}\nu(\mathrm{Ber}(p))$.

This concludes the proof of Theorem 4.1. □

4.4 Spectrum of Unimodular Trees

4.4.1 Stability of unimodularity

In the sequel, we will use a few times that unimodularity is stable by weight mappings, global conditioning and invariant percolation. More precisely, let (G, o) be a unimodular random weighted rooted graph with distribution ρ. The weights on G are denoted by $\omega : V^2 \to \mathbb{Z}$. The following trivially holds :

Weight mapping : let $\psi : \mathcal{G}^* \to \mathbb{Z}$ and $\phi : \mathcal{G}^{**} \to \mathbb{Z}$ be two measurable functions. We define $\bar{G}$ as the weighted graph with weights $\bar{\omega}$, obtained from G by setting for $u \in V$, $\omega(u, u) = \psi(G, u)$ and for $u, v \in V^2$ with $\{u, v\} \in E(G)$, $\omega(u, v) = \psi(G, u, v)$. The random rooted weighted graph $(\bar{G}, o)$ is unimodular. Indeed, the $\mathcal{G}^* \to \mathcal{G}^*$ map $G \mapsto \bar{G}$ is measurable and we can apply (13) to $f(G, u, v) = h(\bar{G}, u, v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$.

Global conditioning : let A be a measurable event on $\mathcal{G}^*$ which is invariant by re-rooting: i.e. for any (G, o) and (G', o) in $\mathcal{G}^*$ such that G and G' are isomorphic, we have $(G, o) \in A$ iif $(G', o) \in A$. Then, if $\rho(A) > 0$, the random rooted weighted graph (G, o) conditioned on $(G, o) \in A$ is also unimodular (apply (13) to $f(G, u, v) = \mathbf{1}((G, u) \in A))h(G, u, v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$).

Invariant percolation : let $B \subset \mathbb{Z}$. We may define a random weighted graph $\hat{G}$ with edge set $E(\hat{G}) \subset E(G)$ by putting the edge $\{u, v\} \in E(G)$ in $E(\hat{G})$ if both $\omega(u, v)$ and $\omega(v, u)$ are in B. We leave the remaining weights unchanged. Then the random weighted rooted graph $(\hat{G}(o), o)$ is also unimodular (apply (13) to $f(G, u, v) = h(\hat{G}(u), u, v)$ for any measurable $h : \mathcal{G}^{**} \to \mathbb{R}_+$).

As an application the measure ρ' defined in the statement of Theorem 4.2 is unimodular. Indeed, consider the weight mapping for $v \in V$, $\omega(v, v) = \mathbf{1}(v \in L)$ and for $\{u, v\} \in E$, $\omega(u, v) = \omega(v, u) = \mathbf{1}(\omega(u, u) = \omega(v, v))$. Then we perform an invariant percolation with $B = \{1\}$ and finally a global conditioning by $A = \{\text{all vertices in } G \text{ satisfy } \omega(v, v) = 0\}$.

4.4.2 Proof of Theorem 4.2

Consider the unimodular weighted tree (T, L, o). Our main strategy will be to construct a suitable invariant labeling on T using the invariant line ensemble L and then apply Theorem 4.12.

We may identify L with a disjoint union of countable lines $(\ell_i)_i$. Each such line $\ell \subset L$ has two topological ends. We enlarge our probability space and associate to each line an independent Bernoulli variable with parameter $1/2$. This allows to orient each line $\ell \subset L$. This can be done by choosing the unique vertex on the line ℓ whose distance from the root o is minimum and then by picking one of its two neighbors on the line using the Bernoulli coin toss (check that this preserves the unimodularity).

Let us denote by $(\overrightarrow{\ell_i})_i$ the oriented lines. We obtain this way a unimodular weighted graph (T, ω, o) where $\omega(u, v) = 1$ if the oriented edge $(u, v) \in \overrightarrow{\ell_i}$ for some k, $\omega(u, v) = -1$ if $(v, u) \in \overrightarrow{\ell_i}$, and otherwise $\omega(u, v) = 0$.

Now, we fix some integer $k \geq 1$. There are exactly k functions $\eta : V \mapsto \mathbb{Z}/k\mathbb{Z}$ such that the discrete gradient of η is equal to ω (i.e. such that for any $u, v \in V$ with $\{u, v\} \in E$, $\eta(u) - \eta(v) = \omega(v, u) \bmod(k)$) since given the gradient ω, the function η is completely determined by its value at any vertex. We may enlarge our probability space in order to sample, given (T, ω, o), such a function η uniformly at random. Then the vertex-weighted random rooted graph (T, η, o) is unimodular (check that this preserves the unimodularity).

In summary, we have obtained an invariant labelling η of (T, o) such that all vertices $v \in V$ outside L are level, all vertices in L such that $\eta(v) \neq \{0, 1\}$ are prodigy, and vertices in L such that $\eta(v) \in \{0, 1\}$ are bad. By Theorem 4.12, we deduce that for any real λ,

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \text{ is bad}) + \sum_j \ell_j,$$

where $\ell_j = \mathbb{E}\langle e_o, P_j e_o \rangle$ and P_j is the projection operator of the eigenspace of λ in the adjacency operator A_j spanned by vertices with label j. Now, observe that the set of level vertices with label j are at graph distance at least 2 from the set of level vertices with label $i \neq j$. It implies that the operators A_j commute and A', the adjacency operator of $T' = T \backslash L$, can be decomposed as a direct sum of the operators A_j. It follows that, if P' is the projection operator of the eigenspace of λ in A'

$$\sum_j \ell_j = \mathbb{E}\langle e_o, P' e_o \rangle = \mathbb{P}(o \notin L) \mu_{\rho'}(\lambda).$$

Also, by construction, $\mathbb{P}(o \text{ is bad})$ is upper bounded by $2/k$. Since k is arbitrary, we find

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \notin L) \mu_{\rho'}(\lambda).$$

This concludes the proof of Theorem 4.2. □

Remark 4.16. In the proof of Theorem 4.2, we have used our tool Theorem 4.12. It is natural to ask if we could have used Theorem 4.14 together with some finite graphs sequence (G_n) having local weak limit (T, o) instead. We could match the set of $v \in L$ such that $\eta(v) = 1$ to the set of $v \in L$ such that $\eta(v) = k - 1$ forbidding the set of $v \in L$ with $\eta(v) = 0$. Note however that if the weighted graph (G_n, η_n) has local weak limit (T, η, o) then the boundary of $\eta_n^{-1}(j)$ for $j \in \mathbb{Z}/k\mathbb{Z}$ has cardinal $(2/k + o(1))\mathbb{P}(o \in L)|V(G_n)|$. In particular, the sequence (G_n) must have a small Cheeger constant. It implies for example that we could not use the usual random graphs as finite approximations of infinite unimodular Galton-Watson trees since they have a Cheeger constant bounded away from 0, see Durrett [38].

4.4.3 Construction of invariant line ensemble on unimodular tree

We will say that a unimodular tree (T, o) is *Hamiltonian* if there exists an invariant line ensemble L such that $\mathbb{P}(o \in L) = 1$. As the first example, we show that d-regular infinite tree is Hamiltonian.

