The Weierstrass subgroup of a curve has maximal rank.

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Abstract We show that the Weierstrass points of the generic curve of genus g over an algebraically closed field of characteristic 0 generate a group of maximal rank in the Jacobian.

The Weierstrass points are a set of distinguished points on curves, which are geometrically intrinsic. In particular, the group these points generate in the Jacobian is a geometric invariant of the curve. A natural question is to determine the structure of this group. For some particular curves with large automorphisms groups (for instance, Fermat curves [15]), these groups have been found to be torsion. The first author provided the first examples where this group has positive rank ([7], [8]) and obtained a lower bound of 11 on the rank of the generic genus 3 curve. The motivation of this paper was to bridge the gap between this bound and the expected bound of 23 – meaning that there are no relations between the Weierstrass points on the generic genus 3 curve. The result we obtain is valid for generic curves of any genus. More precisely, let the *Weierstrass subgroup* of a curve $\mathcal{C}$ be the group generated by the Weierstrass points in the Jacobian of the curve $\mathcal{C}$. We show that

Theorem 1. *The Weierstrass subgroup of the generic curve of genus $g \geq 3$ is $\mathbb{Z}^{g(g^2-1)-1}$.*

As a consequence of this theorem, we deduce the following corollaries.

Corollary 2. *For any field K of characteristic zero, the group generated by the Weierstrass points of a curve over K in its Jacobian is $\mathbb{Z}^{g(g^2-1)-1}$, outside of a set of curves whose moduli lie in a thin set in $\mathcal{M}_g(K)$.*

Corollary 3. *For every $g \leq 13$ there exist infinitely many curves of genus g defined over $\mathbb{Q}$, up to isomorphism over $\bar{\mathbb{Q}}$, for which the group generated by the Weierstrass points in its Jacobian is isomorphic to $\mathbb{Z}^{g(g^2-1)-1}$.*

We start by recalling some basic definitions and properties of Weierstrass points, then some results concerning the behaviour of Weierstrass points under specialisation. We then describe the fundamental tools in our study, which are the natural Galois module structure of the subgroup of divisors with support on the Weierstrass points and the geometric characterisation of the Galois group. Using the specialisation of this Galois module in families, we obtain the main result.

1 The Weierstrass subgroup of a curve.

We recall in this section the definition and some properties of Weierstrass points (see [10] exercise A.4.14).

Let $\mathcal{C}/K$ be a smooth projective curve of genus $g \geq 2$ over a field K of characteristic 0, and let P be any point on $\mathcal{C}$. We say that P is a *Weierstrass point* if and only if there exists a differential form $\omega \in H^\circ(C, \Omega_C)$, such that $\mathrm{ord}_P(\omega) \geq g$. Let $\mathcal{W}$ be the set of Weierstrass points on $\mathcal{C}$.

Alternatively we may characterise the Weierstrass points in terms of the dimensions of Riemann-Roch spaces. For any divisor D on $\mathcal{C}$, let $\mathcal{L}(D)$ be the Riemann-Roch space

$$\{f \in K(\mathcal{C})^* \, : \, \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

and let $\ell(D)$ be its dimension.

Proposition 4. *A point P is Weierstrass if and only if $\ell(gP) \geq 2$.*

The gap sequence associated to a point P is defined to be the set

$$G(P) = \{n \in \mathbb{N} \, : \, \ell(nP) = \ell((n-1)P)\}.$$

We can define the *weight* of a point to be

$$w(P) = \Big(\sum_{n \in G(P)} n \Big) - g(g+1)/2.$$

Positive weight provides yet another characterisation of Weierstrass points and we have a formula for the number of Weierstrass points, counted with multiplicities equal to their weights.

Proposition 5. *A point P is Weierstrass if and only if $w(P) \geq 1$, and $\sum w(P)P$ belongs to the complete linear system*

$$\left| \frac{g(g+1)}{2} K_C \right|.$$

In particular, the sum of the weights of all Weierstrass points is $g(g^2 - 1)$.

We define the *Weierstrass subgroup W* to be the group generated by the differences of Weierstrass points in the Jacobian of the curve identified with $\mathrm{Pic}^\circ(\mathcal{C})$.

2 Specialisation of Weierstrass points.

In this section, we recall two theorems on the behaviour of Weierstrass points under specialisation. For a family $\mathcal{C} \to X$ of smooth projective curves of genus g over an irreducible base X. We define W_η and W_s to be the group generated by the Weierstrass points in the generic fibre $\mathcal{C}_\eta$ and a special $\mathcal{C}_s$, respectively.

Theorem 6. *The group of Weierstrass points form an algebraic family such that W_η surjects on W_s and is injective on torsion.*

Proof: For the first part, see Hubbard [11] or Laksov-Thorup [13], the second part is classic (see [10] Theorem C.1.4). ∎

We furthermore need the following theorem of Néron which provides constraints on the set of rational points for which the generic rank can decrease under specialisation.

