# The geometry of efficient arithmetic on elliptic curves

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# Elliptic curve cryptography

In 1985, Miller and Koblitz introduced the use of elliptic curves in cryptography. This replaced $\mathbb{F}_p^* = \mathbb{G}_m(\mathbb{F}_p)$ (in the protocols of Diffie and Hellman or ElGamal) with the group $E(\mathbb{F}_p)$ of rational points on an elliptic curve $E/\mathbb{F}_p$.

This was made possible by the introduction of a polynomial-time algorithm of Schoof for computing the cardinality $|E(\mathbb{F}_p)|$ in the same year.

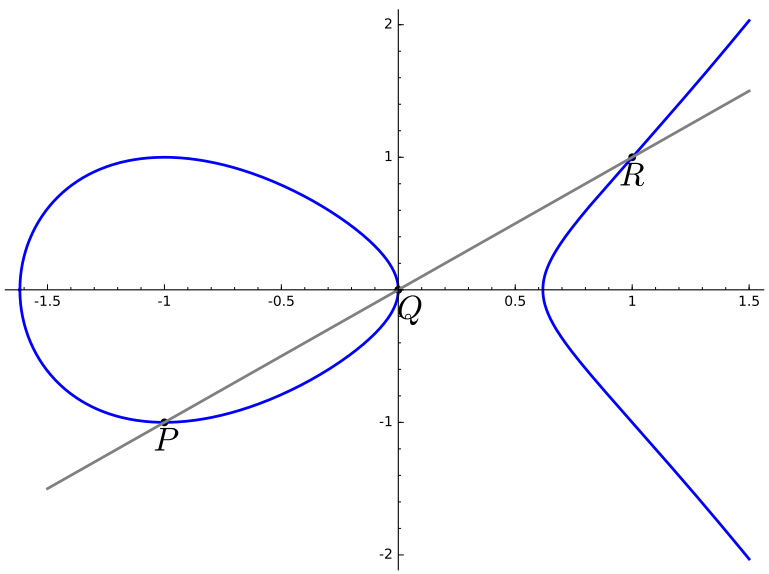
The default model for computing in $E(\mathbb{F}_p)$ involved embedding $E$ as a Weierstrass model $Y^2Z = X^3 + aXZ^2 + bZ^3$ in $\mathbb{P}^2$, with identity $O = (0 : 1 : 0)$. We focus on the role of the choice of model on the algorithms for elliptic curve arithmetic.

# Addition morphism

On a Weierstrass model $E$, the addition morphism is defined by the rule "three points on a line $L$ sum to $O$". This interprets the relation

$$L.E = (P) + (Q) + (R) \sim 3(O) = L_\infty.E,$$

where $L_\infty = V(Z)$ is the line at infinity, in the Picard group of $E$.



# Rational addition law on Weierstrass model

This rule determines the addition morphism $\mu : E \times E \to E$ on all affine points

$$P_1 = (x_1, y_1) = (x_1 : y_1 : 1) \text{ and } P_2 = (x_2, y_2) = (x_2 : y_2 : 1)$$

with $x_1 \neq x_2$, by setting

$$\lambda = \frac{y_1 - y_2}{x_1 - x_2}, \quad \nu = \frac{x_1 y_2 - y_1 x_2}{x_1 - x_2}.$$

Then $P_1 + P_2 = P_3 = (x_3, y_3) = (\lambda^2 - x_1 - x_2, -\lambda x_3 - \nu)$.

**N.B.** This defines $\mu$ outside of the *diagonal* $\Delta$, the *antidiagonal* $\nabla$, the *horizontal* $H = E \times \{O\}$ and *vertical* $V = \{O\} \times E$ divisors.

# A projective addition law on Weierstrass model

Projectively, in $(X_1, Y_1, Z_1)$ and $(X_2, Y_2, Z_2)$, we have

$$\lambda = \frac{Y_1 Z_2 - Z_1 Y_2}{X_1 Z_2 - Z_1 X_2}, \quad \nu = \frac{X_1 Y_2 - Y_1 X_2}{X_1 Z_2 - Z_1 X_2},$$

and then

$$x_3 = \left(\frac{Y_1 Z_2 - Z_1 Y_2}{X_1 Z_2 - Z_1 X_2}\right)^2 - \frac{X_1 Z_2 + Z_1 X_2}{Z_1 Z_2},$$

$$y_3 = -\left(\frac{Y_1 Z_2 - Z_1 Y_2}{X_1 Z_2 - Z_1 X_2}\right) x_3 - \frac{X_1 Y_2 - Y_1 X_2}{X_1 Z_2 - Z_1 X_2}.$$

Clearing denominators we obtain bihomogeneous polynomial expressions $(X_3, Y_3, Z_3)$ of bidegree $(4, 4)$ in the input points, or $(3, 3)$ after exploiting a cancellation of $Z_1 Z_2$ in $x_3$.

