Constructive p-adic CM for genus 2 curves

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Classical CM Constructions

By *CM construction* we refer to an algorithm for constructing invariants of abelian varieties with complex multiplication. The traditional approach in genus 1 has been through evaluation of the j-function on an upper half complex plane at special points τ corresponding to lattices with complex multiplication.

The output of this construction will be a minimal polynomial $H_D(x)$ for $j(\tau)$ over $\mathbb{Q}$, called the *Hilbert class polynomials*. This polynomial should be thought of as the defining polynomial for a zero-dimensional subscheme in the j-line $X(1) \cong \mathbb{P}^1$.

Several mathematicians have used functions f on the modular curves $X = X(N)$ or $X = X_0(N)$ such that

$$X \xrightarrow{f} \mathbb{P}^1 \xrightarrow{j} X(1),$$

to find a *class polynomial* $F_D(x)$ which vanishes on $f(\tau)$ for special CM points τ.

An example of this is the class polynomial $F_{-23}(x)$

$$x^3 - x^2 + 1,$$

defined in terms of a Weber function f on $X(48)$, such that

$$j = \frac{(f^{24} - 16)^3}{f^{24}},$$

which generates the same field as the Hilbert class polynomial

$$x^3 + 3491750x^2 - 5151296875x + 12771880859375.$$

For any $j = j(u) \neq 0, 12^3$, coming from a root u of $F_D(x)$ gives rise to an elliptic curve:

$$E : y^2 + xy = x^3 - \frac{36}{j - 12^3}x - \frac{1}{j - 12^3},$$

with complex multiplication. Once $F_D(x)$ has been computed, this provides a construction of elliptic curves of known group order over any finite field (for use in cryptograpy or primality proving).

CM Constructions in Genus 2

This analytic construction has been extended to genus 2 curves, using theta functions evaluated on the Siegel upper half plane, taking values in the moduli space of genus 2 curves $\mathcal{M}_2$ (which we identify with the moduli space of principally polarized abelian surfaces).

A point on this space is a $(j_1, j_2, j_3) \in \mathcal{M}_2(\mathbb{C})$, where j_k are the special values of absolute Igusa invariants. The output of our algorithm will be a set of defining equations for the zero-dimensional scheme of the Galois orbit of this point over $\mathbb{Q}$.

E.g. The curves $y^2 = x^5+1$ and $y^2 = x^6+1$, respectively, have Igusa invariants $(0, 0, 0)$ and $(6400000/3, 440000/9, -32000/81)$, whose defining ideals are:

$$(j_1, j_2, j_3) \text{ and } (3j_1 - 6400000, 9j_2 - 440000, 81j_3 + 32000).$$

It will be our interest today to describe the p-adic canonical lifts to construct the same moduli.

Effective Canonical Lifting

An algorithm for constructing the canonical lift of an elliptic curve was introduced by Satoh in 1999, as a new algorithm for point counting on elliptic curves over field of small characteristic. Various improvements were introduced, and Mestre introduced an alternative algorithm based on the AGM, arising from theta functions. Efficient versions of this latter construction naturally correspond to canonical lifting invariants on $X_0(8)$.

We extract from this algorithm the canonical lift of the j-invariant (see Couveignes–Henocq (2002)), which we may apply to determine high p-adic approximations to the lifted j-invariant.

The algorithm requires successive solution to a modular equation $\Phi_p(x, y)$, defining the image of

$$X_0(p) \to X(1) \times X(1).$$

(See example 1.)

Moduli of Genus 1 Curves

The j-invariant of an elliptic curve E/k classifies E up to isomorphism over an algebraically closed field k. If $\mathrm{char}(k) = \ell \neq 2$, then E can be defined by a Weierstrass equation

$$y^2 = f(x) = x(x-1)(x-t),$$

where t is an invariant of triples (E, P, Q) where P and Q generate the 2-torsion subgroup of E. There are six possible such t associated to each given $j = j(E)$, each is a solution to:

$$j = \frac{2^8(t^2-t+1)^3}{(t-1)^2t^2}.$$

This defines a Galois cover $X(2) \to X(1)$ with group $S_3 \cong \mathrm{PSL}(\mathbb{F}_2)$. This and other modular curves give rise to variants of Satoh's p-adic lifting construction, e.g. Mestre's AGM (Gaudry), AGM-$X_0(N)$ (K.), or Broker–Stevenhagen, for constructing class invariants on these curves. (See example 2.)