Theorem 7 (Néron [14], see Serre [17], p.152). *Let $\pi : \mathcal{A} \to X$ be a family of abelian varieties over a field K. Then the group $\mathcal{A}(K(X))$ is finitely generated, and the set*

$$\{P \in X(K) \mid \mathcal{A}(K(X)) \to \mathcal{A}(K(P)) \text{ is not injective}\}$$

is a thin set.

N.B This form of the theorem appears in Serre [17] with X any open subvariety of $\mathbb{P}^n$, but the proof holds more generally. However, for X of dimension 1 or of general type, the full set $X(K)$ of rational points may be thin for any field K.

3 The generic Galois group.

Let $\mathcal{M}_g$ be the moduli space of curves of genus g, let $\mathcal{C}_g \to \mathcal{M}_g$ be the universal curve over $\mathcal{M}_g$ (i.e. the moduli space of pointed curves), let and $\pi : \mathcal{W}_g \to \mathcal{M}_g$ the restriction to the locus of Weierstrass points. Eisenbud and Harris [5] study the geometric monodromy group of this cover, which can be identified (see Harris [9]) with the geometric Galois group of π, i.e. the group of automorphisms of the normal closure of $\mathbb{C}(\mathcal{W}_g)/\mathbb{C}(\mathcal{M}_g)$. They prove that the monodromy group is as large as possible.

Theorem 8 (Eisenbud-Harris [5]). *The monodromy group of π equals the full symmetric group $S_{g(g^2-1)}$ acting on the $g(g^2 - 1)$ generic Weierstrass points.*

Since $\mathcal{M}_g$ can be defined over $\mathbb{Q}$, and the geometric Galois group is maximal, we conclude that the Galois group of the normal closure of $\mathbb{Q}(\mathcal{W}_g)/\mathbb{Q}(\mathcal{M}_g)$ must also be the full symmetric group. We apply this theorem to the Weierstrass subgroup of the Jacobian, as a Galois module, in order to prove Theorem 1.

4 Galois module structure.

Let $\mathcal{C}/K$ be a curve of genus g, and let $\mathcal{W}$ be its set of Weierstrass points in $\mathcal{C}(\bar{K})$. Then the absolute Galois group $\mathcal{G} = \mathrm{Gal}(\bar{K}/K)$ acts on the set $\mathcal{W}$. Thus the Weierstrass divisor group,

$$V = \bigoplus_{P \in \mathfrak{P}} \mathbb{Z}.P.$$

is equipped with a natural $\mathbb{Z}[\mathcal{G}]$-module structure, which acts through $\mathbb{Z}[G]$, where G is the image of $\mathcal{G}$ in $\mathrm{Aut}(\mathcal{W}) = S_{g(g^2-1)}$ acting as permutations of $\mathcal{W}$.

Theorem 9. *The Weierstrass subgroup of the Jacobian $\mathcal{J}ac(\mathcal{C}_g)$ of the generic curve $\mathcal{C}_g$ is either a free group of rank $g(g^2 - 1) - 1$ or a torsion subgroup.*

Proof: Both the "degree zero" submodule V° of V, generated by differences of Weierstrass points, and its submodule P of principal divisors with support in $\mathcal{W}$ are $\mathbb{Z}[\mathcal{G}]$-submodules. From Theorem 8, we know that $\mathcal{W}$ consists of one orbit of $\mathcal{G}$, which acts through the full symmetric group S_n, where $n = g(g^2 - 1) = |\mathcal{W}|$. Thus $P_\mathbb{Q} = P \otimes_\mathbb{Z} \mathbb{Q}$ is a $\mathbb{Q}[S_n]$-submodule of $V_\mathbb{Q}^\circ = V^\circ \otimes_\mathbb{Z} \mathbb{Q}$. Since $V_\mathbb{Q}^\circ$ is simple as a $\mathbb{Q}[S_n]$-module, it follows that $P_\mathbb{Q}$ is either trivial or equal to $V_\mathbb{Q}^\circ$, and the theorem follows. ∎

5 Weierstrass subgroups of cyclic covers of $\mathbb{P}^1$.

In this section, we find curves of any genus g such that the subgroup generated by the difference of two Weierstrass points P and Q has odd order. Comparing this with the Weierstrass subgroup of a hyperelliptic curve, we establish that the generic Weierstrass subgroup can not be a torsion subgroup. First, we state the classical result for the Weierstrass subgroup of a hyperelliptic curve.

Proposition 10. *The Weierstrass points of an hyperelliptic curve of genus g generate the 2-torsion subgroup of its Jacobian.*

In any genus, there exists cyclic trigonal covers of the projective line. Such covers can be described (see [3] and [12]) by the plane model

$$C : y^3 = \prod_{i=1}^{s}(x - \alpha_i)\prod_{j=1}^{t}(x - \beta_j)^2$$

where α_i and β_j are distinct complex numbers and s and t satisfy $s + 2t \equiv 0 \bmod 3$ and $t < s$. The genus of C is then equal to $g = s + t - 2$.

