Sato-Tate and notions of generality

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Explicit point counting and cryptography

The original motivation for this talk is to understand the properties of RM curves with a view towards cryptography. For example, in the family considered by Tautz, Top, and Verberkmoes:

$$\mathcal{C} : y^2 = x^5 - 5x^3 + 5x + t,$$

over $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Q}[t])$, the fibers have Jacobians with real multiplication by $\mathbb{Z}[\phi] = \mathbb{Z}[(1+\sqrt{5})/2]$. Moreover, the real endomorphism ϕ is explicitly computable (K. & Smith), and with Gaudry and Smith we developed:

Theorem (Gaudry,K. & Smith)

There exists an algorithm for the point counting problem in a family of genus 2 curves with efficiently computable RM of class number 1, whose complexity is in $\widetilde{O}((\log q)^5)$.

Explicit point counting example

Let $q = 2^{512} + 1273$ (prime), and consider the curve over $\mathbb{F}_q$ from the family $\mathcal{C}$, specialized at (random)

$$\begin{aligned} t = \; & 29085666333787272437998261129919801749 \\ & 74533003680957762232569868073752702720 \\ & 14471477919882845604269700820270816721 \\ & 53243497592108531656059083265912235127 8. \end{aligned}$$

Let π be the Frobenius endomorphism, and set $\pi + \bar{\pi} = m + n\phi$, so $a_1 = 2m + n$ and $a_2 = (a_1^2 - n^2 5)/4 = m^2 + mn - n^2$, where

$$\chi(t) = x^4 - a_1 x^3 - (a_2 + 2q)x^2 - a_1 q x + q^2.$$

The values of m and n for this curve are

$$\begin{aligned} m = & -33785412452026953770179131338679489596 \\ & 56475470770107628408426925155470967294, \\ n = & -37786020778198256317368570028183842800 \\ & 47374979214207223099354900103509328849 2. \end{aligned}$$

Cryptographic notions of generality

We can compute zeta functions as efficiently as for elliptic curves (and even better — unconditionally), but can we in good faith recommend the use of such curves for cryptographic applications?

Q1: In what way is an RM family special?

Q2: How special is the (one dimensional) family of Tautz, Top, and Verberkmoes inside of the (two dimensional) moduli space of genus 2 curves with RM by $\mathbb{Z}[(1+\sqrt{5})/2]$?

Motivation: Serre's talk at AGCT in Luminy, 2011, explaining and motivating work of Kedlaya and Sutherland for higher dimensional Sato–Tate conjectures (particularly $g = 2$).

First we recall some standard families of interest.

Examples of families of curves

We consider $\mathcal{C} \to S$ a family of curves, such that each fiber over a closed point x of S is a curve $C/k = \mathbb{F}_q$.

Examples. Elliptic curves.

1. $\mathcal{E} : y^2 = x^3 + ax + b$ over S, where

$$S = \mathrm{Spec}(\mathbb{Z}[a, b, \frac{1}{6ab}]) \subset \mathbb{A}^2/\mathbb{Z}[\frac{1}{6}],$$

a family of dimension 3.

2. $\mathcal{E} : y^2 + xy = x^3 + ax^2 + b$ over S, where

$$S = \mathrm{Spec}(\mathbb{F}_2[a, b, \frac{1}{b}]) \subset \mathbb{A}^2/\mathbb{F}_2,$$

a family of dimension 2.

3. $\mathcal{E} : y^2 = x^3 + x^2 - 3x + 1$ over $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, a CM family with endomorphism ring $\mathbb{Z}[\sqrt{-2}]$, of dimension 1.

Examples of families of genus 2

Examples. Genus 2 curves.

4. $\mathcal{C} : y^2 = x^5 + a_3x^3 + a_2x^2 + a_1x + a_0$, over

$$S = \mathrm{Spec}(\mathbb{Z}[a_0, \ldots, a_3][\frac{1}{\Delta}]) \subset \mathbb{A}^4/\mathbb{Z}[\frac{1}{10}],$$

a 5-dimensional family.

5. $\mathcal{C} : y^2 = x^5 + 5x^3 + 5x + t$, the RM family of Tautz, Top, and Verberkmoes, over

$$S = \mathrm{Spec}(\mathbb{Z}[t, \frac{1}{10(t^2+4)}]) \subset \mathbb{A}^1/\mathbb{Z}[\frac{1}{10}],$$

a 2-dimensional family with real multiplication by $\mathbb{Z}[(1+\sqrt{5})/2]$.

