The Mathematics of Sudoku
Hippocampe

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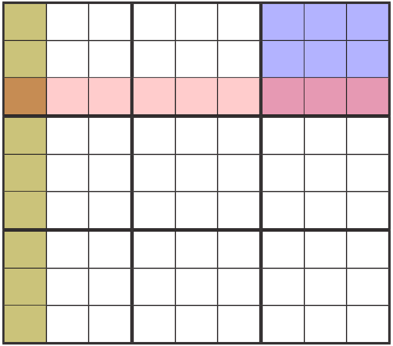
17 octobre 2016

The Sudoku grid

A Sudoku puzzle is a game played in a 9×9 grid (or array), divided into nine 3×3 blocks, nine rows, and nine columns



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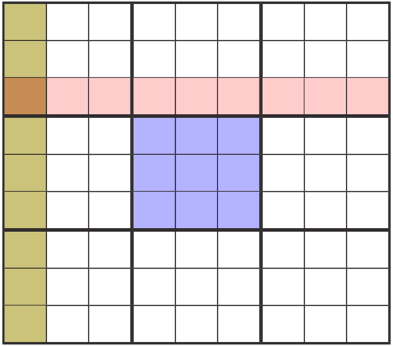

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Sudoku rules

A Sudoku puzzle $\mathcal{S}$ is a partically filled 9×9 grid is to be filled with the numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ or from any other *alphabet* $\mathcal{A}$ of 9 symbols, such as

$$\mathcal{A} = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}\},$$

such that

1. there is no repetition in any Sudoku component, and
2. the completed puzzle respects the *initial conditions* — symbols of $\mathcal{A}$ already placed in the grid.

If at least one such solution $\mathcal{S}_0$ exists the Sudoku puzzle is said to be *consistent*. If a unique solution exist $\mathcal{S}$ is said to be *valid*.

Sudoku example

An example of a valid Sudoku puzzle, which we use as illustration of the methods of solving Sudoku throughout this document, is the following:

5				8			4	9
			5				3	
	6	7	3					1
1	5							
			2		8			
							1	8
7					4	1	5	
	3				2			
4	9			5				3

Sudoku example

An example of a valid Sudoku puzzle, which we use as illustration of the methods of solving Sudoku throughout this document, is the following:

E				H			D	I
			E				C	
	F	G	C					A
A	E							
			B		H			
							A	H
G					D	A	E	
	C				B			
D	I			E				C

over the alphabet $\mathcal{A} = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}\}$.

Binary Sudoku mask

For each k in the alphabet $\mathcal{A}$, if we make a substitution

$$k \mapsto 1 \text{ and } \ell \mapsto 0 \text{ for each } \ell \neq k \text{ in } \mathcal{A},$$

we obtain the *binary Sudoku k-mask*. Taking the above example, the masks for $k = \texttt{A}$ (or 1) and $k = \texttt{E}$ (or 5) are:

0				0			0	0
			0				0	
	0	0	0					1
1	0							
			0		0			
							1	0
0					0	1	0	
	0				0			
0	0			0				0

1				0			0	0
			1				0	
	0	0	0					0
0	1							
			0		0			
							0	0
0					0	0	1	
	0				0			
0	0			1				0

Sudoku logic

Denote by $\mathcal{S}_0(i,j)$ the entry of $\mathcal{S}_0$ in the i-th row and j-th column, and

let $\mathcal{B}_0(i,j)$ be the entry of $\mathcal{B}_0$ in the i-th row and j-th column.

We want to interpret the construction of a k-mask as a logical statement. We set $\mathbb{B} = \{0, 1\}$ and identify

$$0 = \texttt{False} \text{ and } 1 = \texttt{True}.$$

If $\mathcal{B}_0$ is the k-mask of $\mathcal{S}_0$ then the entry $\mathcal{B}_0(i,j)$ is a response to the question:

$$\text{“Is } \mathcal{S}_0(i,j) = k\text{?”}$$

Sudoku solutions and compatible masks

Consider the following solution $\mathcal{S}_0$ to a Sudoku puzzle, with rows and columns labelled.

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	4	5	6	7	8	9	1	2	3
3	7	8	9	1	2	3	4	5	6
4	9	1	2	3	4	5	6	7	8
5	3	4	5	6	7	8	9	1	2
6	6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6	7
8	2	3	4	5	6	7	8	9	1
9	5	6	7	8	9	1	2	3	4

Write down the 9 Sudoku masks. What properties do you observe?

Boolean arithmetic

We call $\mathbb{B} = \{0, 1\}$ the boolean domain, interpretting as above 0 as `False` (or `F`) and 1 as `True` (or `T`), reflecting their roles in logic. Its elements are called booleans, after the English mathematician George Boole (1815–1864).

We now define how to do arithmetic on the elements of $\mathbb{B}$.

Addition. An addition, denoted by '$+$', is determined by the *exclusive or* operator `XOR` on booleans. This gives the addition table:

$+$	0	1
0	0	1
1	1	0

equivalent to

XOR	F	T
F	F	T
T	T	F

In particular we note that $1 + 1 = 0$.

Boolean arithmetic

Multiplication. The operation of multiplication, denoted by '$\cdot$', is given by the logical operator `AND`,

$$\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \text{equivalent to} \quad \begin{array}{c|cc} \texttt{AND} & \texttt{F} & \texttt{T} \\ \hline \texttt{F} & \texttt{F} & \texttt{F} \\ \texttt{T} & \texttt{F} & \texttt{T} \end{array}$$

The notation $x \cdot y$ for the product of x and y is abbreviated xy.

Binary Sudoku

A *binary Sudoku puzzle* $\mathcal{B}$ is played on a the same 9×9 Sudoku grid, with the same Sudoku components of rows, columns, and blocks, to be filled with elements of $\mathbb{B}$. In particular, the binary Sudoku puzzle is a partially filled grid such that

1. there is exactly one 1 in any Sudoku component, and
2. the completed puzzle respects the *initial conditions* — symbols of $\mathbb{B}$ already placed in the grid.

A *solution* $\mathcal{B}_0$ is a completed grid satisfying the above conditions for $\mathcal{B}$.

Binary Sudoku supports

The *support* $S_1(\mathcal{B}_0)$ of a mask solution $\mathcal{B}_0$ is the set of positions (i, j) for which there is a 1 in the i-th row and j-th column.

We denote $S_0(\mathcal{B}_0)$ its *complement* — the positions for which there is a 0 in the i-th row and j-th column.

Group projects

Solving strategies. Consider the combinatorial constraints on the supports $S_1(\mathcal{B}_0)$ and their complements $S_0(\mathcal{B}_0)$ of binary Sudoku masks, and algebraic relations that they satisfy.

Reverse Shidoku and Sudoku. A *Shidoku* is a version of Sudoku on a 4×4 grid, with an alphabet $\mathcal{A}$ of four elements. Consider the problem of constructing valid Shidoku and Sudoku and counting the numbers of Shidoku solutions.

Bipartite graphs. Let $\mathcal{B}_0$ be a binary Sudoku mask. When there exists an pair of positions (i_0, j_0) and (i_1, j_1) at which the mask is unknown, and such that all other positions in that component are 0, then

$$(\mathcal{B}_0(i_0, j_0), \mathcal{B}_0(i_1, j_1)) = (0, 1) \text{ or } (\mathcal{B}_0(i_0, j_0), \mathcal{B}_0(i_1, j_1)) = (1, 0).$$

Connecting such pairs we find that traversing any path in the resulting graph must alternate values.