Lemma 4.17. *For any integer $d \geq 2$, the d-regular infinite tree is Hamiltonian.*

Proof. The case $d = 2$ is trivial : in this case $T = (V, E)$ itself is a line ensemble. Let us assume $d \geq 3$. On a probability space, we attach to each oriented edge (u, v) independent variables, $\xi(u, v)$ uniformly distributed on $[0, 1]$. With probability one, for each $u \in V$, we may then order its d neighbours according to value of $\xi(u, \cdot)$. This gives a weighted graph (T, ω, o) such that, for each $u \in V$ with $\{u, v\} \in V$, $\omega(u, v) \in \{1, \cdots, d\}$ is the rank of vertex v for u. Note that $\omega(u, v)$ may be different from $\omega(v, u)$. We now build a line ensemble as follows. The root picks its first two neighbours, say u_1, u_2, and we set $L(u_1, o) = L(u_2, o) = 1$, for its other neighbours, we set $L(u, o) = 0$. To define further L, let us introduce some notation. For $u \neq v$, let T_u^v be the tree rooted at u spanned by the vertices whose shortest path in T to v meets u, and let $a^v(u)$ be the first visited vertex on the shortest path from u to v (see Figure 6). Then, we define iteratively the line ensemble (we define $L(u, \cdot)$ for a vertex u for which $L(a_u^o, \cdot)$ has already been defined) according to the rule : if $L(u, a^o(u)) = 1$ then u picks its first neighbour in T_u^o, say v_1, and we set $L(u, v_1) = 1$, otherwise $L(u, a^o(u)) = 0$ and u picks its two first neighbours in T_u^o, say v_1, v_2, and we set $L(u, v_1) = L(u, v_2) = 1$. In both cases, for the other neighbours of u in T_u^o, we set $L(u, v) = 0$.

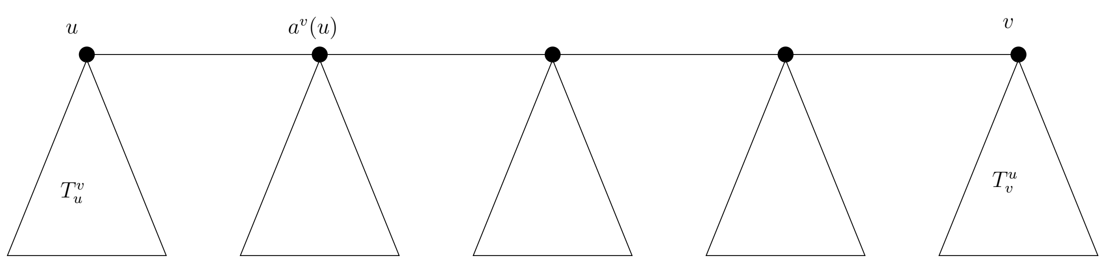

Figure 6: Definition of $a^v(u)$ and T_u^v.

Iterating this procedure gives a line ensemble which covers all vertices. It is however not so clear that this line ensemble is indeed invariant since, in the construction, the root seems to play a special role. In order to verify (13), it is sufficient to restrict to functions $f(G, L, u, v)$ such that $f(G, L, u, v) = 0$ unless $\{u, v\} \in E$ (see [4, Proposition 2.2]). Let us denote $v_1, \cdots, v_d$ the neighbours of the root, we have

$$\mathbb{E} \sum_{k=1}^{d} f(T, L, o, v_k) = (d-2)\mathbb{E}[f(T, L, o, v_1)|L(v_1, o) = 0] + 2\mathbb{E}[f(T, L, o, v_1)|L(v_1, o) = 1].$$

We notice that the rooted trees T_u^v, $u \neq v$, are isomorphic (T_u^v is a $(d-1)$-ary tree) and that, given the value of $L(o, v_1)$, the restriction of L to $T_{v_1}^o$ and $T_o^{v_1}$ have the same law (and are independent).

Since $L(u, v) = L(v, u)$, it follows that, for $\varepsilon \in \{0, 1\}$,

$$\mathbb{E}[f(T, L, o, v_1) | L(v_1, o) = \varepsilon] = \mathbb{E}[f(T, L, v_1, o) | L(o, v_1) = \varepsilon].$$

We have thus checked that L is an invariant line ensemble. ☐

Lemma 4.18. *Let $k \geq 3$. Every unimodular tree with all degrees either 2 or k has an invariant line ensemble of density $\mathbb{E}\deg(o)/k$.*

Proof. Sample the unimodular random tree (T, o). Consider the k-regular labeled tree T' that one gets by contracting each induced subgraph which is a path to a single edge labeled by the number of vertices. This tree has an invariant line ensemble L' with density 1; this corresponds to a line ensemble L in T. Since each edge in T' is contained in L' with probability $2/k$, it follows that each edge of T is contained in L with probability $2/k$. Thus the expected degree of L at the root of T given T is $\frac{2}{k}\deg(o)$. The claim follows after averaging over T. ☐

The following proves Proposition 4.3, part 2 for the case $q = 0$ (i.e. when there are no "bushes").

Proposition 4.19. *Let T be a unimodular tree with degrees in $\{2, 3, \ldots, d\}$. Then T contains an invariant line ensemble with density at least $\frac{1}{3}\mathbb{E}\deg(o)/d$. In fact, when $d \geq 6$ the density is at least $\frac{1}{3}\mathbb{E}\deg(o)/(d-4)$.*

A tree constructed of d-stars with paths of length m emanating shows that in some cases the optimal density can be arbitrary close to $\mathbb{E}\deg(o)/d$. In this sense our bound is sharp up to a factor of 2/3.

Proof of Proposition 4.19. If $d \geq 6$ we argue as follows. For each k, we split all vertices of degree $3k + 2j$ with $j = 0, 1, 2$ into k groups of vertices of degree 3 and j groups of vertices of degree 2. We can perform this in an unimodular fashion by ordering the adjacent edges of a vertex uniformly at random (see the proof of Lemma 4.17). This way we obtain a countable collections of trees $(T_n)_{n\geq 1}$.

By Lemma 4.18 each of these trees contains invariant line ensembles with expected degree $\frac{2}{3}\mathbb{E}\deg_{T_n}(o)$. In particular, the expected degree of their union F_1 in T is $\frac{2}{3}\mathbb{E}\deg(o)$. We thus have found an invariant subforest F_1 of $F_0 = T$ with degrees in $\{0, 2, 4 \ldots, 2k + 2j\}$ and expected degree $\frac{2}{3}\mathbb{E}\deg(o)$.