6. $\mathcal{C} : y^2 = x^5 + 1$, a CM family over $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{10}])$.

Notions of generality in cryptography

We address the question: "What is special about special curves?" Here we distinguish certain geometric and arithmetic properties.

Geometric speciality. If $\mathcal{C} \to S$ is a family (of genus g curves), what is the induced image $S \to \mathcal{X}$ in the moduli space (in $\mathcal{M}_g$)?

Arithmetic speciality. Here we distiguish the (local) level structure and the (global or geometric) Galois distributions.

a. What level structure is fixed by the family? — Is there an exceptional N such that the Galois representation

$$\bar{\rho}_N : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{2g}(\mathbb{Z}/N\mathbb{Z})$$

is smaller than expected?

b. What is the image of the Galois action on the Tate module?

$$\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(T_\ell(J)) \cong \mathrm{GL}_{2g}(\mathbb{Z}_\ell).$$

Frobenius angles and normalized traces

Let $E/\mathbb{Q}$ be an elliptic curve, with discriminant Δ, viewed as a scheme over $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{\Delta}])$. The Sato–Tate conjecture concerns the distribution of the Frobenius *angles* at primes p.

For each p, let $\pi = \pi_p$ be the Frobenius endomorphism on $\bar{E}/\mathbb{F}_p$ and

$$\chi(T) = T^2 - a_p T + p$$

its characteristic polynomial of Frobenius. Set t_p equal to the normalized Frobenius trace

$$t_p = a_p/\sqrt{p},$$

and denote by θ_p in $[0, \pi]$ the Frobenius angle, defined by $t_p = 2\cos(\theta_p)$. We set $\mu_p = e^{i\theta_p}$ (the unit Frobenius), and

$$\widetilde{\chi}(T) = T^2 - t_p T + 1 = (T - \mu_p)(T - \bar{\mu}_p).$$

Sato–Tate Conjecture

Sato–Tate Conjecture. Suppose that $E/\mathbb{Q}$ is a non-CM elliptic curve. For $[\alpha, \beta] \subset [0, \pi]$,

$$\lim_{N\to\infty} \frac{|\{p \leq N \mid \alpha \leq \theta_p \leq \beta\}|}{|\{p \leq N\}|} = \int_\alpha^\beta \frac{2\sin^2(\theta)}{\pi} d\theta,$$

or equivalently for $[a, b] \subset [-2, 2]$,

$$\lim_{N\to\infty} \frac{|\{p \leq N \mid a \leq t_p \leq b\}|}{|\{p \leq N\}|} = \int_a^b \frac{\sqrt{4-t^2}}{2\pi} dt.$$

The analogous distributions for CM elliptic curves is classical:

$$\lim_{N\to\infty} \frac{|\{p \leq N \mid \alpha \leq \theta_p \leq \beta\}|}{|\{p \leq N\}|} = \frac{1}{\pi}\int_\alpha^\beta d\theta = \frac{\beta-\alpha}{\pi}.$$

Sato–Tate distributions

We call the distributions $\mu(\theta)$ on $[0,\pi]$ and $\mu(t)$ and $[-2,2]$, defined by

$$\mu(\theta) = \frac{2\sin^2(\theta)}{\pi} d\theta \quad \text{and} \quad \mu(t) = \frac{\sqrt{4-t^2}}{2\pi} dt,$$

the *Sato–Tate distributions* for non-CM E/S.

For a CM curve E/S, the analogous Sato–Tate distributions are classical:

$$\mu(\theta) = \frac{1}{2}\left(\frac{d\theta}{\pi} + \delta_{\pi/2}\right) \quad \text{and} \quad \mu(t) = \frac{1}{2}\left(\frac{dt}{\pi\sqrt{4-t^2}} + \delta_0\right),$$

