Equilibrium points of systems of differential equations

David Kohel
Institut de Mathématiques de Marseille

Marseille
January 2022

Systems of differential equations

Equilibrium points and flow lines

Stability of equilibrium points

Systems of differential equations

We consider the *system of differential equations*:

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = f_n(x_1, \ldots, x_n).$$

and will study the geometry of this system, in particular when $f_1, \ldots, f_n$ are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. A point $(a_1, \ldots, a_n)$ is said to be an *equilibrium point* if

$$f_1(a_1, \ldots, a_n) = \cdots = f_n(a_1, \ldots, a_n) = 0.$$

A particular system

For the purpose of the talk, we fix $n = 2$, and consider the particular system,

$$\frac{dx}{dt} = x^2 + y^2 - r^2, \ \frac{dy}{dt} = xy - 1.$$

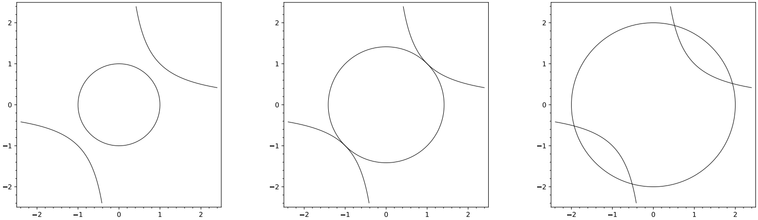
for a parameter $r \in \mathbb{R}$. The equilibrium points of this system are determined by the system of equations,

$$x^2 + y^2 = r^2, xy = 1,$$

i.e. the intersection of a circle and a parabola.

Equilibrium points

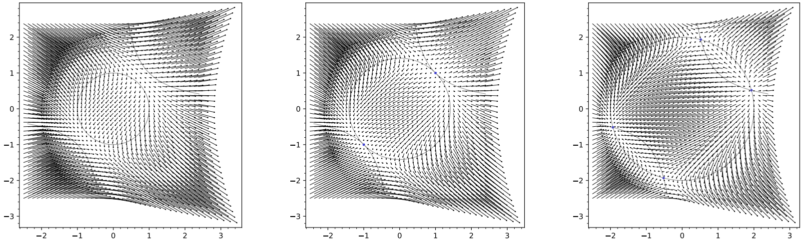
The equilibrium points are then the intersection points of the circle and parabola:



shown here for $r = 1$, $r = \sqrt{2}$, and $r = 2$, with respectively 0, 2, and 4 stable points.

Flow diagrams

We can visualize the flow diagram by graphing the vectors (x', y') in the (x, y)-plane:



represented here for the three cases $r = 1$, $r = \sqrt{2}$ and $r = 2$.

Stability of equilibrium points

The flow diagram in the neighborhood of an equilibrium point (a, b) determines its behavior when the point is deformed slightly. This is determined by the eigenvalues of the Jacobian matrix:

$$\begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_n/\partial x_1 & \cdots & \partial f_n/\partial x_n \end{pmatrix}$$

For our particular system the Jacobian matrix is

$$\begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}$$

with determinant $2(x^2 - y^2)$. What are the possible eigenvalues? Are they real or complex? And what does this say about the stability of the equilibrium points?

THE END

…but it keeps going…

On stability of the particular system

The Jacobian matrix J at a stable point (a, b):

$$J = \begin{pmatrix} df_1/dx & df_1/dy \\ df_2/dx & df_2/dy \end{pmatrix}(a, b) = \begin{pmatrix} 2a & 2b \\ b & a \end{pmatrix}$$

has characteristic polynomial

$$\chi_J(\lambda) = \lambda^2 - 3a\lambda + 2(a^2 - b^2),$$

of discriminant $a^2 + 8b^2 > 0$. Therefore the eigenvalues are real, and of the same sign if $|a| > |b|$, since then

$$\det(J) = \lambda_1\lambda_2 = 2(a^2 - b^2) > 0,$$

and otherwise, if $|a| < |b|$, of opposite sign.

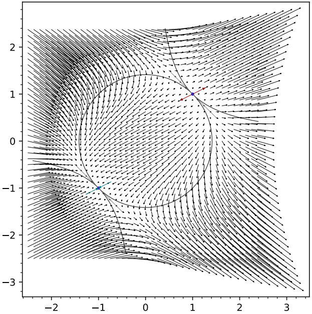
What happens when $\det(J) = 0$? This implies the equilibrium points (a, b) are on the lines $x = \pm y$, which only occurs when $r = \sqrt{2}$.

Stability of the critical case $r = \sqrt{2}$

When $r = \sqrt{2}$, the equilibrium points are $\pm(1,1)$, with eigenspaces:

(a,b)	λ	eigenvector
$(1,1)$	0	$(1,-1)$
$(1,1)$	3	$(2,1)$

(a,b)	λ	eigenvector
$-(1,1)$	0	$(1,-1)$
$-(1,1)$	-3	$(2,1)$



The eigenspace in maroon (■) repells, the eigenspace in teal (■) attracts, and the zero eigenspace is in grey (■).

Stability of the case $r = 2$

The behavoir of the system is similar for any $r > \sqrt{2}$, and so we consider the case $r = 2$ as a model, namely

$$\frac{dx}{dt} = x^2 + y^2 - 4, \ \frac{dy}{dt} = xy - 1.$$

The equilibrium points consist of the four points $\{\pm(a, b), \pm(b, a)\}$, where

$$a = \sqrt{2 + \sqrt{3}} \text{ and } b = \sqrt{2 - \sqrt{3}},$$

noting that $ab = (2 + \sqrt{3})(2 - \sqrt{3}) = 1$, with $0 < b < 1 < a$.

Stability of the case $r = 2$

Let λ_1 and λ_2 be roots of the characteristic polynomials

$$\chi_J(\lambda) = \lambda^2 - 3a\lambda + 2(a^2 - b^2), \text{ and } \chi_J(\lambda) = \lambda^2 - 3b\lambda - 2(a^2 - b^2).$$

of the Jacobian matrices $J(a, b)$ and $J(b, a)$, respectively. Then we find the following table of eigenvectors at the equilibrium points,

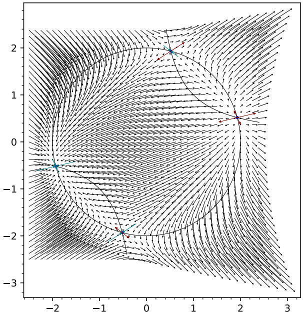
point	λ	eigenvector
(a, b)	λ_1	$(2b, \lambda_1 - 2a)$
	$3a - \lambda_1$	$(2b, a - \lambda_1)$
(-a,-b)	$-\lambda_1$	$(2b, \lambda_1 - 2a)$
	$\lambda_1 - 3a$	$(2b, a - \lambda_1)$

point	λ	eigenvector
(b, a)	λ_2	$(2a, \lambda_2 - 2b)$
	$3b - \lambda_2$	$(2a, b - \lambda_2)$
(-b,-a)	$-\lambda_2$	$(2a, \lambda_2 - 2b)$
	$\lambda_2 - 3b$	$(2a, b - \lambda_2)$

noting that $J(-a, -b) = -J(a, b)$ and $J(-b, a) = -J(b, a)$.

Stability of the case $r = 2$

The point (a, b) with positive eigenvalues is called a *source*, and the point $-(a, b)$ with negative eigenvalues is called a *sink*.



The two equilibrium points $\pm(b, a)$ with eigenvalues of different signs are called *saddle points*.