ORBIFOLD CHERN CLASSES INEQUALITIES AND APPLICATIONS

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ABSTRACT. In this paper we prove that given a pair (X, D) of a threefold X and a boundary divisor D with mild singularities, if $(K_X + D)$ is movable, then the orbifold second Chern class c_2 of (X, D) is pseudoeffective. This generalizes the classical result of Miyaoka on the pseudoeffectivity of c_2 for minimal models. As an application, we give a simple solution to Kawamata's effective non-vanishing conjecture in dimension 3, where we prove that $H^0(X, K_X + H) \neq 0$, whenever $K_X + H$ is nef and H is an ample, effective, reduced Cartier divisor. Furthermore, we study Lang-Vojta's conjecture for codimension one subvarieties and prove that minimal threefolds of general type have only finitely many Fano, Calabi-Yau or Abelian subvarieties of codimension one that are mildly singular and whose numerical classes belong to the movable cone.

CONTENTS

1. INTRODUCTION

It is well known that the Chern classes of nef vector bundles over smooth projective varieties satisfy certain inequalities [DPS94]. More generally, a theorem of Miyaoka [Miy87] states that over a normal, projective variety (that is smooth in codimension two) any torsion free, coherent sheaf $\mathscr{E}$ that is *semipositive* with respect to the tuple of ample divisors $(H_1, \ldots, H_{n-1})$ and whose determinant $\det(\mathscr{E})$ is nef, verifies the inequality

$$c_2(\mathscr{E}) \cdot H_1 \ldots H_{n-2} \geq 0.$$

On the other hand, thanks to Miyaoka's celebrated *generic semipositivity* result, cf. [Miy87], and the result of Boucksom, Demailly, Păun and Peternell ([BDPP13]),

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when K_X is pseudoeffective, the cotangent bundle Ω^1_X of a smooth projective variety is generically semipositive. As a result, for a smooth projective variety X with K_X nef, the inequality

$$c_2(X) \cdot H_1 \dots H_{n-2} \geq 0 \tag{1.0.1}$$

holds, for any tuple of ample divisors $(H_1, \dots, H_{n-2})$.

Recent works of Campana and Păun ([CP15], [CP16]) have generalized some parts of Miyaoka's results, showing in particular that if X is a smooth projective variety with K_X pseudoeffective, then Ω^1_X is semipositive with respect to any *movable* class $\alpha \in \mathrm{Mov}_1(X)$.

Our first result is a natural generalization of the inequality (1.0.1) to the setting of pairs with movable log-canonical divisors.

Theorem 1.1. *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, then for any ample divisor A, the inequality*

$$c_2\big((\Omega^1_X \log(D))^{**}\big) \cdot A \geq 0$$

holds.

The second result is another generalization of an inequality established by Miyaoka [Miy87], which is sometimes referred to as the Miyaoka-Yau inequality.

Theorem 1.2. *Let X be a normal projective threefold that is smooth in codimension two and D a reduced effective divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, then*

$$c_1^2\big((\Omega^1_X \log(D))^{**}\big) \cdot A \leq 3c_2\big((\Omega^1_X \log(D))^{**}\big) \cdot A,$$

for any ample divisor A.

There are two main ingredients in the proof of the above inequalities. The first one is a restriction result for semistable sheaves with respect to some *strongly movable* curves. This is described in section 3. The second component involves the semipositivity of the *orbifold cotangent sheaves* and is treated in section 4.

The rest of the paper is devoted to two applications of Theorems 1.1 and 1.2. The first one concerns the so-called effective non-vanishing conjecture.

Conjecture 1.3 (Effective non-vanishing conjecture of Kawamata)**.** *Let Y be a normal projective variety and D_Y an effective $\mathbb{R}$-divisor such that (Y, D_Y) is klt. Let H be an ample, or more generally big and nef, divisor such that $(K_Y + D_Y + H)$ is Cartier and nef. Then $H^0(X, K_Y + D_Y + H) \neq 0$.*

Using Theorem 1.1, in Section 6, we obtain a simple proof of the following weak version of Conjecture 1.3 in dimension three.

Theorem 1.4 (Non-vanishing for canonical threefolds)**.** *Let Y be a normal projective threefold with only canonical singularities. Let H be a very ample divisor. If $(K_Y + H)$ is a nef and Cartier divisor, then $H^0(Y, K_Y + H) \neq 0$.*

We note that Theorem 1.4 is stated in [Hör12] under the weaker assumption that H is a nef and big Cartier divisor. The proof relies on an inequality similar to that of Theorem 1.1 but under the weaker assumption that the first Chern class is *nef in codimension one*. It seems that there is a gap in the proof of that inequality, but according to the author one can get rid of this assumption and use only the classical result of Miyaoka where c_1 is assumed to be nef (cf. the inequality 1.0.1).

A second application is given in section 8 vis-à-vis Lang-Vojta's conjectures on subvarieties of varieties of general type:

> **Geometric Lang-Vojta conjecture:** *In a projective variety of general type X, subvarieties that are not of general type are contained in a proper algebraic subvariety of X.*

In particular, a variety of general type should have only finitely many codimension one subvarieties that are not of general type. We partially establish this conjecture in the setting of the following theorem.

Theorem 1.5. *Let X be a normal projective $\mathbb{Q}$-factorial threefold such that $K_X \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$. If X is of general type then X has only a finite number of movable codimension one, normal subvarieties D verifying the following conditions.*

(1.5.1) The subvariety D has only canonical singularities.
(1.5.2) The anticanonical divisor $-K_D$ is pseudoeffective.
(1.5.3) The pair (X, D) has only isolated lc singularities.

In particular, there are only finitely many such Fano, Abelian and Calabi-Yau subvarieties.

Here, by a variety of general type, we mean a normal variety whose resolution has a big canonical bundle.

We remark that—in the smooth setting—a stronger version of Theorems 1.5 and 1.1 has been claimed in [LM97], where the authors establish these results under the weaker assumption that $(K_X + D)$ is pseudoeffective. Unfortunately the arguments in [LM97] are not complete. We refer to Remark 8.2 for a detailed discussion of these problems.

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2. Basic definitions and background

2.1. **Movable cone.** We introduce the *movable* cone of divisors; one of the important cones of divisors that is ubiquitous in birational geometry.

Let X be a normal projective variety and D a $\mathbb{Q}$-divisor on X. The stable base locus of D is defined by

$$\mathbb{B}(D) := \bigcap_m \mathrm{Bs}(|mD|).$$

The restricted base locus is given by

$$\mathbb{B}_-(D) = \bigcup_{A \text{ ample}} \mathbb{B}(D + A).$$

Definition 2.1. Let X be a normal variety. The movable cone $\mathrm{Mov}^1(X) \subset \mathrm{N}^1(X)$ is the closure of the cone generated by the classes of all effective divisors D such that $\mathbb{B}_-(D)$ has no divisorial components.

The following proposition gives a more geometric picture of this definition.

Proposition 2.2 ([Bou04], Proposition 2.3)**.** *Given any α in the interior of $\mathrm{Mov}^1(X)$, there is a birational map $\phi : Y \to X$ and an ample divisor A on Y such that $[\phi_* A] = \alpha$.*

2.2. **Stability with respect to movable classes.** Now we introduce the notion of movable curves which generate the cone dual to the pseudoeffective cone.

Definition 2.3. A class $\gamma \in \mathrm{N}_1(X)$ is movable if $\gamma.D \geq 0$ for all effective divisors D. We define $\mathrm{Mov}_1(X)$ to be the closed convex cone of such 1-cycles.

Movable classes form a natural setting for the notion of stability of coherent sheaves (see [CP11] and [GKP15]). We shall now recall the basic definitions and properties.

Definition 2.4. Assume that X is $\mathbb{Q}$-factorial and let $\gamma \in \mathrm{Mov}_1(X)$. The slope of a coherent sheaf $\mathscr{E}$ with respect to γ is given by

$$\mu_\gamma(\mathscr{E}) := \frac{1}{r} \cdot (\det(\mathscr{E})) \cdot \gamma.$$

Definition 2.5. We say that $\mathscr{E}$ is semistable with respect to γ if $\mu_\gamma(\mathcal{F}) \leq \mu_\gamma(\mathscr{E})$ for any coherent subsheaf $0 \subsetneq \mathcal{F} \subset \mathscr{E}$.

Proposition 2.6 ([GKP15], Corollary 2.27)**.** *Let X be a normal, $\mathbb{Q}$-factorial, projective variety and $\gamma \in \mathrm{Mov}_1(X)$. There exists a unique Harder-Narasimhan filtration i.e. a filtration $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r = \mathscr{E}$ where each quotient $\mathcal{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free, γ-semistable, and where the sequaence of slopes $\mu_\gamma(\mathcal{Q}_i)$ is strictly decreasing.*

2.3. **$\mathbb{Q}$-twisted sheaves.** It will be quite useful in the sequel to work in the more general setting of $\mathbb{Q}$-twisted sheaves as introduced in [Miy87].

Definition 2.7 ($\mathbb{Q}$-twisted sheaves)**.** A $\mathbb{Q}$-twisted sheaf is a pair $\mathscr{E}\langle B\rangle$, where $\mathscr{E}$ is a coherent sheaf and B is a $\mathbb{Q}$-Cartier divisor.

Notation 2.8. Let X be a normal projective variety and $\mathscr{F}$ a coherent sheaf on X of rank r. Let D be a Weil divisor in X such that $\det(\mathscr{F}) \cong \mathscr{O}_X(D)$. When D is $\mathbb{Q}$-Cartier, we define $[\mathscr{F}]$ to denote the numerical class $[D] \in \mathrm{N}^1(X)_{\mathbb{Q}}$ of D. For any $\mathbb{Q}$-Cartier divisor A, we set $[\mathscr{F}\langle A\rangle] = [\mathscr{F}] + r \cdot [A]$. Here by $\det(\mathscr{F})$ we always mean the reflexive hull of the determinant of $\mathscr{F}$.