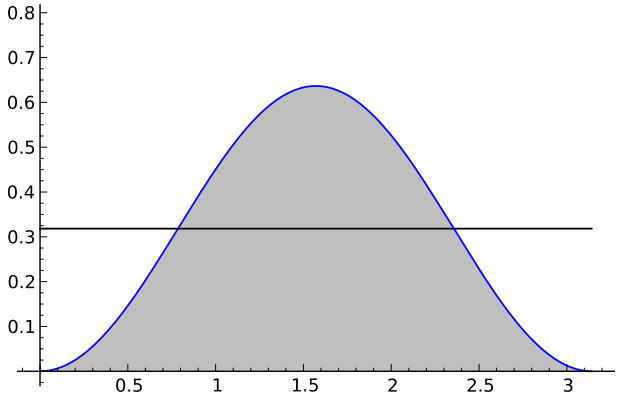
where δ_x is the Dirac distribution. Restricting to the 50% of ordinary primes, we have distributions

$$\mu_0(\theta) = \frac{d\theta}{\pi} \quad \text{and} \quad \mu_0(t) = \frac{dt}{\pi\sqrt{4-t^2}}.$$

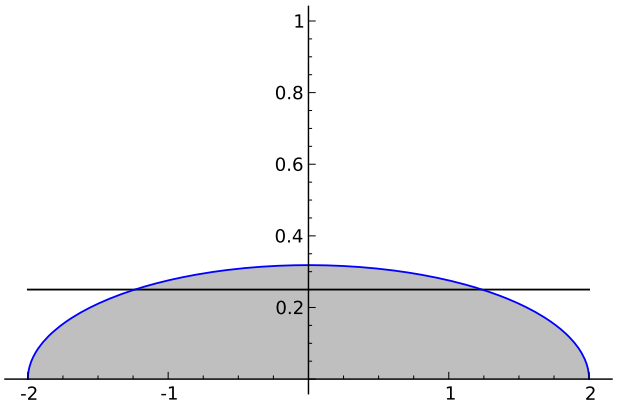
Sato–Tate plots

Generic curve

$$\frac{2\sin^2(\theta)}{\pi}d\theta$$

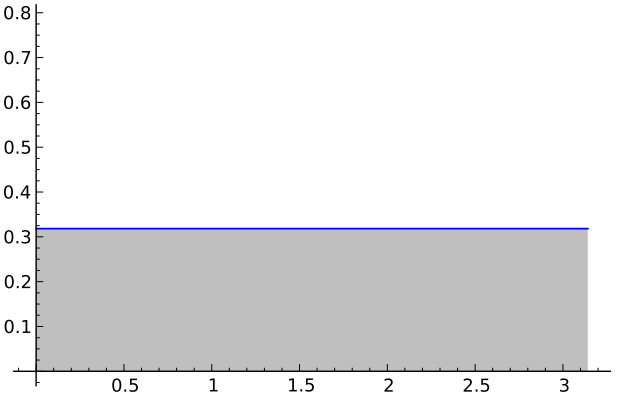


$$\frac{\sqrt{4-t^2}}{2\pi}dt$$

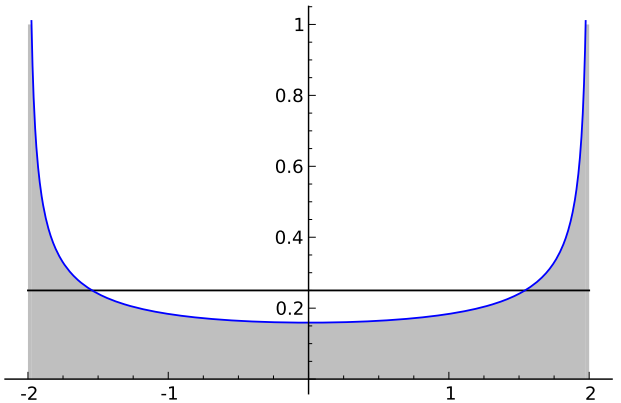


CM curve

$$\frac{1}{\pi}d\theta$$



$$\frac{1}{\pi\sqrt{4-t^2}}dt$$



Galois representation groups

Where do these come from?

The CM case is easy: the ordinary Frobenius endomorphisms π_p lie in a CM field K and their unit normalizations μ_p in $K \otimes \mathbb{R} \cong \mathbb{R}^2$ are uniformly distributed around the unit circle

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \right\} \cong S^1.$$

The supersingular Frobenius endomorphisms lie in a coset of the normalizer in $\mathrm{USp}(2) = \mathrm{SU}(2)$:

$$\mathrm{SO}(2) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \left\{ \begin{pmatrix} i\cos(\theta) & i\sin(\theta) \\ i\sin(\theta) & -i\cos(\theta) \end{pmatrix} \right\}.$$

The ordinary distribution $d\theta/\pi$ arises from the uniform distribution on the unit circle (hence of $\theta \in [0, \pi]$); the supersingular coset has uniform trace zero.