Moduli of Genus 2 Curves

A genus 2 curve X/k (in $\mathrm{char}(k) = \ell \neq 2$) is defined by a Weierstrass equation

$$y^2 = f(x),$$

where $f(x)$ is a polynomial of degree 6. Over an algebraic closure, we have

$$X : f(x) = \prod_{i=1}^{6} (x - u_i).$$

The points $(u_i, 0)$ are then both the Weierstrass points and the fixed points of the hyperelliptic involution.

The absolute *Igusa invariants* (j_1, j_2, j_3) of X are defined either in terms of $f(x)$ or, equivalently, by symmetric functions on the set $\{u_i\}$ of roots.

Igusa Invariants

N.B. The projective Igusa invariants are weighted invariants

$$J_2, J_4, J_6, J_8, J_{10},$$

where

$$4J_8 = J_2J_6 - J_4^2$$

and J_{10} is the discriminant of $f(x)$. The absolute invariants are defined by

$$j_1 = \frac{J_2^5}{J_{10}},\ j_2 = \frac{J_2^3J_4}{J_{10}},\ j_3 = \frac{J_2^2J_6}{J_{10}}.$$

The triple (j_1, j_2, j_3) determines a point of the moduli space $\mathcal{M}_2$ of genus 2 curves, a space birational to $\mathbb{A}^3$.

Moduli of Genus 2 Curves with Level Structure

Beginning with the curve $X : y^2 = \prod(x - u_i)$ as above, a linear fractional transformation of the x-line $\mathbb{P}^1$ sends three of the u_i to 0, 1, and ∞. This determines an isomorphism with a curve in *Rosenhain* from:

$$y^2 = x(x-1)(x-t_0)(x-t_1)(x-t_2).$$

The triple (t_0, t_1, t_2) is determined by an ordering on the Weierstrass points, and such a linear fractional transformation. The Weierstrass points generate the 2-torsion subgroup, and an ordered 6-tuple of Weierstrass points determines a full 2-level structure on the Jacobian of X.

N.B. A map $\mathcal{M}_g \to \mathcal{A}_g$, from the moduli space of genus g curve to the moduli space of principally polarised abelian varieties of dimension g is induced by sending a curve to its Jacobian. The map to $\mathcal{A}_g$ allows us to define moduli spaces of curves with level structure.

2 Level Structure

For $g = 2$ this map is a birational isomorphism, and we identify the triple (t_0, t_1, t_2) with a point in the moduli space $\mathcal{M}_2(2)$, classifying genus 2 curves together with a full 2-level structure. The forgetful morphism

$$\begin{aligned} \mathcal{M}_2(2) &\longrightarrow \mathcal{M}_2 \\ (t_0, t_1, t_2) &\longmapsto (j_1, j_2, j_3) \end{aligned}$$

is a Galois covering of degree 720, with Galois group $S_6 \cong \mathrm{Sp}_4(\mathbb{F}_2)$. The former group naturally acts on the Weierstrass points (the first three of which must then be renormalised to $(0, 1, \infty)$). The isomorphic group $\mathrm{Sp}_4(\mathbb{F}_2)$ is that which naturally acts on the 2-torsion subgroup.

The Richelot Correspondence

Given a genus two curve

$$X_1 : y^2 = G_0(x)G_1(x)G_2(x)$$

where each $G_i(x)$ has degree at most 2, we define a second curve

$$X_2 : t^2 = \delta H_0(z)H_1(z)H_2(z),$$

by the equations

$$H_i(x) = G'_{i+1}(x)G_{i+2}(x) - G_{i+1}(x)G'_{i+2}(x),$$

and an explicit constant δ. Then there exists a *Richelot correspondence*

$$C \longrightarrow X_1 \times X_2,$$

where the curve C is defined by

$$C : \begin{cases} G_0(x)H_0(z) + G_1(x)H_1(z) = 0, \\ y^2 = G_0(x)G_1(x)G_2(x), \\ t^2 = \delta H_0(z)H_1(z)H_2(z), \\ yt = G_0(x)H_0(z)(x - z). \end{cases}$$