# Projective addition laws on Weierstrass models

It was well-known (after Lange & Ruppert) that there exists a finite dimensional space of bidegree $(2,2)$ *addition laws* (homogeneous polynomial maps for $\mu$), and that any nonzero addition law fails to be defined on some divisor on $E \times E$, its *exceptional divisor*.

Bosma & Lenstra computed explicit bidegree $(2,2)$ addition laws for a Weierstrass model $E$, which span a 3-dimensional space, and showed that two addition laws suffice to define $\mu$ globally.

However, compared to the bidegree $(4,4)$ addition law defined by

$$(X_3(X_1Z_2 - Z_1X_2),\ Y_3,\ (X_1Z_2 - Z_1X_2)^3 Z_1Z_2),$$

where

$$\begin{aligned} X_3 &= (Y_1Z_2 - Z_1Y_2)^2 Z_1Z_2 - (X_1Z_2 + Z_1X_2)(X_1Z_2 - Z_1X_2)^2, \\ Y_3 &= -(Y_1Z_2 - Z_1Y_2)X_3 - (X_1Y_2 - Y_1X_2)(X_1Z_2 - Z_1X_2)^2 Z_1Z_2, \end{aligned}$$

the bidegree $(2,2)$ polynomials are too cumbersome to be practical.

# Scalar multiplication

The principal operation in elliptic curve cryptography is scalar multiplication. To carry out $[n] : E \to E$, the doubling morphism [2] plays an important role. If $n_r \ldots n_0$ is the binary representation of $n$, then

$$[n]P = \sum_{i=0}^{r} n_i [2^i]P,$$

and we can determine $[n]P (= Q_r)$ by calculating in parallel the sequences $(P_i)$ and $(Q_i)$ with $P_0 = P$, $Q_0 = n_0 P$,

$$P_i = [2^i]P = [2]P_{i-1}, \text{ and } Q_i = Q_{i-1} + n_i P_i.$$

Using windowing methods, the number of calls to [2] exceeds additions, and it becomes important to have efficient doubling.

# Arithmetic on Weierstrass models

If $E$ has a rational 2-torsion point, then a speed-up can be obtained by an isogeny decomposition

$$[2] = \varphi \circ \hat{\varphi}.$$

Optimally the isogenies $\varphi$ and $\hat{\varphi}$ are given by quadratic polynomails.

Unfortunately for Weierstrass models, no quadratic polynomials defining a 2-isogeny of Weierstrass models can exist.

Worse, any polynomial map (necessarily of degree $\geq 3$) must fail on some subset of points.

In contrast, quartic models in $\mathbb{P}^3$ admit 2-isogenies defined by quadratic polynomials.

# Edwards model of elliptic curves

In 2007, Edwards introduced a model of elliptic curve with remarkable properties. Bernstein, Lange, et al. carried out a descent of the base field and twist , we obtain the *twisted Edwards model*

$$E : ax^2 + y^2 = 1 + dz^2,\ z = xy,$$

with identity $O = (0, 1, 0)$. This embeds via $(1 : x : y : z)$ as the project model in $\mathbb{P}^3$

$$aX_1^2 + X_2^2 = X_0^2 + dX_3^3,\ X_0X_3 = X_1X_2.$$

This model combines features of efficient doubling and addition laws, combined with arithmetic completeness — if $d$ and $d/a$ are nonsquares in $k^*$, then a single addition law is valid for all points over $k$.