Galois representation groups

The generic normalized Frobenius representations lie in

$$\mathrm{USp}(2) = \mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \;\middle|\; |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

This group is isomorphic to the unit quaternions:

$$(\mathbb{H}^*)^1 = \{a + bi + (c + di)j \mid a^2 + b^2 + c^2 + d^2 = 1\} \cong S^3$$

on identifying $\alpha = a + bi$ and $\beta = c + di$. The Sato–Tate distribution arises from the Haar measure on $\mathrm{SU}(2)$. Setting

$$\begin{aligned} \alpha = a + bi &= \cos(\rho)(\cos(\sigma) + i\sin(\sigma)), \\ \beta = c + di &= \sin(\rho)(\cos(\tau) + i\sin(\tau)), \end{aligned}$$

the conjugacy class (on which trace is a class function) is

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

with trace $2\cos(\theta) = 2\cos(\rho)\cos(\sigma)$.

Alternative Sato–Tate domains

Noting that $D = a_p^2 - 4p$ is the discriminant of the ring $\mathbb{Z}[\pi]$, in the case that $E/\mathbb{Q}$ has CM by an order $\mathcal{O}$, we have $D = n^2 D_{\mathcal{O}}$ for some integer n.

In order to study the distribution of Frobenius discriminants, this motivates setting

$$u^2 = \frac{D}{p} = t^2 - 4 \left(= \frac{n^2 D_{\mathcal{O}}}{p} \right)$$

and considering the Frobenius distribution in terms of u.

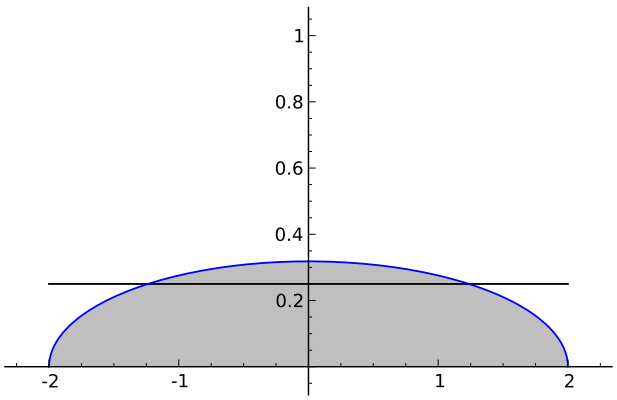
In the non-CM case, the coordinate $u = \sqrt{D/p}$ measures the distribution of normalized square root discriminants (of $\mathbb{Z}[\pi]$).

In the CM case, $\sqrt{D_{\mathcal{O}}}$ remains fixed, and u gives information about the normalized conductors $n/\sqrt{p} = [\mathcal{O} : \mathbb{Z}[\pi]]/\sqrt{p}$ at ordinary primes.