Let X be a normal projective variety of dimension n which is smooth in codimension 2 and $\mathscr{E}$ a reflexive sheaf on X. Then, one can define the second Chern class $c_2(\mathscr{E})$, cycle-theoretically, as a multilinear form on

$$\underbrace{\mathrm{N}^1(X)_{\mathbb{Q}} \times \ldots \times \mathrm{N}^1(X)_{\mathbb{Q}}}_{(n-2)\text{-times}}.$$

We now recall the usual formulas for Chern classes of $\mathbb{Q}$-twisted sheaves.

Definition 2.9 (Chern classes of $\mathbb{Q}$-twisted sheaves)**.** Let $\mathscr{E}\langle B\rangle$ be a $\mathbb{Q}$-twisted locally-free sheaf of rank r.

$$c_1(\mathscr{E}\langle B\rangle) = c_1(\mathscr{E}) + rc_1(B),$$

$$c_2(\mathscr{E}\langle B\rangle) = c_2(\mathscr{E}) + (r-1)c_1(\mathscr{E}) \cdot c_1(B) + \frac{r(r-1)}{2} c_1(B)^2.$$

The semipositivity property for sheaves also naturally extends to this setting.

Definition 2.10 (Semipositive $\mathbb{Q}$-twisted sheaves)**.** Let X be a normal projective variety and $\gamma \in \mathrm{Mov}(X)$. A $\mathbb{Q}$-twisted, torsion-free sheaf $\mathscr{E}\langle B\rangle$ is said to be semipositive with respect to γ, if for every torsion-free, $\mathbb{Q}$-twisted, quotient sheaf $\mathscr{E}\langle B\rangle \twoheadrightarrow \mathscr{F}\langle B\rangle$ we have $[\mathscr{F}\langle B\rangle] \cdot \gamma \geq 0$.

We also have the Bogomolov-Gieseker inequality for semistable $\mathbb{Q}$-twisted sheaves.

Proposition 2.11 (Bogomolov-Gieseker inequality for semistable $\mathbb{Q}$-twisted sheaves)**.** *Take S to be a smooth projective surface. Let $\mathscr{E}\langle B\rangle$ be a $\mathbb{Q}$-twisted locally-free sheaf on S of rank r and $A \in \mathrm{Amp}(X)_{\mathbb{Q}}$. If $\mathscr{E}\langle B\rangle$ is semistable with respect to A, then $\mathscr{E}\langle B\rangle$ verifies the Bogomolov-Gieseker inequality:*

$$2r \cdot c_2(\mathscr{E}\langle B\rangle) - (r-1) \cdot c_1^2(\mathscr{E}\langle B\rangle) \geq 0. \tag{2.11.1}$$

Proof. Let $h : T \to S$ be the morphism adapted to B so that $\mathscr{E}_T := h^*(\widehat{\mathscr{E}}\langle (B)\rangle)$ is locally-free. Define $K := \mathrm{Gal}(T/S)$. Notice that as the maximal destablizing subsheaf of $h^*(\mathscr{E}_T)$ is unique, it is K-invariant. As a result, $\mathscr{E}_T$ is semistable with respect to $A_T := h^*A$. The inequality 2.11.1 now follows from the standard Bogomolov-Gieseker inequality for semistable locally free sheaves.

□

2.4. **Orbifold basics.** Following the terminology of Campana [Cam04], an *orbifold* is simply a pair (X, D), consisting of a normal projective variety and a boundary divisor $D = \sum d_i \cdot D_i$, where $d_i = (1 - b_i/a_i) \in [0,1] \cap \mathbb{Q}$.

Our aim is now to define a notion of cotangent sheaf, adapted to an orbifold. To this end, and since we will not be exclusively working with smooth varieties, we will need a notion of pull-back for Weil divisors (that are not necessarily $\mathbb{Q}$-Cartier).

Definition 2.12 (Pull-back of Weil divisors)**.** Let $f : Y \to X$ be a finite morphism between quasi-projective normal varieties X and Y. We define pull-back $f^*(D)$ of a $\mathbb{Q}$-Weil divisor $D \subset X$ by the Zariski closure of $(f|_{Y_{\mathrm{reg}}})^*(D)$.

To define classical objects for orbifolds, it is quite convenient to use *adapted morphisms*.

Definition 2.13 (Adapted and strongly adapted morphisms)**.** Let (X, D) be an orbifold. A finite, surjective, Galois morphism $f : Y \to X$ is called adapted (to D) if, f^*D is an integral Weil divisor. We say that a given adapted morphism $f : Y \to X$ is *strictly adapted*, if we have $f^*D_i = a_i \cdot D_i'$, for some Weil divisor $D_i' \subset Y$. Furthermore, we call a strictly adapted morphism f, *strongly adapted*, if the branch locus of f only consists of $\mathrm{supp}\big(D - \lfloor D \rfloor + A\big)$, where A is a general member of a linear system of a very ample invertible sheaf on X.

Remark 2.14. For a pair (X, D), where X is smooth, and D is $\mathbb{Q}$-effective divisor with simple normal crossing support, the existence of a strongly adapted morphism $f : Y \to X$ was established by Kawamata, cf. [Laz04, Prop. 4.1]. A similar strategy can be applied to construct strongly adpapted morphisms $f : Y \to X$ when all the irreducible components of D are $\mathbb{Q}$-Cartier; in particular when X is assumed to be $\mathbb{Q}$-factorial.

Notation 2.15. Let $f : Y \to X$ be a a morphism adapted to D, where $D = \sum d_i \cdot D_i$, $d_i = 1 - \frac{b_i}{a_i} \in (0,1] \cap \mathbb{Q}$. For every prime component D_i of $(D - \lfloor D \rfloor)$, let $\{D_{ij}\}_{j(i)}$ be the collection of prime divisors that appear in $f^*(D_i)$. We define new divisors in Y by

$$D_Y^{ij} := b_i \cdot D_{ij} \tag{2.15.1}$$

$$D_f := f^*(\lfloor D \rfloor). \tag{2.15.2}$$

Now, let us explain how to define the cotangent sheaf of an orbifold.

Definition 2.16 (Orbifold cotangent sheaf)**.** In the situation of Notation 2.15, denote Y° to be the snc locus of the pair $(Y, \sum D_{ij} + D_f)$ and define $D_Y^{ij^{\circ}} := D_Y^{ij}|_{Y^{\circ}}$. Set $\Omega^1_{(Y^{\circ},f,D)}$ to be the kernel of the sheaf morphism

$$(f|_{Y^{\circ}})^*(\Omega_X \log(\ulcorner D \urcorner)) \longrightarrow \bigoplus_{i,j(i)} \mathscr{O}_{D_Y^{ij^{\circ}}}$$

induced by the natural residue map. We define the orbifold cotangent sheaf $\Omega^{[1]}_{(Y,f,D)}$ by the coherent extension $(i_{Y^{\circ}})_*(\Omega^1_{(Y^{\circ},f,D)})$, where $i_{Y^{\circ}}$ is the natural inclusion. We define the *orbifold tangent sheaf* $\mathscr{T}_{(Y,f,D)}$ by $(\Omega^{[1]}_{(Y,f,D)})^*$.

3. Restriction results for semistable sheaves

Let $h = (H_1, \ldots, H_{n-1})$ be a tuple of ample divisors on a normal projective variety X of dimension n and $\mathscr{E}$ a torsion free sheaf. A theorem of Mehta-Ramanathan [MR82] states that if m is large enough and $Y \in |mH_{n-1}|$ is a generic hypersurface, then the maximal destabilizing subsheaf of $\mathscr{E}|_Y$ is the restriction of the maximal destabilizing subsheaf of $\mathscr{E}$.

It is natural to try to extend this restriction theorem to movable polarization. Unfortunately, in general, such results are not valid for movable curve. For example, when X is a projective $K3$ surface then its cotangent bundle Ω^1_X is not pseudoeffective, which gives rise to the existence of movable curves for which the restriction theorem does not hold (cf. [BDPP13, Sect. 7]).

In this section, we will prove a restriction theorem for some strongly movable curves (see Proposition 3.2 below). The following lemma will serve as the key technical ingredient in the proof of this result.

Lemma 3.1 (Induced destablizing subsheaves of small rank on higher birational models)**.** *Let $\pi : \widetilde{S} \to S$ be a birational morphism of smooth projective surfaces $\widetilde{S}$ and S. Let $\widetilde{A}_{\widetilde{S}} \subset \widetilde{S}$ be an ample divisor and define $P_S := [\pi_*(\widetilde{A}_{\widetilde{S}})] \in \mathrm{N}^1(S)_{\mathbb{Q}}$. Let $\mathscr{E}_S$ be a P_S-unstable locally free sheaf of rank 3 on S. Then, at least one of the following assertions holds.*

(3.1.1) The maximal destablizing subsheaf $\widetilde{\mathscr{G}}_{\widetilde{S}}$ of $\pi^*\mathscr{E}_S$ verifies the isomorphism $(\pi_*(\widetilde{\mathscr{G}}_{\widetilde{S}}))^{**} \cong \mathscr{L}_S$, where $\mathscr{L}_S \subset \mathscr{E}_S$ is a reflexive subsheaf, with $\mu_{P_S}(\mathscr{L}_S) > \mu_{P_S}(\mathscr{E}_S)$.

(3.1.2) The maximal destablizing subsheaf $\widetilde{\mathscr{G}}_{\widetilde{S}}$ of $\bigwedge^2 \pi^*\mathscr{E}_S$ verifies the isomorphism $(\pi_*(\widetilde{\mathscr{G}}_{\widetilde{S}}))^{**} \cong \mathscr{L}_S$, where $\mathscr{L}_S \subset \bigwedge^2 \mathscr{E}_S$ is a reflexive subsheaf, with $\mu_{P_S}(\mathscr{L}_S) > \mu_{P_S}(\bigwedge^2 \mathscr{E}_S)$.

