HIGHER QUASI-CATEGORIES VS HIGHER REZK SPACES

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Abstract. We introduce a notion of n-quasi-categories as fibrant objects of a model category structure on presheaves on Joyal's n-cell category Θ_n. Our definition comes from an idea of Cisinski and Joyal. However, we show that this idea has to be slightly modified to get a reasonable notion. We construct two Quillen equivalences between the model category of n-quasi-categories and the model category of Rezk Θ_n-spaces, showing that n-quasi-categories are a model for (∞, n)-categories. For $n = 1$, we recover the two Quillen equivalences defined by Joyal and Tierney between quasi-categories and complete Segal spaces.

Introduction

Quasi-categories provide a simplicial model for a certain type of weak ∞-categories. They were initially introduced as "simplicial sets satisfying the restricted Kan condition" by Boardman and Vogt in [19], and their theory has been developed extensively by Joyal in [28], [30] and [31], and Lurie in [35] and [36], among others.

Weak ∞-categories are a generalization of the strict ∞-categories that can be defined algebraically as consisting of sets of k-arrows in all dimensions, with compositions and identities at all levels, satisfying some standard axioms. In a weak ∞-category, these axioms are only asked to be satisfied "up to coherence", that is, up to higher arrows themselves satisfying some identities up to higher arrows, and so on. For instance, weak ∞-categories of dimension 2 can be defined as Bénabou's bicategories [10], and in this case, the coherences are given by an "associator" and two "unitors" that have to satisfy Mac Lane's pentagon and triangle axioms.

Quasi-categories only model $(\infty, 1)$-categories, that is, weak ∞-categories in which the k-arrows, for $k > 1$, are invertible up to higher arrows. These $(\infty, 1)$-categories play an important role in homotopy theory as one can associate an $(\infty, 1)$-category to any abstract homotopy theory. For instance, from the homotopy theory of spaces, we get an $(\infty, 1)$-category where objects are spaces, 1-arrows are maps of spaces, 2-arrows are homotopies, 3-arrows are homotopies of homotopies, and so on. Notice that k-arrows, for $k \geq 1$, are indeed invertible in this ∞-category since homotopies are invertible up to higher homotopies. More generally, an (∞, n)-category is a weak ∞-category in which all k-arrows are weakly invertible for $k > n$.

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A priori, to make sense of this definition, one first has to make precise the notion of a weak ∞-category. This was done by Grothendieck in [24] (actually, Grothendieck defined a notion of ∞-groupoid but Maltsiniotis noticed in [38] and [39] that his definition can be adapted). Variations on this definition are studied by the author in [1], [3] and [4] (see also [40]). There are now plenty of competing definitions (see for instance [33] and [34] for some of them).

A second approach, which is the one this paper is about, is to define directly (∞, n)-categories, without references to weak ∞-categories, by means of homotopical algebra. There are numerous such models for $(\infty, 1)$-categories. The four most popular are probably those explained in Bergner's survey [15], namely, quasi-categories, complete Segal spaces, Segal categories and categories enriched in Kan complexes. These four models are equivalent in the following sense: Joyal ([31]), Rezk ([42]), Hirschowitz and Simpson ([26]) and Bergner ([13]) constructed Quillen model category structures for which the fibrant objects are precisely these four classes of objects, and these model category structures were shown to be Quillen equivalent by Bergner ([14]), Joyal ([29]) and Joyal and Tierney ([32]). Another model, relative categories, was introduced and compared to the other models by Barwick and Kan in [6].

Several of these models for $(\infty, 1)$-categories have been generalized to models for (∞, n)-categories. Hirschowitz and Simpson introduced a notion of higher Segal categories in [26]. This notion is the main topic of the book [45] of Simpson. In [43] and [44], Rezk introduced a notion of higher Segal spaces. We will call these objects Rezk Θ_n-spaces in this paper. Another model based on Rezk Θ_n-spaces has been introduced by Bergner and Rezk in [17]. A second generalization of complete Segal spaces called n-fold Segal spaces has been introduced by Barwick (see Section 12 of [8]). The model of relative categories has been generalized by Barwick and Kan in [7]. Several of these models are explained in Bergner's survey [16].

In this paper, we introduce a notion of n-quasi-categories for $n \geq 1$. Our notion is based on an idea of Cisinski and Joyal. In Section 45 of [30], Joyal writes that he and Cisinski conjecture that the Θ_n-localizer generated by some kind of higher Segal maps gives rise to a model for (∞, n)-categories. Let us briefly explain the terminology. If A is a small category, an (accessible) A-localizer is a class of morphisms of presheaves on A which is the class of weak equivalences of a combinatorial model category structure on presheaves on A whose cofibrations are the monomorphisms. This notion is here applied to Θ_n, the n-truncation of Joyal's cell category introduced in [27]. In this paper, we show that the idea of Cisinski and Joyal has to be slightly modified. More precisely, we exhibit equivalent n-categories which are not weakly equivalent in the sense of Cisinski and Joyal.

We suggest a modification consisting of adding new generators. These generators are essentially the same as the ones given by Rezk to define his Θ_n-spaces. We obtain this way a model category structure on presheaves on Θ_n and we define n-quasi-categories as the fibrant objects of this model category. For $n = 1$, by a Theorem of Joyal, we recover the usual notion of quasi-categories.

We then show that the model category of Rezk Θ_n-spaces is in some sense canonically associated to our model category of n-quasi-categories. More precisely, we show that the localizer of Rezk Θ_n-spaces is the simplicial completion in the sense of Cisinski of the localizer of n-quasi-categories. We deduce from this fact, using Cisinski's theory of simplicial completion, the existence of two Quillen equivalences between the model category of n-quasi-categories and the model category of Rezk Θ_n-spaces. In particular, the homotopy categories of n-quasi-categories and of Rezk Θ_n-spaces are equivalent. For $n = 1$, we recover the two Quillen equivalences between quasi-categories and complete Segal spaces given by Joyal and Tierney in [32].

The tools we use in this work are of two kinds. First, we use the machinery of A-localizers developed by Cisinski in [20]. In particular, our work relies heavily on the notion of simplicial completion of a localizer. For $n = 1$, this theory simplifies the work of Joyal and Tierney. Second, we use the techniques that Joyal and Tierney have developed to prove that their two adjunctions between quasi-categories and complete Segal spaces are Quillen equivalences. Many of our arguments are very similar (if not identical) to the ones used in their proofs. We have tried to make that clear by citing very precisely their work.

After we made public the first version of this paper, we were informed that J. Hahn has obtained related results in his ongoing Ph.D. thesis under the supervision of Barwick and that Gindi has developed a related homotopy theory of Θ-sets. Since then, Gindi's work [23] has been made public.

Organization of the paper. In Section 1, we introduce some preliminary terminology. In Section 2, we give a short introduction to Cisinski's theory of A-localizers, which is the language we will use throughout this paper. In particular, we present the notion of simplicial completion of an A-localizer. As explained above, this notion will play a crucial role in this paper. Everything from this section is extracted from [20]. In Section 3, we introduce tools developed by Joyal and Tierney in [32]. Unfortunately, we will need these tools in a more general setting than the one used in ibid. Nevertheless, everything adapts trivially and we do not claim any originality for this section. In Section 4, we study the simplicial completion of an A-localizer when A is a regular skeletal Reedy category. The techniques of this section are still based on [32] even though they have to be adapted since we do not have a notion of "mid anodyne map" in this context. For this purpose, we introduce the notion of formal Rezk A-spaces. In Section 5, we introduce the n-truncation Θ_n of Joyal's cell category as a full subcategory of the category of strict n-categories and we define our Θ_n-localizer of n-quasi-categories. In Section 6, we explain why the idea of Cisinski and Joyal has to be modified. More precisely, we show that our new generators, which come from equivalences of strict n-categories, are not weak equivalences in the sense of Cisinski and Joyal. In Section 7, we explain why nerves of strict n-categories are not n-quasi-categories in general (contrary to what happens for quasi-categories). More precisely, we show that the nerve of a strict n-category is an n-quasi-category if and only if this n-category has no non-trivial invertible k-arrows for $k > 1$. In Section 8, we recall the definition of Rezk Θ_n-spaces and we show that their localizer is the simplicial completion of our localizer of n-quasi-categories. We obtain two Quillen equivalences between n-quasi-categories and Rezk Θ_n-spaces. We deduce that the model category of n-quasi-categories is cartesian closed from the analogous result for Rezk Θ_n-spaces. Finally, in an appendix, we compare the language of localizers to the language of Bousfield localization.

Notation. Let C be a category. The class of objects of C will be denoted by $\mathsf{Ob}(C)$ and if X, Y is a pair of objects of C, the set of morphisms of C from X to Y will be denoted by $\mathsf{Hom}_C(X, Y)$. The opposite category of C will be denoted by C^{o}. If D is a second category, we will denote by $\underline{\mathsf{Hom}}(C, D)$ the category of functors from C to D.

If A is a small category, the category of presheaves on A will be denoted by $\widehat{A}$. If X is a presheaf on A and a is an object of A, we will sometimes write X_a for $X(a)$. We will denote by $e_{\widehat{A}}$ the terminal object of $\widehat{A}$ and by $\varnothing_{\widehat{A}}$ the initial object of $\widehat{A}$.

If $u : A \to B$ is a functor between small categories, we will denote by u^* the functor from $\widehat{B}$ to $\widehat{A}$ given by precomposition by u. We will denote by $u_!$ its left adjoint and by u_* its right adjoint.

Finally, we will denote by $\mathcal{C}\mathrm{at}$ the category of small categories.

1. Preliminaries

In this section, we gather some categorical preliminaries.

1.1. Let $\mathcal{C}$ be a category. Let $i : A \to B$ and $p : X \to Y$ be two morphisms of $\mathcal{C}$. Recall that the morphism i has the *left lifting property with respect to* p (or that the morphism p has the *right lifting property with respect to* i) if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd},$$

there exists a *lift*, i.e., a morphism $B \to X$ making the two triangles commute. We will then write $i \pitchfork p$. If the lift is unique, one says that the lifting property is a *unique lifting property*.

Let C be a class of morphisms of $\mathcal{C}$. We will denote by $l(C)$ (resp. by $r(C)$) the class of morphisms having the left lifting property (resp. the right lifting property) with respect to C (i.e., with respect to every morphism of C). We define the *saturation* $\mathrm{Sat}(C)$ of C as

$$\mathrm{Sat}(C) = lr(C).$$

The class C is said to be *saturated* if $C = \mathrm{Sat}(C)$.

One easily checks that

$$r(\mathrm{Sat}(C)) = r(C).$$

Dually, we have $\mathrm{Sat}(l(C)) = l(C)$. In other words, the class of morphisms having the left lifting property with respect to a fixed class of morphisms is saturated.

If $\mathcal{C}$ is a presheaf category and S is a *set* of morphisms of $\mathcal{C}$, the small object argument shows that $\mathrm{Sat}(S)$ is the class of retracts of transfinite compositions of pushouts of morphisms of S.

1.2. Let $\mathcal{C}$ be a cartesian closed category with an internal Hom functor $\underline{\mathsf{Hom}}$ and let $u : A \to B$, $v : C \to D$ and $w : E \to F$ be three morphisms of $\mathcal{C}$. We will denote by

$$u \times' v : A \times D \amalg_{A \times C} B \times C \to B \times D$$

the canonical morphism induced by the commutative square

$$\begin{tikzcd} A \times C \arrow[r] \arrow[d] & A \times D \arrow[d] \\ B \times C \arrow[r] & B \times D \end{tikzcd} \quad ;$$

and by

$$\underline{\mathsf{Hom}}'(v,w) : \underline{\mathsf{Hom}}(D,E) \to \underline{\mathsf{Hom}}(C,E) \times_{\underline{\mathsf{Hom}}(C,F)} \underline{\mathsf{Hom}}(D,F)$$

the canonical morphism induced by the commutative square

$$\begin{tikzcd} \underline{\mathsf{Hom}}(D,E) \arrow[r] \arrow[d] & \underline{\mathsf{Hom}}(D,F) \arrow[d] \\ \underline{\mathsf{Hom}}(C,E) \arrow[r] & \underline{\mathsf{Hom}}(C,F) \end{tikzcd} \quad .$$

By adjunction (see for instance Proposition 7.6 of [32]), we have

$$u \times' v \pitchfork w \quad \Leftrightarrow \quad u \pitchfork \underline{\mathsf{Hom}}'(v,w).$$

1.3. Let A be a small category. A *cellular model* of $\widehat{A}$ is a *set* $\mathfrak{M}$ of monomorphisms of $\widehat{A}$ such that $\mathrm{Sat}(\mathfrak{M})$ is the class of monomorphisms of $\widehat{A}$. Such a cellular model always exists by Proposition 1.2.27 of [20].

A morphism of $\widehat{A}$ is a *trivial fibration* if it has the right lifting property with respect to monomorphisms of $\widehat{A}$. If $\mathfrak{M}$ is a cellular model of $\widehat{A}$, then a morphism of $\widehat{A}$ is a trivial fibration if and only if it has the right lifting property with respect to $\mathfrak{M}$.

1.4. A *Reedy category* is a category A endowed with two subcategories A_+ and A_- satisfying the following properties:

(1) there exists a map $d : \mathsf{Ob}(A) \to \mathbb{N}$, assigning to every object of A an non-negative integer, such that:
   (a) if $a \to a'$ is a morphism of A_+ which is not an identity, then $d(a) < d(a')$;
   (b) if $a \to a'$ is a morphism of A_- which is not an identity, then $d(a) > d(a')$;
(2) every morphism of A factors uniquely as a morphism of A_- followed by a morphism of A_+.

We will often denote a Reedy category simply by its underlying category.

Let A be a Reedy category. It is obvious that A contains no non-trivial automorphisms. Moreover, if f is a monomorphism (resp. an epimorphism) of A, then f belongs to A_+ (resp. to A_-).

A Reedy category A is said to be *skeletal* if it satisfies the following additional property:

(3) every morphism of A_- admits a section; two parallel morphisms of A_- are equal if and only if they admit the same set of sections.

A skeletal Reedy category A is said to be *regular* if it satisfies the following additional property:

(4) every morphism of A_+ is a monomorphism.

These two notions come from Chapter 8 of [20] where they are called normal skeletal categories and regular skeletal categories.

By a remark above, if A is a regular skeletal Reedy category, then A_+ is exactly the class of monomorphisms of A. In particular, being a regular skeletal Reedy category is a property of a category and not an additional structure.

Let A be a Reedy category and let a be an object of A. We will denote by ∂a the subpresheaf of a obtained by taking the union of the images of all the morphisms $a' \to a$ of A_+ different from the identity. We will denote by $\delta_a : \partial a \to a$ the inclusion morphism.

Proposition 1.5. *Let A be a skeletal Reedy category. Then the set*

$$\{\delta_a : \partial a \to a;\ a \in \mathsf{Ob}(A)\}$$

is a cellular model of $\widehat{A}$.

Proof. Since A is a Reedy category, it contains no non-trivial automorphisms. The result thus follows from Proposition 8.1.37 of [20]. □

1.6. Let Δ be the simplex category. Recall that its objects are the ordered sets

$$\Delta_n = \{0, \dots, n\}, \quad n \geq 0,$$

and its morphisms are the order preserving maps between them. The category Δ carries a Reedy category structure where Δ_+ is the set of injections and Δ_- is the set of surjections. This Reedy category structure is regular skeletal. For $n \geq 0$, we will denote by δ_n the morphism $\delta_{\Delta_n} : \partial\Delta_n \to \Delta_n$ of simplicial sets (i.e., of presheaves on Δ).

Let $n \geq 1$ and let k such that $0 \leq k \leq n$. Recall that the *horn* Λ^k_n is the sub-simplicial set obtained by taking the union of the images of all the injections $\Delta_m \to \Delta_n$ except the identity and the unique injection $\Delta_{n-1} \to \Delta_n$ avoiding k. We will denote by $h^k_n : \Lambda^k_n \to \Delta_n$ the inclusion morphism.

Set

$$\Lambda = \{h^k_n;\ n \geq 1,\ 0 \leq k \leq n\}.$$

The class of *simplicial anodyne extensions* is the saturation of the set Λ. A morphism of simplicial sets is a *Kan fibration* if it has the right lifting property with respect to Λ and hence with respect to every simplicial anodyne extension.

Recall that the category of simplicial sets admits a combinatorial model category structure, defined by Quillen in [41], in which cofibrations are the monomorphisms and fibrations are the Kan fibrations. We will call the weak equivalences of this model category the *simplicial weak homotopy equivalences*.

2. Cisinski's theory of A-localizers

The purpose of this section is to introduce Cisinski's theory of A-localizers. In the language of localizers, the main theorem of this paper can be stated by saying that the localizer of Rezk Θ_n-spaces is the simplicial completion of the localizer of n-quasi-categories.

Let us explain roughly what this means. If A is a small category, an A-localizer is a class W of maps of $\widehat{A}$ satisfying some conditions that are satisfied when W is the class of weak equivalences of a model category structure on $\widehat{A}$ whose cofibrations are the monomorphisms. Conversely, by a theorem of Cisinski, if W is an (accessible) A-localizer, then W is the class of weak equivalences of such a model category structure on $\widehat{A}$.

To every A-localizer W, there is an associated $(A\times\Delta)$-localizer W_Δ called the simplicial completion of W. By a theorem of Cisinski, the model category structures associated to W and W_Δ are Quillen equivalent. In particular, what we called above the main theorem of this paper implies that n-quasi-categories and Rezk Θ_n-spaces are Quillen equivalent.

Throughout the section, we fix a small category A.

2.1. An *A-localizer* is a class W of morphisms of $\widehat{A}$ such that the following conditions hold:

(1) W satisfies the 2-out-of-3 property;
(2) every trivial fibration of $\widehat{A}$ is in W;
(3) the class of monomorphisms of $\widehat{A}$ belonging to W is stable under pushout and transfinite composition.