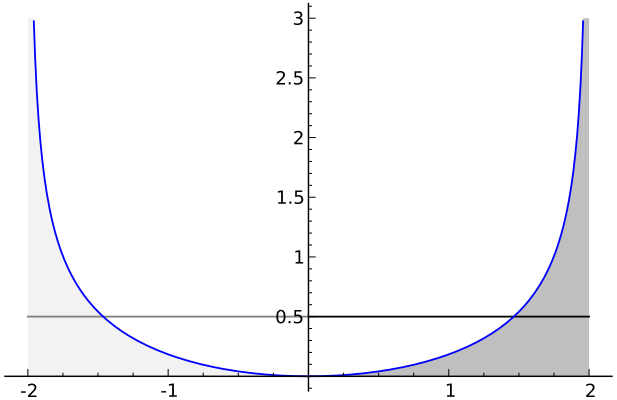
Sato–Tate plots

Generic curve

$$\frac{\sqrt{4-t^2}}{2\pi}dt$$

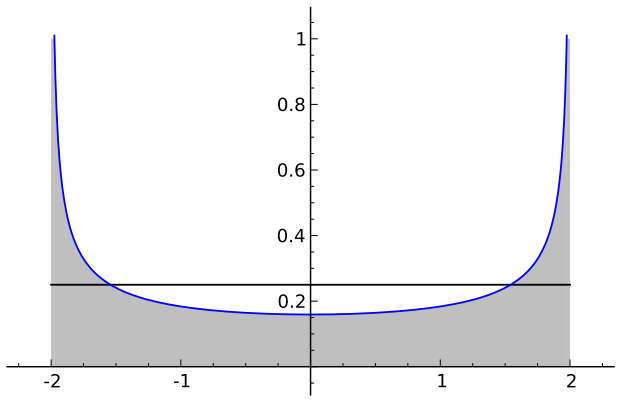


$$\frac{u^2}{\pi\sqrt{4-u^2}}du$$

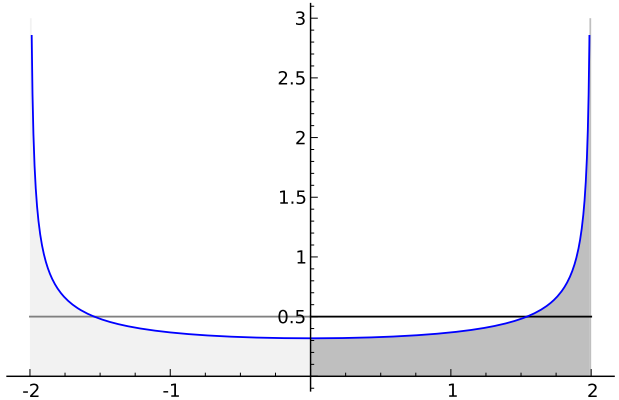


CM curve

$$\frac{1}{\pi\sqrt{4-t^2}}dt$$



$$\frac{2}{\pi\sqrt{4-u^2}}du$$



Generalized Sato–Tate framework

Conjecturally, there exists a compact subgroup H of $\mathrm{USp}(2g)$, with connected component H_0,

$$H_0 \triangleleft H \subseteq \mathrm{USp}(2g),$$

such that the unit Frobenius elements are equidistributed in H.

Remark. The partition into the cosets in $G = H/H_0$ is explained by the Chebotarev density theorem. In general one has a decomposition

$$\mu = \frac{|C_0|}{|G|}\mu_0 + \frac{|C_1|}{|G|}\mu_1 + \cdots \frac{|C_r|}{|G|}\mu_r,$$

where $C_0, C_1, \ldots C_r$ are the conjugacy classes of G.

Here we focus on the distribution $\mu = \mu_0$ in the principle coset H_0 (a vast simplification), and the case $g = 2$ (see work of Kedlaya & Sutherland). We also simplify (experimentally and theoretically) by averaging over fibres over a base scheme.

Sato–Tate domains

Let $C/\mathbb{F}_q$ be a curve and $\chi(T)$ its Frobenius characteristic polynomial

$$\chi(T) = T^{2g} - a_1 T^{2g-1} + \cdots - a_1 q^{g-1} T + q^g.$$

and define the unit Frobenius characteristic polynomial by

$$\begin{aligned}\widetilde{\chi}(T) = \frac{\chi(\sqrt{q}T)}{q^g} &= T^{2g} - s_1 T^{2g-1} + \cdots - s_1 T + 1 \\ &= \prod_{j=1}^{g} (T^2 - t_j T + 1).\end{aligned}$$

By the Weil conjectures, the roots α_j of $\chi(T)$ satisfy $|\alpha_j| = \sqrt{q}$, so we write

$$\mu_j = \frac{\alpha_j}{\sqrt{q}} = e^{i\theta_j},$$

and $t_j = \mu_j + \bar{\mu}_j = 2\cos(\theta_j)$, where $\mu_j \bar{\mu}_j = 1$.

Domains for Sato–Tate distributions

Rather than defining s_j to be the j-th coefficient of $\widetilde{\chi}(T)$, we let the s_j be the normalized symmetric products not including any terms (as factors of summands) of the form $\mu_j\bar{\mu}_j(=1)$. These are the coefficients of the real Weil polynomial. Thus for $g = 2$

$$\widetilde{\chi}(T) = T^4 - s_1T^3 + (s_2 + 2)T^2 - s_1T + 1,$$

and for $g = 3$:

$$\widetilde{\chi}(T) = T^6 - s_1T^5 + (s_2 + 3)T^4 - (s_3 + 2s_1)T^3 + \cdots$$

A naïve application of the Weil bounds gives bounds on the symmetric sums and s_j, equal to their respective number of monomials:

$$|s_j| \le 2^j \binom{g}{j} \text{ vs. } |\mathrm{sym}_j(\{\mu_1, \bar{\mu}_1, \ldots, \mu_g, \bar{\mu}_g\})| \le \binom{2g}{j}.$$