Proof. Let $\mathscr{F}_S \subset \mathscr{E}_S$ be a saturated subsheaf properly destablizing $\mathscr{E}_S$ with the exact sequence

$$0 \longrightarrow \mathscr{F}_S \longrightarrow \mathscr{E}_S \longrightarrow \mathscr{Q}_S \longrightarrow 0.$$

Based on the rank of $\mathscr{F}_S$ we divide the proof into two cases.

Case. 1. (rank$(\mathscr{F}_S) = 2$)*.* Let $\overline{\pi^*\mathscr{F}_S}$ denote the saturation of $\pi^*\mathscr{F}_S$ in $\pi^*\mathscr{E}_S$ with the resulting exact sequence

$$0 \longrightarrow \overline{\pi^*\mathscr{F}_S} \longrightarrow \pi^*\mathscr{E}_S \longrightarrow \widetilde{\mathscr{Q}}_{\widetilde{S}} \longrightarrow 0.$$

Since $\overline{\pi^*\mathscr{F}_S}$ destablizes $\pi^*\mathscr{E}_S$ we have:

$$\mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{Q}}_{\widetilde{S}}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathscr{E}_S) < \mu_{\widetilde{A}_{\widetilde{S}}}(\overline{\pi^*\mathscr{F}_S}). \tag{3.1.3}$$

Subcase. 1. $(\operatorname{rank}(\widetilde{\mathscr{G}}_{\widetilde{S}}) = 1)$. The inequality (3.1.3) implies that there is an injection

$$\widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \overline{\pi^*\mathscr{F}_S},$$

otherwise there is an injective sheaf morphism $\widetilde{\mathscr{G}}_{\widetilde{S}} \to \widetilde{\mathscr{Q}}_{\widetilde{S}}$ implying that $\mu(\widetilde{Q}_{\widetilde{S}}) > \mu(\widetilde{\mathscr{G}}_{\widetilde{S}})$; contradicting the assumption that $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is destablizing.

Let $\Delta_B(\cdot)$ denote the Bogomolov-Gieseker discriminant

$$2r \cdot c_2(\cdot) - (r-1) \cdot c_1^2(\cdot)$$

for a torsion free sheaf of rank r on the smooth projective surface S (or $\widetilde{S}$).

Now, as the semipositivity of $\mathscr{E}$ characterizes the semistability of rank two bundles on a smooth projective surface [Rei77], if $\mathscr{F}_S$ is semistable, then so is $\pi^*\mathscr{F}_S$, for $\Delta_B(\pi^*\mathscr{F}_S) = \Delta_B(\mathscr{F}_S) \geq 0$.

Let $\mathscr{M}$ be the torsion free quotient of the injection $\widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \pi^*\mathscr{E}_S$ with the exact sequence

$$0 \longrightarrow \widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \pi^*\mathscr{E}_S \longrightarrow \mathscr{M} \longrightarrow 0.$$

The resulting inequality between slopes

$$\mu_{\widetilde{A}_{\widetilde{S}}}(\mathscr{M}) < \mu_{\widetilde{A}_{\widetilde{S}}}(\pi^*\mathscr{E}_S) < \mu_{\widetilde{A}_{\widetilde{S}}}(\widetilde{\mathscr{G}}_{\widetilde{S}}), \tag{3.1.4}$$

now implies that the induced non-trivial morphism $\pi^*\mathscr{F}_S \longrightarrow \mathscr{M}$ is not injective. Let us denote the image by $\mathscr{N}$ and the kernel by $\widetilde{\mathscr{K}}$ and consider the resulting the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\mathscr{G}}_{\widetilde{S}} & \longrightarrow & \pi^*\mathscr{E}_S & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \sigma} & & \\
0 & \longrightarrow & \widetilde{\mathscr{K}} & \longrightarrow & \pi^*\mathscr{F}_S & \longrightarrow & \mathscr{N} & \longrightarrow & 0.
\end{array}$$

For simplicity, but with no loss of generality, we may assume that the exceptional center of π consists of a single point p. Let U be a trivialization neighbourhood for both $\mathscr{F}$ and $\mathscr{E}$ containing p. Set $\widetilde{U} = \pi^{-1}(U)$.

According to the above constructions we have the following properties.

- (3.1.5) Both $\pi^*\mathscr{F}$ and $\pi^*\mathscr{E}$ are trivial over $\widetilde{U}$.
- (3.1.6) There is a morphism $\sigma : \mathscr{N} \to \mathscr{M}$, that is injective at least in codimension one.
- (3.1.7) We have $\widetilde{\mathscr{G}}_{\widetilde{S}}|_{\widetilde{S}\setminus \mathrm{Exc}(\pi)} \cong \widetilde{\mathscr{K}}|_{\widetilde{S}\setminus \mathrm{Exc}(\pi)}$.

By construction there are line bundles $\mathscr{G}_S$ and $\mathscr{K}$ on S such that $\widetilde{\mathscr{G}}_{\widetilde{S}} \cong \pi^*\mathscr{G}_S \otimes \mathscr{O}_{\widetilde{S}}(E_1)$ and $\widetilde{\mathscr{K}} \cong \pi^*\mathscr{K} \otimes \mathscr{O}_{\widetilde{S}}(E_2)$, where E_1 and E_2 are effective exceptional divisors with $E_1 \geq E_2$. On the other hand, by using (3.1.5), (3.1.6) and (3.1.7), we find that $E_1 = E_2$ so that

$$\widetilde{\mathscr{K}} \cong \widetilde{\mathscr{G}}_{\widetilde{S}}. \tag{3.1.8}$$

But this contradicts the semistability of $\pi^*\mathscr{F}_S$.

So we may assume that $\mathscr{F}_S$ is not semistable. Let $\mathscr{B} \subset \mathscr{F}_S$ be the maximal destablizing subsheaf so that, by definition, $\mathscr{B}$ satisfies the inequality $\mu(\mathscr{B}) > \mu(\mathscr{E}_S)$.

Now, given the saturation $\overline{\pi^*\mathscr{B}}$ of $\pi^*\mathscr{B}$ in $\pi^*\mathscr{F}_S$, we have the exact sequence

$$0 \longrightarrow \overline{\pi^*\mathscr{B}} \longrightarrow \pi^*\mathscr{F}_S \longrightarrow \widetilde{\mathscr{A}} \longrightarrow 0$$

with the inequalities

$$\mu_{P_S}(\widetilde{\mathscr{A}}) < \mu_{P_S}(\pi^*\mathscr{F}_S) < \mu_{P_S}(\overline{\pi^*\mathscr{B}}).$$

Together with the isomorphism (3.1.8) this implies that there is an injection

$$\widetilde{\mathscr{G}}_{\widetilde{S}} \hookrightarrow \overline{\pi^*\mathscr{B}}.$$

It follows that $\widetilde{\mathscr{G}}_{\widetilde{S}} \cong \overline{\pi^*\mathscr{B}}$, i.e. $\widetilde{\mathscr{G}}_{\widetilde{S}}$ descends to the destablizing invertible subshaaf $\mathscr{B} \subset \mathscr{E}_S$. We can now take $\mathscr{L}_S$ to be equal to $\mathscr{B}$.

Subcase. 2. (rank$(\widetilde{\mathscr{G}}_{\widetilde{S}}) = 2$). We consider the exact sequence

$$0 \longrightarrow \overline{\pi^*\mathscr{F}_S} \longrightarrow \pi^*\mathscr{E}_S \longrightarrow \widetilde{\mathscr{Q}} \longrightarrow 0.$$

If the induced morphism $\widetilde{\mathscr{G}}_{\widetilde{S}} \to \overline{\pi^*\mathscr{F}_S}$ is an injection then, as $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is the maximal destablizing subsheaf of $\pi^*\mathscr{E}_S$ we have $\widetilde{\mathscr{G}}_{\widetilde{S}} \cong \overline{\pi^*\mathscr{F}_S}$. So that

$$(\pi_*\widetilde{\mathscr{G}}_{\widetilde{S}})^{**} \cong \mathscr{F}_S.$$

Taking $\mathscr{L}_S$ to be equal to $\mathscr{F}_S$ finishes the proof in this case.

Next, we assume that there is a non-zero morphism $\widetilde{\mathscr{G}}_{\widetilde{S}} \to \widetilde{\mathscr{Q}}$ and let

$$0 \longrightarrow \mathscr{C}_1 \longrightarrow \widetilde{\mathscr{G}}_{\widetilde{S}} \longrightarrow \mathscr{C}_2 \subseteq \widetilde{\mathscr{Q}} \longrightarrow 0$$

be the resulting exact sequence. As $\widetilde{\mathscr{G}}_{\widetilde{S}}$ is semistable we have

$$\mu(\mathscr{C}_1) \leq \mu(\widetilde{\mathscr{G}}_{\widetilde{S}}) \leq \mu(\widetilde{\mathscr{Q}}).$$

On the other hand, we have $\mu(\widetilde{\mathscr{Q}}) < \mu(\overline{\pi^*\mathscr{F}_S})$, a contradiction.

Case. 2. (rank$(\mathscr{F}_S) = 1$). Again, we consider the exact sequence

$$0 \longrightarrow \mathscr{F}_S \longrightarrow \mathscr{E}_S \longrightarrow \mathscr{Q} \longrightarrow 0$$

and the slope inequalities

$$\mu(\mathscr{Q}) < \mu(\mathscr{E}_S) < \mu(\mathscr{F}_S). \tag{3.1.9}$$

The surjective morphism $\mathscr{E}_S \to \mathscr{Q}$ naturally induces the map

$$\bigwedge^2 \mathscr{E}_S \to \bigwedge^2 \mathscr{Q}. \tag{3.1.10}$$

We may assume with no loss of generality that $\bigwedge^2 \mathscr{Q}$ is torsion free and that the morphism (3.1.10) is surjective. Let

$$0 \longrightarrow \mathscr{G} \longrightarrow \bigwedge^2 \mathscr{E}_S \longrightarrow \bigwedge^2 \mathscr{Q} \longrightarrow 0$$

be the resulting exact sequence.