Iterating this construction i times we get a sequence of subforests F_i with expected degree $\left(\frac{2}{3}\right)^i \mathbb{E}\deg(o)$. The maximal degree of F_i is bounded above by some d_i (with $d_0 = d$), which satisfy the following recursion: if $d_i = 3k + 2j$ with $j = 0, 1, 2$, then $d_{i+1} = 2k + 2j$. In particular, d_i is even for $i \geq 1$, and

$$d_{i+1} \leq \frac{2}{3}d_i + \frac{4}{3}. \tag{29}$$

Let k be the first value so that $d_k \leq 4$; by checking cases we see that $d_k = 4$, and that $d_{k-1} = 5$ or $d_{k-1} = 6$. Assuming $k > 1$ we also know that d_{k-1} is even, so $d_{k-1} = 6$. Otherwise, $k = 1$ and then $d_0 = d$. However the assumption $d \geq 6$ yields to $d_0 = d = 6$. Hence in any case $d_{k-1} = 6$. Now using the inequality (29) inductively we see that for $1 \leq i \leq k$ we have $d_{k-i} \geq \frac{4}{3}\left(\frac{3}{2}\right)^i + 4$. Setting $i = k$ and rearranging we get

$$\left(\frac{2}{3}\right)^k \geq \frac{4}{3}\frac{1}{d-4}.$$

The forest F_k has degrees in $\{0,2,4\}$. Another application of Lemma 4.18 (with $k = 4$ there) gives an invariant line ensemble with density

$$\frac{1}{4}\left(\frac{2}{3}\right)^k \mathbb{E}\deg(o) \geq \frac{1}{3}\frac{\mathbb{E}\deg(o)}{d-4}.$$

If $d = 5$, then $k = 1$, and the above argument gives an invariant line ensemble with density $\frac{1}{4}\left(\frac{2}{3}\right)\mathbb{E}\deg(o)$.

The only cases left are $d = 3, 4$. In the first case, just use Lemma 4.18 with $k = 3$. In the second, split each degree 4 vertex in 2 groups of degree 2 vertices as above. Then apply Lemma 4.18 with $k = 3$ to get a subforest with degrees in $0, 2, 4$. Then apply the Lemma again with $k = 4$. The density lower bounds are given by $\frac{1}{3}\mathbb{E}\deg(o)$, $\frac{1}{6}\mathbb{E}\deg(o)$ respectively, and this proves the remaining cases. $\square$

Recall that the core C of a tree T is the induced subgraph of vertices whose removal breaks T into at least two infinite components. The following is a reformulation of part (ii) of Proposition 4.3.

Corollary 4.20 (Removing bushes)**.** *Let (T, o) be an infinite unimodular tree, with core C and maximal degree d. Then Proposition 4.19 holds with $\mathbb{E}\mathrm{deg}(o)$ replaced by $\mathbb{E}\mathrm{deg}(o) - 2\mathbb{P}(o \notin C)$.*

Proof. We clarify that $\deg_C(o) = 0$ if $o \notin C$. It suffices to to show that $\mathbb{E}\mathrm{deg}_C(o) = \mathbb{E}\mathrm{deg}(o) - 2\mathbb{P}(o \notin C)$. For this, let every vertex v with $\deg_C(v) = 0$ send unit mass to the unique neighbor vertex closest to C (or closest to the single end of T in case C is empty). We have

$$\deg_C(o) = \deg(o) - r - \mathbf{1}(o \notin C)$$

where r is the amount of mass o receives. The claim now follows by mass transport : (13) applied to $f(G, o, v)$ equal to the amount of mass send by o to v gives $\mathbb{P}(o \notin C) = \mathbb{E}r$. $\square$

We are now ready to prove the main assertion of Proposition 4.3, repeated here as follows.

Corollary 4.21. *Let (T, o) be a unimodular tree with at least 2 ends with positive probability. Then T contains an invariant line ensemble with positive density.*

Proof. We may decompose the measure according to whether T is finite or infinite and prove the claim separately. The finite case being trivial, we now assume that T is infinite.

Consider the core C of T. If T has more than one end, then C has the same ends as T, in particular it is not empty. Thus for the purposes of this corollary we may assume that $T = C$, or in other words all degrees of T are at least 2.

If $\mathbb{E}\deg(o) = 2$, then T is a line and we are done. So next we consider the case $\mathbb{E}\mathrm{deg}(o) > 2$.

Let F_d be a subforest where all edges incident to vertices of degree more than d are removed. Then $\deg_{F_d}(o) \to \deg_T(o)$ a.s. in a monotone way. Thus by the Monotone Convergence Theorem $\mathbb{E}\mathrm{deg}_{F_d}(o) \to \mathbb{E}\mathrm{deg}_T(o) > 2$. Pick a d so that $\mathbb{E}\mathrm{deg}_{F_d}(o) > 2$. Corollary 4.20 applied to the components of F_d now yields the claim. $\square$

Part (i) of Proposition 4.3 is restated here as follows.

Corollary 4.22. *Let T be a unimodular tree and assume that $\mathbb{E}\mathrm{deg}(o)^2$ is finite. Then T contains an invariant line ensemble L with density*

$$\mathbb{P}(o \in L) \geq \frac{1}{6}\frac{(\mathbb{E}\deg(o) - 2)_+^2}{\mathbb{E}\deg(o)^2}.$$

Proof. Let $d \geq 1$ be an integer. For each vertex v we mark $(\deg(v) - d)_+$ incident edges at random. To set up a mass transport argument, we also make each vertex to send mass one along every one of its marked edges. The unmarked edges form a forest F_d with the same vertices as T and maximal degree d: we now bound its expected degree. Note that the degree of the root in F_d is bounded below by the same in T minus the total amount of mass sent or received. These two quantities are equal in expectation, so we get

$$\mathbb{E}\deg_{F_d}(o) \geq \mathbb{E}\deg(o) - 2\mathbb{E}(\deg(o) - d)_+.$$

By Proposition 4.19 applied to components of F_d, as long as $d \geq 6$ we get an invariant line ensemble L with density

$$\mathbb{P}(o \in L) \geq \frac{1}{3}\frac{1}{d-4}\left(\mathbb{E}\deg(o) - 2 - 2\mathbb{E}(\deg(o) - d)_+\right).$$

To bound the last term, note that setting $c = \deg(o) - d$, the inequality $4(\deg(o) - d)_+ d \leq \deg(o)^2$ reduces to $4cd \leq (c + d)^2$, which certainly holds. Thus we can bound

$$\mathbb{P}(o \in L) \geq \frac{1}{3}\frac{1}{d-4}\left(\mathbb{E}\deg(o) - 2 - \frac{\mathbb{E}\mathrm{deg}(o)^2}{2d}\right).$$

Now set $d = \lceil \mathbb{E}\mathrm{deg}(o)^2/(\eta - 2)\rceil \geq \eta^2/(\eta - 2) \geq 8$, where $\eta = \mathbb{E}\mathrm{deg}(o)$ can be assumed to be more than 2. Using the bound $\lceil x \rceil - 4 \leq x$ we get the claim. $\square$

4.4.4 Maximal invariant line ensemble

Let (T, o) be a unimodular rooted tree with distribution ρ. In view of Theorem 4.2 and Proposition 4.3, we may wonder what is the value

$$\Sigma(\rho) = \sup \mathbb{P}(o \in L),$$

where the supremum runs over all invariant line ensembles L of (T, o). Recall that a line ensemble L of (T, o) is a weighted graph (T, L, o) with weights $L(u, v)$ in $\{0, 1\}$. By diagonal extraction, the set of $\{0, 1\}$-weighted graphs of a given (locally finite) rooted graph $G = (G, o)$ is compact for the local topology. Hence, the set of probability measures on rooted $\{0, 1\}$-weighted graphs such that the law of the corresponding unweighted rooted graph is fixed is a compact set for the local weak topology. Recall also that the set of unimodular measures in closed for the local weak topology. By compactness, it follows that there exists an invariant line ensemble, say L^*, such that

$$\Sigma(\rho) = \mathbb{P}(o \in L^*).$$

It is natural to call such invariant line ensemble a maximal invariant line ensemble.