If W is an A-localizer, the elements of W will be called *W-equivalences*. It is immediate that an intersection of A-localizers is again an A-localizer. If C is a class of morphisms of $\widehat{A}$, the *A-localizer generated by C* is by definition the intersection of all the A-localizers containing C. We will denote it by $\mathsf{W}(C)$. A localizer is *accessible* if it is generated by a *set*.

Theorem 2.2. *Let W be a class of morphisms of $\widehat{A}$. Then the following conditions are equivalent:*

(1) *W is an accessible A-localizer;*
(2) *there exists a cofibrantly generated model category structure on $\widehat{A}$ whose weak equivalences are the elements of W and whose cofibrations are the monomorphisms.*

Proof. See Theorem 1.4.3 of [20]. □

2.3. We will denote by W_∞ the Δ-localizer of simplicial weak homotopy equivalences. Simplicial weak homotopy equivalences will thus also be called W_∞-equivalences.

2.4. By the above theorem, from an accessible A-localizer W, we obtain a model category structure on $\widehat{A}$. We will call this model category structure the *W-model category structure*. The weak equivalences of the W-model category structure are the elements of W and the cofibrations are the monomorphisms. The fibrations (resp. the fibrant objects) will be called *W-fibrations* (resp. *W-fibrant objects*).

Remark 2.5. The language of localizers is very related to the language of left Bousfield localization. In particular, we prove in an appendix to this paper that if W is an accessible A-localizer and W' is an accessible A-localizer generated by W and a class of morphisms C, then the W'-model category is the left Bousfield localization of the W-model category with respect to C.

2.6. We will say that a localizer W is *cartesian* if it is closed under binary product.

Proposition 2.7. *Let W be an accessible A-localizer. The following conditions are equivalent:*

(1) *the localizer* W *is cartesian;*
(2) *the class of monomorphisms of* $\widehat{A}$ *belonging to* W *is closed under binary product;*
(3) *the* W*-model category is cartesian closed.*

Proof. By definition, the class of monomorphisms of $\widehat{A}$ belonging to W is the class of trivial cofibrations of the W-model category. The equivalence (1) $\Leftrightarrow$ (2) thus follows immediately from the fact that monomorphisms and trivial fibrations are stable under product.

Let us prove the implication (2) $\Rightarrow$ (3). Let $U \to V$ and $S \to T$ be two monomorphisms of $\widehat{A}$. Consider the diagram

$$\begin{tikzcd}
U \times S \arrow[r] \arrow[d] & U \times T \arrow[d] \arrow[ddr, bend left] & \\
V \times S \arrow[r] \arrow[drr, bend right] & U \times T \amalg_{U \times S} V \times S \arrow[dr] & \\
& & V \times T
\end{tikzcd}\quad .$$

The morphism $U \times T \amalg_{U \times S} V \times S \to V \times T$ is a monomorphism (it is nothing but the inclusion of $U \times T \cup V \times S$ into $V \times T$). Suppose moreover that the morphism $U \to V$ is a W-equivalence. Then $U \times S \to V \times S$ and $U \times T \to V \times T$ are trivial cofibrations by (2). It follows that $U \times T \to U \times T \amalg_{U \times S} V \times S$ is a trivial cofibration and, by the 2-out-of-3 property, that $U \times T \amalg_{U \times S} V \times S \to V \times T$ is a W-equivalence, thereby proving (3).

Let us show the converse. Let $U \to V$ be a trivial cofibration and let T be a presheaf on A. It clearly suffices to show that $U \times T \to V \times T$ is a trivial cofibration. By applying (3) to $U \to V$ and to the unique morphism $\varnothing_{\widehat{A}} \to T$, we obtain that the morphism $U \times T \amalg_{U \times \varnothing} V \times \varnothing_{\widehat{A}} \to V \times T$ is a trivial cofibration. But this morphism is nothing but $U \times T \to V \times T$. $\square$

Proposition 2.8. *Let* W *be an* A*-localizer, let* W' *be an* A'*-localizer and let* $F : \widehat{A} \to \widehat{A'}$ *be a functor. Suppose that* F *respects binary products and that* $\mathsf{W} = F^{-1}(\mathsf{W}')$*. Then if the localizer* W' *is cartesian, so is the localizer* W*.*

Proof. This is an immediate consequence of the definition of cartesian localizers. $\square$

2.9. An *interval of* $\widehat{A}$ consists of a presheaf I on A and two morphisms $\partial^0, \partial^1 : e_{\widehat{A}} \to I$. We will often denote such an interval simply by I. If I is an interval of $\widehat{A}$, we will denote by $\{\varepsilon\}$, where $\varepsilon = 0, 1$, the image of ∂^ε in I. We will denote by ∂I the union $\{0\} \cup \{1\}$ and by $\delta_I : \partial I \to I$ the canonical inclusion. If X is a presheaf on A and $\varepsilon = 0, 1$, we will denote by $\partial^\varepsilon_X : X \to X \times I$ the morphism $X \times \partial^\varepsilon$.

An interval I is *separating* if the intersection of $\{0\}$ and $\{1\}$ is $\varnothing_{\widehat{A}}$. The interval I is *injective* if the morphism $I \to e_{\widehat{A}}$ is a trivial fibration of $\widehat{A}$.

2.10. Let I be a separating interval of $\widehat{A}$. A class of *anodyne I-extensions* is a class An of monomorphisms of $\widehat{A}$ satisfying the following conditions:

(1) there exists a *set* S such that $\mathsf{An} = \mathrm{Sat}(S)$;

(2) the canonical inclusion
$$U \times I \cup V \times \{\varepsilon\} \to V \times I$$
is in An for every monomorphism $U \to V$ of $\widehat{A}$ and $\varepsilon = 0, 1$;

(3) the canonical inclusion
$$U \times I \cup V \times \partial I \to V \times I$$
is in An for every $U \to V$ in An.

Let S be a set of monomorphisms of $\widehat{A}$. By Proposition 1.3.13 of [20], there exists a smallest class of anodyne I-extensions containing S. We will denote this class by $\mathsf{An}_I(S)$ and we will call its elements *anodyne* (S, I)*-extensions.*

The class of anodyne (S, I)-extensions can be described in the following way. We define by induction on $k \geq 0$ a set $\Lambda^k_I(S)$ of monomorphisms of $\widehat{A}$ by setting
$$\Lambda^0_I(S) = S, \quad \Lambda^{k+1}_I(S) = \Lambda_I(\Lambda^k_I(S)),$$
where
$$\Lambda_I(T) = \{U \times I \cup V \times \partial I \to V \times I;\ U \to V \in T\}.$$
By definition, the set $\Lambda^\infty_I(S)$ is the union of the $\Lambda^k_I(S)$, $k \geq 0$. Let $\mathfrak{M}$ be any cellular model of $\widehat{A}$. Then the class of anodyne (S, I)-extensions is the saturated class generated by
$$\Lambda^\infty_I(S) \cup \{U \times I \cup V \times \{\varepsilon\} \to V \times I;\ U \to V \in \mathfrak{M},\ \varepsilon = 0, 1\}.$$
Note that this description is not exactly the one given in paragraph 1.3.12 of [20]. Nevertheless, it follows easily from Remark 1.3.15 of ibid. that it is correct.

Remark 2.11. By Section 2 of Chapter IV of [22], in the case where $A = \Delta$, $I = \Delta_1$ and S is empty, the class of anodyne (S, I)-extensions is precisely the class of simplicial anodyne extensions. See also paragraph 2.1.3 of [20].

Lemma 2.12. *Let S be a set of monomorphisms of $\widehat{A}$ and let I be a separating interval. Let C be a class of monomorphisms of $\widehat{A}$ satisfying the following conditions:*

(1) *C contains S;*
(2) *C is saturated;*
(3) *if $u : X \to Y$ and $v : Y \to Z$ are monomorphisms of $\widehat{A}$ such that vu and u are in C, then v is in C;*
(4) *the morphisms $\partial^\varepsilon_X : X \to X \times I$ belong to C for every presheaf X on A and $\varepsilon = 0, 1$.*

Then C contains the class of anodyne (S, I)-extensions.

Proof. See Lemma 1.3.16 of [20]. □

2.13. Let S be a set of monomorphisms of $\widehat{A}$ and let I be a separating interval of $\widehat{A}$. A morphism of $\widehat{A}$ will be said to be a *naive* (S, I)*-fibration* if it has the right lifting property with respect to the class of anodyne (S, I)-extensions. A presheaf X on A will be said to be (S, I)*-fibrant* if the morphism $X \to e_{\widehat{A}}$ is a naive (S, I)-fibration.

Theorem 2.14. *Let S be a set of monomorphisms of $\widehat{A}$ and set $\mathsf{W} = \mathsf{W}(S)$. Let J be an injective separating interval of $\widehat{A}$. Then, if f is a morphism whose target is a W-fibrant object, then f is a W-fibration if and only if it is a naive (S, J)-fibration. In particular, the class of W-fibrant objects and of (S, J)-fibrant objects coincide.*

Proof. By Corollary 1.4.18 of [20], the W-model category is the model category associated to S and J in the sense of Theorem 1.3.22 of ibid. The result then follows from Proposition 1.3.36 of ibid. □

From now on, we fix an A-localizer W.

2.15. We will denote by
$$A \xleftarrow{p} A \times \Delta \xrightarrow{q} \Delta$$
the two canonical projections. They induce functors
$$\widehat{A} \xrightarrow{p^*} \widehat{A \times \Delta} \xleftarrow{q^*} \widehat{\Delta}.$$
These functors admit left and right adjoints and hence respect limits and colimits. In particular, they preserve monomorphisms.

We will denote by
$$i_0 : A \to A \times \Delta$$
the functor defined by
$$i_0(a) = (a, \Delta_0).$$
It follows from the fact that Δ_0 is a terminal object of Δ that the functor
$$i_0^* : \widehat{A \times \Delta} \to \widehat{A}$$
is right adjoint to p^*.

2.16. We will say that a morphism $f : X \to Y$ of $\widehat{A \times \Delta}$ is a *horizontal equivalence* if for every $n \geq 0$, the morphism $f_{\bullet,n} : X_{\bullet,n} \to Y_{\bullet,n}$ is a W-equivalence. We will denote by $\mathsf{W}_{\mathrm{hor}}$ the class of horizontal equivalences. If follows from the existence of the injective model category structure for combinatorial model categories that if W is accessible, then $\mathsf{W}_{\mathrm{hor}}$ is an accessible $(A \times \Delta)$-localizer.

We will say that a morphism $f : X \to Y$ of $\widehat{A \times \Delta}$ is a *vertical equivalence* if for every object a of A, the morphism $f_{a,\bullet} : X_{a,\bullet} \to Y_{a,\bullet}$ is a W_∞-equivalence. We will denote by $\mathsf{W}_{\mathrm{vert}}$ the class of vertical equivalences. If follows again from the existence of the injective model category structure for combinatorial model categories that if W is accessible, then $\mathsf{W}_{\mathrm{vert}}$ is an accessible $(A \times \Delta)$-localizer.

The *simplicial completion* of W is the $(A \times \Delta)$-localizer generated by
$$\mathsf{W}_{\mathrm{hor}} \cup \{X \times q^*(\Delta_1) \to X;\ X \in \mathsf{Ob}(\widehat{A \times \Delta})\}.$$
We will denote it by W_Δ.

Remark 2.17. To make sense of the terminology “vertical weak equivalences” and “horizontal weak equivalences”, one has to think of a presheaf on $A \times \Delta$ as a grid of sets whose columns are indexed by objects of A and whose rows are indexed by integers $n \geq 0$.

Proposition 2.18. *If the A-localizer W is accessible, then the $(A \times \Delta)$-localizer W_Δ is accessible.*

Proof. See Proposition 2.3.24 of [20]. □

Proposition 2.19. *If A is a regular skeletal Reedy category, then the simplicial completion of W is generated by W_{hor} and W_{vert}.*

Proof. This is a direct consequence of Proposition 8.2.9 and Corollary 3.4.37 of [20]. □

Proposition 2.20. *Suppose W is accessible. Then the adjunction*

$$p^* : \widehat{A} \rightleftarrows \widehat{A \times \Delta} : i_0^*,$$

where $\widehat{A}$ (resp. $\widehat{A \times \Delta}$) is endowed with the W-model category structure (resp. with the W_Δ-model category structure), is a Quillen equivalence.

Proof. We have already noticed that the functor p^* preserves monomorphisms and hence cofibrations. It also preserves weak equivalences by definition of W_Δ. The pair (p^*, i_0^*) is hence a Quillen adjunction. By Proposition 2.3.27 of [20], the functor p^* induces an equivalence on the homotopy categories. The pair (p^*, i_0^*) is hence a Quillen equivalence. □

Corollary 2.21. *Suppose W is accessible. If the simplicial completion of W is cartesian, then so is the localizer W.*

Proof. The functor p^* respects limits and in particular binary products. Since every object is cofibrant in the W-model category structure, it follows from the above proposition that it preserves and reflects weak equivalences. The result thus follows from Proposition 2.8. □

2.22. Let D be a cosimplicial object in $\widehat{A}$, i.e., a functor $D : \Delta \to \widehat{A}$. For $n \geq 0$, we will denote $D(\Delta_n)$ by D_n.

Consider the functor

$$\underline{D} : A \times \Delta \to \widehat{A}$$

defined by

$$\underline{D}(a, \Delta_n) = a \times D_n.$$

Since $\widehat{A}$ is cocomplete, this functor induces an adjunction

$$\mathrm{Real}_D : \widehat{A \times \Delta} \rightleftarrows \widehat{A} : \mathrm{Sing}_D,$$

where Real_D is the unique extension of $\underline{D}$ to $\widehat{A \times \Delta}$ which respects colimits, and Sing_D is defined by

$$\mathrm{Sing}_D(X)_{a,n} = \mathsf{Hom}_{\widehat{A}}(a \times D_n, X),$$

where a is an object of A and $n \geq 0$.

The cosimplicial object D is a *cosimplicial W-resolution* if it satisfies the following conditions:

(1) the morphism $D_0 \amalg D_0 \to D_1$ induced by $\delta_1 : \partial\Delta_1 = \Delta_0 \amalg \Delta_0 \to \Delta_1$ is a monomorphism;
(2) for every $n \geq 0$ and every presheaf X on A, the canonical projection $X \times D_n \to X$ is a W-equivalence.

Proposition 2.23. *Suppose* W *is accessible and let* D *be a cosimplicial* W*-resolution. Then the adjunction*

$$\mathit{Real}_D : \widehat{A \times \Delta} \rightleftarrows \widehat{A} : \mathit{Sing}_D,$$

where $\widehat{A \times \Delta}$ *(resp.* $\widehat{A}$*) is endowed with the* W_Δ*-model category structure (resp. with the* W*-model category structure), is a Quillen equivalence.*

Proof. By Lemma 2.3.10 of [20], the functor Real_D preserves monomorphisms and hence cofibrations. By Proposition 2.3.27 of ibid., it also preserves weak equivalences. The pair $(\mathrm{Real}_D, \mathrm{Sing}_D)$ is hence a Quillen pair. By the same proposition, the functor Real_D induces an equivalence on the homotopy categories. The pair $(\mathrm{Real}_D, \mathrm{Sing}_D)$ is hence a Quillen equivalence. □

3. The vertical and the horizontal model categories

Let A be a skeletal Reedy category and let W be an accessible A-localizer. We have two different Reedy model category structures on the category

$$\widehat{A \times \Delta} = \underline{\mathsf{Hom}}(A^o, \widehat{\Delta}) = \underline{\mathsf{Hom}}(\Delta^o, \widehat{A}).$$

One is coming from the Reedy structure of A and the classical model category structure on simplicial sets; the second is coming from the Reedy structure of Δ and the model category structure associated to the localizer W (see Theorem 2.2). Following Section 2 of [32], the former model category structure we will called the vertical model category structure, and the latter will be called the horizontal model category structure (see Remark 2.17 on this terminology). The purpose of this section is to introduce and study these two model category structures. (Our case of interest in this paper is the case where $A = \Theta_n$ and W is the localizer of n-quasi-categories defined in paragraph 5.17.)

Throughout the section, we fix two small categories A and B. (We will soon specialize to the case $B = \Delta$.)

3.1. We will denote by

$$\Box : \widehat{A} \times \widehat{B} \to \widehat{A \times B}$$

the functor defined in the following way: if X is a presheaf on A, Y is a presheaf on B, a is an object of A and b is an object of B, then

$$(X \Box Y)_{a,b} = X_a \times Y_b.$$

If X is a fixed presheaf on A, then the functor

$$X \Box - : \widehat{B} \to \widehat{A \times B}$$

admits a right adjoint

$$X\backslash - : \widehat{A \times B} \to \widehat{B}$$

given by

$$(X\backslash Z)_b = \mathsf{Hom}_{\widehat{A\times B}}(X \Box b, Z),$$

where Z is a presheaf on $A \times B$ and b is an object of B. Similarly, if Y is a fixed presheaf on B, the functor

$$- \Box Y : \widehat{A} \to \widehat{A \times B}$$

admits a right adjoint

$$-/Y : \widehat{A \times B} \to \widehat{A}.$$

given by

$$(Z/Y)_a = \mathsf{Hom}_{\widehat{A\times B}}(a \,\square\, Y, Z),$$

where Z is a presheaf on $A \times B$ and a is an object of A.