The inequality (3.1.9) together with

$$\mu(\bigwedge^2 \mathscr{E}_S) = 2 \cdot \mu(\mathscr{E}_S) \qquad \text{and} \qquad \mu(\bigwedge^2 \mathscr{Q}) = 2 \cdot \mu(\mathscr{Q})$$

imply that $\mu(\bigwedge^2 \mathscr{E}_S) < \mu(\mathscr{G})$. That is, $\mathscr{G}$ is a destablizing subsheaf of $\bigwedge^2 \mathscr{E}_S$ of rank two. The arguments in Case. 1 above now applies, showing that the maximal destablizing subsheaf $\widetilde{\mathscr{G}}_{\widetilde{S}}$ of $\pi^*(\bigwedge^2 \mathscr{E}_S)$ verifies the isomorphism

$$\pi_*(\widetilde{\mathscr{G}}_{\widetilde{S}})^{**} \cong \mathscr{L}_S,$$

for some properly destablizing subsheaf of $\mathscr{L}_S \subset \bigwedge^2(\mathscr{E}_S)$.

□

The next proposition is the main result in this section, proving a restriction theorem for semistable sheaves with respect to a particular set of movable classes. As we shall see later in Section 5, these classes naturally arise in the context of positivity problems for second Chern classes.

Proposition 3.2 (A restriction theorem for movable classes)**.** *Let X be a normal $\mathbb{Q}$-factorial projective threefold that is smooth in codimension two. Let $P \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$ and $H_1, H_2 \in \mathrm{Amp}(X)_{\mathbb{Q}}$. Let $\mathscr{E}$ be a torsion free sheaf on X of rank 3. There exists a positive integer M_1 such that for all sufficiently divisible integers $m_1 \geq M_1$, there is a Zariski open subset $V_{m_1} \subset |m_1 \cdot H_1|$ for which the following properties holds.*

- *(3.2.1) Every member $S \in V_{m_1}$ is smooth, irreducible and that $S \subset X_{\mathrm{reg}}$.*
- *(3.2.2) The restriction $\mathscr{E}|_S$ torsion free.*
- *(3.2.3) The divisor $P|_S$ is nef.*
- *(3.2.4) For every such S, there exists $M_2 \in \mathbb{N}^+$ such that every sufficiently divisible integer $m_2 \geq M_2$ gives rise to a Zariski open subset $V_{m_2} \subset |m_2 \cdot (P + H_2)|_S|$, where every $\gamma \in V_{m_2}$ is a smooth, irreducible curve in S verifying the following property:*
  (*) The formation of the HN-filtration of $\mathscr{E}$ with respect to $(H_1, P + H_2)$ commutes with restriction to γ, i.e. $\mathrm{HN}_\bullet(\mathscr{E})|_\gamma = \mathrm{HN}_\bullet(\mathscr{E}|_\gamma)$.

Proof. Let $\pi : \widetilde{X} \to X$ be the birational morphism and $\widetilde{X}$ the smooth projective variety with ample ample divisor $\widetilde{A} \subset \widetilde{X}$ in Proposition 2.2 associated to the Fujita approximation of the big divisor $P + H_2$, i.e.

$$\pi_*[\widetilde{A}] = [(P + H_2)].$$

Now, let $N_1 \in \mathbb{N}^+$ be a sufficiently large and divisible integer such that for every $n_1 \geq N_1$, there are open subsets $U_{n_1} \subset |n_1 \cdot \pi^* H_1|$ and $\widetilde{U}_{n-1} \subset |n_1 \cdot \widetilde{A}|$, where for every subscheme $\widetilde{S} := \widetilde{D}_{n_1}$ and $\widetilde{C} := \widetilde{D}_{n_1} \cap D_{n_1}$, with $\widetilde{D}_{n_1} \in U_{n_1}$ and $D_{n_1} \in \widetilde{U}_{n_1}$, we have:

- (3.2.5) Both $\widetilde{S}$ and $\widetilde{C}$ are smooth and irreducible.
- (3.2.6) The restrictions $(\pi^{[*]}\mathscr{E})|\widetilde{S}$ is locally free.
- (3.2.7) The HN-filtration of $\pi^{[*]}\mathscr{E}$ with respect to $(H_1, P + H_2)$ verifies: $\mathrm{HN}_\bullet\big((\pi^{[*]}\mathscr{E})|_{\widetilde{S}}\big) = \mathrm{HN}_\bullet(\pi^{[*]}\mathscr{E})|_{\widetilde{S}}$.

The positive integer N_1 exists, thanks to Bertini theorem and Langer's restriction theorem for stable sheaves, cf. [Lan04].

Step. 1. (Reflexivity assumption). By the Bertini theorem and [DG65, Thm. 12.2.1], and as $P \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, there exists a positive integer N_2 such that for every sufficiently divisible $n_2 \geq N_2$ there exists a Zariski open subset $V_{n_2} \subset |n_2 \cdot H_1|$ where every $S \in V_{n_2}$ satisfies the three Properties (3.2.1), (3.2.2) and (3.2.3). We can also ensure that every $S \in V_{n_2}$ is transversal to the exceptional centre of π. Furthermore, as $P|_S$ is nef, we can find $N_3 \in \mathbb{N}^+$ such that for each sufficiently divisible $n_3 \geq N_3$, the general member of $\gamma \in |n_3 \cdot (P + H_2)|_S|$ is smooth and is contained in an open subset of X over which the HN-filtration of $\mathscr{E}$ (with respect to

$(H_1, P + H_2)$) is a filtration of $\mathscr{E}$ by locally-free sheaves. Therefore, to prove that Property (*) is verified by γ, we may assume, without loss of generality, that $\mathscr{E}$ is reflexive.

Step. 2. (Construction of S and γ). Let $m \in \mathbb{N}^+$ be a sufficiently divisible integer verifying the inequality $m_1 \geq M_1 := \max\{N_1, N_2\}$. After shrinking V_{m_1}, if necessary, we have, for every $S \in V_{m_1}$ (defined in Step. 1), that $\widetilde{S} := \pi^*(S) \in U_m$.

Let $M_2 \geq N_1$ be a sufficiently large and divisible integer such that for every $m_2 \geq M_2$ there exists a Zariski open subset $V_{m_2} \subset |m_2(P + H_2)|_S|$, where every curve $\gamma \in V_{m_2}$ is smooth and if $\mathscr{E}|_\gamma$ is not semistable, then $\mathscr{E}_S := \mathscr{E}|_S$ is not semistable with respect to $(P + H_2)|_S$ and that $\mathrm{HN}_\bullet(\mathscr{E}_S)|_\gamma = \mathrm{HN}_\bullet(\mathscr{E}|_\gamma)$. The existence of such M_2 us guaranteed by Mehta-Ramanathan restriction Theorem, cf. [MR82].

Now, to prove the proposition, it suffices to show that if $\mathscr{E}$ is semistable with respect to $(H_1, P + H_2)$, then so is $\mathscr{E}|_\gamma$. So let us now assume that $\mathscr{E}$ is indeed semistable. The next step is devoted to proving that $\mathscr{E}|_\gamma$ is also semistable.

Step. 3. (Extension of maximal destablizing subsheaves). Aiming for a contradiction, assume that $\mathscr{E}|_\gamma$ is not semistable. Then, by our construction in Step. 2, it follows that $\mathscr{E}_S$ is not semistable with respect to $(P + H_2)|_S \equiv (1/m_2) \cdot \gamma$ and that the maximal destablizing subsheaf $\mathscr{F}_S \subset \mathscr{E}_S$ restricts to the one for $\mathscr{E}|_\gamma$. Note that $\mathscr{F}_S$, being saturated inside $\mathscr{E}|_S$, is locally-free.

By applying Lemma 3.1 to $\pi|_{\widetilde{S}} : \widetilde{S} \to S$, with $\widetilde{A}_{\widetilde{S}} := \widetilde{A}|_{\widetilde{S}}$, we find that the maximal destablizing subsheaf $\widetilde{\mathscr{G}}_{\widetilde{S}}$ of $(\pi|_{\widetilde{S}})^*(\mathscr{E}_S)$ or $\bigwedge^2(\pi|_S)^*\mathscr{E}_S$ with respect to $\widetilde{A}_{\widetilde{S}}$ satisfies the isomorphism

$$\big((\pi|_S)_*(\widetilde{\mathscr{G}}_{\widetilde{S}})\big)^{**} \cong \mathscr{L}_S$$

where $\mathscr{L}_S$ is destablizing subsheaf of $\mathscr{E}_S$, or respectively $\bigwedge^2 \mathscr{E}_S$.

By the constructions in Step. 1, it now follows that $\widetilde{\mathscr{G}}_{\widetilde{S}} = \widetilde{\mathscr{G}}|_{\widetilde{S}}$, where $\widetilde{\mathscr{G}}$ is the maximal destablizing subsehaf of $\pi^*(\mathscr{E})$, or respectively $\bigwedge^2 \pi^*(\mathscr{E})$, with respect to $(\pi^*H_1, \widetilde{A})$.