Question 4.23. *What is the value of $\Sigma(\rho)$ for ρ a unimodular Galton-Watson tree ?*

Let L^* be an maximal invariant line ensemble and assume $\mathbb{P}(o \in L^*) < 1$. Then ρ', the law of $(T \backslash L^*, o)$ conditioned on $o \notin L^*$, is unimodular. Assume for simplicity that ρ is supported on rooted trees with uniformly bounded degrees. Then, by Proposition 4.3 and the maximality of L^*, it follows that, if (T', o) has law ρ', then a.s. T' has either 0 or 1 topological end. Theorem 4.2 asserts that the atoms of μ_ρ are atoms of $\mu_{\rho'}$. We believe that the following is true.

Question 4.24. *Is it true that if ρ is a unimodular Galton-Watson tree then ρ' is supported on finite rooted trees ?*

4.4.5 Two examples

Ring graphs. With Theorem 4.2, we can give many examples of unimodular rooted trees (T, o) with continuous expected spectral measure. Indeed, by Theorem 4.2 all Hamiltonian trees have continuous spectrum.

An example of a Hamiltonian unimodular tree is the *unimodular ring tree* obtained as follows. Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with finite positive mean. We build a multi-type Galton-Watson tree with three types $\{o, a, b\}$. The root o has type-o and has two type-a children and a number of type-b children sampled according P. Then, a type-b vertex has two type-a children and a number of type-b sampled independently according to $\widehat{P}$ given by (24). A type-a vertex has 1 type-a child and a number of type-b sampled according to P. We then remove the types and obtain a rooted tree. By construction, it is Hamiltonian : the edges connecting type-a vertices to their genitor is a line ensemble covering all vertices. We can also check easily that it is unimodular.

If P has two finite moments, consider a graphic sequence $\underline{d}(n) = (d_1(n), \cdots, d_n(n))$ such that the empirical distribution of $\underline{d}(n)$ converges weakly to P and whose second moment is uniformly integrable. Sample a graph G_n with vertex set $\mathbb{Z}/(n\mathbb{Z})$ uniformly on graphs with degree sequence $\underline{d}(n)$ and, if they are not already present, add the edges $\{k, k+1\}$, $k \in \mathbb{Z}/(n\mathbb{Z})$. The a.s. weak limit of G_n is the above ring tree. This follows from the known result that the uniform graph with degree sequence $\underline{d}(n)$ has a.s. weak limit the unimodular Galton-Watson tree with degree distribution P (see [38, 33, 17])

Alternatively, consider a random graph G_n on $\mathbb{Z}/(n\mathbb{Z})$ with the edges $\{k, k+1\}$, $k \in \mathbb{Z}/(n\mathbb{Z})$ and each other edge is present independently with probability c/n. Then the a.s. weak limit of G_n will be the unimodular ring tree with $P = \mathrm{Poi}(c)$. Note that G_n is the Watts-Strogatz graph [78].

Stretched regular trees. Let us give another example of application of Theorem 4.2. Fix an integer $d \geq 3$. Consider a unimodular rooted tree (T, o) with only vertices of degree 2 and degree d. Denote its law by ρ. For example a unimodular Galton-Watson tree with degree distribution $P = p\delta_2 + (1-p)\delta_d$, $0 < p < 1$. Then, arguing as in Proposition 4.3, a.s., all segments of degree 2 vertices are finite. Contracting these finite segments, we obtain a d-regular infinite tree. Hence, by Lemma 4.17, there exists an invariant line ensemble L of (T, o) such that a.s. all degree d vertices are covered. By Theorem 4.2, the atoms of μ_ρ are contained in set of atoms in the expected spectral measure of rooted finite segments. Eigenvalues of finite segments of length n are of the form $\lambda_{k,n} = 2\cos(\pi k/(n+1))$, $1 \leq k \leq n$. This proves that the atomic part of μ_ρ is contained in $\Lambda = \cup_{k,n}\{\lambda_{k,n}\} \subset (-2, 2)$.

On the other hand, if ρ is a unimodular Galton-Watson tree with degree distribution $P = p\delta_2 + (1-p)\delta_d$, $0 < p < 1$, the support of μ_ρ is equal to $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. Indeed, recall that $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$ and

$$\int x^{2k} \mu_A^{e_o} = \langle e_o, A^{2k} e_o \rangle$$

is equal to the number of path in T of length $2k$ starting and ending at the root. An upper bound is certainly the number of such paths in the infinite d-regular tree. In particular, from Kesten [55],

$$\int x^{2k} \mu_A^{e_o} \leq (2\sqrt{d-1} + o(1))^{2k}.$$

It implies that the convex hull of the support of μ_ρ is contained $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. The other way around, recall first that if μ is the spectral measure of the infinite d-regular tree then $\mu(I) > 0$ if I is an open interval in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$, see [55]. Recall also that for the local topology on rooted graphs with degrees bounded by d, the map $G \mapsto \mu_{A(G)}^{e_o}$ is continuous in $\mathcal{P}(\mathbb{R})$ equipped with the weak topology (e.g. it follows from Reed and Simon [70, Theorem VIII.25(a)]). Hence, there exists $t > 0$ such that if $(T, o)_t$ is d-regular then $\mu_{A(T)}^{e_o}(I) > 0$. Observe finally that under ρ the probability that $(T, o)_t$ is d-regular is positive. Since $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$, it implies that $\mu_\rho(I) > 0$.

We thus have proved that for a unimodular Galton-Watson tree with degree distribution $P = p\delta_2 + (1-p)\delta_d$, μ_ρ restricted to the interval $[2, 2\sqrt{d-1}]$ is continuous.

5 Local laws and delocalization of eigenvectors

In this section, we consider a finite graph $G = (V, E)$ with $|V| = n$ and study the behavior of $o(n)$ eigenvalues and the delocalization of the eigenvectors.