Thus, if X is a presheaf on A, Y is a presheaf on B and Z is a presheaf on $A \times B$, we have natural bijections

$$\mathsf{Hom}_{\widehat{A\times B}}(X \,\square\, Y, Z) \cong \mathsf{Hom}_{\widehat{B}}(Y, X\backslash Z) \cong \mathsf{Hom}_{\widehat{A}}(X, Z/Y).$$

3.2. Let $u : U \to V$ be a morphism of $\widehat{A}$ and let $v : S \to T$ be a morphism of $\widehat{B}$. We will denote by

$$u \,\square\text{'}\, v : U \,\square\, T \amalg_{U \,\square\, S} V \,\square\, S \to V \,\square\, T$$

the morphism induced by the commutative square

$$\begin{array}{ccc} U \,\square\, S & \longrightarrow & U \,\square\, T \\ \downarrow & & \downarrow \\ V \,\square\, S & \longrightarrow & V \,\square\, T \end{array} \quad .$$

If $f : X \to Y$ is a morphism of $\widehat{A \times B}$, we will denote by

$$\langle u\backslash f\rangle : V\backslash X \to V\backslash Y \times_{U\backslash Y} U\backslash X$$

the morphism induced by the commutative square

$$\begin{array}{ccc} V\backslash X & \longrightarrow & V\backslash Y \\ \downarrow & & \downarrow \\ U\backslash X & \longrightarrow & U\backslash Y \end{array} \quad ;$$

and by

$$\langle f/v\rangle : X/T \to Y/T \times_{Y/S} X/S$$

the morphism induced by the commutative square

$$\begin{array}{ccc} X/T & \longrightarrow & Y/T \\ \downarrow & & \downarrow \\ X/S & \longrightarrow & Y/S \end{array} \quad .$$

Proposition 3.3. *If u is a morphism of $\widehat{A}$, v is a morphism of $\widehat{B}$ and f is a morphism of $\widehat{A \times B}$, then we have*

$$(u \,\square\text{'}\, v) \pitchfork f \quad \Leftrightarrow \quad u \pitchfork \langle f/v\rangle \quad \Leftrightarrow \quad v \pitchfork \langle u\backslash f\rangle.$$

Proof. See Proposition 7.6 of [32]. □

From now on, we set $B = \Delta$.

Proposition 3.4. *If $\mathfrak{M}$ is a cellular model of $\widehat{A}$, then*

$$\{\delta \,\Box'\, \delta_n : U \,\Box\, \Delta_n \amalg_{U \,\Box\, \partial\Delta_n} V \,\Box\, \partial\Delta_n \to V \,\Box\, \Delta_n;\ \delta : U \to V \in \mathfrak{M},\ n \geq 0\}$$

is a cellular model of $\widehat{A \times \Delta}$.

Proof. See Lemma 2.3.2 of [20]. □

Proposition 3.5. *A morphism f of $\widehat{A \times \Delta}$ is a trivial fibration if and only if the following equivalent conditions are satisfied:*

1. *$\langle \delta \backslash f \rangle$ is a trivial fibration for every δ in a fixed cellular model of $\widehat{A}$;*
2. *$\langle u \backslash f \rangle$ is a trivial fibration for every monomorphism u of $\widehat{A}$;*
3. *$\langle f / \delta_n \rangle$ is a trivial fibration for all $n \geq 0$;*
4. *$\langle f / v \rangle$ is a trivial fibration for every monomorphism v of simplicial sets.*

Proof. The proof is essentially the same as the one of Proposition 2.3 of [32]. Fix a cellular model $\mathfrak{M}$ of $\widehat{A}$. By the previous proposition, a morphism f of $\widehat{A \times \Delta}$ is a trivial fibration if and only if for every δ in $\mathfrak{M}$ and every $n \geq 0$, we have $\delta \,\Box'\, \delta_n \pitchfork f$. But by Proposition 3.3, we have

$$\delta \,\Box'\, \delta_n \pitchfork f \quad \Leftrightarrow \quad \delta_n \pitchfork \langle \delta \backslash f \rangle \quad \Leftrightarrow \quad \delta \pitchfork \langle f / \delta_n \rangle,$$

thereby proving that f is a trivial fibration if and only if one of the conditions (1) and (3) is satisfied. The other equivalences follow from the fact that the class of morphisms having the left lifting property with respect to a fixed class of morphisms is saturated. □

From now on, we assume that A is a skeletal Reedy category. In particular, if a is an object of A, we have a morphism $\delta_a : \partial a \to a$.

3.6. Recall that we have defined in paragraph 2.16 a class $\mathsf{W}_{\mathrm{vert}}$ of morphisms of $\widehat{A \times \Delta}$ called vertical equivalences. In the language of this section, a morphism f of $\widehat{A \times \Delta}$ is a vertical equivalence if and only if, for every object a of A, the map $a \backslash f$ is a W_∞-equivalence.

Theorem 3.7 (Reedy, Kan)**.** *The class W_{vert} is an accessible $(A \times \Delta)$-localizer. The fibrations of the W_{vert}-model category structure are the morphisms f such that for every object a of A, the morphism $\langle \delta_a \backslash f \rangle$ is a Kan fibration. Moreover, this model category structure is proper, simplicial and cartesian closed.*

Proof. The proof is essentially the same as the one of Theorem 2.6 of [32]. Consider the Reedy model category structure on $\widehat{A \times \Delta}$ seen as $\underline{\mathsf{Hom}}(A^o, \widehat{\Delta})$, where $\widehat{\Delta}$ is endowed with the W_∞-model category structure. By definition, the weak equivalences of this model category structure are the vertical equivalences. Recall that a morphism $f : X \to Y$ of $\widehat{A \times \Delta}$ is a fibration (resp. a trivial fibration) if and only if, for every object a of A, the morphism

$$X_a \to Y_a \times_{M_a(Y)} M_a(X),$$

where $M_a(Z)$ denotes the a-th latching object of a presheaf Z, is a Kan fibration (resp. a trivial fibration). But this morphism is nothing but $\langle \delta_a \backslash f \rangle$. In particular, by Proposition 3.5, the trivial fibration of this Reedy structure are the trivial fibrations in the sense of paragraph 1.3. It follows that the cofibrations of this Reedy structure are the monomorphisms. This Reedy structure is hence a model category structure whose weak equivalences are the $\mathsf{W}_{\mathrm{vert}}$-equivalences and whose cofibrations are the monomorphisms. By Theorem 2.2, the class $\mathsf{W}_{\mathrm{vert}}$ is hence an accessible localizer and the $\mathsf{W}_{\mathrm{vert}}$-model category structure is this Reedy model category structure.

The fact that this model category structure, which is nothing but the injective model category structure on simplicial presheaves, is proper, simplicial and cartesian closed is well-known (see the proof of Theorem 2.6 of [32] for a proof). □

3.8. We will call the $\mathsf{W}_{\mathrm{vert}}$-model category structure on $\widehat{A \times \Delta}$ the *vertical model category structure.* A fibration of this structure will be called a *vertical fibration.*

Proposition 3.9. *A morphism f of $\widehat{A \times \Delta}$ is a vertical fibration if and only if the following equivalent conditions are satisfied:*

(1) *$\langle \delta_a \backslash f \rangle$ is a Kan fibration for every object a of A;*
(2) *$\langle u \backslash f \rangle$ is a Kan fibration for every monomorphism u of $\widehat{A}$;*
(3) *$\langle f / h_n^k \rangle$ is a trivial fibration for all $n \geq 1$ and $0 \leq k \leq n$;*
(4) *$\langle f / v \rangle$ is a trivial fibration for every simplicial anodyne extension v.*

Proof. The proof is essentially the same as the one of Proposition 2.5 of [32]. The morphism $\langle \delta_a \backslash f \rangle$ is a Kan fibration if and only if for every $n \geq 1$ and $0 \leq k \leq n$, we have $h_n^k \pitchfork \langle \delta_a \backslash f \rangle$. But

$$h_n^k \pitchfork \langle \delta_a \backslash f \rangle \quad \Leftrightarrow \quad \delta_a \pitchfork \langle f / h_n^k \rangle,$$

and the result follows from the fact that the δ_a's form a cellular model of $\widehat{A}$ (Proposition 1.5). □

From now on, we fix an accessible A-localizer W.

3.10. Recall that we have defined in paragraph 2.16 a class $\mathsf{W}_{\mathrm{hor}}$ of morphisms of $\widehat{A \times \Delta}$ called horizontal equivalences. In the language of this section, a morphism f of $\widehat{A \times \Delta}$ is a horizontal equivalence if and only if, for every $n \geq 0$, the morphism f / Δ_n is a W-equivalence.

Theorem 3.11 (Reedy)**.** *The class W_{hor} is an accessible $(A \times \Delta)$-localizer. The fibrations of the W_{hor}-model category structure are the morphisms f such that for every $n \geq 0$, the morphism $\langle f / \delta_n \rangle$ is a W-fibration.*

Proof. The proof is essentially the same as the one of Proposition 2.10 of [32]. Consider the Reedy model category structure on $\widehat{A \times \Delta}$ seen as $\underline{\mathsf{Hom}}(\Delta^{o}, \widehat{A})$, where $\widehat{A}$ is endowed with the W-model category structure. By definition, the weak equivalences of this model category structure are the horizontal equivalences. For the same reasons as in the proof of Theorem 3.7, a morphism f of $\widehat{A \times \Delta}$ is a fibration (resp. a trivial fibration) for this Reedy structure if and only if for every $n \geq 0$, the morphism $\langle f / \delta_n \rangle$ is a W-fibration (resp. a trivial fibration). It follows as in the proof of Theorem 3.7 that the cofibrations of this Reedy structure are the monomorphisms. The class $\mathsf{W}_{\mathrm{hor}}$ is hence an accessible localizer and the $\mathsf{W}_{\mathrm{hor}}$-model category structure is this Reedy model category structure. □

3.12. We will call the $\mathsf{W}_{\mathrm{hor}}$-model category structure on $\widehat{A \times \Delta}$ the *horizontal model category structure.* A fibration of this structure will be called a *horizontal fibration.*

Remark 3.13. All the results of this section have appropriate generalizations when Δ is replaced by a skeletal Reedy category B endowed with an accessible B-localizer. This is easily seen once Proposition 3.4 has been generalized. An inspection of the proof of this proposition, that is of the proof of Lemma 2.3.2 of [20], reveals that all the properties of Δ used in this proof are shared by skeletal Reedy categories (see Section 8.1 of [20]).

4. The model category of formal Rezk spaces

In this section, we associate to any accessible A-localizer W, where A is a skeletal Reedy category, an $(A \times \Delta)$-localizer $\mathsf{W}_{\mathrm{fRezk}}$ of formal Rezk spaces. The purpose of the section is then to show that when the skeletal Reedy category A is regular, the localizer $\mathsf{W}_{\mathrm{fRezk}}$ is the simplicial completion of the localizer W.

Throughout the section, we fix a skeletal Reedy category A, a set S of monomorphisms of $\widehat{A}$ and an injective separating interval J of $\widehat{A}$. We denote by W the A-localizer generated by S.

4.1. We will denote by $\mathsf{W}_{\mathrm{fRezk}}$ the $(A \times \Delta)$-localizer generated by

$$\mathsf{W}_{\mathrm{vert}} \cup p^*(S) \cup \{p^*(X \times J \to X);\ X \in \mathsf{Ob}(\widehat{A})\}.$$

Note that by the 2-out-of-3 property, $\mathsf{W}_{\mathrm{fRezk}}$ is also generated by

$$\mathsf{W}_{\mathrm{vert}} \cup p^*(S) \cup \{p^*(\partial^\varepsilon_X : X \to X \times J);\ X \in \mathsf{Ob}(\widehat{A}),\ \varepsilon = 0, 1\}.$$

Proposition 4.2. *The $(A \times \Delta)$-localizer $\mathsf{W}_{\mathit{fRezk}}$ is accessible.*

Proof. By Theorem 3.7, the localizer $\mathsf{W}_{\mathrm{vert}}$ is accessible. Let T be a generating set of $\mathsf{W}_{\mathrm{vert}}$. We claim that the set

$$T \cup p^*(S) \cup \{p^*\big(\partial a \times J \cup a \times \{\varepsilon\} \to a \times J\big);\ a \in \mathsf{Ob}\, A,\ \varepsilon = 0, 1\}$$

generates the localizer $\mathsf{W}_{\mathrm{fRezk}}$.

Let W' be the localizer generated by this set. We first show that W' is included in $\mathsf{W}_{\mathrm{fRezk}}$. Let a be an object of A and let $\varepsilon = 0, 1$. Consider the diagram

$$\begin{tikzcd}
p^*(\partial a \times \{\varepsilon\}) \arrow[r] \arrow[d] & p^*(\partial a \times J) \arrow[d] \arrow[ddr, bend left] & \\
p^*(a \times \{\varepsilon\})) \arrow[r] \arrow[drr, bend right] & p^*(\partial a \times J \cup a \times \{\varepsilon\}) \arrow[dr] & \\
& & p^*(a \times J)
\end{tikzcd} .$$

The morphisms

$$p^*\big(\partial a \times \{\varepsilon\} \to \partial a \times J\big) \quad \text{and} \quad p^*\big(a \times \{\varepsilon\} \to a \times J\big)$$

are $\mathsf{W}_{\mathrm{fRezk}}$-equivalences. The upper horizontal morphism is hence both a $\mathsf{W}_{\mathrm{fRezk}}$-equivalence and a monomorphism. Since the square of the diagram is cocartesian, the lower horizontal morphism is also a $\mathsf{W}_{\mathrm{fRezk}}$-equivalence. It follows from the 2-out-of-3 property that the morphism

$$p^*\big(\partial a \times J \cup a \times \{\varepsilon\} \to a \times J\big)$$

is a $\mathsf{W}_{\mathrm{fRezk}}$-equivalence.

Let us now show that $\mathsf{W}_{\mathrm{fRezk}}$ is included in W'. Let C be the class of monomorphisms $U \to V$ of $\widehat{A}$ such that

$$p^*\big(U \times J \cup V \times \{\varepsilon\} \to V \times J\big)$$

belongs to W' for $\varepsilon = 0, 1$. It is easy to check that this class is saturated. But by definition of W', the class C contains a cellular model of $\widehat{A}$. It hence contains every monomorphism of $\widehat{A}$. If X is a presheaf on A, the morphism $f : \varnothing_{\widehat{A}} \to X$ thus belongs to C. This means that

$$p^*\big(X \times \{\varepsilon\} \to X \times J\big)$$

is in W', thereby proving the result. □

4.3. We will call the $\mathsf{W}_{\mathrm{fRezk}}$-model category structure on $\widehat{A \times \Delta}$ the *model category structure of formal Rezk A-spaces.* A $\mathsf{W}_{\mathrm{fRezk}}$-fibrant object will be called a *formal Rezk A-space.*

Proposition 4.4. *A presheaf X on $A \times \Delta$ is a formal Rezk A-space if and only if it satisfies the following conditions:*

(1) *X is vertically fibrant;*
(2) *$s\backslash X$ is a trivial fibration for every s in S;*
(3) *$\partial^\varepsilon_Y\backslash X$ is a trivial fibration for every presheaf Y on A and $\varepsilon = 0, 1$.*

Proof. Let

$$C = p^*(S) \cup \{p^*(\partial^\varepsilon_X : X \to X \times J);\ X \in \mathsf{Ob}(\widehat{A}),\ \varepsilon = 0, 1\}.$$

By definition, the localizer $\mathsf{W}_{\mathrm{fRezk}}$ is generated by $\mathsf{W}_{\mathrm{vert}}$ and C. It follows from Proposition A.9 that the formal Rezk A-spaces are the fibrant C-local objects of the vertical model category, i.e., the vertically fibrant objects X of $\widehat{A \times \Delta}$ such that for every morphism $f : K \to L$ in C, the morphism

$$\mathrm{Map}(f, X) : \mathrm{Map}(L, X) \to \mathrm{Map}(K, X),$$

where Map denotes the simplicial enrichment of $\widehat{A \times \Delta}$, is a W_∞-equivalence. (Recall that the vertical model category is a simplicial model category.) But if a morphism f of $\widehat{A \times \Delta}$ is equal to $p^*(f_0)$ for some morphism $f_0 : K_0 \to L_0$ of $\widehat{A}$, then $\mathrm{Map}(f, X)$ is nothing but the morphism

$$f_0\backslash X : L_0\backslash X \to K_0\backslash X.$$

Thus, a presheaf X on $A \times \Delta$ is a formal Rezk A-space if and only if it satisfies the following conditions:

(1) X is vertically fibrant;
(2) $s\backslash X$ is a W_∞-equivalence for every s in S;
(3) $\partial^\varepsilon_Y\backslash X$ is a W_∞-equivalence for every presheaf Y on A and $\varepsilon = 0, 1$.

But by Proposition 3.9, under the assumption that X is vertically fibrant, the morphisms $s\backslash X$ and $\partial^\varepsilon_Y\backslash X$, where s is in S and Y is a presheaf on A, are Kan fibrations. They are hence W_∞-equivalences if and only if they are trivial fibrations, thereby proving the result. □

Proposition 4.5. *Let X be a vertically fibrant presheaf on $A \times \Delta$. Then the following conditions are equivalent:*

1. *X is a formal Rezk A-space;*
2. *$u\backslash X$ is a trivial fibration for every anodyne (S, J)-extension u;*
3. *X/δ_n is a naive (S, J)-fibration for all $n \geq 0$;*
4. *X/v is a naive (S, J)-fibration for every monomorphism v of simplicial sets.*

Proof. The proof is similar to the one of Proposition 3.4 of [32].