Let $\mathscr{L}$ be the reflexive sheaf on X defined by the extension of the sheaf $(\pi|_{\widetilde{X}\setminus \mathrm{Exc}(\pi)})_*(\widetilde{\mathscr{G}})$ onto X. We have, by the construction of the sheaves $\widetilde{\mathscr{G}}$, $\widetilde{\mathscr{G}}_{\widetilde{S}}$, $\mathscr{L}$, $\mathscr{L}_S$, and the fact that S is transversal to the exceptional centre $Y \subset X$, that

$$\mathscr{L}|_{(S\setminus Y)} \cong (\mathscr{L}_S)|_{(S\setminus Y)}. \tag{3.2.8}$$

As the construction of $\widetilde{\mathscr{G}}$, and hence $\mathscr{L}$, is independent of the choice of S, by shrinking V_{m_1}, if necessary, we can ensure that $\mathscr{L}|_S$ is reflexive. The isomorphism in (3.2.8), together with the fact that $\mathscr{L}|_S$ and $\mathscr{L}_S$ are both reflexive, imply that $\mathscr{L}|_S \cong \mathscr{L}_S$. As $\mathscr{L}_S$ destablizes $\mathscr{E}|_S$, or respectively $\bigwedge^2 \mathscr{E}_S$, with respect to $(P + H_2)$, it follows that $\mathscr{L}$ is a properly destablizing subsheaf with respect to $(H_1, P + H_2)$. On the other hand, thanks to [GKP15, Thm. 4.2], we know that semistable sheaves with respect to movable classes over normal $\mathbb{Q}$-factorial varieties form a tensor category. As a result we get a contradiction to the semistability assumption on $\mathscr{E}$.

□

Remark 3.3 (Restriction of HN-filtration for $\mathbb{Q}$-twisted sheaves). We note that the consequences of Proposition 3.2 are still valid for $\mathbb{Q}$-twisted torsion-free sheaves. More precisely, given a $\mathbb{Q}$-twisted, torsion-free sheaf $\mathscr{E}\langle B\rangle$ and $H_i \in \mathrm{Amp}(X)_{\mathbb{Q}}$, $P \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, there is a complete intersection surface S and $\gamma \subset S$, as in Proposition 3.2, such that $\mathrm{HN}_\bullet(\mathscr{E}\langle B\rangle)|_\gamma = \mathrm{HN}_\bullet(\mathscr{E}\langle B\rangle|_\gamma)$. To see this, let $\mathscr{F}\langle B\rangle$ be a $\mathbb{Q}$-twisted reflexive sheaf, semistable with respect to $(H_1, \ldots, P + H_{n-1})$. Let

$f : Y \to X$ be a finite morphism, adapted to B so that the reflexive pull-back $f^{[*]}(\mathscr{F}\langle B\rangle)$ is a coherent reflexive sheaf on Y. Semistability of $f^{[*]}(\mathscr{F}\langle B\rangle)$ is guaranteed by [HL10, Lem. 3.2.2]. According to Proposition 3.2 the reflexive sheaf $f^{[*]}(\mathscr{E}\langle B\rangle)$ verifies the Restriction Theorem, and therefore so does $\mathscr{E}\langle B\rangle$.

Remark 3.4 (Restriction result in higher dimensions). Following the same arguments as those of the proof of Proposition 3.2, we can remove the restriction on the dimension, that is the consequences of Proposition 3.2 are still valid, if X is of dimension $n \geq 3$ and the polarization is $(H_1, H_2, \ldots, (P + H_{n-1}))$, for any $H_1, \ldots, H_{n-1} \in \mathrm{Amp}(X)_{\mathbb{Q}}$, as long as $\mathrm{rank}(\mathscr{E}) = 3$.

As an immediate consequence we establish a Bogomolov-Gieseker inequality for ($\mathbb{Q}$-twisted) sheaves that are semistable with respect to movable classes of the form that appear in Proposition 3.2. Although we do not use this inequality in the rest of the paper, we find it to be of independent interest.

Proposition 3.5 (Bogomolov-Gieseker inequality in higher dimensions)**.** *Let X be an n-dimensional, normal projective variety that is smooth in codimension two and $\mathscr{E}\langle B\rangle$ a $\mathbb{Q}$-twisted, reflexive sheaf of rank at most equal to 3 on X. If $\mathscr{E}\langle B\rangle$ is semistable with respect to $(H_1, P + H_2)$, where $H_1, H_2 \in \mathrm{Amp}(X)_{\mathbb{Q}}$ and $P \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, then*

$$\big(2r \cdot c_2(\mathscr{E}\langle B\rangle) - (r-1) \cdot c_1^2(\mathscr{E}\langle B\rangle)\big) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0.$$

Proof. This is an immediate consequence of the restriction result in Proposition 3.2 together with Proposition 2.11 (and Remark 3.3).

□

4. Semipostivity of adapted sheaf of forms

In [CP16] Campana and Păun remarkably prove that the orbifold cotangent sheaf of a log-smooth pair (X, D) is semipositive with respect to movable curve classes on X (see Theorem 4.1 below). Currently it is not clear if this result can be easily extended to the case of singular pairs. In the present section we show that, for a special subset of movable classes, the generalization to singular pairs can be achieved by essentially reducing to the smooth case.

Theorem 4.1 (Orbifold semipositivity with respect to movable classes, cf. [CP16, Thm. 1.2])**.** *Given an snc pair (X, D), if $(K_X + D)$ is pseudoeffective, then for any movable class $\gamma \in \mathrm{Mov}_1(X)$ and any adapted morphism $f : Y \to X$, where Y is smooth, the adapted cotangent sheaf $\Omega^1_{(Y,f,D)}$ is semipositive with respect to $f^*(\gamma)$.*

In the next proposition we slightly refine Theorem 4.1 for a class of movable 1-cycles that we call *complete intersection* 1*-cycles*. We say that $\gamma \in \mathrm{Mov}_1(X)_{\mathbb{Q}}$ is a complete intersection 1-cycle, if there are classes $B_1, \ldots, B_{n-1} \in \mathrm{N}^1(X)_{\mathbb{Q}}$ such that γ is numerically equivalent to the cycle defined by $(B_1 \cdot \ldots \cdot B_{n-1}) \in \mathrm{N}_1(X)_{\mathbb{Q}}$. As we will see later in Section 5, such classes appear naturally in our treatment of the pseudoeffectivity of c_2.

Proposition 4.2 (A refinement of the orbifold semipositivity result)**.** *Let (X, D) be an snc pair and $\gamma \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$ a complete intersection movable cycle. If $(K_X + D)$ is pseudoeffective, then for any strictly adapted morphism $g : Z \to X$, the adapted cotangent sheaf $\Omega^{[1]}_{(Z,g,D)}$ is semipositive with respect to $g^*\gamma$.*

Proof. Assume that Z is not smooth, otherwise the claim follows from the arguments of Campana and Păun, cf. [CP16]. Let $D = \sum d_i \cdot D_i$, where D_i are Weil divisors and $d_i \in [0, 1] \cap \mathbb{Q}$. For every D_i, let $g^*(D_i) = n_i \cdot D_{Z,i}$, for some $D_{Z,i} \in \mathrm{Div}(X)$ and $n_i \in \mathbb{N}^+$.

Now, set $f : Y \to X$ to be a strictly adapted morphism, where, thanks to Kawamata's construction, cf. [Laz04, Prop. 4.1.12], the variety Y is smooth. Let W be the irreducible component of the normalization of fibre product $Y \times_X Z$ with the resulting commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ {\scriptstyle u}\downarrow & \overset{h}{\searrow} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X. \end{array}$$

Aiming for a contradiction, assume that $\Omega^{[1]}_{(Z,g,D)}$ is not semipositive with respect to $g^*\gamma$, that is there exists a reflexive subsheaf $\mathscr{G}_Z \subset \Omega^{[1]}_{(Z,g,D)}$ such that

$$\big(\gamma^*(K_X + D) - [\mathscr{G}_Z]\big) \cdot g^*\gamma < 0. \tag{4.2.1}$$

We consider $v^{[*]}(\mathscr{G}_Z) \subset \Omega^{[1]}_{(W,h,D)}$. As γ is, numerically, a complete intersection cycle, we can use the projection formula to conclude that

$$\big(h^*(K_X + D) - [v^{[*]}\mathscr{G}_Z]\big) \cdot h^*\gamma < 0, \tag{4.2.2}$$

which implies that $\Omega^{[1]}_{(W,h,D)}$ is not semipostive with respect to $h^*\gamma$. Now, let $\Omega^{[1]}_{(W,h,D)} \twoheadrightarrow \mathscr{F}_W$ be the torsion free quotient having the minimal slope with the kernel $\mathscr{G}_W$:

$$0 \to \mathscr{G}_W \to \Omega^{[1]}_{(W,h,D)} \to \mathscr{F}_W \to 0. \tag{4.2.3}$$

Let $G := \mathrm{Gal}(W/Y)$. Notice that by the construction of f, we have $\Omega^{[1]}_{(W,h,D)} = u^*(\Omega^1_{(Y,f,d)})$. Now, as the inclusion $\mathscr{G}_W \subset \Omega^1_{(W,h,D)}$ is saturated, and since $\mathscr{G}_W$ is a G-subsheaf (thanks to its uniqueness), according to [HL10, Thm. 4.2.15] or [GKPT15, Prop. 2.16], there exists a reflexive subsheaf $\mathscr{G}_Y \subset \Omega^1_{(Y,f,D)}$ $\mathscr{G}_Y \subset \Omega^1_{(Y,f,D)}$ such that $u^{[*]}(\mathscr{G}_Y) = \mathscr{G}_W$.

Now by taking the G-invariant sections of Sequence 4.2.3 we find

$$0 \to \mathscr{G}_Y \to \Omega^1_{(Y,f,D)} \to \big(u_*(\mathscr{F}_W)\big)^G \to 0. \tag{4.2.4}$$

Again, by using the projection formula we find that $\Omega^1_{(Y,f,D)}$ is not semipositive with respect to $f^*\gamma$, contradicting Theorem 4.1.

□

The next proposition is the extension of Theorem 4.1 to a special class of complete intersection, movable 1-cycles on a mildly singular X.