To be more precise, assume that (G_n) is a sequence of finite graphs, with vertex set $V(G_n) = \{1, \ldots, n\}$ such that $U(G_n) \to \rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$. Then Theorem 2.5 asserts that for any fixed interval $I \subset \mathbb{R}$,

$$\lim_{n\to\infty} \mu_{G_n}(I) = \mu_\rho(I).$$

We would like to have a more quantitative statement. Notably, assume that μ_ρ has a bounded density f in a neighborhood of $\lambda \in \mathbb{R}$ so that $\mu_\rho([\lambda + t/2, \lambda + t/2]) = tf(\lambda) + o(t)$. We would like to find an explicit sequence $t_n \to 0$ such that, if $I_n = [x + t_n/2, x + t_n/2]$,

$$\lim_{n\to\infty} \frac{\mu_{G_n}(I_n) - \mu_\rho(I_n)}{t_n} = \lim_{n\to\infty} \frac{\mu_{G_n}(I_n)}{t_n} - f(\lambda) = 0. \tag{30}$$

This type of statement is usually called a *local limit spectral law*. In many important cases, we expect that the above convergence holds as soon as $t_n \gg 1/n$.

It is also important to understand the *localization properties* of the eigenvectors in the canonical basis of $\mathbb{R}^n$. More precisely, let $(v_k)_{1\le k\le n}$ be an orthonormal basis of eigenvectors of the adjacency matrix A of $G = G_n$. For an eigenvector v_k, we may wonder whether the probability vector $(v_k^2(1), \ldots, v_k^2(n))$ has most of its mass concentrated on few coordinates (or vertices in our context) or whether its mass is well spread out. This can be measured by studying ratio of ℓ^p-norms. Namely, a form of delocalization occurs if for some $p \in (2, \infty]$,

$$\left(\sum_{x=1}^{n} |v_k(x)|^p\right)^{1/p} = \frac{\|v_k\|_p}{\|v_k\|_2} = o(1). \tag{31}$$

Note that this notion of delocalization depends on the choice of the basis of $\mathbb{R}^n$. Physicists call the above quantities *inverse participation ratios*. The logarithm of the left hand side of (31) is, up to a constant, the Rényi entropy of the probability vector $(v_k^2(1), \ldots, v_k^2(n))$ with parameter $p/2$. If $v_k = (1, \ldots, 1)/\sqrt{n}$, then $\|v_k\|_p = n^{1/p-1/2}$. It is often easier to study the *average* of inverse participation ratios over eigenvectors associated to close eigenvalues. If $\Lambda_I = \{k : \lambda_k \in I\}$ is not empty, we set

$$P_I = \frac{1}{|\Lambda_I|} \sum_{k\in\Lambda_I} \left(\sum_{x=1}^{n} |v_k(x)|^p\right) \in [n^{1-p/2}, 1]. \tag{32}$$

A form of delocalization occurs if the above expression goes to 0.

If the graph has enough homogeneity, then we may even expect that stronger forms of delocalization of the eigenvectors occur. We could then try to compare the orthonormal basis of eigenvectors $(v_1, \ldots, v_n)$ to the columns of a Haar distributed orthogonal matrix in $\mathbb{R}^n$. A weaker form of this question is to look at the distance between the probability vector $(v_k^2(1), \ldots, v_k^2(n))$ with a random vector sampled uniformly on the simplex $\sum_x p_x = 1$, $p_x \ge 0$. Unfortunately, this type of questions on eigenvectors are currently out of reach for most graphs.

Nevertheless, motivated by the study of quantum chaos, a delocalization criterion in this spirit is studied by Anantharaman and Le Masson in [6, 5]. For a fixed k, the aim is to compare the probability measures

$$\sum_{x=1}^{n} v_k^2(x)\delta_x \quad \text{and} \quad \frac{1}{n}\sum_{x=1}^{n}\delta_x.$$

That is, over a reasonable class of functions f, we want to upper bound

$$\left|\sum_x f(x)\left(v_k^2(x) - \frac{1}{n}\right)\right|.$$

If the above expression goes to 0, this is usually referred as unique quantum ergodicity (see [25, 6, 5]). Again, in practice, it is easier to study an average

$$Q_I(f) = \frac{1}{|\Lambda_I|}\sum_{k\in\Lambda_I}\left|\sum_x f(x)\left(v_k^2(x) - \frac{1}{n}\right)\right|. \tag{33}$$

If the above expression goes to 0, this is referred as quantum ergodicity.

The focus of these notes is the spectral measures at vectors. We can easily estimate (32) in terms of the spectral measures $\mu_G^{e_x}$ as follows. Using for $p \geq 2$,

$$\sum_{k\in\Lambda_I}|v_k(x)|^p \leq \left(\sum_{k\in\Lambda_I}|v_k(x)|^2\right)^{p/2} = \left(\mu_G^{e_x}(I)\right)^{p/2},$$

we find that

$$P_I \leq \frac{1}{n}\sum_{x=1}^{n}\frac{\left(\mu_G^{e_x}(I)\right)^{p/2}}{\mu_G(I)}.$$

Hence, if $\mu_G^{e_x}(I) \leq c|I|$ for 'most' $x \in V$ and $\mu_G(I) \geq |I|/c$ then $P_I = O(|I|^{p/2-1})$ goes to 0 if I has a vanishing length. The expression (33) is not directly related to the spectral measures. However, if we remove the absolute value in (33), we find

$$\frac{1}{|\Lambda_I|}\sum_{k\in\Lambda_I}\sum_{x=1}^{n} f(x)\left(v_k^2(x) - \frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} f(x)\left(\frac{\mu_G^{e_x}(I)}{\mu_G(I)} - 1\right).$$

Notably, we see that the quantum ergodicity implies a form of concentration of the spectral measures $\mu_G^{e_x}$ around their spatial average μ_G, see (9).

In this section, we are going to see that the above expressions can be controlled from fine estimates on the resolvent matrix. In the context of random matrices, these methods have been introduced by Erdős, Yau and Schlein, see [42, 43]. For random graphs ensemble with growing average degree, we refer to [40, 41, 13]. This section is partly adapted from [18].

5.1 Cauchy-Stieltjes transform

Let μ be a finite positive measure on $\mathbb{R}$. Define its *Cauchy-Stieltjes transform* as for all $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$,

$$g_\mu(z) = \int \frac{1}{\lambda - z}d\mu(\lambda).$$

Note that if μ has bounded support

$$g_\mu(z) = -\sum_{k\geq 0} z^{-k-1} \int \lambda^k d\mu(\lambda).$$

The Cauchy-Stieltjes transform is thus essentially the generating function of the moments of the measure μ. It is straightforward that the function g_μ is an analytic function from $\mathbb{C}_+ \to \mathbb{C}_+$ and for any $z \in \mathbb{C}_+$, $|g_\mu(z)| \leq \mu(\mathbb{R})(\Im(z))^{-1}$.

The Cauchy-Stieltjes transform characterizes the measure. More precisely, the following holds.

Lemma 5.1 (Inversion of Cauchy-Stieltjes transform)**.** *Let μ be a finite measure on $\mathbb{R}$.*

(i) For any $f \in \mathcal{C}_0(\mathbb{R})$,
$$\int f d\mu = \lim_{t\downarrow 0} \frac{1}{\pi} \int f(x) \Im g_\mu(x+it) dx.$$

(ii) If $f = \mathbf{1}_I$ with I is interval and $\mu(\partial I) = 0$ the above formula holds.

