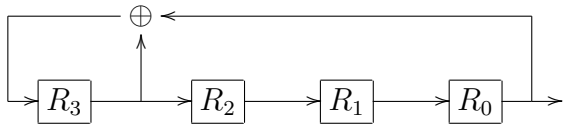
Semester 1 **Exercises for Week 8** 2004

Linear Feedback Shift Registers

Linear feedback shift registers (LFSR's) are an efficient way of describing and generating certain sequences in hardware implementations. We derive and work with equivalent mathematical descriptions of the sequences produced by a LFSR, along with some generalized sequences which do not arise in this way.

A linear feedback shift register is composed of a shift register R which contains a sequence of bits and a feedback function f which is the bit sum (`xor`) of a subset of the entries of the shift register. The shift register contains n memory cells, or stages, labelled $R_{n-1},\ldots, R_1, R_0$, each holding one bit. Each time a bit is needed the entry in stage R_0 is output while the entry in cell R_i is passed to cell R_{i-1} and the top stage R_{n-1} is updated with the value $f(R)$.

The following is a schematic of a linear feedback shift register:



1. In the above LFSR, let the initial entries of stages R_i be s_i, for $0 \le i \le n$. For each of the following initial entries below:

	s_3	s_2	s_1	s_0
a)	0	1	1	0
b)	1	1	1	0
c)	1	0	1	0
d)	1	1	0	0

compute the first 16 bits in the output sequence. Show that the output sequence is defined by the initial entries and the recursion $s_{i+4} = s_{i+3} + s_i$.

2. Show that every linear feedback register defines and is defined by a recursion of the form $s_{i+n} = \sum_{j=0}^{n-1} a_j s_{i+j}$, where the a_j are bits in $\mathbb{Z}/2\mathbb{Z}$; the products $a_j s_{i+j}$ and the summation are operations in $\mathbb{Z}/2\mathbb{Z}$.

N.B. The ring $\mathbb{Z}/2\mathbb{Z}$ is also referred to as $\mathbb{F}_2$, the unique finite field of two elements. Note that the addition operation is the same `xor` that we have been using and the multiplication operation is the logical `and` operation.)

3. For a linear feedback register of length n, define a power series

$$s(x) = \sum_{i=1}^{\infty} s_i x^i$$

from the output sequence s_i. Suppose that the linear feedback register defines the recursion $s_{i+n} = \sum_{j=0}^{n-1} a_j s_{i+j}$. Define a polynomial $g(x) = \sum_{j=0}^{n-1} a_j x^{n-j} + 1$. Show that $f(x) = g(x)s(x)$ is a polynomial, that is, all of its coefficients are eventually zero. What is the polynomial $f(x)$?

4. In the previous exercise we showed that the power series $s(x)$ has the form $f(x)/g(x)$ in the power series ring $\mathbb{F}_2[[x]]$. In `Magma` it is possible to form power series rings in the following way

```
> F2 := FiniteField(2);
> PS<x> := PowerSeriesRing(F2);
> f := x^2 + x;
> g := x^3 + x + 1;
> f/g + O(x^16);
x + x^4 + x^5 + x^6 + x^8 + x^11 + x^12 + x^13 + x^15 + O(x^16)
```

Consider the linear feedback shift register at the beginning of the worksheet. Construct the corresponding power series and verify that these are the same of the output sequences that you computed.