$2 \Leftrightarrow 3 \Leftrightarrow 4$) Let u be a morphism of $\widehat{A}$. The morphism $u\backslash X$ is a trivial fibration if and only if, for all $n \geq 0$, we have $\delta_n \pitchfork u\backslash X$. But we have

$$\delta_n \pitchfork u\backslash X \quad \Leftrightarrow \quad u \pitchfork X/\delta_n.$$

$1 \Rightarrow 2$) Let C be the class of monomorphisms u of $\widehat{A}$ such that $u\backslash X$ is a trivial fibration. We have just seen that u belongs to C if and only if, for all $n \geq 0$, we have $u \pitchfork X/\delta_n$. The class C is thus saturated. Moreover, by Proposition 3.9, if u is a monomorphism of $\widehat{A}$, then $u\backslash X$ is a Kan fibration. In particular, u belongs to C if and only if $u\backslash X$ is a W_∞-equivalence. It follows that the class C satisfies the 2-out-of-3 property. Since X is a formal Rezk A-space, the morphisms of S and the ∂^ε_Y's belong to C by the previous proposition. It follows from Lemma 2.12 that the class C contains the class of anodyne (S, J)-extensions.

$2 \Rightarrow 1$) By definition, the morphisms of S and the ∂^ε_Y's are anodyne (S, J)-extensions. $\square$

Corollary 4.6. *If X is a formal Rezk A-space, then for every simplicial set U, the presheaf X/U is W-fibrant. In particular, $X_{\bullet,n}$ is W-fibrant for every $n \geq 0$.*

Proof. Consider the morphism of simplicial sets $v : \varnothing_{\widehat{\Delta}} \to U$. By the previous proposition, the morphism $X/v = X/U \to X/\varnothing_{\widehat{\Delta}} = e_{\widehat{A}}$ is a naive (S, J)-fibration. By Theorem 2.14, the object X/U is hence W-fibrant. The second assertion follows from the fact that $X/\Delta_n = X_{\bullet,n}$. $\square$

Proposition 4.7. *Let $f : X \to Y$ be a vertical fibration between formal Rezk A-spaces. Then, for every monomorphism $v : S \to T$ of simplicial sets, the morphism*

$$\langle f/v \rangle : X/T \to Y/T \times_{Y/S} X/S$$

is a W-fibration between W-fibrant objects.

Proof. The proof is similar to the one of Proposition 3.10 of [32]. Let $v : S \to T$ be a monomorphism of simplicial sets. By the above corollary, the presheaves X/S and X/T are W-fibrant. By proposition 4.5, the morphism $Y/T \to Y/S$ is a naive (S, J)-fibration. It follows that $Y/T \times_{Y/S} X/S \to X/S$ is a naive (S, J)-fibration. The presheaf $Y/T \times_{Y/S} X/S$ is thus W-fibrant by Theorem 2.14.

By the same theorem, it suffices to show that $\langle f/v \rangle$ is a naive (S, J)-fibration, i.e., that for every anodyne (S, J)-extension u, we have $u \pitchfork \langle f/v \rangle$. But we have

$$u \pitchfork \langle f/v \rangle \quad \Leftrightarrow \quad v \pitchfork \langle u\backslash f \rangle,$$

and it thus suffices to show that for every anodyne (S, J)-extension $u : U \to V$, the morphism

$$\langle u\backslash f \rangle : V\backslash X \to V\backslash Y \times_{U\backslash Y} U\backslash X$$

is a trivial fibration. Let u be such an anodyne (S, J)-extension. Since f is a vertical fibration, by Proposition 3.9, the morphism $\langle u\backslash f\rangle$ is a Kan fibration. It thus suffices to prove that $\langle u\backslash f\rangle$ is a W_∞-equivalence. Consider the commutative square

$$\begin{array}{ccc} V\backslash X & \longrightarrow & V\backslash Y \\ \big\downarrow & & \big\downarrow \\ U\backslash X & \longrightarrow & U\backslash Y \end{array} \quad .$$

Since X and Y are formal Rezk A-spaces, by Proposition 4.5, the vertical morphisms are trivial fibrations. This square is hence homotopically cartesian and the result follows. $\square$

Corollary 4.8. *Let $f : X \to Y$ be a vertical fibration between formal Rezk A-spaces. Then f is a horizontal fibration.*

Proof. By the previous proposition, for every $n \geq 0$, the morphism $\langle f/\delta_n\rangle$ is a W-fibration. The results hence follows from Theorem 3.11. $\square$

Theorem 4.9. *The $(A \times \Delta)$-localizer $\mathsf{W}_{\mathrm{fRezk}}$ contains $\mathsf{W}_{\mathrm{hor}}$.*

Proof. The proof is similar to the one of Theorem 4.5 of [32]. We have an adjunction

$$F : \widehat{A \times \Delta}_{\mathrm{hor}} \rightleftarrows \widehat{A \times \Delta}_{\mathrm{fRezk}} : G,$$

where $\widehat{A \times \Delta}_{\mathrm{hor}}$ and $\widehat{A \times \Delta}_{\mathrm{fRezk}}$ denote the $\mathsf{W}_{\mathrm{hor}}$-model category and the $\mathsf{W}_{\mathrm{fRezk}}$-model category, respectively, and F, G both denote the identity functor. The functor F clearly preserves monomorphisms and hence cofibrations. Moreover, by the above corollary, the functor G preserves fibrations between fibrant objects. It follows from a lemma of Dugger (Corollary A.2 of [21]) that (F, G) is a Quillen pair. In particular, by Ken Brown's lemma, F preserves weak equivalences between cofibrant objects, and hence all the weak equivalences since every object of $\widehat{A \times \Delta}_{\mathrm{hor}}$ is cofibrant. This exactly means that $\mathsf{W}_{\mathrm{fRezk}}$ contains $\mathsf{W}_{\mathrm{hor}}$. $\square$

From now on, we suppose that A is a regular skeletal Reedy category.

Theorem 4.10. *The $(A \times \Delta)$-localizer $\mathsf{W}_{\mathrm{fRezk}}$ is the simplicial completion of W.*

Proof. Since A is a regular skeletal Reedy category, by Proposition 2.19, the localizer W_Δ is generated by $\mathsf{W}_{\mathrm{vert}}$ and $\mathsf{W}_{\mathrm{hor}}$. By definition, $\mathsf{W}_{\mathrm{vert}}$ is included in $\mathsf{W}_{\mathrm{fRezk}}$ and by the above theorem, $\mathsf{W}_{\mathrm{hor}}$ is included in $\mathsf{W}_{\mathrm{fRezk}}$. The localizer W_Δ is thus included in $\mathsf{W}_{\mathrm{fRezk}}$.

Conversely, let us show that $\mathsf{W}_{\mathrm{fRezk}}$ is included in W_Δ. By Proposition 2.19, $\mathsf{W}_{\mathrm{vert}}$ is included in W_Δ. Moreover, the morphisms of S are in W by definition and the morphisms $X \times J \to X$, where X is a presheaf on A, are in W since J is injective. But by definition, p^* sends W into $\mathsf{W}_{\mathrm{hor}}$ and hence into W_Δ. The generators of $\mathsf{W}_{\mathrm{fRezk}}$ are hence included in W_Δ, thereby proving the result. $\square$

Theorem 4.11. *Let us endow $\widehat{A}$ (resp. $\widehat{A \times \Delta}$) with the W-model category structure (resp. with the $\mathsf{W}_{\mathrm{fRezk}}$-model category structure).*

(1) *Then the adjunction*

$$p^* : \widehat{A} \rightleftarrows \widehat{A \times \Delta} : i_0^*$$

is a Quillen equivalence.

(2) *Let $D : \Delta \to \widehat{A}$ be a cosimplicial $\mathcal{W}$-resolution. Then the adjunction*

$$Real_D : \widehat{A \times \Delta} \rightleftarrows \widehat{A} : Sing_D,$$

is a Quillen equivalence.

Proof. This follows from Propositions 2.20 and 2.23, and from the above theorem. □

5. n-QUASI-CATEGORIES

The purpose of this section is to introduce our notion of n-quasi-categories.

Throughout the section, we fix an integer $n \geq 1$.

5.1. We will denote by $\mathbb{G}_n$ the category generated by the graph

$$\mathrm{D}_0 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} \mathrm{D}_1 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} \cdots \underset{\tau_{n-1}}{\overset{\sigma_{n-1}}{\rightrightarrows}} \mathrm{D}_{n-1} \underset{\tau_n}{\overset{\sigma_n}{\rightrightarrows}} \mathrm{D}_n$$

under the relations

$$\sigma_{i+1}\sigma_i = \tau_{i+1}\sigma_i \quad \text{and} \quad \sigma_{i+1}\tau_i = \tau_{i+1}\tau_i, \qquad 1 \leq i < n.$$

For i, j such that $0 \leq j \leq i \leq n$, we will denote by σ^i_j and τ^i_j the morphisms from D_j to D_i defined by

$$\sigma^i_j = \sigma_i \cdots \sigma_{j+2}\sigma_{j+1} \quad \text{and} \quad \tau^i_j = \tau_i \cdots \tau_{j+2}\tau_{j+1}.$$

By definition, the category of *n-graphs* is the category $\widehat{\mathbb{G}_n}$ of presheaves on $\mathbb{G}_n$. An n-graph X thus consists of a diagram of sets

$$X_n \underset{t_n}{\overset{s_n}{\rightrightarrows}} X_{n-1} \underset{t_{n-1}}{\overset{s_{n-1}}{\rightrightarrows}} \cdots \underset{t_2}{\overset{s_2}{\rightrightarrows}} X_1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_0$$

satisfying the relations

$$s_i s_{i+1} = s_i t_{i+1} \quad \text{and} \quad t_i s_{i+1} = t_i t_{i+1}, \qquad 1 \leq i < n.$$

If X is an n-graph, we will call X_0 the set of *objects* of X and X_k, for $0 \leq k \leq n$, the set of *k-arrows* of X. If f is a k-arrow of X for $k \geq 1$, the $(k-1)$-arrow $s_i(f)$ (resp. $t_i(f)$) will be called the *source* (resp. the *target*) of f. We will often denote an arrow f of X whose source is x and whose target is y by $f : x \to y$.

We will say that two k-morphisms f, g of an n-graph X are *parallel* if, either $k = 0$, or $k \geq 1$ and these morphisms satisfy

$$s_k(f) = s_k(g) \quad \text{and} \quad t_k(f) = t_k(g).$$

If f, g is a pair of parallel k-arrows for $k < n$, we will denote by $\mathsf{Hom}_X(f, g)$ the set of $(k+1)$-arrows of X from f to g.

5.2. Let m be a positive integer. A *table of dimensions* of *width* m consists of a table

$$\begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & i'_1 & & i'_2 & \cdots & i'_{m-1} & \end{pmatrix}$$

filled with integers satisfying

$$i_k > i'_k \quad \text{and} \quad i_{k+1} > i'_k, \qquad 1 \leq k < m.$$

The *dimension* of a table of dimensions is the greatest integer appearing in it.

Let C be a category under $\mathbb{G}_n$, i.e., a category endowed with a functor $F : \mathbb{G}_n \to C$. We will often denote in the same way the objects and morphisms of $\mathbb{G}_n$ and their image by the functor F. Let

$$T = \begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & i'_1 & & i'_2 & \cdots & i'_{m-1} & \end{pmatrix}$$

be a table of dimensions of dimension at most n. The *globular sum* in C associated to T (if it exists) is the iterated pushout

$$(\mathrm{D}_{i_1}, \sigma^{i_1}_{i'_1}) \amalg_{\mathrm{D}_{i'_1}} (\tau^{i_2}_{i'_1}, \mathrm{D}_{i_2}, \sigma^{i_2}_{i'_2}) \amalg_{\mathrm{D}_{i'_2}} \cdots \amalg_{\mathrm{D}_{i'_{m-1}}} (\tau^{i_m}_{i'_{m-1}}, \mathrm{D}_{i_m})$$

in C, i.e., the colimit of the diagram

$$\begin{tikzcd}[column sep=small] \mathrm{D}_{i_1} & & \mathrm{D}_{i_2} & & \mathrm{D}_{i_3} & \cdots & \mathrm{D}_{i_{m-1}} & & \mathrm{D}_{i_m} \\ & \mathrm{D}_{i'_1} \arrow[ul, "\sigma^{i_1}_{i'_1}"] \arrow[ur, "\tau^{i_2}_{i'_1}"'] & & \mathrm{D}_{i'_2} \arrow[ul, "\sigma^{i_2}_{i'_2}"] \arrow[ur, "\tau^{i_3}_{i'_2}"'] & & & & \mathrm{D}_{i'_{m-1}} \arrow[ul, "\sigma^{i_{m-1}}_{i'_{m-1}}"] \arrow[ur, "\tau^{i_m}_{i'_{m-1}}"'] & \end{tikzcd}$$

in C. We will denote it simply by

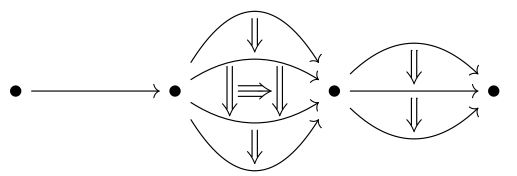
$$\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \mathrm{D}_{i_2} \amalg_{\mathrm{D}_{i'_2}} \cdots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m}.$$

We will always see $\widehat{\mathbb{G}_n}$ as a category under $\mathbb{G}_n$ by using the Yoneda functor. If T is a table of dimensions of dimension at most n, we will denote by G_T the globular sum associated to T in $\widehat{\mathbb{G}_n}$.

Example 5.3. If T is the table of dimensions

$$\begin{pmatrix} 2 & & 2 & & 2 & & 3 & & 2 & & 1 \\ & 1 & & 0 & & 1 & & 1 & & 0 & \end{pmatrix},$$

then the associated 3-graph G_T is



.

Remark 5.4. The G_T's are exactly the n-graphs associated to finite planar rooted trees by Batanin in [9].

5.5. We will denote by n-$\mathcal{C}$at the category of strict n-categories and strict n-functors. This category can be defined by induction by saying that n-$\mathcal{C}$at is the category of categories enriched in $(n\text{-}1)$-$\mathcal{C}$at. Thus, a strict n-category C is given by

- a set of objects $\mathsf{Ob}(C)$;
- for every pair x, y of objects of C, an $(n-1)$-category $\mathrm{Map}_C(x, y)$;
- for every triple x, y, z of objects of C, a strict $(n-1)$-functor

$$*_0 : \mathrm{Map}_C(y, z) \times \mathrm{Map}_C(x, y) \to \mathrm{Map}_C(x, z);$$

- for every object x, a distinguished object 1_x of $\mathrm{Map}_C(x, x)$, satisfying the associativity and unit axioms. This definition can be unpacked to a more explicit definition (see for instance Section 1.2 of [5]). If C and D are two strict n-categories, a strict n-functor $u : C \to D$ is given by

- a map $u_0 : \mathsf{Ob}(C) \to \mathsf{Ob}(D)$;
- for every pair x, y of objects of C, a strict $(n-1)$-functor

$$u_{x,y} : \mathrm{Map}_C(x, y) \to \mathrm{Map}_D(u_0(x), u_0(y)),$$

satisfying some obvious axioms. In the sequel, by "strict n-functor" we will mean "strict n-functor between strict n-categories".

We will denote by

$$U_n : n\text{-}\mathcal{C}\mathrm{at} \to \widehat{\mathbb{G}_n}$$

the forgetful functor sending a strict n-category to its underlying n-graph. We will often implicitly apply this forgetful functor to transfer notation and terminology from n-graphs to strict n-categories. The functor U_n admits a left adjoint

$$L_n : \widehat{\mathbb{G}_n} \to n\text{-}\mathcal{C}\mathrm{at}$$

sending an n-graph G to the free strict n-category on G.

The category n-$\mathcal{C}$at will always be seen as a category under $\mathbb{G}_n$ by using the functor L_n. In particular, for $k \leq n$ we have a strict n-category D_k. Note that D_0 is the terminal object of n-$\mathcal{C}$at.

Recall that the category n-$\mathcal{C}$at is cartesian closed. If C and D are two n-categories, we will denote by $\underline{\mathsf{Hom}}(C, D)$ the corresponding internal Hom. A k-arrow of $\underline{\mathsf{Hom}}(C, D)$ is given by a strict n-functor $C \times \mathrm{D}_k \to D$. In particular, the set of objects of $\underline{\mathsf{Hom}}(C, D)$ is in canonical bijection with $\mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(C, D)$.

5.6. We will denote by Θ_n the category defined in the following way:

- the objects of Θ_n are the tables of dimensions of dimension at most n;
- if S and T are two objects of Θ_n, then

$$\mathsf{Hom}_{\Theta_n}(S, T) = \mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(L_n(G_S), L_n(G_T));$$

- the composition and the identities of Θ_n are induced by those of n-$\mathcal{C}$at.

By definition of Θ_n, we have a fully faithful functor $\Theta_n \to n$-$\mathcal{C}$at sending an object T of Θ_n to $L_n(G_T)$. This functor is injective on objects and thus identifies Θ_n to a full subcategory of n-$\mathcal{C}$at.

The functor

$$\mathbb{G}_n \to \widehat{\mathbb{G}_n} \xrightarrow{L_n} n\text{-}\mathcal{C}\mathrm{at}$$

factors through Θ_n and the category Θ_n will always be seen as a category under $\mathbb{G}_n$ by using this functor. It follows from the fact that L_n commutes with colimits that if

$$T = \begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & i'_1 & & i'_2 & \cdots & i'_{m-1} & \end{pmatrix}$$

is an object of Θ_n, then

$$T = \mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \mathrm{D}_{i_2} \amalg_{\mathrm{D}_{i'_2}} \cdots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m},$$

where the globular sum is taken in Θ_n.

For $n = 1$, the category Θ_1 is canonically isomorphic to the simplex category Δ. The object Δ_m corresponds to the table of dimensions of width m

$$\begin{pmatrix} 1 & & 1 & & \cdots & & 1 \\ & 0 & & 0 & \cdots & 0 & \end{pmatrix}$$

and we indeed have

$$\Delta_m = \Delta_1 \amalg_{\Delta_0} \cdots \amalg_{\Delta_0} \Delta_1$$

in Δ. In the sequel, we will identify Θ_1 and Δ.