(iii) For any $x \in \mathbb{R}$,
$$\mu(\{x\}) = \lim_{t\downarrow 0} t \Im g_\mu(x+it).$$

(iv) If μ admits a density at $x \in \mathbb{R}$, then its density is equal to

$$\lim_{t\downarrow 0} \frac{1}{\pi} \Im g_\mu(x+it).$$

Proof. By linearity, we can assume that μ is probability measure. We have the identity

$$\Im g(x+it) = \int \frac{t}{(\lambda - x)^2 + t^2} d\mu(\lambda).$$

Hence $\frac{1}{\pi}\Im g(x+it)$ is the equal to density at x of the distribution $(\mu * P_t)$, where P_t is a Cauchy distribution with density
$$P_t(x) = \frac{t}{\pi(x^2+t^2)}.$$

In other words,
$$\frac{1}{\pi} \int f(x) \Im g_\mu(x+it) dx = \mathbb{E} f(X+tY),$$

where X has law μ and is independent of Y with distribution P_1. Since $X+tY$ converges weakly to X as $t \to 0$, the statements follow easily. (Alternatively, it suffices to use that in $\mathcal{D}'(\mathbb{C})$, $\partial(\frac{1}{z}) = -\frac{1}{\pi}\delta_0$ where $\partial f(z) = \frac{1}{2}(\partial_{\Re(z)} f(z) - i \partial_{\Im(z)} f(z))$ denotes the Cauchy derivative). $\square$

There are more quantitative inversion or deconvolution formulas which are useful, notably for the local laws (30). For example, the following holds (for a proof see [20]).

Lemma 5.2 (Quantitative inversion of Stieltjes transform)**.** *There exists a constant c such that the following holds. Let $L \geq 1$, K be an interval of $\mathbb{R}$ and μ be a probability measure on $\mathbb{R}$. We assume that for some $t > 0$ and all $\lambda \in K$, either*

$$\Im g_\mu(\lambda + it) \leq L \quad \text{or} \quad \mu\left(\left[\lambda - \frac{t}{2}, x + \frac{t}{2}\right]\right) \leq Lt.$$

Then, for any interval $I \subset K$ of size at least t and such that $\mathrm{dist}(I, K^c) \geq 1/L$, we have

$$\left|\mu(I) - \frac{1}{\pi}\int_I \Im g_\mu(\lambda + it) d\lambda\right| \leq cLt \log\left(1 + \frac{|I|}{t}\right).$$

5.2 Bounds using the resolvent

If $A \in \mathcal{H}_n(\mathbb{C})$ is an Hermitian matrix and $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$, then $A - zI$ is invertible. Recall that *the resolvent* of A is the function $R : \mathbb{C}_+ \mapsto \mathcal{M}_n(\mathbb{C})$,

$$R(z) = (A - zI)^{-1}.$$

For $\phi \in \mathbb{C}^n$, we have the identity

$$\langle \phi, R(z)\phi \rangle = \int \frac{1}{\lambda - z} d\mu_A^\phi(\lambda) = g_{\mu_A^\phi}(z), \tag{34}$$

where μ_A^ϕ is the spectral measure with vector ϕ. We also find

$$g_{\mu_A}(z) = \frac{1}{n}\mathrm{tr}(R(z)). \tag{35}$$

For any $1 \leq x, y \leq n$, $z \mapsto R(z)_{xy}$ is an analytic function on $\mathbb{C}_+ \to \mathbb{C}$. Moreover the operator norm of $R(z)$ is at most $\Im(z)^{-1}$.

We see from (35) and Lemma 5.2 that the local law (30) can be rephrased in terms of the resolvent matrix.

Lemma 5.3. *Let $A \in \mathcal{H}_n(\mathbb{C})$ be an Hermitian matrix with resolvent $R(z) = (A - zI_n)^{-1}$. Let $L \geq 1$, K be an interval of $\mathbb{R}$ and $\mu \in \mathcal{P}(\mathbb{R})$ be as in Lemma 5.2. We assume that for some $t > 0$, $0 < \delta < 1/2$ and all $\lambda \in K$,*

$$\left|\frac{1}{n}\mathrm{tr}R(\lambda + it) - g_\mu(\lambda + it)\right| \leq \delta.$$

Then for any interval $I \subset K$ of length $|I| \geq t\big(\frac{1}{\delta}\log\frac{1}{\delta}\big)$ such that $\mathrm{dist}(I, K^c) > 1/L$ we have

$$\frac{|\mu_A(I) - \mu(I)|}{|I|} \leq CL\delta,$$

where C is a universal constant.

Let $\phi \in \mathbb{C}^n$ with $\|\phi\| = 1$. Obviously, from (34), the same statement holds by replacing μ_A by μ_A^ϕ and $\frac{1}{n}\mathrm{tr}R(z)$ by $\langle \phi, R(z)\phi \rangle$.

Proof of Lemma 5.3. Let $s = |I|/t$. By Lemma 5.2, applied with $L' = L + \delta \leq 2L$, for some constant $c > 0$,

$$\frac{|\mu_A(I) - \mu(I)|}{|I|} \leq \frac{2\delta}{\pi} + \frac{2cL}{s}\log(1+s).$$

Now, if $\delta < 1/2$ and $s \geq \frac{1}{\delta}\log\frac{1}{\delta}$, it is easy to check that $\frac{1}{s}\log(1+s) \leq c_0\delta$. $\square$

Simpler bounds are also available. For example, if $I = [\lambda_0 - t, \lambda_0 + t]$ and $z = \lambda_0 + it$ then $\Im((\lambda - z)^{-1}) = t/((\lambda-\lambda_0)^2 + t^2) \geq (1/2t)\mathbf{1}(\lambda \in I)$. We deduce that

$$\mu(I) \leq 2t\Im(g_\mu(z)).$$

In particular,

$$\sum_{k \in \Lambda_I} |v_k(x)|^2 \leq 2t\Im(R_{xx}(z)), \tag{36}$$

where $\Lambda_I = \{k : \lambda_k(A) \in I\}$ and $(v_k)_{1\leq k\leq n}$ is an orthonormal basis of eigenvectors of A. It follows that bounds on the diagonal coefficients of the resolvent when z is close to the real axis will give information on the eigenvectors. Notably, if $k \in \Lambda_I$ and $p \geq 2$,

$$\|v_k\|_p \leq \sqrt{2t}\left(\sum_{x=1}^{n} \Im(R_{xx}(z))^{p/2}\right)^{1/p} \quad \text{and} \quad \|v_k\|_\infty \leq \sqrt{2t \max_{1\leq x\leq n} \Im(R_{xx}(z))}.$$

These bounds could thus be used to check that (31) holds. Similarly, for $p \geq 2$, we find

$$\sum_{k\in\Lambda_I} |v_k(x)|^p \leq \left(\sum_{k\in\Lambda_I} |v_k(x)|^2\right)^{p/2} \leq (2t\Im(R_{xx}(z)))^{p/2}.$$

Hence,

$$\sum_{k\in\Lambda_I}\sum_{x=1}^{n} |v_k(x)|^p \leq (2t)^{p/2}\sum_{x=1}^{n}(\Im(R_{xx}(z)))^{p/2}. \tag{37}$$

It follows that once a local law has been established (to lower bound $|\Lambda_I| = n\mu_A(I)$), the above inequality could be used to upper bound the average of inverse participation ratios defined in (32).