The Richelot Correspondence

The correspondence $\varphi \times \psi : C \to X_1 \times X_2$ determines a $(2,2)$-isogeny

$$\psi_* \varphi^* : J_1 \to J_2$$

of Jacobians. More importantly, from our point of view, it will let use determine a correspondence of moduli:

$$\mathcal{X} \longrightarrow \mathcal{M}_2(2) \times \mathcal{M}_2(2).$$

The Richelot Correspondence on Moduli

Associated to a point $(t_0, t_1, t_2) \in \mathcal{M}_2(2)$, we can write down a curve

$$X_1 : y^2 = f(x) = x(x-1)(x-t_0)(x-t_1)(x-t_2).$$

A Richelot isogeny is determined setting $f(x) = G_0(x)G_1(x)G_2(x)$, where the $G_i(x)$ are:

$$G_0(x) = x(x-t_0),\ G_1(x) = (x-1)(x-t_1),\ G_2(x) = x - t_2.$$

The curve $X_2 : y^2 = \delta H_0(x)H_1(x)H_2(x)$ is then determined by the triple of polynomials:

$$\begin{aligned} H_0(x) &= x^2 - 2t_2x + t_1t_2 - t_1 + t_2, \\ H_1(x) &= -(x^2 - 2t_2x + t_0t_2), \\ H_2(x) &= (t_0 - t_1 - 1)x^2 + 2t_1x - t_0t_1, \end{aligned}$$

and $\delta = t_0t_2 - t_1t_2 + t_1 - t_2$.

The Richelot Correspondence on Moduli

Let (u_0, u_1, u_2) be a triple of solutions to $H_i(u_i) = 0$, and set

$$(v_0, v_1, v_2) = (2t_2 - u_0, 2t_2 - u_1, 2t_1/(t_0 - t_1 - 1) - u_2)$$

equal to the conjugate solutions. Then

$$X_2 : y^2 = \delta H_0(z)H_1(z)H_2(z) = \delta \prod_{i=0}^{2} (x - u_i)(x - v_i),$$

and a linear fraction transformation sending (u_0, u_1, u_2) to $(0, 1, \infty)$, maps (v_0, v_1, v_2) to a new triple $(s_0, s_1, s_2) \in \mathcal{M}_2(2)$.

The Richelot Modular Correspondence

We summarise by writing down the defining set of polynomials for the previous correspondence. First we have the relations between the t_i's and u_i's:

$$\Phi_0(T_0, T_1, T_2, U_0, U_1, U_2) = U_0^2 - 2T_2U_0 + T_1T_2 - T_1 + T_2,$$
$$\Phi_1(T_0, T_1, T_2, U_1) = U_1^2 - 2T_2U_1 + T_0T_2,$$
$$\Phi_2(T_0, T_1, T_2, U_2) = (T_0 - T_1 - 1)U_2^2 + 2T_1U_2 - T_0T_1.$$

That is, we find $\mathcal{X} \to \mathcal{M}_2(2) \times \mathbb{A}^3$

$$\Phi(x, u) = (\Phi_0(x, u), \Phi_1(x, u), \Phi_2(x, u)) = (0, 0, 0).$$

where $x = (t_0, t_1, t_2)$ and $u = (u_0, u_1, u_2)$.

The Richelot Modular Correspondence

Then we define the second projection to $\mathcal{M}_2(2)$:

$$\psi : \mathcal{M}_2(2) \times \mathbb{A}^3 \xrightarrow{\psi} \mathcal{M}_2(2)$$

by letting ψ be the map

$$((t_0, t_1, t_2), (u_0, u_1, u_2)) \mapsto (u_0, u_1, u_2, v_0, v_1, v_2),$$

followed by the transformation $(s_0, s_1, s_2) = (S(v_0), S(v_1), S(v_3))$, where

$$S(z) = \frac{(u_1 - u_2)(z - u_0)}{(u_1 - u_0)(z - u_2)}.$$

Then the image of $\mathcal{X}$ in $\mathcal{M}_2(2) \times \mathbb{A}^3 \times \mathcal{M}_2(2)$ is defined by

$$\Phi_i(T_0, T_1, T_2, U_0, U_1, U_2) = \Psi_j(T_0, T_1, T_2, U_0, U_1, U_2, S_j) = 0.$$