Remark 5.7. The category Θ_n is canonically isomorphic to (a truncation of) the cell category introduced by Joyal in [27]. This was proved independently by Makkai and Zawadowski in [37] and by Berger in [11]. Alternative definitions of this category are given in [11] and [12]. See also Proposition 3.11 of [2] for a definition by universal property.

5.8. Since the category n-$\mathcal{C}\mathrm{at}$ is cocomplete, the inclusion functor $\iota : \Theta_n \to n$-$\mathcal{C}\mathrm{at}$ induces an adjunction

$$\tau_n : \widehat{\Theta_n} \rightleftarrows n\text{-}\mathcal{C}\mathrm{at} : N_n,$$

where τ_n is the unique extension of ι to $\widehat{\Theta_n}$ preserving colimits and N_n is given by the formula

$$N_n(C)_T = \mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(\iota(T), C),$$

where C is a strict n-category and T is an object of Θ_n. It follows formally from the fact that $\iota : \Theta_n \to n$-$\mathcal{C}\mathrm{at}$ is fully faithful and that n-$\mathcal{C}\mathrm{at}$ is cartesian closed that the functor τ_n commutes with binary products (see the proof of Proposition B.0.15 of [31]). Moreover, abstract (but non-trivial) considerations (see Example 4.24 of [46]) show that the functor N_n is fully faithful.

For $n = 1$, the functor N_1 is the usual nerve functor $N : \mathcal{C}\mathrm{at} \to \widehat{\Delta}$.

Remark 5.9. The fact that N_n is fully faithful was first proved by Berger starting from a combinatorial definition of Θ_n (see Theorem 1.12 of [11]).

Theorem 5.10 (Berger)**.** *The category Θ_n is a regular skeletal Reedy category.*

Proof. By Lemma 2.4 and Remark 2.5 of [11], there is a structure of skeletal Reedy category on Θ_n. It is not hard to show that this structure is regular (i.e., that the level preserving cellular operators in Berger's terminology are monomorphisms). $\square$

Remark 5.11. Another point of view on the Reedy structure on Θ_n can be found in [18].

5.12. Since Θ_n is a Reedy category, for every object T we have a presheaf ∂T on Θ_n endowed with a monomorphism $\delta_T : \partial T \to T$. The presheaf ∂T is obtained by taking the union of the images of all the monomorphisms $S \to T$ of Θ_n except the identity.

Note that a morphism of Θ_n is a monomorphism if and only if its underlying n-graph morphism is a monomorphism, that is, if and only if it induces injections on k-arrows for every k such that $0 \leq k \leq n$.

5.13. The category $\widehat{\Theta_n}$ will always be seen as the category under $\mathbb{G}_n$ by using the functor

$$\mathbb{G}_n \to \Theta_n \to \widehat{\Theta_n}.$$

Let

$$T = \begin{pmatrix} i_1 & & i_2 & & \cdots & & i_m \\ & i'_1 & & i'_2 & \cdots & i'_{m-1} & \end{pmatrix}$$

be an object of Θ_n. We will denote by I_T the globular sum

$$\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \mathrm{D}_{i_2} \amalg_{\mathrm{D}_{i'_2}} \dots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m}$$

taken in $\widehat{\Theta_n}$. There is a canonical morphism

$$i_T : I_T \to T$$

in $\widehat{\Theta_n}$ coming from the universal property of I_T. One easily checks that this morphism is a monomorphism.

When $n = 1$, the object I_{Δ_k} will be denoted by I_k. This is the sub-simplicial set of Δ_k obtained by taking the union of all the 1-simplices of Δ_k whose vertices are consecutive integers. This object is called the *spine* of Δ_k by Joyal in [31]. We will denote by $i_k : I_k \to \Delta_k$ the inclusion morphism.

Remark 5.14. Let T be an object of Θ_n. The restriction of the morphism $I_T \to T$ of $\widehat{\Theta_n}$ to $\widehat{\mathbb{G}_n}$ is nothing but the inclusion morphism $G_T \to U_n L_n(G_T)$ given by the unit of the adjunction (L_n, U_n).

5.15. Let $k \geq 1$ and let C be a strict $(k-1)$-category. We define a strict k-category $\Delta_1 \wr C$ as a category enriched in strict $(k-1)$-categories in the following way:

- the objects of $\Delta_1 \wr C$ are 0 and 1;
- for every objects ε and ε' of $\Delta_1 \wr C$, we have

$$\mathrm{Map}_{\Delta_1 \wr C}(\varepsilon, \varepsilon') = \begin{cases} C & \text{if } \varepsilon = 0 \text{ and } \varepsilon' = 1, \\ * & \text{if } \varepsilon = \varepsilon', \\ \varnothing & \text{if } \varepsilon = 1 \text{ and } \varepsilon' = 0. \end{cases}$$

A priori, we have only defined a graph enriched in strict $(k-1)$-categories. It is obvious that there is a unique structure of enriched category on this enriched graph and the strict n-category $\Delta_1 \wr C$ is thus well-defined. The construction $\Delta_1 \wr C$ is clearly functorial in C.

Let J be the simply connected groupoid on two objects 0 and 1. In other words, J is defined in the following way:

- the objects of J are 0 and 1;
- for every objects ε and ε' of J, we have $\mathsf{Hom}_J(\varepsilon, \varepsilon') = *$.

We will denote by ∂J the discrete subcategory of J consisting of the objects 0 and 1, and by $\delta_J : \partial J \to J$ the inclusion functor.

We define by induction on $k \geq 1$ a strict k-category J_k in the following way:

$$J_1 = J \quad \text{and} \quad J_k = \Delta_1 \wr J_{k-1}, \quad k \geq 2.$$

This k-category is equipped with a strict k-functor $j_k : J_k \to \mathrm{D}_{k-1}$. For $k = 1$, the functor j_1 is the unique functor $J \to \mathrm{D}_0$. We will also denote this functor simply by j. For $k \geq 2$, we have

$$\mathrm{D}_{k-1} = \Delta_1 \wr \mathrm{D}_{k-2}.$$

This allows us to define j_k by induction setting

$$j_k = \Delta_1 \wr j_{k-1} : \Delta_1 \wr J_{k-1} \to \Delta_1 \wr \mathrm{D}_{k-2}.$$

This k-functor admits two sections s_k^0 and s_k^1. For $k = 1$ and $\varepsilon = 0, 1$, the section s_1^ε corresponds to the object ε of J. It will also be denoted by ∂^ε. For $k \geq 2$ and $\varepsilon = 0, 1$, we define s_k^ε by induction in the following way:

$$s_k^\varepsilon = \Delta_1 \wr s_{k-1}^\varepsilon : \Delta_1 \wr \mathrm{D}_{k-2} \to \Delta_1 \wr J_{k-1}.$$

Remark 5.16. Here are (the underlying graph without the identities of) J_1, J_2 and J_3:

$$J_1 = \begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd} \quad , \quad J_2 = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \Uparrow"] & 1 \end{tikzcd} \quad \text{and} \quad J_3 = \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \substack{\Rrightarrow \\ \Lleftarrow} \Downarrow"] & 1 \end{tikzcd} \quad .$$

The two arrows of maximal dimension are inverse of each other. The two sections s_k^ε correspond to the two non-trivial $(k-1)$-arrows of J_k. The k-functor j_k is the unique strict k-functor from J_k to D_{k-1} which sends these two $(k-1)$-arrows to the unique non-trivial $(k-1)$-arrow of D_{k-1}.

5.17. Let

$$\mathcal{I}_n = \{i_T : \ I_T \to T; \ T \in \mathsf{Ob}(\Theta_n)\}$$

and

$$\mathcal{J}_n = \{N_n(j_k) : N_n(J_k) \to N_n(\mathrm{D}_{k-1}); \ 1 < k \leq n\}.$$

The *localizer of n-quasi-categories* is the Θ_n-localizer generated by $\mathcal{I}_n$ and $\mathcal{J}_n$. We will denote it by $\mathsf{W}_{\mathrm{QCat}_n}$. By the 2-out-of-3 property, this localizer is also generated by $\mathcal{I}_n$ and $\mathcal{J}'_n$ where

$$\mathcal{J}'_n = \{N_n(s_k^\varepsilon) : N_n(\mathrm{D}_{k-1}) \to N_n(J_k); \ 1 < k \leq n, \ \varepsilon = 0, 1\}.$$

The $\mathsf{W}_{\mathrm{QCat}_n}$-model category structure on $\widehat{\Theta_n}$ will be called the *model category of n-quasi-categories*. By definition, an *n-quasi-category* is a $\mathsf{W}_{\mathrm{QCat}_n}$-fibrant object.

Remark 5.18. We will show in the next section that $N_n(j) : N_n(J) \to N_n(\mathrm{D}_0)$ is a trivial fibration and hence belongs to any Θ_n-localizer, and that, on the contrary, none of the morphisms of $\mathcal{J}_n$ belong to the Θ_n-localizer generated by $\mathcal{I}_n$.

5.19. Recall that a simplicial set X is a *quasi-category* if the unique morphism from X to Δ_0 has the right lifting property with respect to $h_n^k : \Lambda_n^k \to \Delta_n$ for every $n \geq 2$ and every $0 < k < n$. Joyal defined in [31] (see Theorem 6.12) a model category structure on simplicial sets, the so-called *model category of quasi-categories*. This model category is uniquely defined by the fact that its cofibrations are the monomorphisms and its fibrant objects are the quasi-categories.

Theorem 5.20 (Joyal)**.** *The model category of* 1*-quasi-categories coincide with the model category of quasi-categories.*

Proof. Let us denote by $\mathsf{W}_{\mathrm{Joyal}}$ the Δ-localizer associated to the model category of quasi-categories. By Proposition 2.13 of [31], the morphisms of $\mathcal{I}_1$ are $\mathsf{W}_{\mathrm{Joyal}}$-equivalences. The localizer $\mathsf{W}_{\mathrm{QCat}_n}$ is thus contained in $\mathsf{W}_{\mathrm{Joyal}}$. It follows that the $\mathsf{W}_{\mathrm{Joyal}}$-fibrant objects, i.e., the quasi-categories, are $\mathsf{W}_{\mathrm{QCat}_n}$-fibrant objects.

Let us show the converse. It suffices to prove that for every $n \geq 2$ and every $0 < k < n$, the morphism $h_n^k : \Lambda_n^k \to \Delta_n$ is a $\mathsf{W}_{\mathrm{QCat}_n}$-equivalence. This follows from Lemma 3.5 of [32] applied to the class of trivial cofibrations of the $\mathsf{W}_{\mathrm{QCat}_n}$-model category.

We have shown that the two model categories have the same fibrant objects. Since they also have the same cofibrations, they coincide. □

Remark 5.21. If C is a category, it is well-known that its nerve $N_1(C)$ is a quasi-category. As we will see in Section 7, it is not true that the nerve of a strict n-category is an n-quasi-category in general. For instance, $N_n(J_k)$ is not an n-quasi-category for $1 < k \leq n$ (see Corollary 7.11). This is the reason why we have chosen the terminology "n-quasi-category" rather than "quasi-n-category" which was used in a preliminary version of this paper: strict n-categories should be quasi-n-categories.

6. On our generators of the localizer of n-quasi-categories

In this section, we study our generators of the localizer of n-quasi-categories. We first show that the morphism $N_n(j) : N_n(J) \to N_n(\mathrm{D}_0)$ is a trivial fibration and hence belongs to any Θ_n-localizer. This is the reason why in dimension 1 (the case $\Theta_1 = \Delta$) the spines are sufficient to generate the localizer of quasi-categories. This is probably also the reason why Cisinski and Joyal conjectured that the higher spines would generate a Θ_n-localizer which would model (∞, n)-categories. We show in this section that it is not the case: more precisely, we show that none of the morphisms of $\mathcal{J}_n$ (which are equivalences of strict n-categories and hence should be equivalences of (∞, n)-categories) belong to the Θ_n-localizer generated by $\mathcal{I}_n$.

Throughout the section, we fix an integer $n \geq 1$.

6.1. Consider the inclusion functor $i : \mathcal{C}\mathrm{at} \to n\text{-}\mathcal{C}\mathrm{at}$. This functor admits a left adjoint $t : n\text{-}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ and a right adjoint $t_r : n\text{-}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$. The functor t will be called the *truncation functor*. It sends a strict n-category C to the category whose objects are the same as those of C and whose arrows are the 1-arrows of C up to 2-arrows. The functor t_r will be called the *right truncation functor*. It sends a strict n-category to the category whose objects and 1-arrows are the same as those of C.

The adjunction

$$t : n\text{-}\mathcal{C}\mathrm{at} \rightleftarrows \mathcal{C}\mathrm{at} : i$$

restricts to an adjunction

$$t : \Theta_n \rightleftarrows \Delta : i.$$

This new adjunction induces a third adjunction

$$t^* : \widehat{\Delta} \rightleftarrows \widehat{\Theta_n} : i^*.$$

An immediate calculation shows that the square

$$\begin{tikzcd} \widehat{\Delta} \arrow[r, "t^*"] & \widehat{\Theta_n} \\ \mathcal{C}\mathrm{at} \arrow[u, "N_1"] \arrow[r, "i"'] & n\text{-}\mathcal{C}\mathrm{at} \arrow[u, "N_n"'] \end{tikzcd}$$

commutes.

On the contrary, the functor $t_r : n\text{-}\mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}$ does not restrict to a functor $\Theta_n \to \Delta$. Nevertheless, we can consider the functor

$$\Theta_n \to n\text{-}\mathcal{C}\mathrm{at} \xrightarrow{t_r} \mathcal{C}\mathrm{at} \to \widehat{\Delta}.$$

Since $\widehat{\Delta}$ is cocomplete, this functor induces an adjunction

$$t_{r!} : \widehat{\Theta_n} \rightleftarrows \widehat{\Delta} : t_r^*,$$

where $t_{r!}$ is the unique extension of this functor $\Theta_n \to \widehat{\Delta}$ to $\widehat{\Theta_n}$ preserving colimits and t_r^* is given by the formula

$$t_r^*(X)_T = \mathsf{Hom}_{\widehat{\Delta}}(t_r(T), X),$$

where X is a simplicial set and T is an object of Θ_n. The functor $t_{r!}$ and i^* both preserve colimits and coincide on objects of Θ_n. It follows that they are isomorphic and hence that their right adjoints are isomorphic. In particular, if X is a simplicial set and T is an object of Θ_n, we have

$$i_*(X)_T \cong t_r^*(X)_T = \mathsf{Hom}_{\widehat{\Delta}}(t_r(T), X).$$

Proposition 6.2. *Let f be a morphism of simplicial sets. Then $i_*(f)$ is a trivial fibration of $\widehat{\Theta_n}$ if and only if f is a trivial fibration of simplicial sets.*

Proof. The functor i^* admits a left adjoint and hence preserves monomorphisms. It follows that its right adjoint i_* preserves trivial fibrations.

The same argument shows that i^* preserves trivial fibrations (its left adjoint t^* admits a left adjoint). Suppose $i_*(f)$ is a trivial fibration. We have just seen that $i^*i_*(f)$ is a trivial fibration. But since i is fully faithful, we have $i^*i_*(f) \cong f$ and so f is a trivial fibration. □

6.3. We will say that a category is a *preorder* if there is at most one arrow between every pair of objects.

Corollary 6.4. *Let u be a functor between preorders. Then $N_n(u)$ is a trivial fibration of $\widehat{\Theta_n}$ if and only if $N_1(u)$ is a trivial fibration of simplicial sets.*

Proof. If u is any functor, by paragraph 6.1, we have $t^*N_1(u) = N_n(u)$. On the other hand, if C is a preorder, we have

$$t^*N_1(C)_T = \mathsf{Hom}_{\mathcal{C}\mathrm{at}}(t(T), C) \cong \mathsf{Hom}_{\mathcal{C}\mathrm{at}}(t_r(T), C) \cong i_*N_1(C)_T$$

for every object T of Θ_n. If follows that if u is a functor between preorders, we have

$$N_n(u) \cong i_*N_1(u)$$

and the result follows from the above proposition. □

Proposition 6.5. *Let u be a functor. Then $N_1(u)$ is a trivial fibration of simplicial sets if and only if u is an equivalence of categories surjective on objects.*

Proof. The morphism $N_1(u)$ is a trivial fibration if and only if for every $n \geq 0$, we have $\delta_n \pitchfork N_1(u)$. By adjunction, we have

$$\delta_n \pitchfork N_1(u) \quad \Leftrightarrow \quad \tau_1(\delta_n) \pitchfork u.$$

But it is well-known that $\tau_1(\delta_n)$ is an isomorphism for $n \geq 3$ (see for instance the lemma page 32 of [22]). Thus, the morphism $N_1(u)$ is a trivial fibration if and only if it satisfies

$$\tau_1(\delta_n) \pitchfork u, \quad n = 0, 1, 2.$$

The condition for $n = 0$ (resp. $n = 1$, resp. $n = 2$) is equivalent to the surjectivity on objects of u (resp. the fullness of u, resp. the faithfulness of u), thereby proving the result. □

Corollary 6.6. *Let u be a functor between preorders. Then $N_n(u)$ is a trivial fibration of $\widehat{\Theta_n}$ if and only if u is an equivalence of categories surjective on objects.*

Proof. This follows from Proposition 6.4 and the above proposition. □

Corollary 6.7. *The morphism $N_n(j) : N_n(J) \to N_n(\mathrm{D}_0)$ is a trivial fibration of $\widehat{\Theta_n}$.*

Proof. The functor $J \to \mathrm{D}_0$ is an equivalence of categories surjective on objects between preorders. The result thus follows from the above corollary. □

6.8. Consider $N_n(J)$ endowed with the two morphisms

$$\partial^\varepsilon = N_n(\partial^\varepsilon) : N_n(\mathrm{D}_0) \to N_n(J), \quad \varepsilon = 0, 1.$$

The presheaf $N_n(\mathrm{D}_0)$ is the terminal object of $\widehat{\Theta_n}$ and $N_n(J)$ is thus endowed with the structure of an interval. It is immediate that this interval is separating. Moreover, by the above lemma, this interval is injective.