Proposition 11. *There exists a curve of genus g with two Weierstrass points whose difference is a point of order 3 in the Jacobian.*

Proof: We take C a trigonal curve as defined above, of genus $g > 2$, with t in $\{0, 1, 2\}$ such that $t \equiv -g + 1 \pmod 3$, and with $s = g - t + 2 \geq 2$. Then there exist two nonsingular points $P_1 = (\alpha_1, 0)$ and $P_2 = (\alpha_2, 0)$. The functions $f = (x - \alpha_1)/(x - \alpha_2)$ and $1/f$ are respectively in $\mathcal{L}(3P_2)$ and in $\mathcal{L}(3P_1)$, and thus the points P_1, P_2 are Weierstrass points. Moreover, since $\mathrm{div}(f) = 3(P_1 - P_2)$ it follows that $P_1 - P_2$ is a 3-torsion point in the Weierstrass subgroup of the Jacobian of C. ∎

6 Proof of the main theorem.

We are now in a position to prove:

Theorem 1. *The Weierstrass subgroup of the generic curve of genus $g \geq 3$ is $\mathbb{Z}^{g(g^2-1)-1}$.*

Proof: By Corollary 9, the generic Weierstrass subgroup is either a free group or is purely torsion. In the latter case, Theorem 6 implies that the generic Weierstrass subgroup is isomorphic with the Weierstrass subgroup of every special curve. We first consider the moduli space $T = \mathcal{M}_g^{(m)}$ with m-level structure. For $m \geq 3$, the space T is a fine moduli space, with universal cover $\mathcal{D}_g \to T$ such that each fibre C_t is a curve of genus g whose isomorphism class determines the moduli point $\pi(t)$ on $\mathcal{M}_g$ (see e.g. [6]). On the other hand, the finite covers $\mathcal{D}_g \to \mathcal{C}_g$ and $T \to \mathcal{M}_g$ determine a birational morphism of $\mathcal{D}_g$ to the fibre product $\mathcal{C}_g \times_{\mathcal{M}_g} T$, by which we may identify the generic Weierstrass subgroup W_η with the generic Weierstrass subgroup over T (since then $\mathcal{D}_g/\mathbb{Q}(T)$ is isomorphic to $\mathcal{C}_g/\mathbb{Q}(T)$). By specialising to a hyperelliptic curve C_t, Proposition 10 implies that, if torsion, the generic Weierstrass subgroup must equal the 2-torsion subgroup. This contradicts the result of Proposition 11 which implies that it must surject on a subgroup of order 3. We conclude that W_η contains a point of infinite order, and thus is free of maximal rank. ∎

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Using Néron's specialisation Theorem 7, we obtain the following corollary (this requires a statement of Néron's theorem for arbitrary base).

Corollary 2. *For any field K of characteristic zero, the group generated by the Weierstrass points of a curve over K in its Jacobian is $\mathbb{Z}^{g(g^2-1)-1}$, outside of a set of curves whose moduli lie in a thin set in $\mathcal{M}_g(K)$.*

For $g \leq 6$ the moduli space $\mathcal{M}_g$ is rational, and the complement of a thin set in $\mathcal{M}_g(\mathbb{Q})$ provides a dense set in $\mathcal{M}_g$ consisting of moduli of curves whose Weierstrass subgroup has maximal rank. More generally, this latter property holds in $\mathcal{M}_g(\mathbb{Q})$ for all g up to 13.

Corollary 3. *For each $g \leq 13$, the curves of genus g over $\mathbb{Q}$ for which the group generated by the Weierstrass points in its Jacobian is isomorphic to $\mathbb{Z}^{g(g^2-1)-1}$, determine a Zariski dense set of moduli in $\mathcal{M}_g$.*

Proof: For each $g \leq 13$ the moduli space $\mathcal{M}_g$ is unirational [1, 2, 16], i.e. $\mathcal{M}_g$ is covered by a dominant map $\pi : \mathbb{P}^N \to \mathcal{M}_g$ for some N. We base extend by π to form the family $\mathcal{C}_g \times_\pi \mathbb{P}^N \to \mathbb{P}^N$, with generic Weierstrass group isomorphic to $\mathbb{Z}^{g(g^2-1)}$. Applying Theorem 7 of Néron, the Weierstrass group remains free of rank $g(g^2-1)$ outside of a thin set Z in $\mathbb{P}^N(\mathbb{Q})$. Then the set $\pi(\mathbb{P}^N(\mathbb{Q}) - Z)$ is a dense set in $\mathcal{M}_g$ consisting of moduli of curves for which the Weierstrass subgroup attains the maximal rank. ∎

We note that for $g \geq 24$, the variety $\mathcal{M}_g$ is known to be of general type, thus Lang's Conjecture would imply that the set $\mathcal{M}_g(K)$ of rational points over any number field K is contained in a proper closed subvariety. An analogous result to Theorem 8 for the Galois group of Weierstrass points of plane curves of degree d would be desirable in order to establish the maximality of the rank of the Weierstrass subgroup for plane curves.

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