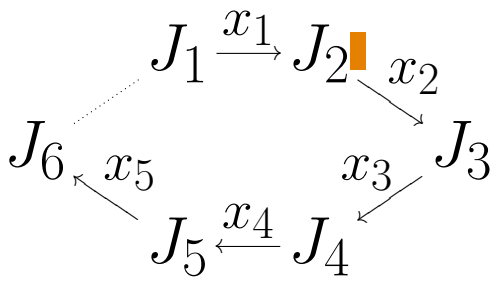
The Richelot Modular Correspondence

In summary, for $(x, u, y) \in \mathcal{M}_2(2) \times \mathbb{A}^3 \times \mathcal{M}_2(2)$ we have

$$\Phi(x,u) = (\Phi_0(x,u), \Phi_1(x,u), \Phi_2(x,u)) = (0,0,0),$$
$$\Psi(x,u,y) = (\Psi_0(x,u,y), \Psi_0(x,u,y), \Psi_0(x,u,y)) = (0,0,0),$$

whose zero set $\mathcal{X}$ admits two finite covers of $\mathcal{M}_2(2)$.

We are interested in pairs $x = (t_0, t_1, t_2)$ and $y = (s_0, s_1, s_2)$ such that $y = x^\sigma$ for some automorphism σ of the base field k. We then set $x_i = x$ and $x_{i+1} = y$. Since each x_i corresponds to a Richelot isogeny $J_i \to J_{i+1}$, the Galois action determines a cycle of isogenies:



The Richelot Modular Correspondence

In fact, with the map ψ as defined above, the point $y = (s_0, s_1, s_2)$ determines the dual isogeny, which can only be the Galois conjugate of $x = (t_0, t_1, t_2)$ if it determines a 2-cycle:

$$J_1 \underset{y}{\overset{x}{\rightleftarrows}} J_2$$

Instead we modify ψ, and Ψ_i, by composing with a permutation of $(u_0, u_1, u_2, v_0, v_1, v_2)$ to find a Galois conjugate 2-level structure. This corresponds to an isomorphism of the curve

$$X_2 : y^2 = \delta x(x-1)(x-s_0)(x-s_1)(x-s_2),$$

to another in Rosenhain form.

Canonical Lifting and Complex Multiplication

Suppose that A/k is an ordinary, simple abelian variety over a finite field of characteristic ℓ. Let R be the unramified extension of $\mathbb{Z}_\ell$ such that $[R : \mathbb{Z}_\ell] = [k : \mathbb{F}_\ell]$. A *canonical lift* is an abelian variety $\tilde{A}/R$ such that

$$\tilde{A}/R \times_R k = A/k \text{ and } \mathrm{End}(\tilde{A}) = \mathrm{End}(A).$$

The main theorem of complex multiplication describes the relation between (certain) ideal classes of a maximal order O_K, isogenies of an abelian variety $A/\overline{\mathbb{Q}}$ with $\mathrm{End}(A) = O_K$, and and the action of Galois on the conjugates of A.

Canonical Lifting and Complex Multiplication

We say that an isogeny φ *splits* $A[n]$ if $\ker(\varphi)$ is a proper subgroup of $A[n]$ and $\ker(\varphi) \not\subseteq A[m]$ for any $m \mid n$. The canonical lift of A/k is determined by:

- A cycle of isogenies $\tilde{A}_1 \to \tilde{A}_2 \to \cdots \to \tilde{A}_r \to \tilde{A}_1$ with $\tilde{A}_1 \times_R k = A$ such that the compositum is an endomorphism of $\tilde{A}_1$ whose kernel splits $\tilde{A}_1[n]$; or
- An isogeny $\varphi : \tilde{A}_1 \to \tilde{A}_2$ with $\tilde{A}_1 \times_R k = A$ such that $\tilde{A}_2 = \tilde{A}_1^{\sigma}$ with $\ker(\varphi)$ splitting $\tilde{A}_1[n]$.

The latter condition, exploiting the Galois action, yeilds a better algorithmic solution to the construction of the canonical lift. As a constructive CM method, we only need to solve for the canonical lift of a moduli point in $\mathcal{M}_2(R)$, and solve a system of equations $\Phi(x, x^{\sigma}) = 0$ for $x \in \mathcal{M}_2(R)$.