Proposition 6.9. *For every object T of Θ_n, the n-functor*

$$\tau_n(i_T) : \tau_n(I_T) \to \tau_n(T)$$

is an isomorphism of n-categories.

Proof. Let

$$T = \mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \dots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m}.$$

It is immediate that, with the notation of paragraphs 5.2 and 5.5, we have

$$T = N_n L_n(G_T).$$

Using the fact that the functor N_n is fully faithful, we obtain

$$\begin{aligned}
\tau_n(T) &\cong \tau_n N_n L_n(G_T) \\
&\cong L_n(G_T) \\
&\cong L_n\big(\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \dots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m}\big) \\
&\cong L_n(\mathrm{D}_{i_1}) \amalg_{L_n(\mathrm{D}_{i'_1})} \cdots \amalg_{L_n(\mathrm{D}_{i'_{m-1}})} L_n(\mathrm{D}_{i_m}) \\
&\cong \mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \dots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m} \\
&\cong \tau_n(\mathrm{D}_{i_1}) \amalg_{\tau_n(\mathrm{D}_{i'_1})} \cdots \amalg_{\tau_n(\mathrm{D}_{i'_{m-1}})} \tau_n(\mathrm{D}_{i_m}) \\
&\cong \tau_n\big(\mathrm{D}_{i_1} \amalg_{\mathrm{D}_{i'_1}} \dots \amalg_{\mathrm{D}_{i'_{m-1}}} \mathrm{D}_{i_m}\big) \\
&\cong \tau_n(I_T),
\end{aligned}$$

thereby proving the result. □

Lemma 6.10. *Let T be an object of Θ_n different from D_0. Then the map $\delta_T : \partial T \to T$ induces a bijection $\delta_{T,\mathrm{D}_0} : \partial T_{\mathrm{D}_0} \to T_{\mathrm{D}_0}$.*

Proof. By definition, the morphism δ_T is a monomorphism. The map δ_{T,D_0} is thus injective. Let us show that it is also surjective. Let $x : \mathrm{D}_0 \to T$ be a morphism of Θ_n. Since D_0 is the terminal object of Θ_n, the morphism x is a monomorphism. The object T being different from D_0, the morphism x is not an identity and it thus factors through ∂T by definition of ∂T. The map δ_{T,D_0} is thus surjective. □

Proposition 6.11. *Let X be a presheaf on Θ_n. Then the set of objects of $\tau_n(X)$ is in canonical bijection with X_{D_0}.*

Proof. We have seen in paragraph 6.1 that the square

$$\begin{tikzcd} \widehat{\Delta} \arrow[r, "t^*"] & \widehat{\Theta_n} \\ \mathcal{C}\mathrm{at} \arrow[u, "N_1"] \arrow[r, "i"'] & n\text{-}\mathcal{C}\mathrm{at} \arrow[u, "N_n"'] \end{tikzcd}$$

is commutative. By taking left adjoints, we obtain that the square

$$\begin{tikzcd} \widehat{\Delta} \arrow[d, "\tau_1"'] & \widehat{\Theta_n} \arrow[l, "t_!"'] \arrow[d, "\tau_n"] \\ \mathcal{C}\mathrm{at} & n\text{-}\mathcal{C}\mathrm{at} \arrow[l, "t"] \end{tikzcd}$$

is also commutative (up to isomorphism). We thus have

$$\mathsf{Ob}(\tau_n(X)) = \mathsf{Ob}(t\tau_n(X)) \cong \mathsf{Ob}(\tau_1 t_!(X)) = t_!(X)_0,$$

where the last equality comes from the case $n = 1$ of the proposition which is well-known (see for instance the proposition page 33 of [22]). By the theory of Kan extensions, we have

$$t_!(X)_0 \cong \varinjlim_{(S,\Delta_0 \to t(S)) \in (\Delta_0 \backslash \Theta_n)^{\mathrm{o}}} X_S,$$

where $\Delta_0 \backslash \Theta_n$ denotes the category whose objects are pairs $(S, \Delta_0 \to t(S))$ consisting of an object S of Θ_n and a morphism $\Delta_0 \to t(S)$ of Δ, and whose morphisms are the obvious ones. It follows from the canonical bijection

$$\mathsf{Hom}_\Delta(\Delta_0, t(S)) \cong \mathsf{Hom}_{\Theta_n}(\mathrm{D}_0, S)$$

that the category $\Delta_0 \backslash \Theta_n$ admits $(\mathrm{D}_0, 1_{\Delta_0})$ as an initial object. The above colimit is thus canonically isomorphic to X_{D_0}, thereby proving the result. □

Corollary 6.12. *Let T be an object of Θ_n different from D_0. Then the n-functor*

$$\tau_n(\delta_T) : \tau_n(\partial T) \to \tau_n(T)$$

is bijective on objects.

Proof. This follows from Lemma 6.10 and the above proposition. □

6.13. Let $u : C \to D$ be a strict n-functor. We will say that u is *fully faithful* if for every pair x, y of objects of C, the $(n-1)$-functor

$$u_{x,y} : \mathrm{Map}_C(x, y) \to \mathrm{Map}_D(u(x), u(y))$$

is an isomorphism of strict $(n-1)$-categories. Explicitly, this means that for every k such that $0 < k \leq n$ and every pair f, g of parallel $(k-1)$-arrows of C, the n-functor u induces a bijection

$$\mathsf{Hom}_C(f, g) \to \mathsf{Hom}_D(u(f), u(g)).$$

Lemma 6.14. *Let C be a strict n-category and let $\varepsilon = 0, 1$. Then the n-functor*

$$\underline{\mathsf{Hom}}(\partial^\varepsilon, C) : \underline{\mathsf{Hom}}(J, C) \to \underline{\mathsf{Hom}}(\{\varepsilon\}, C) \cong C$$

is fully faithful.

Proof. We will prove the case $\varepsilon = 1$. The case $\varepsilon = 0$ will follow by duality. Recall that we denote by $*_0$ the composition in dimension 0 of two k-arrows of an n-category.

By definition, objects of $\underline{\mathsf{Hom}}(J, C)$ are invertible 1-arrows of C. Let $f : x \to y$ and $f' : x' \to y'$ be two invertible 1-arrows of C. A morphism from f to f' in $\underline{\mathsf{Hom}}(J, C)$ is given by a pair $h : x \to x', k : y \to y'$ of morphisms of C making the square

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y \arrow[r, "k"'] & y' \end{tikzcd}$$

commute. Since f and f' are invertible, the pair (h, k) is uniquely determined by k. This exactly means that the map

$$\mathsf{Hom}_{\underline{\mathsf{Hom}}(J,C)}(f, f') \to \mathsf{Hom}_C(y, y')$$

induced by u is a bijection.

Let us now describe the k-arrows of $\underline{\mathsf{Hom}}(J, C)$ for k such that $0 < k \leq n$. By definition, a k-arrow of $\underline{\mathsf{Hom}}(J, C)$ is given by a strict k-functor $J \times \mathrm{D}_k \to C$. Such a k-functor is given by a tuple (f, g, α, β) where f, g are two invertible 1-arrows of C and α, β are two k-arrows of C such that the compositions $\beta *_0 f$ and $g *_0 \alpha$ make sense and are equal. Two k-arrows (f, g, α, β) and $(f', g', \alpha', \beta')$ of $\underline{\mathsf{Hom}}(J, C)$ are parallel if and only if $f = f'$, $g = g'$, α and α' are parallel, and β and β' are parallel.

Suppose now k is such that $1 < k \leq n$ and let (f, g, α, β) and (f, g, α', β') be two parallel k-arrows of $\underline{\mathsf{Hom}}(J, C)$. We have to show that the map

$$\mathsf{Hom}_{\underline{\mathsf{Hom}}(J,C)}((f, g, \alpha, \beta), (f, g, \alpha', \beta')) \to \mathsf{Hom}_C(\beta, \beta')$$

induced by u is a bijection. The same argument as in dimension 0 applies. Formally, an inverse of this map is given by

$$\Gamma : \beta \to \beta' \mapsto (f, g, g^{-1} *_0 \Gamma *_0 f, \Gamma). \qquad \square$$

Proposition 6.15. *Let v be a fully faithful strict n-functor. Then v has the unique right lifting property with respect to strict n-functors bijective on objects.*

Proof. The assertion is a special case of a standard result in enriched category theory. Let us prove our particular case. We will use the notation of paragraph 5.5. Consider a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & C \arrow[d, "v"] \\ B \arrow[r, "g"'] & D \end{tikzcd} ,$$

where u is a strict n-functor bijective on objects and v is a fully faithful strict n-functor. Let $h : B \to C$ be a lift. The condition $hu = f$ imposes that if x is an object of B, then $h(x) = f(u_0^{-1}(x))$. On the other hand, the condition $hv = g$ imposes that if x, y is a pair of objects of B, then $h_{x,y} = v^{-1}_{fu_0^{-1}(x), fu_0^{-1}(y)} g$. The strict n-functor h is thus unique. One immediately checks that the formulas given above define a strict n-functor, thereby proving the result. □

Corollary 6.16. *Let $u : C \to D$ be a strict n-functor bijective on objects. Then the strict n-functors*

$$u \times' \partial^{\varepsilon}_{J} : C \times J \amalg_{C \times \{\varepsilon\}} D \times \{\varepsilon\} \to D \times J, \quad \varepsilon = 0, 1,$$

are isomorphisms of strict n-categories.

Proof. Let A be a strict n-category. By the Yoneda lemma, it suffices to prove that

$$\mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(D \times J, A) \to \mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(C \times J \amalg_{C \times \{\varepsilon\}} D \times \{\varepsilon\}, A)$$

is a bijection, or, in other words, that the functor of the statement has the unique left lifting property with respect to the unique strict n-functor $A \to \mathrm{D}_0$. By adjunction, this is equivalent to saying that the functor

$$\underline{\mathsf{Hom}}(\partial^{\varepsilon}, A) : \underline{\mathsf{Hom}}(J, A) \to \underline{\mathsf{Hom}}(\{\varepsilon\}, A) \cong A$$

has the unique right lifting property with respect to the functor $C \to D$. This follows from Lemma 6.14 and the above proposition. □

6.17. Let $u : C \to D$ be a functor. Recall that u is said to be an *iso-fibration* if for every invertible arrow $f' : x' \to y'$ of D and every object y of C such that $u(y) = y'$, there exists an invertible arrow $f : x \to y$ of C such that $u(f) = f'$. In other words, the functor u is an iso-fibration if it has the right lifting property with respect to $\partial^1 : \mathrm{D}_0 \to J$. Note that u is an iso-fibration if and only if $u^{\mathrm{o}} : C^{\mathrm{o}} \to D^{\mathrm{o}}$ is an iso-fibration, that is, if and only if u has the right lifting property with respect to $\partial^0 : \mathrm{D}_0 \to J$.

Proposition 6.18. *Let u be a strict n-functor and denote by W the Θ_n-localizer generated by $\mathcal{I}_n$. Then the morphism $N_n(u)$ is a W-fibration if and only if the right truncation $t_r(u)$ of u is an iso-fibration. In particular, for any strict n-category C, its nerve $N_n(C)$ is W-fibrant.*

Proof. In this proof, by "naive fibration" we will mean "naive $(\mathcal{I}_n, N_n(J))$-fibration". (By paragraph 6.8, $N_n(J)$ is an injective separating interval and we will thus be able to apply Theorem 2.14.)

By paragraph 2.10, the morphism $N_n(u)$ is a naive fibration if and only if it has the right lifting property with respect to

$$\Lambda^{\infty}_{N_n(J)}(\mathcal{I}_n) \cup \{\partial T \times N_n(J) \cup T \times \{\varepsilon\} \to T \times N_n(J);\ T \in \mathsf{Ob}(\Theta_n),\ \varepsilon = 0, 1\}.$$

By adjunction and using the fact that τ_n commutes with colimits and binary products, we obtain that $N_n(u)$ is a naive fibration if and only if u has the right lifting property with respect to

$$\Lambda^{\infty}_J(\tau_n(\mathcal{I}_n)) \cup \{\tau_n(\partial T) \times J \amalg_{\tau_n(\partial T)\times\{\varepsilon\}} \tau_n(T) \times \{\varepsilon\} \to \tau_n(T) \times J;\ T \in \mathsf{Ob}(\Theta_n),\ \varepsilon = 0, 1\}.$$

(A priori, we have only defined the Λ^{∞} construction in presheaf categories. Nevertheless, it is clear that the definition still makes sense in any category admitting finite products and pushouts.) By Proposition 6.9, the n-functors of $\tau_n(\mathcal{I}_n)$ are isomorphisms. Since isomorphisms are stable under pushout and binary product, it follows that the n-functors of $\Lambda^{\infty}_J(\tau_n(\mathcal{I}_n))$ are also isomorphisms. Moreover, by Corollaries 6.12 and 6.16, the n-functors of

$$\{\tau_n(\partial T) \times J \amalg_{\tau_n(\partial T)\times\{\varepsilon\}} \tau_n(T) \times \{\varepsilon\} \to \tau_n(T) \times J;\ T \in \mathsf{Ob}(\Theta_n)\backslash\{\mathrm{D}_0\},\ \ \varepsilon = 0, 1\}$$

are again isomorphisms. It follows that $N_n(u)$ is a naive fibration if and only if u has the right lifting property with respect to

$$\tau_n(\partial\mathrm{D}_0) \times J \amalg_{\tau_n(\partial\mathrm{D}_0)\times\{\varepsilon\}} \tau_n(\mathrm{D}_0) \times \{\varepsilon\} \to \tau_n(\mathrm{D}_0) \times J.$$

But this n-functor is nothing but $\partial^{\varepsilon} : \mathrm{D}_0 \to J$ and $N_n(u)$ is thus a naive fibration if and only if $t_r(u)$ is an iso-fibration. In particular, every strict n-category C is $(\mathcal{I}_n, N_n(J))$-fibrant and hence W-fibrant by Theorem 2.14. The result follows from the same theorem since $N_n(u)$ is a morphism between W-fibrant objects. □

Remark 6.19. Let W be the Θ_n-localizer generated by $\mathcal{I}_n$. By the above proposition, nerves of strict n-categories are W-fibrant objects. We will see in the introduction of the next section that this implies that when $n > 1$ the functor N_n cannot have the property that a strict n-functor u is an equivalence of strict n-categories if and only if $N_n(u)$ is a W-equivalence. This shows that we have to add other generators to obtain a model for (∞, n)-categories.

Proposition 6.20. *Let k be an integer such that $1 < k \le n$. Then the morphism*

$$N_n(j_k) : N_n(J_k) \to N_n(\mathrm{D}_{k-1})$$

is not a trivial fibration of $\widehat{\Theta_n}$.

Proof. Let T be the object $\mathrm{D}_{k-1} \amalg_{\mathrm{D}_{k-2}} \mathrm{D}_{k-1}$ of Θ_n. We claim that $N_n(j_k)$ does not have the right lifting property with respect to $\delta_T : \partial T \to T$, or, by adjunction, that j_k does not have the right lifting property with respect to $\tau_n(\delta_T) : \tau_n(\partial T) \to \tau_n(T)$. Since τ_n commutes with colimits, we have

$$\tau_n(\partial T) = \varinjlim_{S \overset{\neq}{\hookrightarrow} T} S,$$

the colimit being taken over the category of non-trivial monomorphisms of Θ_n whose target is T. Using this formula, one easily obtains that a strict n-functor from $\tau_n(\partial T)$ to a strict n-category C is given by a triangle of $(k-1)$-arrows composable in dimension $k-2$.

It follows that a strict n-functor $u : C \to D$ has the right lifting property with respect to $\tau_n(\delta_T) : \tau_n(\partial T) \to \tau_n(T)$ if and only if every such triangle in C which is sent to a commutative triangle in D is already commutative in C, or equivalently, if and only if u is injective on parallel $(k-1)$-arrows, i.e., if two parallel $(k-1)$-arrows f, g of C are sent to the same $(k-1)$-arrow in D, then $f = g$. This condition is not fulfilled by j_k, thereby proving that $N_n(j_k)$ is not a trivial fibration. □

Corollary 6.21. *Let k be an integer such that $1 < k \leq n$. Then the morphism*

$$N_n(j_k) : N_n(J_k) \to N_n(\mathrm{D}_{k-1})$$

is not in the Θ_n-localizer generated by $\mathcal{I}_n$.

Proof. Denote by W the Θ_n-localizer generated by $\mathcal{I}_n$. The functor $t_r(j_k)$ is an isofibration and the morphism $N_n(j_k)$ is thus a W-fibration by Proposition 6.18. By the above proposition, this morphism is not a trivial fibration. It follows that it is not a W-equivalence. □

7. Nerves of strict n-categories and n-quasi-categories

If C is a category, it is well-known that its nerve $N_1(C)$ is a quasi-category. In this section, we show that it is not true that nerves of strict n-categories are n-quasi-categories as soon as $n > 1$. More precisely, we will show that the nerve $N_n(C)$ of a strict n-category C is an n-quasi-category if and only if C has no non-trivial invertible k-arrows for $k > 1$.

There is actually an abstract reason for which nerves of strict n-categories cannot be n-quasi-categories. Indeed, if they were n-quasi-categories, they would be cofibrant-fibrant objects in the model category of n-quasi-categories. That would imply that any weak equivalence between nerves of strict n-categories has an inverse up to homotopy. Assuming that N_n has the property that a strict n-functor u is an equivalence of strict n-categories if and only if $N_n(u)$ is a weak equivalence of n-quasi-categories (which is expected but not proved in this paper) and using the fully faithfulness of N_n, we would get that if a strict n-functor $C \to D$ is an equivalence of strict n-categories, then there exists a strict n-functor $D \to C$ which is an equivalence of strict n-categories, which is false when $n > 1$.

