Classification of Genus 3 Curves in Special Strata of the Moduli Space

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Generic Curves of Genus 3

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3. Arithmetic classification by explicit Galois cohomology.

In what follows we assume that K is a field of characteristic 0; we describe the invariant theory in terms of $K = \mathbb{C}$.

The implicit computational side can be carried out over any ring in which 6 is invertible.

A careful analysis at these exceptional characteristics should yield integral invariants as Igusa carried out for genus 2.

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Invariant theory concerns the classification of forms in $\bar{K}[X_1, \ldots, X_n]_r$ up to $\mathrm{GL}_n(\bar{K})$-equivalence, from which we derive $\mathrm{PGL}_n(\bar{K})$-invariance of an associated projective variety.

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	$Y^2 =$ Binary quartic	where	
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Genus 2			3
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Hyperelliptic locus:			5
	$Y^2 =$ Binary octavic	Shioda (1967)	

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The last two conditions imply that ψ is homogeneous of degree r in the coefficients of a form F.

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An *invariant* is a covariant (or contravariant) of order 0.

Invariant Theory Constructions

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Lemma 1. *Let φ be a covariant and ψ be a contravariant, then $D_\varphi(\psi)$ is a contravariant and $D_\psi(\varphi)$ a covariant, of degree and order summarised in the table below.*

Co/contravariant	*Degree*	*Order*
$D_\varphi(\psi)$	$\mathrm{ord}(\psi) - \mathrm{ord}(\varphi)$	$\deg(\varphi) + \deg(\psi)$
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2. Let φ be a covariant of order 2, and define

$$D(\varphi) = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right)_{ij}$$

and let $D(\varphi)^*$ be the classical adjoint. Define $D(\psi)$ similarly for a contravariant ψ of order 2.

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$$J_{ij} : \mathbb{C}[x_1, \dots, x_n]_2 \times \mathbb{C}[u_1, \dots, u_n]_2 \longrightarrow \mathbb{C},$$

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$$\begin{aligned} J_{11}(\varphi, \psi) &= \langle D(\varphi), D(\psi)\rangle, & J_{n0}(\varphi, \psi) &= \det D(\varphi), \\ J_{n-1,n-1}(\varphi, \psi) &= \langle D(\varphi)^*, D(\psi)^*\rangle, & J_{0n}(\varphi, \psi) &= \det D(\psi), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the vector dot product.

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Lemma 2. *For φ and ψ as above, $J_{ij}(\varphi, \psi)$ is an invariant of degree $i \deg(\varphi) + j \deg(\psi)$.*

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Lemma 2. *For φ and ψ as above, $J_{ij}(\varphi, \psi)$ is an invariant of degree $i \deg(\varphi) + j \deg(\psi)$.*

These constructions give *almost* all of the tools needed to define and compute invariants of plane quartics.

Plane Quartic Invariants

Let $F : \mathbb{C}[x, y, z]_4 \to \mathbb{C}[x, y, z]_4$ be the identity covariant, and define $H = \det D(F)$ to be the Hessian covariant (of degree 3 and order 6).

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With these definitions, we can state:

Theorem 1 (Dixmier, 1987)**.** *Let I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27} be defined by*

$$I_3 = 144^{-1} D_\sigma(F), \qquad I_9 = J_{11}(\tau, \rho), \qquad I_{15} = J_{30}(\tau),$$
$$I_6 = 4608^{-1}(D_\psi(H) - 8 I_3^2), \qquad I_{12} = J_{03}(\rho), \qquad I_{18} = J_{22}(\tau, \rho),$$

with I_{27} the discriminant of the plane quartic. Then $I_3, \ldots I_{27}$ are algebraically independent, and generate a subring of index 50 in the ring of invariants of ternary quartics.

Plane Quartic Invariants [continued]

Theorem/Conjecture 2 (Ohno)**.** *Let J_9, J_{12}, J_{15}, J_{18}, I_{21}, J_{21} be defined by*

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The above invariants generate the ring of ternary quartic invariants as an integral extension of $\mathbb{C}[I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}]$.

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Moduli of Plane Quartics

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Here we focus on those strata of dimension one or zero. These arise as irreducible components of the closed subvarieties of $\mathcal{M}_3^\circ$ with 5, 6, 7, 8, 9, or 12 hyperflexes.

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$\mathcal{X}$	#Hyperflexes	Substrata
Z_6	5	$\Theta, \Pi_i, \Omega_i, \Phi$
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Z_8	5	$\Theta, \Pi_i, \Sigma, \Psi$
Z_2	6	$\Pi_i, \Omega_i, \Phi, \Psi$
Z_3	6	$\Theta, \Pi_i, \Omega_i, \Psi$
Z_5	6	Σ, Φ, Ψ
Z_9	6	Ω_i, Φ, Ψ
Z_4	7	Ω_i, Ψ
Z_1	8	Φ, Ψ

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Dimension zero strata

Our objective is to determine modular equations defining each of the strata in Vermeulen's classification, beginning with the strata of small dimension.

Modular Equations for Special Strata

For each such strata we determined an explicit model of a generic curve $\widetilde{\mathcal{C}}$ in the family. Since Vermeulen worked over an algebraically closed field, he freely assumed that the hyperflexes were rational, and mapped three of them to

$$(0 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 1 : 1).$$

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One Dimensional Strata

Among the one-dimensional strata, we found that Z_8 is isomorphic to an elliptic curve over $\mathbb{Q}$, and the remaining Z_j are isomorphic to $\mathbb{P}^1/\mathbb{Q}$. It remains to determine whether there exists an obstruction to the existence of a universal family $\mathcal{C} \to Z_j$. However, this question can be addressed computationally (as we see below).

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Zero Dimensional Strata

The stratum Φ has as representative the Fermat quartic, and the stratum Ψ is represented by the quartic curve

$$X^4 + Y^4 + Z^4 + 3(X^2Z^2 + X^2Y^2 + Y^2Z^2) = 0.$$

The strata Ω_i is defined over its field of moduli $\mathbb{Q}(\sqrt{7})$, which leaves the question open for the strata Θ, Π_i, and Σ.

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By means of an effective Hilbert Theorem 90 for GL_n (reducing to the one-dimensional case), we obtain representative curves for Θ and Π_i over their fields of moduli:

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$$\begin{aligned} \mathcal{C}_{\Pi_i} : 49X^4 &+ (64s^2 + 60s - 38)X^2Y^2 + (-28s^2 + 14)X^2YZ + (-12s^2 - 108s + 49)Y^4 \\ &+ (24s^2 + 216s - 98)Y^3 Z + (-26s^2 - 150s + 70)Y^2 Z^2 + (14s^2 + 42s - 21)YZ^3, \end{aligned}$$

where s has minimal polynomial $x^3 + x^2 + 4x - 2$.

A Curve with Explicit Obstruction

A representative curve for the zero-dimensiona strata Σ exists over $\mathbb{Q}(\sqrt{-7})$,

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