Proposition 4.3 (Semipositivity for mildly singular pairs)**.** *Let X be a normal projective variety. Let $D = \sum d_i \cdot D_i$, $d_i \in [0,1] \cap \mathbb{Q}$, be an effective $\mathbb{Q}$-divisor such that the pair (X, D), in case D is reduced, is at worst lc, and otherwise is assumed to be klt. Let $H_1 \dots, H_{n-1} \in \mathrm{Amp}(X)_{\mathbb{Q}}$ and $P \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$. If $(K_X + D)$ is pseudoeffective, then for any strictly adapted morphism $f : Y \to X$, the adapted cotangent sheaf $\Omega^{[1]}_{(Y,f,D)}$ is semipositive with respect to $f^*(H_1, \dots, H_{n-2}, P + H_{n-1})$.*

Proof. Notice that as (X, D) has simple normal crossing in codimension two. According to the construction of adapted covers, cf. [Laz04, Prop. 4.1.12] there exists an adapted morphism $f : Y \to X$ (which is not unique) such that Y is smooth in codimension two. Now, let $\pi : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a log-resolution and $\widetilde{Y}$ the main component of the normalization of the fibre product $Y \times_X \widetilde{Y}$ with the commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{X} \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ f\ } & X, \end{array}$$

where $\widetilde{\pi} : \widetilde{Y} \to Y$ and $\widetilde{f} : \widetilde{Y} \to Y$ are the naturally induced projections.

For simplicity, and as the arguments are identical in higher dimensions, we only deal with the case when $\dim X = 3$. Denote $H_{Y,i} = f^*(H_i)$, for $i \in \{1, 2\}$ and $P_Y = f^*(P)$.

Now, aiming for a contradiction, assume that $\Omega^{[1]}_{(Y,f,D)}$ is not semipositive with respect to $(H_{Y,1}, P_Y + H_{Y,2})$. This implies that there exists a saturated subsheaf $\mathscr{G} \subset \mathscr{T}_{(Y,f,D)}$ such that $[\mathscr{G}] \cdot (H_{Y,1}, P_Y + H_{Y,2}) > 0$. Define $\widetilde{\mathscr{H}} := (\widetilde{\pi}^{[*]}\mathscr{H}) \cap \mathscr{T}_{(\widetilde{Y},\widetilde{f},D)}$. Let m be a sufficiently large positive integer such that the 1-cycle $\gamma \in \mathrm{Mov}^1(Y)_{\mathbb{Q}}$ that is numerically equivalent to the cycle defined by $m^2(H_{Y,1}, P_Y + H_{Y,2})$ is away from the exceptional centre of $\widetilde{\pi}$. Existence of such γ in particular guarantees that

$$[\widetilde{\mathscr{H}}] \cdot \widetilde{\pi}^*(H_{Y,1}, P_Y + H_{Y,2}) > 0.$$

In other words there exists a torsion-free quotient sheaf

$$\Omega^{[1]}_{(\widetilde{Y},\widetilde{f},\widetilde{D})} \twoheadrightarrow \widetilde{\mathscr{F}} \tag{4.3.1}$$

on $\widetilde{Y}$ such that $\deg(\widetilde{\mathscr{F}}|_{\widetilde{\gamma}}) < 0$, where $\widetilde{\gamma} := \widetilde{\pi}^{-1}(\gamma)$.
Now, let us consider the logarithmic ramification formula

$$K_{\widetilde{X}} + \widetilde{D} = \pi^*(K_X + D) + \sum a_i \cdot E_i - \sum b_i \cdot E'_i,$$

where $a_i \in \mathbb{Q}^+$, and, thanks to the assumptions on the singularities, $b_i \in (0, 1] \cap \mathbb{Q}$. Define $\widetilde{G} := \sum b_i \cdot E'_i$ and let $\widetilde{h} : Z \to \widetilde{X}$ be the morphism adapted to $(\widetilde{X}, \widetilde{D} + \widetilde{G})$, factoring through $\widetilde{f} : \widetilde{Y} \to \widetilde{X}$:

$$Z \xrightarrow{\ r\ } \widetilde{Y} \xrightarrow{\ \widetilde{f}\ } \widetilde{X}, \qquad \widetilde{h} = \widetilde{f} \circ r .$$

Set $B_Z := \widetilde{h}^*(\pi^*(H_1, P + H_2))$ and $B_{\widetilde{Y}} := \widetilde{f}^*(\pi^*(H_1, P + H_2))$. Now, let $\mathscr{G}_{\widetilde{Y}}$ be the kernel of the sheaf morphism (4.3.1) so that

$$\big(\widetilde{f}^*(K_{\widetilde{X}} + \widetilde{D}) - [\mathscr{G}_{\widetilde{Y}}]\big) \cdot B_{\widetilde{Y}} < 0. \tag{4.3.2}$$

As γ is away from the exceptional centre of $\widetilde{\pi}$ and since $\widetilde{G}$ is supported on the exceptional locus of π, we have

$$\begin{aligned} \widetilde{h}^*(K_{\widetilde{X}} + \widetilde{D} + \widetilde{G}) \cdot B_Z &= \widetilde{h}^*(K_{\widetilde{X}} + \widetilde{D}) \cdot B_Z \\ &= r^*(\widetilde{f}^*(K_{\widetilde{X}} + \widetilde{D})) \cdot B_Z. \end{aligned}$$

As a result, for the inclusion $r^{[*]}(\mathscr{G}_{\widetilde{Y}}) \subset \Omega^{[1]}_{(Z,\widetilde{h},\widetilde{D}+\widetilde{G})}$, we find that

$$\begin{aligned}\Big(\big[\Omega^{[1]}_{(Z,\widetilde{h},\widetilde{D}+\widetilde{G})}\big] - r^{[*]}\mathscr{G}_{\widetilde{Y}}\Big)\cdot B_Z &= \Big(r^*\big(\widetilde{f}^*(K_{\widetilde{X}}+\widetilde{D})\big) - r^{[*]}\mathscr{G}_{\widetilde{Y}}\Big)\cdot B_Z \\ &= (\deg r)\Big(\widetilde{f}^*(K_{\widetilde{X}}+\widetilde{D}) - [\mathscr{G}_{\widetilde{Y}}]\Big)\cdot B_{\widetilde{Y}} \\ &< 0, \qquad\qquad \text{by Inequality 4.3.2,}\end{aligned}$$

contradicting Proposition 4.2.

□

5. Pseudoeffectivity of the orbifold c_2

In [Miy87] Miyaoka famously proved that c_2 of a generically semipositive sheaf with nef determinant is pseudoeffective. Thanks to his result on the semipositivity of cotangent sheaves, Miyaoka then established the pseudoeffectivity of $c_2(X)$ for any minimal model X. Our aim in this section is to generalize this result to the case of pairs (X, D) with movable $(K_X + D)$ (Corollary 5.2) by first extending Miyaoka's result on pseudoeffectivity of c_2 for any semipositive sheaf.

Proposition 5.1 (Pseudoeffectivity of c_2 for semipositive sheaves)**.** *Let X be a normal projective threefold with isolated singularities and $A_1 \in \mathrm{Amp}(X)_{\mathbb{Q}}$. Then, the inequality*

$$c_2(\mathscr{E})\cdot A_1 \geq 0$$

holds for any reflexive sheaf $\mathscr{E}$ of rank r verifying the following properties.

(5.1.1) $[\mathscr{E}] \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$.

(5.1.2) For any $A_2 \in Amp(X)_{\mathbb{Q}}$, the sheaf $\mathscr{E}$ is semipositive with respect to $(A_1, [\mathscr{E}] + A_2)$.

Proof. Let c any any positive integer. Consider the $\mathbb{Q}$-twisted reflexive sheaf $\mathscr{E}\langle\frac{1}{c}\cdot H\rangle$. For the choice of polarization $(A_1, [\mathscr{E}\langle\frac{1}{c}\cdot H\rangle])$, the assumptions of Proposition 3.2 are satisfied, for all c.

Now let S be the complete intersection surface defined in Proposition 3.2 (see also Remark 3.3) so that the restriction $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle := (\mathscr{E}\langle\frac{1}{c}\cdot H\rangle)|_S$ is semipositive with respect to

$$\beta := c_1(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) = ([\mathscr{E}] + \frac{r}{c}\cdot[H_S])|_S.$$

Following the arguments of Miyaoka, we now consider two cases based on the stability of $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$.

First, we consider the case where $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$ is semistable with respect to β. Here, the semipositivity of c_2 follows from Bogomolov-Gieseker inequality for for $\mathbb{Q}$-twisted locally-free sheaves (Proposition 2.11).

So we now assume that $\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle$ is not semistable with respect to β. Let

$$0 \neq \mathscr{E}^1_S\langle\frac{1}{m}\cdot H_S\rangle \subset \ldots \subset \mathscr{E}^t_S\langle\frac{1}{c}\cdot H_S\rangle = \mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle \tag{5.1.3}$$

be the the $\mathbb{Q}$-twisted HN-filtration $\mathscr{E}_S\langle\frac{1}{c}H_S\rangle$. Denote the semistable, torison-free, $\mathbb{Q}$-twisted sheaves

$$\mathscr{E}^i_S\langle\frac{1}{c}\cdot H_S\rangle / \mathscr{E}^{i-1}_S\langle\frac{1}{c}\cdot H_S\rangle$$

of rank r_i by $\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle$ and let $\overline{\mathscr{Q}}^i_S\langle\frac{1}{c}\cdot H_S\rangle$ denote its reflexivization. As the second Chern character $ch_2(\cdot)$ is additive, we have

(5.1.4)

$$2\cdot c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)-c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)=\sum\big(2\cdot c_2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)-c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\big)$$
$$\geq\sum\big(2\cdot c_2(\overline{\mathscr{Q}}^i_S\langle\frac{1}{c}\cdot H_S\rangle)-c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\big),$$

where the last inequality follows from the fact that $c_2(\mathscr{Q}^i_S)\geq c_2(\overline{\mathscr{Q}}^i_S)$. Now, by applying the Bogomolov inequality 2.11 to each semistable, $\mathbb{Q}$-twisted sheaf $\overline{\mathscr{Q}}^i_S\langle\frac{1}{c}\cdot H_S\rangle$ we find that each term in the right-hand side of the inequality (5.1.4) verifies the inequality

$$2\cdot c_2(\overline{\mathscr{Q}}^i_S\langle\frac{1}{c}\cdot H_S\rangle)-c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\geq\frac{-1}{r_i}\cdot c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle).$$

Therefore we have

$$2\cdot c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)-c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)\geq\sum\frac{-1}{r_i}\cdot c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle). \tag{5.1.5}$$

Next, we define the rational number $\alpha_i\in\mathbb{Q}$ by the equality

$$r_i\cdot\alpha_i=\frac{c_1(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\cdot\beta}{c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)}=\frac{c_1(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\cdot\beta}{\beta^2}. \tag{5.1.6}$$

It follows that

$$\sum r_i\cdot\alpha_i=1. \tag{5.1.7}$$

Furthermore, according to the definition of α_i, and by using the fact that the slopes of the quotients of the HN-filtration (5.1.3) is strictly decreasing, we know that

$$\alpha_1>\alpha_2>\ldots>\alpha_t\geq 0, \tag{5.1.8}$$

where the last inequality follows from the semipositivity of $\mathscr{E}_S\langle\frac{1}{c}H_S\rangle$.