Throughout the section, we fix an integer $n \geq 1$ and an integer k such that $1 < k \leq n$.

7.1. We will say that a strict n-category C is *rigid in dimension* k if any (strictly) invertible k-arrow of C is the identity of a $(k-1)$-arrow.

Proposition 7.2. *A strict n-category C is rigid in dimension k if and only if the unique n-functor $C \to \mathrm{D}_0$ has the right lifting property with respect to the n-functor*

$$s^0_k \times' \delta_J : \mathrm{D}_{k-1} \times J \amalg_{\mathrm{D}_{k-1} \times \partial J} J_k \times \partial J \to J_k \times J$$

induced by $s^0_k : \mathrm{D}_{k-1} \to J_k$ and $\delta_J : \partial J \to J$.

Proof. Let us denote by i the unique arrow of J from 0 to 1, and by γ the unique k-arrow of J_k whose source (resp. whose target) corresponds to the morphism $s^0_k : \mathrm{D}_{k-1} \to J_k$ (resp. to the morphism $s^1_k : \mathrm{D}_{k-1} \to J_k$).

To any strict n-functor
$$u : J_k \times J \to C,$$
we can associate a tuple (f, g, α, β) given by
$$f = u(0, i), \quad g = u(1, i), \quad \alpha = u(\gamma, 0) \quad \text{and} \quad \beta = u(\gamma, 1).$$
Conversely, if (f, g, α, β) is a tuple where f, g are invertible 1-arrows of C and α, β are invertible k-arrows of C, then the above formulas define a strict n-functor if and only if we have
$$g *_0 \alpha = \beta *_0 f.$$
Similarly, to any strict n-functor
$$u : \mathrm{D}_{k-1} \times J \amalg_{\mathrm{D}_{k-1} \times \partial J} J_k \times \partial J \to C,$$
we can associate a tuple (f, g, α, β) given by
$$f = u(0, i), \quad g = u(1, i), \quad \alpha = u(\gamma, 0) \quad \text{and} \quad \beta = u(\gamma, 1).$$
Conversely, if (f, g, α, β) is a tuple where f, g are invertible 1-arrows of C and α, β are invertible k-arrows of C, then the above formulas define a strict n-functor if and only if we have
$$g *_0 s_k(\alpha) = s_k(\beta) *_0 f.$$
Moreover, if a tuple (f, g, α, β) defines a functor $J_k \times J \to C$, then the induced functor $\mathrm{D}_{k-1} \times J \amalg_{\mathrm{D}_{k-1} \times \partial J} J_k \times \partial J \to C$ is also defined by (f, g, α, β).

The above analysis shows that $C \to \mathrm{D}_0$ has the right lifting property with respect to $s^0_k \times' \delta_J$ if and only if for every tuple (f, g, α, β) where f, g are invertible 1-arrows of C and α, β are invertible k-arrows of C, if
$$g *_0 s_k(\alpha) = s_k(\beta) *_0 f,$$
then
$$g *_0 \alpha = \beta *_0 f.$$
This clearly holds if α and β are identities of $(k-1)$-arrows and it hence holds for any α and β if C is rigid in dimension k. Conversely, suppose C is not rigid in dimension k. By definition, there exists an invertible k-arrow $\alpha : v \to w$ which is not an identity. Denote by x (resp. by y) the source (resp. the target) of v in dimension 0. Consider the tuple $(1_x, 1_y, \alpha, 1_v)$. We indeed have
$$1_y *_0 s_k(\alpha) = v = s_k(1_v) *_0 1_x,$$
but
$$1_y *_0 \alpha = \alpha \neq 1_v = 1_v *_0 1_x,$$
thereby proving the result. □

Remark 7.3. The above proposition is false for $k = 1$: for any category C, the n-functor $C \to \mathrm{D}_0$ has the right lifting property with respect to $s^0_1 \times' \delta_J$.

Remark 7.4. We could have replaced s^0_k by s^1_k in the statement of the proposition. This immediately follows from the existence of an automorphism σ of $J_k \times J$ satisfying
$$s^1_k \times' \delta_J = \sigma \circ (s^0_k \times' \delta_J).$$

7.5. We will say that a strict n-functor $u : C \to D$ is *rigid in dimension* k if it has the right lifting property with respect to the n-functor

$$s_k^0 \times' \delta_J : \mathrm{D}_{k-1} \times J \amalg_{\mathrm{D}_{k-1}\times\partial J} J_k \times \partial J \to J_k \times J.$$

(By the above remark, we could replace s_k^0 by s_k^1.) The above proposition can then be rephrased by saying that a strict n-category C is rigid in dimension k if and only if the n-functor $C \to \mathrm{D}_0$ is rigid in dimension k.

Lemma 7.6. *The n-functor*

$$s_k^0 \times' \delta_J : \mathrm{D}_{k-1} \times J \amalg_{\mathrm{D}_{k-1}\times\partial J} J_k \times \partial J \to J_k \times J$$

is an epimorphism.

Proof. We have to show that for any strict n-category C, the map $\mathsf{Hom}_{n\text{-}\mathcal{C}\mathrm{at}}(s_k^0 \times' \delta_J, C)$ is injective. This is an immediate consequence of the description of this map given in the proof of Proposition 7.2. □

Proposition 7.7. *Let $u : C \to D$ be a strict n-functor. If the n-category C is rigid in dimension k, then so is the n-functor u.*

Proof. If follows formally from the fact that the n-functor $s_k^0 \times' \delta_J$ is an epimorphism (see the above lemma) that if $C \to \mathrm{D}_0$ has the right lifting property with respect to this n-functor, so does any strict n-functor whose source is C. The result thus follows from Proposition 7.2. □

Proposition 7.8. *Let C be a strict n-category. If C is rigid in dimension k, then so is the n-category $\underline{\mathsf{Hom}}(J, C)$.*

Proof. Invertible k-arrows of $\underline{\mathsf{Hom}}(J, C)$ are in canonical bijection with strict n-functors $J_k \times J \to C$. We saw in the proof of Proposition 7.2 that these n-functors are in canonical bijection with tuples (f, g, α, β), where f, g are invertible 1-arrows of C and α, β are invertible k-arrows of C satisfying $g *_0 \alpha = \beta *_0 f$. In this correspondence, such a tuple is an identity in the n-category $\underline{\mathsf{Hom}}(J, C)$ if and only if α and β are identities in C. The result follows immediately. □

Remark 7.9. The previous proposition is easily seen to hold when J is replaced by any strict n-category D: all one has to do to adapt the proof is to describe the strict n-functors $J_k \times D \to C$.

Proposition 7.10. *Let C be a strict n-category. Then $N_n(C)$ is an n-quasi-category if and only if C is rigid in dimension k for $1 < k \leq n$.*

Proof. Since $N_n(J)$ is an injective separating interval (see paragraph 6.8), by Theorem 2.14, $N_n(C)$ is an n-quasi-category if and only if it is $(\mathcal{I}_n \cup \mathcal{J}'_n, N_n(J))$-fibrant. By Proposition 6.18, $N_n(C)$ is always $(\mathcal{I}_n, N_n(J))$-fibrant. We thus have to understand when the unique map $N_n(C) \to \mathrm{D}_0$ has the right lifting property with respect to $\Lambda^\infty_{N_n(J)}(\mathcal{J}'_n)$. By using the same arguments (and notation) than in the proof of Proposition 6.18, this condition is equivalent to the fact that the map $C \to \mathrm{D}_0$ has the right lifting property with respect to

$$\Lambda^\infty_J(\tau_n(\mathcal{J}'_n)) = \bigcup_{1<k\leq n} \Lambda^\infty_J(\{s_k^\varepsilon : \mathrm{D}_{k-1} \to J_k;\ \varepsilon = 0, 1\}).$$

We now fix k such that $1 < k \le n$. We will show that $C \to \mathrm{D}_0$ has the right lifting property with respect to $\Lambda^\infty_J(\{s^0_k, s^1_k\})$ if and only if C is rigid in dimension k.

Given a class S of strict n-functors, define $V^\infty_J(S)$ to be the smallest class of strict n-functors containing S and closed under the operation

$$f : A \to B \quad \longmapsto \quad \underline{\mathsf{Hom}}'(\delta_J, f) : \underline{\mathsf{Hom}}(J, A) \to \underline{\mathsf{Hom}}(\partial J, A) \times_{\underline{\mathsf{Hom}}(\partial J, B)} \underline{\mathsf{Hom}}(J, B).$$

By adjunction (see paragraph 1.2), if S and T are two classes of strict n-functors, we have

$$\Lambda^\infty_J(S) \pitchfork T \quad \Leftrightarrow \quad S \pitchfork V^\infty_J(T).$$

In particular, we have

$$\Lambda^\infty_J(\{s^0_k, s^1_k\}) \pitchfork \{p\} \quad \Leftrightarrow \quad \{s^0_k, s^1_k\} \pitchfork V^\infty_J(\{p\}),$$

where p denotes the unique n-functor $C \to \mathrm{D}_0$. Since the n-functors s^0_k and s^1_k both admit a retraction, they have the left lifting property with respect to $p : C \to \mathrm{D}_0$. By adjunction, we thus have

$$\{s^0_k, s^1_k\} \pitchfork V^\infty_J(\{p\}) \quad \Leftrightarrow \quad \{s^0_k \times' \delta_J, s^1_k \times' \delta_J\} \pitchfork V^\infty_J(\{p\}).$$

This last condition can be rephrased by saying that the n-functors of $V^\infty_J(\{C \to D_0\})$ are rigid in dimension k. In particular, this condition implies that C is rigid in dimension k. Conversely, if C is rigid in dimension k, then by Proposition 7.8, the sources of the n-functors of $V^\infty_J(\{C \to D_0\})$ are rigid in dimension k and the result follows by Proposition 7.7. □

Corollary 7.11. *For every k such that $1 < k \le n$, the presheaf $N_n(J_k)$ is not an n-quasi-category.*

Proof. This follows immediately from the proposition since J_k has non-trivial invertible k-arrows. □

Corollary 7.12. *If C is a category, then $N_n(C)$ is an n-quasi-category.*

Proof. This follows immediately from the proposition since categories have only trivial k-arrows for $k > 1$. □

8. Comparison with Rezk Θ_n-spaces

In [43], Rezk introduced a notion of Θ_n-spaces as fibrant objects of a model category structure on simplicial presheaves on Θ_n. Since the cofibrations of this model category structure are the monomorphisms, the weak equivalences define a $(\Theta_n \times \Delta)$-localizer. The purpose of this section is to show that this localizer is the simplicial completion of the localizer of n-quasi-categories. It will then follow formally from the theory of simplicial completion that we have two Quillen equivalences between the corresponding model category structures.

8.1. The *localizer of Rezk Θ_n-spaces* is the $(\Theta_n \times \Delta)$-localizer generated by

$$\mathsf{W}_{\mathrm{vert}} \cup p^*(\mathcal{I}_n) \cup \{p^*(N_n(j) : N_n(J) \to N_n(\mathrm{D}_0))\} \cup p^*(\mathcal{J}_n),$$

where $\mathsf{W}_{\mathrm{vert}}$ is the class of vertical equivalences defined in paragraph 2.16 (with $A = \Theta_n$). We will denote it by $\mathsf{W}_{\mathrm{Rezk}_n}$. By the 2-out-of-3 property, this localizer is also generated by

$$\mathsf{W}_{\mathrm{vert}} \cup p^*(\mathcal{I}_n) \cup \{p^*(N_n(\partial^\varepsilon) : N_n(\mathrm{D}_0) \to N_n(J));\ \varepsilon = 0,1\} \cup p^*(\mathcal{J}'_n).$$

Since the localizer $\mathsf{W}_{\mathrm{vert}}$ is accessible, so is the localizer $\mathsf{W}_{\mathrm{Rezk}_n}$. It follows from the connection between localizers and Bousfield localization (see Proposition A.11) that the $\mathsf{W}_{\mathrm{Rezk}_n}$-model category structure on $\widehat{\Theta_n \times \Delta}$ is nothing but the model category structure $\Theta_n\mathrm{Sp}_\infty$ introduced by Rezk in [43]. The fibrant objects of this model category are called (∞, n)-Θ-spaces by Rezk. We will call them *Rezk Θ_n-spaces* and the $\mathsf{W}_{\mathrm{Rezk}_n}$-model category will be called *the model category of Rezk Θ_n-spaces.*

Theorem 8.2 (Rezk)**.** *The model category of Rezk Θ_n-spaces is simplicial, left proper and cartesian.*

Proof. The model category of Rezk Θ_n-spaces is a left Bousfield localization of the vertical model category. By Theorem 3.7, the vertical model category is simplicial. It follows from Theorem 4.11 of [25] that the model category of Rezk Θ_n-spaces is simplicial.

By definition, every object is cofibrant in the model category of Rezk Θ_n-spaces. This structure is hence left proper.

The fact that the model category of Rezk Θ_n-spaces is cartesian is highly non-trivial. It is a special case of one of the main results of [43] (Proposition 11.5). □

Theorem 8.3. *The $(\Theta_n \times \Delta)$-localizer of Rezk Θ_n-spaces is the simplicial completion of the Θ_n-localizer of n-quasi-categories.*

Proof. By Theorem 5.10, the category Θ_n is a regular skeletal Reedy category. We can thus apply Theorem 4.10 to the set $S = \mathcal{I}_n \cup \mathcal{J}_n$ and to the injective separating interval $N_n(J)$ (see paragraph 6.8). We obtain that the simplicial completion of $\mathsf{W}_{\mathrm{QCat}_n}$ is the $(\Theta_n \times \Delta)$-localizer $\mathsf{W}_{\mathrm{fRezk}}$ defined in paragraph 4.1. By definition, this localizer is generated by

$$\mathsf{W}_{\mathrm{vert}} \cup p^*(\mathcal{I}_n \cup \mathcal{J}_n) \cup \{p^*(\partial^\varepsilon_X : X \to X \times N_n(J));\ X \in \mathsf{Ob}(\widehat{\Theta_n}),\ \varepsilon = 0,1\}.$$

Since $\partial^\varepsilon = \partial^\varepsilon_{\mathrm{D}_0}$, the localizer $\mathsf{W}_{\mathrm{Rezk}_n}$ is contained in $\mathsf{W}_{\mathrm{fRezk}}$. Conversely, we have to show that $\partial^\varepsilon_X = X \times \partial^\varepsilon$ belongs to $\mathsf{W}_{\mathrm{Rezk}_n}$. But by Theorem 8.2, the $\mathsf{W}_{\mathrm{Rezk}_n}$-model category is cartesian and the localizer $\mathsf{W}_{\mathrm{Rezk}_n}$ is hence closed under binary product, thereby proving the result. □

Theorem 8.4. *Let us endow $\widehat{\Theta_n}$ (resp. $\widehat{\Theta_n \times \Delta}$) with the model category structure of n-quasi-categories (resp. with the model category structure of Rezk Θ_n-spaces).*

(1) *Then the adjunction*

$$p^* : \widehat{\Theta_n} \rightleftarrows \widehat{\Theta_n \times \Delta} : i_0^*$$

is a Quillen equivalence.

(2) *Let $D : \Delta \to \widehat{\Theta_n}$ be a cosimplicial $\mathsf{W}_{\mathrm{QCat}_n}$-resolution. Then the adjunction*

$$\mathit{Real}_D : \widehat{\Theta_n \times \Delta} \rightleftarrows \widehat{\Theta_n} : \mathit{Sing}_D$$

is a Quillen equivalence.

Proof. This follows from Propositions 2.20 and 2.23, and from the above theorem. □

Corollary 8.5. *The model category of n-quasi-categories is cartesian closed.*

Proof. By Theorems 8.2 and 8.3, the simplicial completion of the localizer of n-quasi-categories is cartesian. It follows from Corollary 2.21 that this localizer is cartesian, and from Proposition 2.7 that the associated model category is cartesian closed. □

8.6. Let $\widetilde{\Delta_\bullet} : \Delta \to \mathcal{C}\mathrm{at}$ be the functor

$$\Delta \hookrightarrow \mathcal{C}\mathrm{at} \xrightarrow{\Pi_1} \mathcal{G}\mathrm{pd} \hookrightarrow \mathcal{C}\mathrm{at},$$

where $\mathcal{G}\mathrm{pd}$ denotes the category of groupoids and Π_1 is the fundamental groupoid functor, i.e., the left adjoint to the inclusion functor $\mathcal{G}\mathrm{pd} \to \mathcal{C}\mathrm{at}$. In other words, the functor $\widetilde{\Delta_\bullet}$ sends Δ_k to $\widetilde{\Delta_k}$, the simply connected groupoid with objects $\{0, \dots, k\}$.

Proposition 8.7. *The functor $N_n\widetilde{\Delta_\bullet} : \Delta \to \widehat{\Theta_n}$ is a simplicial W_{QCat_n}-resolution of $\widehat{\Theta_n}$.*

Proof. It is clear that $N_n\widetilde{\Delta_0} \amalg N_n\widetilde{\Delta_0} \to N_n\widetilde{\Delta_1}$ is a monomorphism. Let $k \geq 0$. We have to prove that for every presheaf X on Θ_n, the projection $X \times N_n\widetilde{\Delta_k} \to X$ is a $\mathsf{W}_{\mathrm{QCat}_n}$-equivalence. It suffices to prove that the unique morphism $N_n\widetilde{\Delta_k} \to \mathrm{D}_0$ is a trivial fibration. But this morphism can be obtained by applying the functor N_n to the unique functor $\widetilde{\Delta_k} \to \Delta_0$. The result thus follows from Corollary 6.6 since $\widetilde{\Delta_k} \to \Delta_0$ is an equivalence of categories surjective on objects between preorders. □

Corollary 8.8. *The simplicial W_{QCat_n}-resolution $N_n\widetilde{\Delta_\bullet}$ induces a Quillen equivalence*

$$Real_{N_n\widetilde{\Delta_\bullet}} : \widehat{\Theta_n \times \Delta} \rightleftarrows \widehat{\Theta_n} : Sing_{N_n\widetilde{\Delta_\bullet}},$$

where $\widehat{\Theta_n \times \Delta}$ (resp. $\widehat{\Theta_n}$) is endowed with the model category structure of Rezk Θ_n-spaces (resp. with the model category structure of n-quasi-categories).