Canonical Lifts of Moduli

Recall that we derived a set of defining equations in $\mathcal{M}_2(2) \times \mathbb{A}^3 \times \mathcal{M}_2(2)$,

$$\Phi(x,u) = (\Phi_0(x,u), \Phi_1(x,u), \Phi_2(x,u)) = (0,0,0),$$
$$\Psi(x,u,y) = (\Psi_0(x,u,y), \Psi_0(x,u,y), \Psi_0(x,u,y)) = (0,0,0),$$

where

$$x = (t_0, t_1, t_2) \in \mathcal{M}_2(2),$$
$$u = (u_0, u_1, u_2) \in \mathbb{A}^3,$$
$$y = (s_0, s_1, s_2) \in \mathcal{M}_2(2).$$

In order to preserve the simplicity of these defining equations, we refrain from eliminating $u \in \mathbb{A}^3$ to find relations only in $\mathcal{M}_2(2) \times \mathcal{M}_2(2)$. Also we work with moduli in $\mathcal{M}_2(2)(R)$, with 2-level structure, rather than $\mathcal{M}_2(R)$, and only afterwards compute the image under $\mathcal{M}_2(2) \to \mathcal{M}_2$.

We can solve this system of equations by Hensel's Lemma, first for u, given x, such that

$$\Phi(x, u) = (0, 0, 0),$$

and then for y satisfying

$$\Psi(x, u, y) = (0, 0, 0).$$

However, the resulting y need not converge to x^σ. For this purpose we adapt a method of Harley from the one-dimensional setting (of moduli of genus 1 curves) to higher dimension.

A 3-Adic Example

Let $\mathbb{F}_{27} = \mathbb{F}_3[w]/(w^3 - w + 1)$, and set

$$x = (t_0, t_1, t_2) = (w^{14}, w^8, 2),$$

determining a Galois cycle of length 3. The point

$$y = (s_0, s_1, s_2) = (w^{16}, w^{24}, 2)$$

is the image of x under Frobenius, and defines a second curve related to the first by a Richelot correspondence. Then the 3-adic lifts of these invariants map to a triple of absolute Igusa invariants (j_1, j_2, j_3), satisfying:

$$10460353203j_1^6 - 2064460619497231368 0j_1^5 +$$
$$1584797903444725069000181184j_1^4 -$$
$$57934203669971774729663594299868672j_1^3 -$$
$$4757210399363959986030325710967261851156 48j_1^2 -$$
$$2319410019701066580457483440392962776928771637248j_1 -$$
$$1633610752539414651637667693318669910064037028972986368,$$
$$19683j_2^6 - 3154427913690j_2^5 + 13018458284705642175j_2^4 -$$
$$901184719670502090989387 5j_2^3 -$$
$$4691292251233815299883705732 0000j_2^2 +$$
$$1371934434680672253419375717574400 0000j_2 -$$
$$4251723415703581159078958066726110412 8000000,$$
$$531441j_3^6 - 8007981976085 4j_3^5 + 68165223135645882471 3j_3^4 -$$
$$1621537231026449336569333993j_3^3 -$$
$$1566137192004297839675972173376896j_3^2 -$$
$$1479377322341359891148215922582439772160j_3 -$$
$$9399370213706557076073840873302176 9872651059 2.$$

Appendix: The Method of Harley

In the one-dimensional setting, Harley developed a means of solving a generalised p-adic AGM recursion, determined by a geometric correspondence of moduli:

$$\Phi(x, x^\sigma) = 0,$$

where σ is the Frobenius automorphism. In particular, if x is such a solution, and $x_i \equiv x \bmod p^i$, then we set

$$\delta = \frac{1}{p^i}(x - x_i).$$

We observe that

$$\begin{aligned}\frac{1}{p^i}\Phi(x_i, y_i) + \delta\, \Phi_x(x_i, x_i^\sigma) + \delta^\sigma\, \Phi_x(x_i, x_i^\sigma) \\ \equiv \Phi(x, x^\sigma) \bmod p^i \equiv 0 \bmod p^i.\end{aligned}$$

Thus it comes down to determining δ such that

$$\delta^\sigma \alpha + \delta\beta + \gamma = 0 \bmod p^i.$$

The additional condition $v_p(\beta) > 0$ implies that a unique p-adic solution is determined.

N.B. Such an equation $\Phi(x, y) = 0$ arises as the defining equations for the image modular curve

$$X_0(Np) \longrightarrow X_0(N) \times X_0(N),$$

where $X_0(N)$ is a modular curve of genus 0.