5.3 Local convergence and convergence of the resolvent

The objective of this subsection is to compare the Stieltjes transforms of two measures whose first moments coincide. Roughly speaking, if two probability measures have their first n moments equal then their Cauchy-Stieltjes transform are close for all $z \in \mathbb{C}_+$ such that $\Im(z) \gg 1/n$.

Proposition 5.4. *Let μ_1, μ_2 be two real probability measures such that for any integer $1 \leq k \leq n$,*

$$\int \lambda^k d\mu_1(\lambda) = \int \lambda^k d\mu_2(\lambda).$$

Let $\zeta = e^2\pi$, for any $0 \le a < b$, for all $z \in \mathbb{C}_+$, $|\Re(z)| \le a$ and $\Im(z) \ge \zeta b \lceil \log n \rceil / n$,

$$|g_{\mu_1}(z) - g_{\mu_2}(z)| \le \frac{2}{\zeta n b} + \frac{2}{b-a}.$$

Moreover, if μ_1 and μ_2 have support in $[-b, b]$ then for all $z \in \mathbb{C}^+$ with $\Im(z) \ge \zeta b \lceil \log n \rceil / n$,

$$|g_{\mu_1}(z) - g_{\mu_2}(z)| \le \frac{2}{\zeta n b}.$$

Proof. We set $t = \Im(z)$ and

$$g_z(\lambda) = \frac{1}{\lambda - z}.$$

For integer $k \ge 0$, we have

$$\|\partial^{(k)} g_z\|_\infty = k! t^{-k-1}.$$

From Jackson's theorem [34, Chap. 7, §8], there exists a polynomial p_z of degree n such that for any $\lambda \in [-b, b]$ and $k \le n$,

$$|g_z(\lambda) - p_z(\lambda)| \le \left(\frac{\pi}{2} b\right)^k \frac{(n-k+1)!}{(n+1)!} \|\partial^{(k)} g_z\|_\infty.$$

We take $k = \lceil \log n \rceil \ge 1$ and $t \ge \zeta b \lceil \log n \rceil / n$. We use $k! \le k^k$ and $(\log n)/n \le e^{-1}$ and get,

$$|g_z(\lambda) - p_z(\lambda)| \le \frac{1}{t} \left(\frac{\pi b k}{2t(n+2-k)} \right)^k \le \frac{1}{t} \left(\frac{1}{2e^2} \frac{1}{1 - e^{-1}} \right)^k \le \frac{1}{tn^2} \le \frac{1}{\zeta b n}.$$

The second statement follows.

For the first statement, we use that if $b > |\Re(z)|$, then for any real λ, $|\lambda| \ge b$, we have $|g_z(\lambda)| \le 1/(b - |\Re(z)|)$. In particular, from what precedes,

$$\left| g_\mu(z) - \int_{-b}^{b} p_z(\lambda) d\mu(\lambda) \right| \le \frac{1}{\zeta b n} + \frac{\mu([-b, b]^c)}{b - a}.$$

The conclusion follows. □

As an immediate corollary, we have the following statement.

Corollary 5.5. *For $i = 1, 2$, let (G_i, o) be a finite rooted graph and denote by A_i their adjacency operators which are assumed to be essentially self-adjoint. Assume further that that $(G_1, o)_h$ and $(G_2, o)_h$ are isomorphic. Then for any $b > a$ and all $z \in \mathbb{C}_+$ with $|\Re(z)| \le a$ and $\Im(z) \ge \zeta b \lceil \log 2h \rceil / (2h)$,*

$$\left| \langle e_o, (A_1 - z)^{-1} e_o \rangle - \langle e_o, (A_2 - z)^{-1} e_o \rangle \right| \le \frac{1}{\zeta b h} + \frac{2}{b - a}.$$

Moreover, if for $i = 1, 2$, $\|A_i\| \le b$ then for all $z \in \mathbb{C}_+$, with $\Im(z) = t$,

$$\left| \langle e_o, (A_1 - z)^{-1} e_o \rangle - \langle e_o, (A_1 - z)^{-1} e_o \rangle \right| \le \frac{1}{\zeta b h}.$$

Proof. By assumption and (3), we can apply Proposition 5.4 to $n = 2h$. $\square$

Proposition 5.4 does not require any type of continuity for the measures μ_1 or μ_2. If μ_1 or μ_2 has a bounded support and a bounded density, then it is possible to upper bound the Kolmogorov-Smirnov distance of μ_1 and μ_2. This is a consequence of the Chebyshev-Markov-Stieltjes inequalities, see e.g. Akhiezer [3, Chapter 3] and for their applications in our context see notably [62, 74] and particularly Geisinger [45, Theorem 4].

5.4 Application to tree-like regular graphs

We may now apply the above estimates to study the eigenvectors of tree-like regular graphs. The results of this section are contained in Dumitriu and Pal [37], Brooks and Lindenstrauss [25], Anantharaman and Le Masson, [6, 5] or Geisinger [45].

Let $d \geq 2$, be an integer and let G be a graph with $|V(G)| = n$. We denote by $B(h)$ the number of vertices v in $V(G)$ such that $(G, v)_h$ is not isomorphic $(\mathbb{T}_d, o)_h$ where $\mathbb{T}_d$ is the infinite d-regular tree.

Theorem 5.6 (Local Kesten-McKay law)**.** *Let $0 < \delta < 1$ and assume that there exists $h \geq 1$ such that*

$$\delta \geq \max\left(\frac{hB(h)}{n}, \frac{1}{h}\right).$$

Then, for any interval $I \subset \mathbb{R}$ of length $|I| \geq \frac{20d\log(2h)}{h}\left(\frac{1}{\delta}\log\frac{1}{\delta}\right)$ we have

$$\frac{|\mu_G(I) - \mu_{\mathbb{T}_d}(I)|}{|I|} \leq C\delta,$$

where the constant C depends only on d.

Proof. Let $t = \zeta d\lceil \log 2h\rceil/(2h) \leq \frac{20d\log(2h)}{h}$, $R(z) = (A(G) - zI)^{-1}$, $R'(z) = (A(\mathbb{T}_d) - zI)^{-1}$. We have $R'_{oo}(z) = \langle e_o, R'(z)e_o\rangle = g_{\mu_{\mathbb{T}_d}}(z)$. From Corollary 5.5, we have, if $\Im(z) = t$,

$$\begin{aligned} |g_{\mu_G}(z) - g_{\mu_{\mathbb{T}_d}}(z)| &= \left|\frac{1}{n}\sum_{x=1}^{n} R(z)_{xx} - R'(z)_{oo}\right| \\ &\leq 2\frac{B(h)}{nt} + \frac{1}{\zeta dh} \\ &\leq \frac{4}{\zeta d}\frac{hB(h)}{n} + \frac{1}{\zeta dh}. \end{aligned}$$

By assumption the above expression is bounded $5\delta/(d\zeta) \leq \delta$. It remains to apply Lemma 5.3 and use that there exists a constant c such that $\Im(g_{\mu_{\mathbb{T}_d}}(z)) \leq c$ for all $z \in \mathbb{C}$. $\square$

If G is a uniformly sampled d-regular graph on n vertices (dn even and n large enough), then Theorem 5.6 can be applied with probability tending to one, with $2h = (1 - \varepsilon)\log_{d-1} n$. Indeed, in this case, $B(h) \leq n^{o(1)}(d-1)^{2h} = n^{1-\varepsilon+o(1)}$ with probability tending to one. This follows from known asymptotics on the number of cycles in random regular graphs, see [37, 65].