Proof. By the above proposition, the functor $N_n\widetilde{\Delta_\bullet}$ is a simplicial $\mathsf{W}_{\mathrm{QCat}_n}$-resolution. The result thus follows from Theorem 8.4 applied to $D = N_n\widetilde{\Delta_\bullet}$. □

Remark 8.9. When $n = 1$, we recover from Theorem 8.4 and the above corollary the two Quillen equivalences between quasi-categories and complete Segal spaces defined by Joyal and Tierney in [32].

Appendix A. Localizers and Bousfield localization

In this appendix, we compare the language of localizers to the language of Bousfield localization. This comparison is needed for instance to see that the definition of Rezk Θ_n-spaces we gave in terms of $(\Theta_n \times \Delta)$-localizers coincide with the original definition of Rezk.

A.1. In this appendix, the category $\widehat{\Delta}$ of simplicial sets will always be endowed with its classical model category structure (i.e., the one defined by Quillen in [41]).

If $\mathcal{M}$ is a model category, the category $\underline{\mathsf{Hom}}(\Delta^{\mathrm{o}}, \mathcal{M})$ of simplicial objects in $\mathcal{M}$ will always be endowed with the Reedy model category structure. Similarly, the category $\underline{\mathsf{Hom}}(\Delta, \mathcal{M})$ of cosimplicial objects in $\mathcal{M}$ will always be endowed with the Reedy model category structure. The weak equivalences of these structures are the objectwise weak equivalences. All we will need about their cofibrations is that they only depend on the cofibrations of $\mathcal{M}$. We refer the reader to Chapters 15 and 16 of [25] for details on these model categories.

Recall that the functors

$$\mathcal{M} \to \underline{\mathsf{Hom}}(\Delta^o, \mathcal{M}) \quad \text{and} \quad \mathcal{M} \to \underline{\mathsf{Hom}}(\Delta, \mathcal{M})$$

sending an object of $\mathcal{M}$ to the associated constant simplicial object (resp. to the associated constant cosimplicial object) are fully faithful. Moreover, they induce fully faithful functors

$$\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\underline{\mathsf{Hom}}(\Delta^o, \mathcal{M})) \quad \text{and} \quad \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\underline{\mathsf{Hom}}(\Delta, \mathcal{M}))$$

between homotopy categories. In the sequel, we will often consider these four functors as inclusions.

A.2. Let $\mathcal{M}$ be a model category. The functor

$$\mathsf{Hom}_{\mathcal{M}} : \mathcal{M}^o \times \mathcal{M} \to \mathcal{S}et$$

induces a functor

$$\underline{\mathsf{Hom}}(\Delta, \mathcal{M})^o \times \underline{\mathsf{Hom}}(\Delta^o, \mathcal{M}) \to \widehat{\Delta}$$

that we will denote in the same way. Explicitly, if X is a cosimplicial object in $\mathcal{M}$ and Y is a simplicial object in $\mathcal{M}$, the simplicial set $\mathsf{Hom}_{\mathcal{M}}(X, Y)$ is given by

$$\mathsf{Hom}_{\mathcal{M}}(X, Y)_n = \mathsf{Hom}_{\mathcal{M}}(X_n, Y_n), \quad n \geq 0.$$

It is well-known that this functor preserves weak equivalences between fibrant objects and hence admits a right derived functor

$$\mathsf{RHom}_{\mathcal{M}} : \mathrm{Ho}(\underline{\mathsf{Hom}}(\Delta, \mathcal{M}))^o \times \mathrm{Ho}(\underline{\mathsf{Hom}}(\Delta^o, \mathcal{M})) \to \mathrm{Ho}(\widehat{\Delta}).$$

In particular, we obtain a functor

$$\mathsf{RHom}_{\mathcal{M}} : \mathrm{Ho}(\mathcal{M})^o \times \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\widehat{\Delta}).$$

A.3. Let X and Y be two objects of a model category $\mathcal{M}$. Denote by $p : \widehat{\Delta} \to \mathrm{Ho}(\widehat{\Delta})$ the localization functor. The theory of derived functors gives the formula

$$\mathsf{RHom}_{\mathcal{M}}(X, Y) \cong p(\mathsf{Hom}_{\mathcal{M}}(\widetilde{X}, \widetilde{Y})),$$

where $\widetilde{X}$ is any cofibrant cosimplicial replacement of X and $\widetilde{Y}$ is any fibrant simplicial replacement of Y. It follows from Proposition 17.4.6 of [25] that $\mathsf{RHom}_{\mathcal{M}}(X, Y)$ can be computed using the simpler formula

$$\mathsf{RHom}_{\mathcal{M}}(X, Y) \cong p(\mathsf{Hom}_{\mathcal{M}}(\widetilde{X}, Y')),$$

where $\widetilde{X}$ is any cofibrant cosimplicial replacement of X and Y' is any fibrant replacement of Y in $\mathcal{M}$; or using the dual formula

$$\mathsf{RHom}_{\mathcal{M}}(X, Y) \cong p(\mathsf{Hom}_{\mathcal{M}}(X', \widetilde{Y})),$$

where X' is any cofibrant replacement of X in $\mathcal{M}$ and $\widetilde{Y}$ is any fibrant simplicial replacement of Y. Moreover, if the model category $\mathcal{M}$ is simplicial, it follows from Proposition 16.6.4 of [25] that

$$\mathsf{RHom}_{\mathcal{M}} : \mathrm{Ho}(\mathcal{M})^o \times \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\widehat{\Delta})$$

is the right derived functor of the simplicial enrichment

$$\mathrm{Map} : \mathcal{M}^{\mathrm{o}} \times \mathcal{M} \to \widehat{\Delta}$$

of $\mathcal{M}$. In other words, we have

$$\mathsf{RHom}_{\mathcal{M}}(X, Y) \cong p(\mathrm{Map}(X', Y')),$$

where X' is any cofibrant replacement of X in $\mathcal{M}$ and Y' is any fibrant replacement of Y in $\mathcal{M}$.

Lemma A.4. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two model category structures on the same category such that*

- *the class of cofibrations of $\mathcal{M}$ and $\mathcal{M}'$ are the same;*
- *the class of weak equivalences of $\mathcal{M}$ is included in the class of weak equivalences of $\mathcal{M}'$.*

Then for every object X and every fibrant object Y of $\mathcal{M}'$, we have a natural isomorphism

$$\mathsf{RHom}_{\mathcal{M}}(X, Y) \cong \mathsf{RHom}_{\mathcal{M}'}(X, Y).$$

Proof. Note that the hypothesis on $\mathcal{M}$ and $\mathcal{M}'$ implies that

- every fibrant object of $\mathcal{M}'$ is fibrant in $\mathcal{M}$;
- the class of cofibrations of $\underline{\mathsf{Hom}}(\Delta, \mathcal{M})$ and $\underline{\mathsf{Hom}}(\Delta, \mathcal{M}')$ are the same;
- the class of weak equivalences of $\underline{\mathsf{Hom}}(\Delta, \mathcal{M})$ is included in the class of weak equivalences of $\underline{\mathsf{Hom}}(\Delta, \mathcal{M}')$.

In particular, by the first point, the object Y is fibrant in $\mathcal{M}$. Now let $\widetilde{X}$ be a cofibrant cosimplicial replacement of X in $\underline{\mathsf{Hom}}(\Delta, \mathcal{M})$. By the second and third points, $\widetilde{X}$ is also a cofibrant cosimplicial replacement of X in $\underline{\mathsf{Hom}}(\Delta, \mathcal{M}')$. It follows by paragraph A.3 that

$$\mathsf{Hom}_{\mathcal{M}}(\widetilde{X}, Y) = \mathsf{Hom}_{\mathcal{M}'}(\widetilde{X}, Y)$$

can be used to compute both $\mathsf{RHom}_{\mathcal{M}}(X, Y)$ and $\mathsf{RHom}_{\mathcal{M}'}(X, Y)$, thereby proving the result. □

A.5. Let $\mathcal{M}$ be a model category, let C be a class of morphisms of $\mathcal{M}$ and let D be a class of objects of $\mathcal{M}$.

- An object Z of $\mathcal{M}$ is said to be *C-local* if for every morphism $f : X \to Y$ in C, the morphism

  $$\mathsf{RHom}_{\mathcal{M}}(f, Z) : \mathsf{RHom}_{\mathcal{M}}(Y, Z) \to \mathsf{RHom}_{\mathcal{M}}(X, Z)$$

  is an isomorphism.
- A morphism $f : X \to Y$ of $\mathcal{M}$ is said to be a *D-local equivalence* if for every object Z in D, the morphism

  $$\mathsf{RHom}_{\mathcal{M}}(f, Z) : \mathsf{RHom}_{\mathcal{M}}(Y, Z) \to \mathsf{RHom}_{\mathcal{M}}(X, Z)$$

  is an isomorphism. If the class D consists of a single object Z, we will say that f is a Z-local equivalence.
- A morphism $f : X \to Y$ of $\mathcal{M}$ is said to be a *C-local equivalence* if it is an L-local equivalence, where L is the class of C-local objects.

Remark A.6. By definition, if C is a class of morphisms of a model category $\mathcal{M}$, then the elements of C are C-local equivalences. By functoriality of $\mathsf{RHom}_{\mathcal{M}}$, the weak equivalences of $\mathcal{M}$ are also C-local equivalences and even $\mathsf{Ob}(\mathcal{M})$-local equivalences. This property actually characterizes weak equivalences:

Proposition A.7. *The weak equivalences of a model category $\mathcal{M}$ are precisely the $\mathsf{Ob}(\mathcal{M})$-local equivalences. More generally, they are the D-local equivalences, where D is any class of objects of $\mathcal{M}$ such that every object of $\mathcal{M}$ is weakly equivalent to an object in D.*

Proof. This is a direct consequence of the formula

$$\pi_0(\mathsf{RHom}_{\mathcal{M}}(X,Y)) \cong \mathsf{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y).$$

(See Theorem 17.7.2 of [25].) □

Lemma A.8. *Let A be a small category and let $\mathcal{M}$ be a model category structure on $\widehat{A}$ whose cofibrations are the monomorphisms. Let D be a class of objects of $\widehat{A}$. Then the class of D-local equivalences is an A-localizer. In particular, if C is a class of morphisms of $\widehat{A}$, then the class of C-local equivalences is an A-localizer.*

Proof. The 2-out-of-3 property is immediate by functoriality of $\mathsf{RHom}_{\mathcal{M}}$.

By hypothesis, the trivial fibrations of $\widehat{A}$ are the trivial fibrations of $\mathcal{M}$. In particular, they are weak equivalences in $\mathcal{M}$ and hence D-local equivalences.

Let us now show the stability conditions of the class of morphisms which are both monomorphisms and D-local equivalences. Consider a cocartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

in $\mathcal{M}$, where f is both a monomorphism and a D-local equivalence. Let us show that f' is a D-local equivalence. Let Z be an object of $\mathcal{M}$ and let $\widetilde{Z}$ be a fibrant simplicial replacement of Z. By applying the functor $\mathsf{Hom}_C(-,\widetilde{Z})$, we obtain a cartesian square

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{M}}(Y',\widetilde{Z}) \arrow[r] \arrow[d, "\mathsf{Hom}_{\mathcal{M}}(f',\widetilde{Z})"'] & \mathsf{Hom}_{\mathcal{M}}(Y,\widetilde{Z}) \arrow[d, "\mathsf{Hom}_{\mathcal{M}}(f,\widetilde{Z})"] \\ \mathsf{Hom}_{\mathcal{M}}(X',\widetilde{Z}) \arrow[r] & \mathsf{Hom}_{\mathcal{M}}(X,\widetilde{Z}) \end{tikzcd}$$

of simplicial sets. Since every object of $\mathcal{M}$ is cofibrant, for every object W of $\mathcal{M}$, we have

$$\mathsf{RHom}_{\mathcal{M}}(W,Z) \cong p(\mathsf{Hom}_{\mathcal{M}}(W,\widetilde{Z})),$$

where $p : \widehat{\Delta} \to \mathrm{Ho}(\widehat{\Delta})$ is the localization functor. Moreover, by Theorem 16.5.2 of [25], the objects of this square are fibrant and the morphism $\mathsf{Hom}_{\mathcal{M}}(f,\widetilde{Z})$ is a fibration. This shows that this square is homotopically cartesian. In particular, if the morphism $\mathsf{Hom}_{\mathcal{M}}(f,\widetilde{Z})$ is a weak equivalence, then so is the morphism $\mathsf{Hom}_{\mathcal{M}}(f',\widetilde{Z})$. In other words, if f is a Z-local equivalence, then so is f'. This shows that f' is a D-local equivalence.

The stability under transfinite composition follows from a similar argument, the key point being that a limit defining a transfinite cocomposition in which every object is fibrant and every morphism is a fibration is actually a homotopy limit. □

Proposition A.9. *Let A be a small category, let W be an accessible A-localizer and let W' be an accessible A-localizer generated by W and a class of morphisms C. Then the W'-fibrant objects are the W-fibrant C-local objects.*

Proof. We will denote by $\mathcal{M}$ (resp. by $\mathcal{M}'$) the W-model category (resp. the W'-model category).

Let Z be a W'-fibrant object. Since W is included in W', Z is also W-fibrant. Let us show that Z is C-local. Let $X \to Y$ be an element of C. We have to show that the morphism

$$\mathsf{RHom}_{\mathcal{M}}(Y, Z) \to \mathsf{RHom}_{\mathcal{M}}(X, Z)$$

is an isomorphism. By Lemma A.4, this is the case if and only if the morphism

$$\mathsf{RHom}_{\mathcal{M}'}(Y, Z) \to \mathsf{RHom}_{\mathcal{M}'}(X, Z)$$

is an isomorphism, which is true since $X \to Y$ is a W'-equivalence.

Suppose now Z is a C-local W-fibrant object and let us prove that Z is W'-fibrant. Let $f : U \to V$ be a trivial cofibration of $\mathcal{M}'$. We have to show that

$$\mathsf{Hom}_{\widehat{A}}(V, Z) \to \mathsf{Hom}_{\widehat{A}}(U, Z)$$

is a surjection. By Proposition 16.6.4 of [25], there exist in $\underline{\mathsf{Hom}}(\Delta, \mathcal{M})$ cofibrant cosimplicial replacements $\widetilde{U}$ and $\widetilde{V}$ of U and V, respectively, and a cofibration $\widetilde{f} : \widetilde{U} \to \widetilde{V}$ such that $\widetilde{f}_0 = f$. It thus suffices to show that the morphism of simplicial sets

$$\mathsf{Hom}_{\widehat{A}}(\widetilde{V}, Z) \to \mathsf{Hom}_{\widehat{A}}(\widetilde{U}, Z)$$

is a trivial fibration. Since $\widetilde{f}$ is a cofibration between cosimplicial replacements of objects of $\mathcal{M}$ and Z is W-fibrant, Theorem 16.5.2 of [25] implies that this morphism is a fibration.

Let us show that it is a weak equivalence. By hypothesis, the class of Z-local equivalences contains W and C. Moreover, by Lemma A.8, this class is an A-localizer. It follows by definition of W' that every W'-equivalence is a Z-local equivalence. In particular, this shows that f is a Z-local equivalence and so, by paragraph A.3, that the morphism

$$\mathsf{Hom}_{\widehat{A}}(\widetilde{V}, Z) \to \mathsf{Hom}_{\widehat{A}}(\widetilde{U}, Z)$$

is a weak equivalence, thereby proving the result. □

A.10. Let $\mathcal{M}$ be a model category and let C be a class of morphisms of $\mathcal{M}$. The *left Bousfield localization of $\mathcal{M}$ with respect to C* (if it exists) is the unique model category structure on the underlying category of $\mathcal{M}$ whose weak equivalences are the C-local equivalences and whose cofibrations are the cofibrations of $\mathcal{M}$.

Proposition A.11. *Let A be a small category, let W be an accessible A-localizer and let W' be an accessible A-localizer generated by W and a class of morphisms C. Then the W'-model category is the left Bousfield localization of the W-model category with respect to C.*

Proof. We have to show that the W'-equivalences are exactly the C-local equivalences. We will denote by $\mathcal{M}$ (resp. by $\mathcal{M}'$) the W-model category (resp. the W'-model category).

By Lemma A.8, the class of C-local equivalences is an A-localizer. Since it contains W and C, the definition of W' implies that every W'-equivalence is a C-local equivalence.

Conversely, let $f : X \to Y$ be a C-local equivalence. Let us show that f is a W'-weak equivalence. By Proposition A.7, it suffices to prove that for every W'-fibrant object Z, the morphism

$$\mathsf{RHom}_{\mathcal{M}'}(Y, Z) \to \mathsf{RHom}_{\mathcal{M}'}(X, Z)$$

is an isomorphism. By Lemma A.4, these RHom can be computed in $\mathcal{M}$. Moreover, by Proposition A.9, the object Z is C-local. The above morphism is thus an isomorphism, thereby proving the result. □

Remark A.12. The proof of the above proposition only relies on the easy implication of Proposition A.9. Therefore, we could have proved it directly. Proposition A.9 would then have been a corollary of the theory of Bousfield localization (see Proposition 3.4.1 of [25]).

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