Generalised Method of Harley

In place of a single modular equation $\Phi(x, x^\sigma) = 0$, we need to generalise the method to the multivariate setting. For a solution (x, u, y) with $y = x^\sigma$ to the system of equations

$$\Phi(x, u) = \Psi(x, u, y) = 0,$$

we set $x_i \equiv x \bmod \ell^i$. Then

$$\frac{1}{\ell^i}\Phi(x_i, u_i) + \Delta_x \cdot D_x\Phi(x_i, u_i) + \Delta_u \cdot D_u\Phi(x_i, u_i) \equiv 0 \bmod \ell^i,$$

where

$$\Delta_x = \frac{1}{\ell^i}(x - x_i) \text{ and } \Delta_u = \frac{1}{\ell^i}(u - u_i),$$

$$D_x\Phi(x, u) = \begin{pmatrix} \frac{\partial\Phi_0(x,u)}{\partial t_0} & \frac{\partial\Phi_1(x,u)}{\partial t_0} & \frac{\partial\Phi_2(x,u)}{\partial t_0} \\ \frac{\partial\Phi_0(x,u)}{\partial t_1} & \frac{\partial\Phi_1(x,u)}{\partial t_1} & \frac{\partial\Phi_2(x,u)}{\partial t_1} \\ \frac{\partial\Phi_0(x,u)}{\partial t_2} & \frac{\partial\Phi_1(x,u)}{\partial t_2} & \frac{\partial\Phi_2(x,u)}{\partial t_2} \end{pmatrix}$$

Generalised Method of Harley

and

$$D_u\Phi(x,u) = \begin{pmatrix} \frac{\partial\Phi_0(x,u)}{\partial u_0} & \frac{\partial\Phi_1(x,u)}{\partial u_0} & \frac{\partial\Phi_2(x,u)}{\partial u_0} \\ \frac{\partial\Phi_0(x,u)}{\partial u_1} & \frac{\partial\Phi_1(x,u)}{\partial u_1} & \frac{\partial\Phi_2(x,u)}{\partial u_1} \\ \frac{\partial\Phi_0(x,u)}{\partial u_2} & \frac{\partial\Phi_1(x,u)}{\partial u_2} & \frac{\partial\Phi_2(x,u)}{\partial u_2} \end{pmatrix}.$$

And also

$$\begin{aligned} \tfrac{1}{p^i}\Psi(x_i,u_i,x_i^\sigma) &+ \Delta_x \cdot D_x\Psi(x_i,u_i,x_i^\sigma) \\ &+ \Delta_u \cdot D_u\Psi(x_i,u_i,x_i^\sigma) \\ &+ \Delta_x^\sigma \cdot D_y\Psi(x_i,u_i,x_i^\sigma) \equiv 0 \bmod p^i, \end{aligned}$$

where $D_x\Psi$, $D_u\Psi$, and $D_y\Psi$ are the similarly defined Jacobian matrices.

Generalised Method of Harley

We solve for u_i such that $\Phi(x_i, u_i) \equiv 0 \bmod \ell^{2i}$, then, assuming $D_u\Phi$ is invertible, we may eliminate Δ_u to find an equation

$$\Delta_x^\sigma \cdot A + \Delta_x \cdot B + C \equiv 0 \bmod \ell^i.$$

where

$$\begin{aligned} A &= D_y\Psi, \\ B &= D_x\Psi - D_x\Phi D_u\Phi^{-1}D_u\Psi, \\ C &= \frac{1}{\ell^i}(\Psi) - \frac{1}{\ell^i}(\Phi)D_u\Phi^{-1}D_u\Psi \equiv \frac{1}{\ell^i}(\Psi) \bmod \ell^i \end{aligned}$$

We apply this for input x_i, correct to precision ℓ^i, and u_i such that $\Phi(x_i, u_i) = 0$. This provides a matrix equation which we can solve for the deficiency $\Delta_x \bmod \ell^i$, and set $x_{i+1} = x_i + \ell^i\Delta_x$.

We note that when $B \not\equiv 0 \bmod \ell$, there will generally be multiple solutions to the matrix equation, and we must determine which solution extends to the canonical lift.

This gives a convergent Hensel lifting algorithm for the CM moduli, in which precision doubles with each iteration.

An algebraic relation can be recovered over $\mathbb{Z}$ by means of LLL reduction of the lattice dependency relations betwee powers of j_1, j_2, and j_3.