Domains for Sato–Tate distributions

In higher dimension, the real subring $\mathbb{Z}[\pi + \bar{\pi}]$ is a nontrivial subring of $\mathbb{Z}[\pi, \bar{\pi}]$, and hence

$$\operatorname{disc}(\mathbb{Z}[\pi, \bar{\pi}]) = D_+^2 D_-,$$

where $D_+ = \operatorname{disc}(\mathbb{Z}[\pi + \bar{\pi}])$, and for $g = 2$ we have

$$D_+ = a_1^2 - 4a_2 \text{ and } D_- = -((a_2 - 4q)^2 - 4q(a_1^2 - 4a_2)).$$

where D_- is the norm of the relative discriminant $\mathbb{Z}[\pi, \bar{\pi}]/\mathbb{Z}[\pi, \bar{\pi}]$.

For a family with fixed RM order R, we have $\mathbb{Z}[\pi + \bar{\pi}] \subset R$ of finite index (on any fiber of simple ordinary reduction), hence

$$D_+ = n_+^2 D_R,$$

and additionally for a subfamily with CM by $\mathcal{O}/R$ we have

$$D_- = n_-^2 D_1,$$

where D_1 is the norm of the relative discriminant of $\mathcal{O}/R$.

Domains for Sato–Tate distributions

Generic: $H_0 = H = \mathrm{USp}(4)$

$$\mu_0 = \frac{8(\cos(\theta_1) - \cos(\theta_2))^2 \sin^2(\theta_1) \sin^2(\theta_2)}{\pi^2} d\theta_1 d\theta_2$$

RM: $H_0 = \mathrm{SU}(2) \times \mathrm{SU}(2)$

$$\mu_0 = \frac{4 \sin^2(\theta_1) \sin^2(\theta_2)}{\pi^2} d\theta_1 d\theta_2$$

CM: $H_0 = \mathrm{SO}(2) \times \mathrm{SO}(2)$

$$\mu_0 = \frac{d\theta_1 d\theta_2}{\pi^2}$$

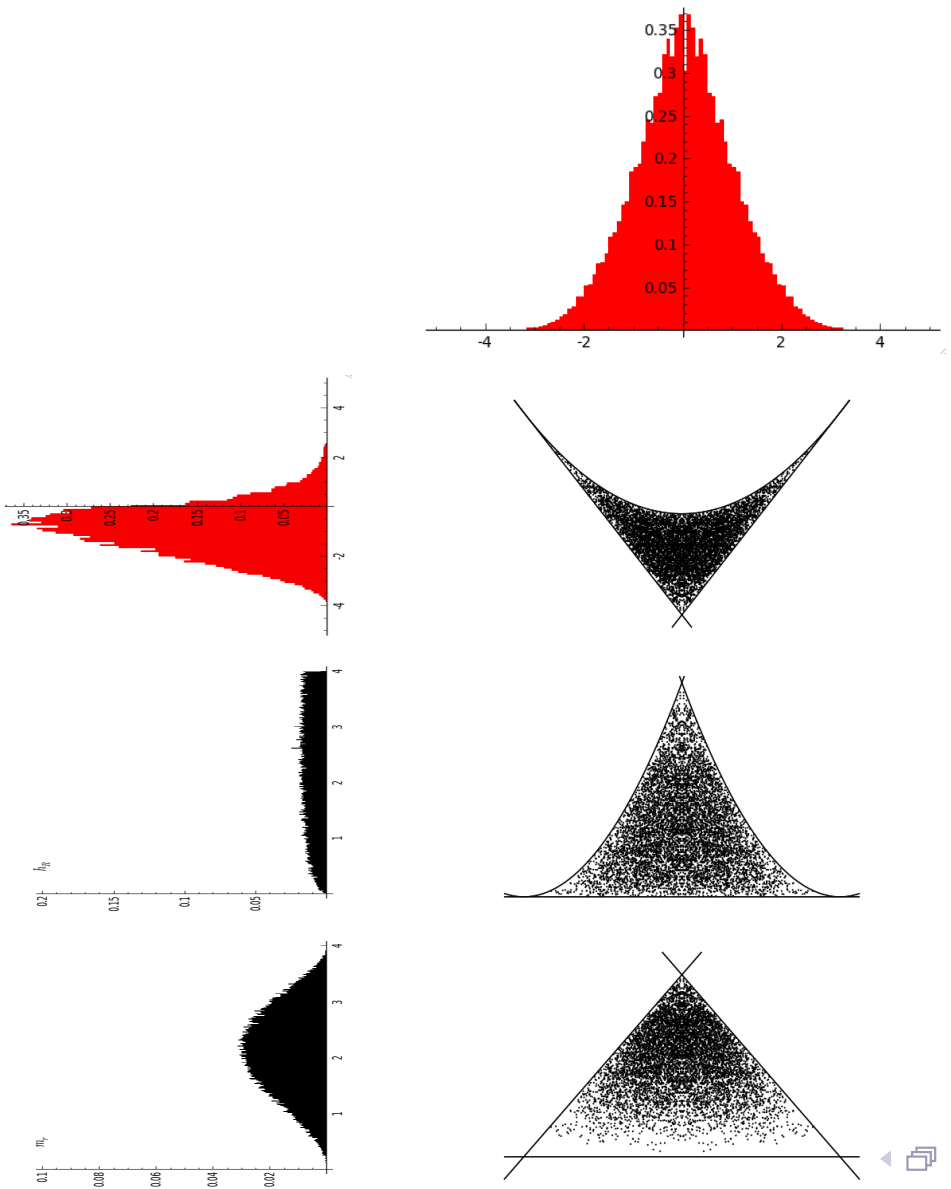
These induced well-defined distributions in terms of the spaces