# Edwards addition law

The interest in Edwards model is the simple rational addition law:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_3, y_3, z_3),$$

given by

$$x_3 = \frac{x_1 y_2 + y_1 x_2}{1 + d z_1 z_2} \text{ and } y_3 = \frac{-a x_1 x_2 + y_1 y_2}{1 - d z_1 z_2} \text{ with } z_3 = x_3 y_3.$$

In terms of the projective model in $\mathbb{P}^3$, this gives

$$P + Q = (U_0 V_0 : U_0 V_1 : U_1 V_0 : U_1 V_1)$$

where each $U_i$, $V_j$ are bilinear in the coordinates of $P$ and $Q$:

$$(U_0, U_1) \in \left\{ \begin{array}{c} (X_2 Y_1 - X_1 Y_2,\ X_0 Y_3 - X_3 Y_0), \\ (X_0 Y_0 - d X_3 Y_3,\ X_2 Y_2 - a X_1 Y_1) \end{array} \right\}$$

$$(V_0, V_1) \in \left\{ \begin{array}{c} (a X_1 Y_1 + X_2 Y_2,\ X_0 Y_3 + X_3 Y_0), \\ (X_0 Y_0 + d X_3 Y_3,\ X_2 Y_1 + X_1 Y_2) \end{array} \right\}$$

# Edwards arithmetic

In comparison to the multipage formulas for bidegree $(2, 2)$ addition laws on the Weierstrass model, these addition laws on Edwards' model is strikingly elegant and efficient.

Moreover, specializing to $X_i = Y_i$, we find simple expressions for doubling as well, which also are everywhere defined.

These results spawned a minor industry of optimization of the pair of elliptic curve model and algorithms for evaluating their addition and doubling laws.

# Elliptic curve models

The previous discussion motivates the study of elliptic curves with given *projective model*, by which we mean $E/k$ with given embedding $\iota : E \to \mathbb{P}^r$.

The addition and doubling laws are polynomial expressions in terms of the coordinate functions $X_0, \ldots, X_r$, which play an intrinsic role in their definition.

To understand elliptic curve arithmetic, rather than classifying curves up to arbitrary isomorphism, we will be interested in elliptic curves up to *projective linear isomorphism*.

Such a classification determines the complexity of arithmetic on elliptic curves, up to additions and multiplications by constants.

# Embeddings by complete linear systems

The classification is simplified by the imposed condition that the curve model is given by a complete linear system with respect to an effective divisor $D = (P_0) + \cdots + (P_r)$.

If the Riemann-Roch space has basis $(1, x_1, \ldots, x_r)$, the map

$$P \longmapsto (1 : x_1(P) : \cdots : x_r(P)).$$

gives an embedding in $E \to \mathbb{P}^r$ $(r = \deg(D) - 1)$ such that the line $X_0 = 0$ then cuts out $D$.

The question of projective linear isomorphism between two models for $E$ is reduced to whether the two divisors are linearly equivalent.

This equivalence class, in turn, is uniquely determined by the degree $d = r + 1$ and the point $P = P_0 + \cdots + P_r \in E(k)$.

# Elliptic curve models

The fastest arithmetic is observed for elliptic curve models with a high degree of symmetry. Such symmetries come from $[-1]$ and translation by points in a finite subgroup $G \subset E[d]$.

This suggests the study of elliptic curve models $E \to \mathbb{P}^r$, embedding with respect to a divisor $D$ such that:

1. $[-1]$ is a projective linear transformation.
2. Translation by $P \in G$ acts by projective linear transformation.

Moreover by choose the basis of coordinate functions, so that $d$-torsion points act by a combination of coordinate permutation and scalar multiplication by roots of unity, the resulting addition laws take a particularly simple form.

# Hessian normal form

For cubic models, it is natural to investigate models with 3-torsion level structure, which acts linearly. The Hessian normal form is an embedded cubic curve $H \subset \mathbb{P}^2$ given by

$$X^3 + Y^3 + Z^3 = dXYZ,$$

and with identity $(0 : 1 : -1)$. The 3-torsion subgroup decomposes $H[3] \cong \boldsymbol{\mu}_3 \times \mathbb{Z}/3\mathbb{Z}$, such that $P = (1 : \zeta_3 : \zeta_3^2) \in \boldsymbol{\mu}_3$ acts by

$$(X : Y : Z) \mapsto (X : \zeta_3 Y : \zeta_3^2 Z),$$

$Q = (1 : -1 : 0) \in \mathbb{Z}/3ZZ$ acts by $(X : Y : Z) \mapsto (Y : Z : X)$, and

$$[-1](X : Y : Z) = (X : Z : Y).$$

The line $X_0 = 0$ cuts out a subgroup

$$\{(0, 1, -1), (0, \zeta_3, -\zeta_3^2), (0, \zeta_3^2, -\zeta_3)\}$$

isomorphic to $\boldsymbol{\mu}_3 = \{1, \zeta_3, \zeta_3^2\}$.