Now, as $\alpha_i\geq 0$, for each i, the equality (5.1.7) implies that $\alpha_i\leq 1$. On the other hand, according to the Hodge index theorem we have

$$-c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\geq\frac{\big(c_1(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H_S\rangle)\cdot\beta\big)^2}{\beta^2},$$

so that

$$-c_1^2(\mathscr{Q}^i_S\langle\frac{1}{c}\cdot H\rangle)\geq\beta^2(r_i\cdot\alpha_i)^2.$$

After substituting back into the inequality (5.1.5) we now find that

$$\begin{aligned}2\cdot c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)&\geq\beta^2(1-\sum r_i\cdot\alpha_i^2)\\&\geq\beta^2(1-\alpha_1\sum r_i\cdot\alpha_i)&&\text{by 5.1.8}\\&=\beta^2(1-\alpha_1)&&\text{by 5.1.7}\\&\geq 0&&\text{as }\alpha_1\leq 1.\end{aligned}$$

The inequality $c_2(\mathscr{E}_S)\geq 0$ now follows by taking the limit $c\to\infty$.

□

As an immediate consequence we can now prove the pseudoeffectivity of c_2 for the orbifold cotangent sheaves of pairs (X,D) in dimension 3, whose K_X+D is movable and has only isolated singularities.

Corollary 5.2 (Positivity of c_2 of orbifold cotangent sheaves)**.** *Let X be a normal projective threefold and D an effective $\mathbb{Q}$-divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, then then for any ample divisors $A \subset X$ and strongly adapted morphism $f : Y \to X$, the inequality*

$$c_2(\Omega^{[1]}_{(Y,f,D)}) \cdot f^*(A) \geq 0$$

holds.

Proof. As $[\Omega^{[1]}_{(Y,f,D)}] = f^*(K_X + D)$, the corollary is a direct consequence of Proposition 5.1 together with Proposition 4.3. □

5.1. **Positivity of orbifold c_2 for log-minimal models.** We would like to point out that once we assume that $(K_X + D)$ is nef, then an easy adaptation of the original results of Miyaoka to the case of orbifold Chern classes, together with semipositivity result of [CP14] leads to the following theorem.

Theorem 5.3. *Let X be a projective klt variety of dimension n and $D = \sum(1 - 1/a_i) \cdot D_i$, $a_i \in \mathbb{N}^+ \cup \{\infty\}$, an effective $\mathbb{Q}$-divisor such that (X, D) is lc. If $(K_X + D)$ is nef, then for any strongly adapted morphism $f : Y \to X$, we have*

$$c_2(\Omega^{[1]}_{(Y,f,D)}) \cdot f^*(A^{n-2}) \geq 0,$$

where $A \subset X$ is any ample divisor.

6. An effective non-vanishing result for threefolds

The goal of this section is to prove Theorem 1.4. The main point of our strategy is to devise an effective lower bound for $\chi(K_Y + H)$, when Y is terminal (and (Y, H) is lc).

Proposition 6.1 (Lower bounds for the Euler characteristic of adjoint bundles)**.** *Let X be a terminal projective threefold and D an effective divisor. Then, the inequality*

$$\chi(X, K_X + D + A) \geq (\frac{1}{12}) \cdot (K_X + D + A) \cdot (D + A) \cdot (D + A + \frac{1}{2}K_X). \tag{6.1.1}$$

holds, for any divisor A satisfying the following conditions.

(6.1.2) The divisor $D + A$ is Cartier and nef and, up to integral linear equivalence, effective and reduced.
(6.1.3) The pair $(X, D + A)$ is lc.
(6.1.4) The divisors $(D + A)$ and $(K_X + D + A)$ are Cartier and nef.

Proof. As usual, a key element in the proof is the Hizerbruch-Riemann-Roch for $(K_X + D + A)$:

$$\begin{aligned}\chi(X, K_X + D + A) = \frac{1}{12} \cdot (K_X + D + A) \cdot (D + A) \cdot \big(2(K_X + D + A) - K_X\big) \\ + \frac{1}{12} \cdot c_2(X) \cdot (K_X + D + A) + \chi(X, \mathscr{O}_X).\end{aligned} \tag{6.1.5}$$

Standard Chern class calculations then show that we have the equality

$$c_2(X) = c_2(\Omega^{[1]}_X \log(D + A)) - (K_X + D + A) \cdot (D + A), \tag{6.1.6}$$

as linear forms on $\mathrm{N}^1(X)_{\mathbb{Q}}$. After substituting back into Equality 6.1.5, we find that the equality

$$\chi(X, K_X + D + A) = (K_X + D + A) \cdot \Big\{(D+A) \cdot (K_X + 2(D+A)) + c_2(\Omega_X^{[1]} \log(D+A)) - (K_X + D + A) \cdot (D+A)\Big\} + \chi(X, \mathscr{O}_X)$$

holds, which then simplifies to

$$\chi(X, K_X + D + A) = (K_X + D + A) \cdot \Big\{(D+A)^2 + c_2(\Omega_X^{[1]} \log(D+A))\Big\} + \chi(X, \mathscr{O}_X). \tag{6.1.7}$$

On the other hand, as X is terminal, we know, thanks to [Kaw81, Lem. 2.3] (see also [KM98, Cor. 5.39]), that

$$\chi(X, \mathscr{O}_X) \geq \frac{-1}{24} K_X \cdot c_2(X). \tag{6.1.8}$$

After substituting 6.1.8 in 6.1.6 we find:

$$\begin{aligned}\chi(X, \mathscr{O}_X) &= \big((K_X + D + A) - (D+A)\big) \cdot c_2(\Omega_X^{[1]} \log(D+A)) \\ &\quad + (K_X) \cdot (K_X + D + A) \cdot (D+A) \\ &\geq (K_X + D + A) \cdot \Big\{c_2(\Omega_X^{[1]} \log(D+A)) - (K_X) \cdot (D+A)\Big\},\end{aligned}$$

where we have used the assumption that $(D+A)$ is nef and the pseudoeffectivity of c_2 (Theorem 5.3). Now, substituting back into Equation 6.1.7, we get

$$\chi(X, K_X + D + A) = (K_X + D + A)\Big\{(D+A)^2 + \frac{1}{2}(K_X) \cdot (D+A) + \frac{1}{2} c_2(\Omega_X^{[1]} \log(D+A))\Big\}. \tag{6.1.9}$$

Again, by using Corollary 5.2 and the nefness assumptions on $(K_X + D + A)$ and $(K_X + A)$, we find that

$$\chi(X, K_X + D + A) \geq (K_X + D + A) \cdot (D+A) \cdot (D + A + \frac{1}{2} K_X), \tag{6.1.10}$$

as required.

□

6.1. **Proof of Theorem 1.4.** According to Kawamata-Viehweg vanishing, it suffices to prove that $\chi(Y, K_Y + H) \neq 0$. The pair (Y, H) satisfies the assumptions of Proposition 6.1 with $D = 0$, except for the terminal condition.

Now, let $\pi : X \to Y$ be a terminalization of Y, cf. [KM98, Sect. 6.3]. Set $A := \pi^*(H)$. Since π is small, the adjoint divisor $(K_X + A)$ is nef and big. As a result, the strict positivity of the right-hand side of the inequality (6.1.1) immediately follows: First we rewrite the right-hand side of (6.1.1) as

$$\frac{1}{2} \cdot (K_X + A) \cdot A \cdot ((K_X + A) + A).$$

Now, according to the basepoint freeness theorem for log-canonical threefolds, cf. [Kol92], the divisor $K_X + A$ is semi-ample. Therefore, for sufficiently large integer m, we can find an irreducible surface $S \in |m \cdot (K_X + 2A)|$ such that $(A|_S)$ is big. On the other hand $(K_X + A)|_S$ is nef. It thus follows that $(K_X + A)|_S \cdot A|_S > 0$. □

7. A Miyaoka-Yau inequality in higher dimensions

In [Miy87], Miyaoka generalized the famous inequality $c_1^2 \leq 3c_2$ from surfaces with pseudoeffective canonical divisor to higher dimensional varieties with nef canonical divisor. We extend this result to the case of movable canonical divisor.

Theorem 7.1. *Let X be a normal projective threefold and D an effective $\mathbb{Q}$-divisor such that (X, D) has only isolated lc singularities. If $(K_X + D) \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$, then for any $A \in \mathrm{Amp}(X)_{\mathbb{Q}}$ and for any strongly adapted morphism $f : Y \to X$,*

$$c_1^2(\Omega^{[1]}_{(Y,f,D)}) \cdot f^*A \leq 3c_2(\Omega^{[1]}_{(Y,f,D)}) \cdot f^*A.$$

Proof. Let $\tilde{H} \in \mathrm{Amp}(X)_{\mathbb{Q}}$, $H := f^*\tilde{H}$ and $\mathscr{E} := \Omega^{[1]}_{(Y,f,D)}$. Let c any any positive integer. Consider the $\mathbb{Q}$-twisted reflexive sheaf $\mathscr{E}\langle \frac{1}{c} \cdot H \rangle$. For the choice of polarization $(f^*A, [\mathscr{E}\langle \frac{1}{c} \cdot H \rangle])$, the assumptions of Proposition 3.2 are satisfied, for all c.