- (s_1, s_2),
- $(s_1, D_+ = s_1^2 - 4s_2)$,
- $(s_1, \sqrt{s_1^2 - 4s_2})$

graphical representations follow . . .

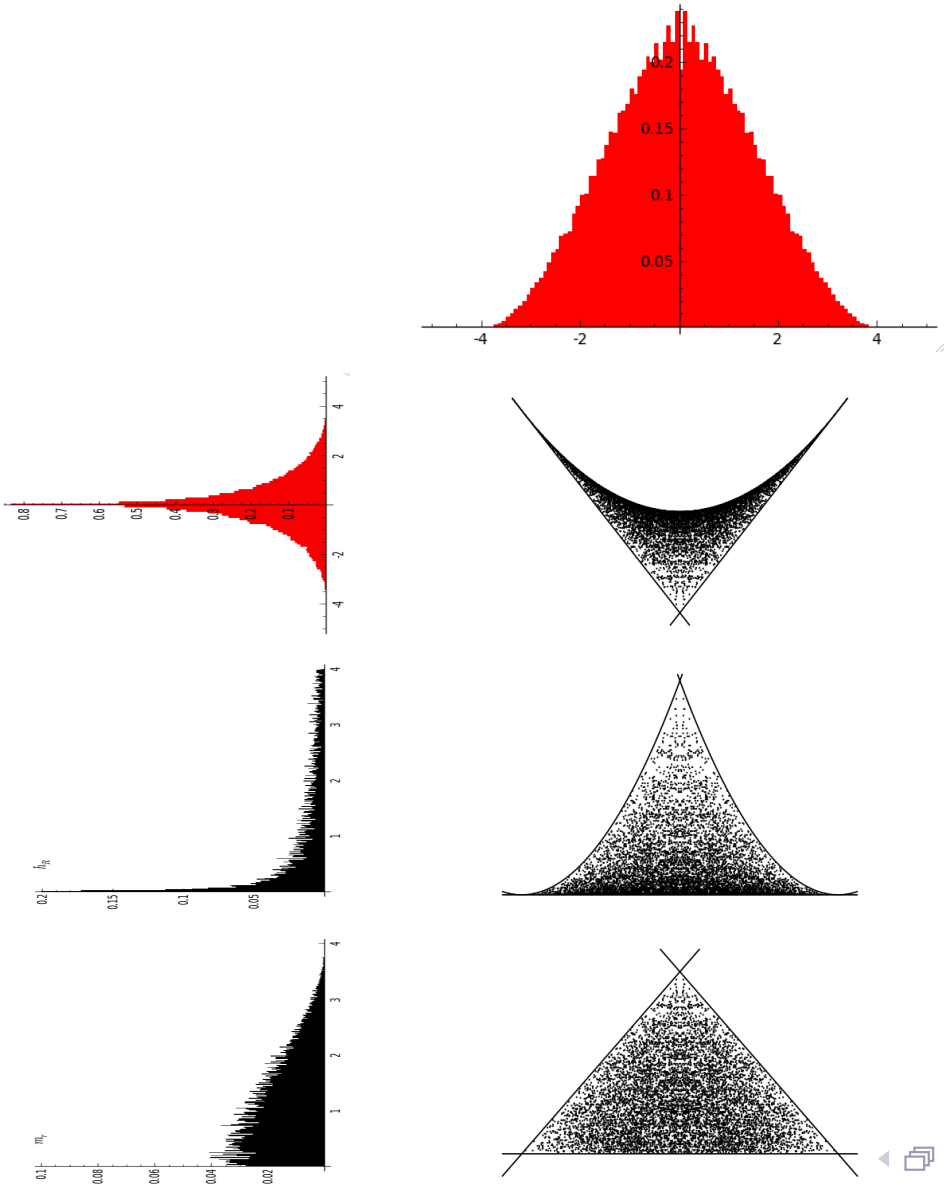
Experimental Sato–Tate: generic family

Generic:



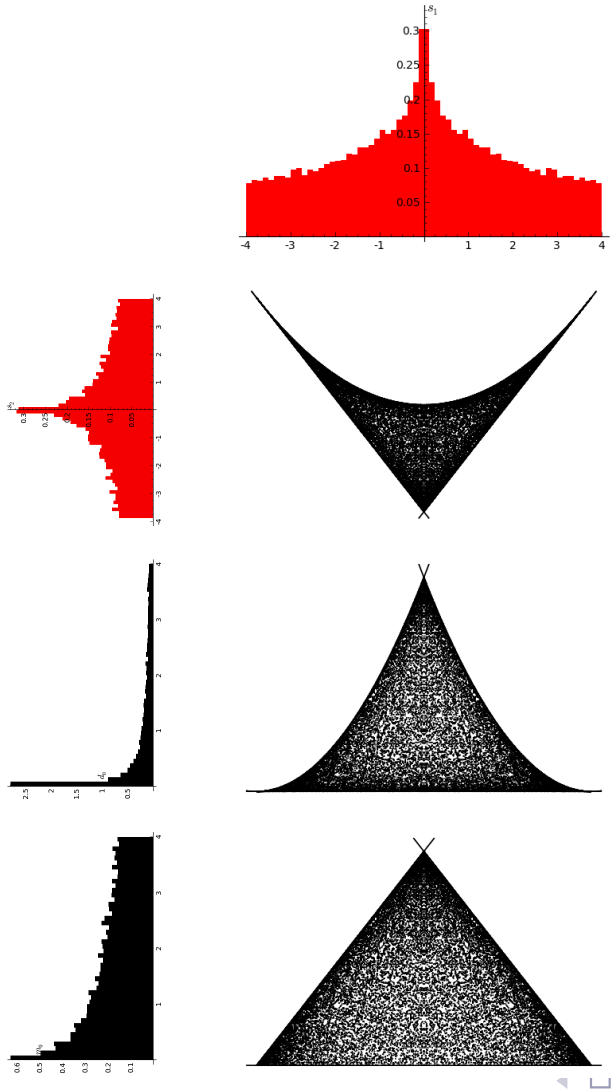
Experimental Sato–Tate: RM family

RM:



Experimental Sato–Tate: CM family

CM:



Conjectural Sato–Tate distributions

What are these distributions?

Note: the real and relative unit discriminants are:

$$D_+ = s_1^2 - 4s_2 \text{ and } D_- = (4 - s_1 + s_2)(4 + s_1 + s_2).$$

Generic: (up to constant scalars)

$$\sqrt{(s_1^2 - 4s_2)(4 - s_1 + s_2)(4 + s_1 + s_2)}\, ds_1 ds_2$$

RM:

$$\frac{\sqrt{(4 - s_1 + s_2)(4 + s_1 + s_2)}\, ds_1 ds_2}{\sqrt{(s_1^2 - 4s_2)}}$$

CM:

$$\frac{ds_1 ds_2}{\sqrt{(s_1^2 - 4s_2)(4 - s_1 + s_2)(4 + s_1 + s_2)}}$$