Theorem 5.6 applies also to d-regular graphs whose girth (lenght of the smallest cycle) is $2h+1$. Indeed, in this case, we simply have $B(h) = 0$.

We can also derive some weak bounds on delocalization of eigenvectors. The main result of Brooks and Lindenstrauss [25] gives however a much stronger statement.

Theorem 5.7 (Weak delocalization of eigenvectors)**.** *For any $\varepsilon > 0$, there exists a subset of eigenvectors $\mathcal{B}^*$ of cardinal at most $B(h)/\varepsilon$ such that for all $k \notin \mathcal{B}^*$ and any subset $S \subset \{1, \cdots, n\}$,*

$$\sum_{x \in S} v_k^2(x) \leq \varepsilon + \frac{C|S|\log h}{h}.$$

where the constant C depends only on d.

Proof. Let $\mathcal{B}$ be the subset of vertices v in $V(G)$ such that $(G, v)_h$ is not isomorphic to $(\mathbb{T}_d, o)_h$. We have

$$\sum_{x \in \mathcal{B}} \sum_{k=1}^{n} v_k^2(x) = B(h).$$

In particular, the set $\mathcal{B}^*$ of eigenvectors such that

$$\sum_{x \in \mathcal{B}} v_k^2(x) \geq \varepsilon$$

has cardinal at most $B(h)/\varepsilon$. Now, take $k \notin \mathcal{B}^*$ and $z = \lambda_k + it$ with $t = 20d\log(2h)/h$, then, from (36) and Corollary 5.5,

$$\begin{aligned}
\sum_{x \in S} v_k^2(x) &\leq \varepsilon + \sum_{x \in S \backslash \mathcal{B}} v_k^2(x) \\
&\leq \varepsilon + \sum_{x \in S \backslash \mathcal{B}} 2t\Im(R_{xx}(z)) \\
&\leq \varepsilon + 2|S|t\Im(g_{\mu_{\mathbb{T}_d}}(z)) + \frac{|S|}{\zeta d h}.
\end{aligned}$$

Now, there exists a constant $c > 0$ such that $\Im(g_{\mu_{\mathbb{T}_d}}(z)) \leq c$ for all $z \in \mathbb{C}$. □

Finally, we can also compute bounds on the average of inverse participation ratios P_I defined by (32).

Theorem 5.8 (Inverse participation ratio)**.** *Let $p > 2$, $L \geq 1$, and $K \subset (-2\sqrt{d-1}, 2\sqrt{d-1})$ be a closed set. There exists a constant h_0 depending on d, K, p, L such that the following holds. If for some $h \geq h_0$,*

$$\frac{h^{p/2} B(h)}{n} \leq L$$

then, for all intervals $I \subset K$ of length at least $C(\log h)/h$,

$$P_I \leq C|I|^{p/2-1},$$

where C is a constant depending d, K, p, L.

Proof. First, since $K \subset (-2\sqrt{d-1}, 2\sqrt{d-1})$, the density of $\mu_{\mathbb{T}_d}$ is lower bounded by some positive constant say $2c_0$ on K. Also,

$$\frac{hB_h}{n} \leq Lh^{1-p/2} = \delta$$

and $h \geq \log(1/\delta) = (p/2-1)\log h - \log L$ if h is large enough. It follows from Theorem 5.6 that

$$\frac{\mu_G(I)}{|I|} \geq 2c_0 - C\delta,$$

for all intervals $I \subset K$ of lenght at least $\frac{20d\log(2h)}{h}\left(\frac{1}{\delta}\log\frac{1}{\delta}\right)$. In particular, if $\delta \leq \delta_0 = c_0/C$ then $\mu_G(I)/|I| \geq c_0$, for all interval $I \subset K$ of length at least $c_1 \log(2h)/h$ with $c_1 = 20d\left(\frac{1}{\delta_0}\log\frac{1}{\delta_0}\right)$. In other words, for all such intervals

$$|\Lambda_I| \geq c_0 n|I|.$$

On the other end, let $h \geq 1$ and $t = 20d\log(2h)/h$. We set $R(z) = (A(G) - zI)^{-1}$, $R'(z) = (A(\mathbb{T}_d) - zI)^{-1}$. We note that $R'_{oo}(z) = \langle e_o, R'(z)e_o\rangle = g_{\mu_{\mathbb{T}_d}}(z)$ is uniformly bounded for all $z \in \mathbb{C}$ by say c. From Corollary 5.5, we have if $(G,x)_h$ and $(\mathbb{T}_d, o)_h$ are isomorphic

$$\left|R_{xx}(z) - R'_{oo}(z)\right| \leq \frac{1}{\zeta dh}.$$

In particular, $|R_{xx}(z)|$ is bounded by $c+1$. We deduce that, for some constant C, C' depending on p, d,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{x=1}^{n}(\Im(R_{xx}(z)))^{p/2} - \left(\Im\left(R'_{oo}(z)\right)\right)^{p/2}\right| &\leq \frac{1}{n}\sum_{x=1}^{n}\frac{p}{2}\left|R_{xx}(z) - R'_{oo}(z)\right|\left(|R_{xx}(z)| \vee |R'_{oo}(z)|\right)^{p/2-1} \\
&\leq C\left(\frac{B(h)}{nt^{p/2}} + \frac{1}{\zeta dh}\right) \\
&\leq C'\left(\frac{h^{p/2}B(h)}{n} + \frac{1}{h}\right).
\end{aligned}$$

By assumption, the above expression is bounded by some constant depending on L, p, d. Hence, from (37), if $I = [\lambda - t, \lambda + t]$ and $z = \lambda + it$, we get

$$\sum_{k\in\Lambda_I}\sum_{x=1}^{n}|v_k(x)|^p \leq |I|^{p/2}\sum_{x=1}^{n}(\Im(R_{xx}(z)))^{p/2} \leq C''n|I|^{p/2}.$$

Putting together this last bound with the lower bound on $|\Lambda_I|$, we conclude the proof. $\square$

Remark that the techniques used here are not really specific to regular graphs. They could be extended to other sequences of graphs G_n with $U(G_n) \to \rho$ for which we have a good understanding of the regularity of the spectral measure $\mu_G^{e_o}$, where (G, o) has distribution ρ. This is done in [18]. We note however that in the present exposition, they are far from being optimal, the bound given by Corollary 5.5 is too rough.

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