# Split $\mu_4$-normal form

The split $\mu_4$-normal form $C \subset \mathbb{P}^3$ is the elliptic curve model

$$X_0^2 - X_2^2 = c^2 X_1 X_3, \;\; X_1^2 - X_3^2 = c^2 X_0 X_2,$$

with identity $O = (c : 1 : 0 : 1)$. A subgroup isomorphic to $\mu_4 = \{1, i, -1, -i\}$ is cut out by $X_2 = 0$:

$$\{(c : 1 : 0 : 1), (c : i : 0 : i), (c : -1 : 0 : -1), (c : -i : 0 : -i)\}.$$

We note that this model is isomorphic to the $-1$-twist of an Edwards curves, which admits addition laws with the most efficient known evaluation algorithm, but has good reduction at 2.

The split $\mu_4$-normal form is analogous to the above Hessian normal form in that it parametrizes a full level-4 structure.

# Explicit addition laws

We recall that, by a theorem of Lange and Ruppert, the minimal bidegree of any addition law is $(2, 2)$, and for an elliptic curve model of degree $d$, the laws of this minimal bidegree span a space of dimension $d$.

We give the form of these addition laws, and the resulting complexity for the above models.

# Addition laws: Hessian model

**Theorem**

*The space of addition laws of bidegree* $(2,2)$ *on* $H$ *is spanned by:*

$$\begin{array}{l}(X_1^2Y_2Z_2 - Y_1Z_1X_2^2,\ Z_1^2X_2Y_2 - X_1Y_1Z_2^2,\ Y_1^2X_2Z_2 - X_1Z_1Y_2^2),\\ (X_1Y_1Y_2^2 - Z_1^2X_2Z_2,\ X_1Z_1X_2^2 - Y_1^2Y_2Z_2,\ Y_1Z_1Z_2^2 - X_1^2X_2Y_2),\\ (X_1Z_1Z_2^2 - Y_1^2X_2Y_2,\ Y_1Z_1Y_2^2 - X_1^2X_2Z_2,\ X_1Y_1X_2^2 - Z_1^2Y_2Z_2).\end{array}$$

As a consequence, the doubling map sends $(X, Y, Z)$ to

$$(X(Y^3 - Z^3), (X^3 - Y^3)Z, Y(Z^3 - X^3)).$$

The best known algorithms for evaluating these maps gives a complexity of $12\mathbf{M}$ for addition and $7\mathbf{M} + 1\mathbf{S}$ for doubling.

# Addition laws: split $\mu_4$-normal form

**Theorem**

*The space of addition laws of bidegree* $(2,2)$ *for* $C$ *is spanned by :*

$$(X_{13}^2 - X_{31}^2,\ c(X_{13}X_{20} - X_{31}X_{02}),\ X_{20}^2 - X_{02}^2,\ c(X_{20}X_{31} - X_{13}X_{02})),$$
$$(c(X_{03}X_{10} + X_{21}X_{32}),\ X_{10}^2 - X_{32}^2,\ c(X_{03}X_{32} + X_{10}X_{21}),\ X_{03}^2 - X_{21}^2),$$
$$(X_{00}^2 - X_{22}^2,\ c(X_{00}X_{11} - X_{22}X_{33}),\ X_{11}^2 - X_{33}^2,\ c(X_{00}X_{33} - X_{11}X_{22})),$$
$$(c(X_{01}X_{30} + X_{12}X_{23}),\ X_{01}^2 - X_{23}^2,\ c(X_{01}X_{12} + X_{23}X_{30}),\ X_{30}^2 - X_{12}^2),$$

*where* $X_{ij} = X_i Y_j$.

and the doubling map sends $(X_0, X_1, X_2, X_3)$ to

$$(X_0^4 - X_2^4, cX_0^2X_1^2 - cX_2^2X_3^2, X_1^4 - X_3^4, -cX_1^2X_2^2 + cX_0^2X_3^2).$$

We deduce the best known complexity for their evaluation: $8\mathbf{M}$ for addition ($7\mathbf{M} + 2\mathbf{S}$ in char. 2) and $4\mathbf{M} + 3\mathbf{S}$ ($2\mathbf{M} + 5\mathbf{S}$ in char. 2).