Now let S be the complete intersection surface defined in Proposition 3.2 (see also Remark 3.3) so that the restriction $\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle := (\mathscr{E}\langle \frac{1}{c} \cdot H \rangle)|_S$ is semipositive with respect to

$$\beta := ([\mathscr{E}] + \frac{r}{c} \cdot H_S)|_S.$$

Let

$$0 \neq \mathscr{E}^1_S\langle \frac{1}{c} \cdot H_S \rangle \subset \ldots \subset \mathscr{E}^s_S\langle \frac{1}{c} \cdot H_S \rangle = \mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle \tag{7.1.1}$$

be the the $\mathbb{Q}$-twisted HN-filtration of $\mathscr{E}_S\langle \frac{1}{c} H_S \rangle$.

The same arguments as those in the proof of Proposition 5.1 show that

$$(2c_2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - c_1^2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq (\sum \frac{-1}{r_i} c_1^2(\mathscr{Q}^i_S)),$$

where $\mathscr{Q}^i_S\langle \frac{1}{c} \cdot H_S \rangle$ is the torsion free, $\mathbb{Q}$-twisted quotient sheaf of rank r_i of the filtration (7.1.1).

Again, as in the proof of Proposition 5.1, for each i, we define α_i by the equation

$$r_i \cdot \alpha_i = \frac{c_1(\mathscr{Q}^i_S\langle \frac{1}{c} \cdot H_S \rangle) \cdot \beta}{\beta^2}.$$

From the definition of α_i it follows that $\sum r_i \cdot \alpha_i = 1$. Moreover, we have $\alpha_1 > \cdots > \alpha_s \geq 0$, where the last inequality is due to the semipositivity of $\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle$.

We now deduce

$$(6c_2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq$$
$$\left(3(\sum_{i>1} \frac{-1}{r_i} c_1^2(\mathcal{G}_i)) + 6c_2(\mathscr{E}^1_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathscr{E}^1_S\langle \frac{1}{c} \cdot H_S \rangle) + c_1^2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle) \right).$$

And finally,

$$\begin{aligned} &(6c_2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle) - 2c_1^2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle)) \geq \\ &\quad ((1 - 3\sum_{i>1} r_i\alpha_i^2).c_1^2(\mathscr{E}_S\langle \frac{1}{c} \cdot H_S \rangle) + 6c_2(\mathscr{E}^1_S\langle \frac{1}{c} \cdot H_S \rangle) - 3c_1^2(\mathscr{E}^1_S\langle \frac{1}{c} \cdot H_S \rangle)). \end{aligned} \tag{7.1.2}$$

There are three possibilities: $r_1 \geq 3$, $r_1 = 2$ and $r_1 = 1$.

If $r_1 \geq 3$, using Bogomolov-Gieseker inequality and the Hodge index theorem, we obtain

$$(6c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 2c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) \geq$$

$$((1-3\sum_{i>1} r_i\alpha_i^2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 3\frac{1}{r_1}c_1^2(\mathscr{E}_1)) \geq$$

$$(1-3\sum_i r_i\alpha_i^2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) \geq (1-3\alpha_1)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) \geq 0.$$

since $3\alpha_1 \leq r_1\alpha_1 \leq \sum_i r_i\alpha_i = 1$.

If $r_1 = 2$, we choose S general enough so that $\mathscr{E}_S^1$ injects into $\Omega_S(\log f^{-1}\lceil D\rceil_{|S})$.

Using the Bogomolov-Miyaoka-Yau inequality, we have either $\kappa(S, c_1(\mathscr{E}_S^1)) \leq 0$ or $c_1^2(\mathscr{E}_S^1) \leq 3c_2(\mathscr{E}_S^1)$.

In the case $\kappa(S, c_1(\mathscr{E}_S^1)) \leq 0$, since $c_1(\mathscr{E}_S^1).\beta > 0$, we have $c_1^2(\mathscr{E}_S^1) \leq 0$.

Applying Bogomolov-Gieseker inequality to 7.1.2:

$$(6c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 2c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) \geq$$

$$((1-3\sum_{i>1} r_i\alpha_i^2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) \geq$$

$$(1-3\sum_{i>1} r_i\alpha_i^2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) \geq$$

$$(1-3\alpha_2\sum_{i>1} r_i\alpha_i)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) =$$

$$(1-3\alpha_2(1-2\alpha_1))\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) \geq$$

$$(1-3\alpha_1(1-2\alpha_1))\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) =$$

$$\left(6(\alpha_1-\frac{1}{4})^2+\frac{5}{8}\right)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - \frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) \geq -\frac{3}{2}c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)).$$

Finally, we obtain $(3c_2(\mathscr{E}_S) - c_1^2(\mathscr{E}_S)) \geq 0$.

In the case $c_1^2(\mathscr{E}_S^1) \leq 3c_2(\mathscr{E}_S^1)$ we have from 7.1.2:

$$(6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle)) \geq$$

$$((1-3\sum_{i>1} r_i\alpha_i^2)c_1^2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}H_S\rangle) + (6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}H_S\rangle) \geq$$

$$((1-4\alpha_1^2-3\sum_{i>1} r_i\alpha_i^2)c_1^2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle)) + (6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}H_S\rangle) \geq$$

$$((1-4\alpha_1^2-3\alpha_2\sum_{i>1} r_i\alpha_i)c_1^2(\mathscr{E}_S\langle\frac{1}{c}\dot{H}_S\rangle)) + (6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}H_S\rangle) =$$

$$((1-4\alpha_1^2-3\alpha_2(1-2\alpha_1))c_1^2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) + (6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle)) =$$

$$(1-2\alpha_1)(1+2\alpha_1-3\alpha_2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle)) + (6c_2(\mathscr{E}_S\langle\frac{1}{c}H_S\rangle) - 2c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}H_S\rangle).$$

As $3\alpha_2 < r_1\alpha_1 + r_2\alpha_2 \leq 1$, we have

$$(6c_2(\mathscr{E}_S) - 2c_1^2(\mathscr{E}_S)) \geq 0.$$

Finally, if $r_1 = 1$, a classical result of Bogomolov and Sommese (the Bogomolov-Sommese vanishing) implies that $\mathscr{E}_S^1 \subset \Omega_S(\log f^{-1}\lceil\Delta\rceil_{|S})$ has Kodaira dimension at most one. Therefore $c_1^2(\mathscr{E}_S^1) \leq 0$. From 7.1.2, one obtains:

$$
\begin{aligned}
(6c_2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 2c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) \geq \\
((1-3\sum_{i>1} r_i\alpha_i^2)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) - 3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle) \geq \\
((1-3\alpha_1\sum_{i>1} r_i\alpha_i)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) - 3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle) = \\
((1-3\alpha_1(1-\alpha_1))\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle)) - 3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle) \geq \\
\left(1-\frac{3}{2}(1-\frac{1}{2})\right)\cdot c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle) = \\
\frac{1}{4}c_1^2(\mathscr{E}_S\langle\frac{1}{c}\cdot H_S\rangle) - 3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle) \geq \\
-3c_1^2(\mathscr{E}_S^1\langle\frac{1}{c}\cdot H_S\rangle).
\end{aligned}
$$

Therefore, we have

$$(6c_2(\mathscr{E}_S) - 2c_1^2(\mathscr{E}_S)) \geq 0.$$

□

We finish this section by pointing out that when $(K_X + D)$ is nef, the original result of Miyaoka can be adapted to the case of orbifold Chern classes. This can then be combined with the semipositivity result of [CP14] to conclude the following result.

Theorem 7.2. *Let X be a projective klt variety of dimension n and $D = \sum(1-1/a_i)\cdot D_i$, $a_i \in \mathbb{N}^+ \cup \{\infty\}$, an effective $\mathbb{Q}$-divisor such that (X, D) is lc. If $(K_X + D)$ is nef, then for arbitrary ample divisors $H_1, \ldots, H_{n-2}$ and any strongly adapted morphism $f : Y \to X$, we have*

$$c_1^2(\Omega^{[1]}_{(Y,f,D)})\cdot f^*(H_1 \ldots H_{n-2}) \leq 3c_2(\Omega^{[1]}_{(Y,f,D)})\cdot f^*(H_1 \ldots H_{n-2}). \tag{7.2.1}$$

8. Remarks on Lang-Vojta's conjecture in codimension one

A classical conjecture of Lang predicts that a variety of general type X, admits a proper algebraic subvariety that contains all subvarieties of X that are *not* of general type. In this section, we will prove a particular case of this conjecture for codimension one subvarieties satisfying certain conditions: The codimension one subvariety will be assumed to be movable and with only canonical singularities.

First, an immediate application of the inequality (7.1) gives the following theorem.

Theorem 8.1. *Let X be a normal projective $\mathbb{Q}$-factorial threefold such that $K_X \in \mathrm{Mov}^1(X)_{\mathbb{Q}}$. Let H be a nef divisor, D a movable, reduced, irreducible, normal divisor such that (X, D) has only isolated lc singularities. If $-K_D$ is pseudoeffective, then*

$$K_X \cdot D \cdot H \leq (3c_2 - c_1^2)\cdot H. \tag{8.1.1}$$

Proof. From the inequality (7.1), we have $c_1^2(\Omega_X(\log D))\cdot H \leq 3c_2(\Omega_X(\log D))\cdot H$. Therefore, $(K_X + D)^2 \cdot H \leq 3(c_2 + (K_X + D)\cdot D)\cdot H$. It follows that

$$2K_X \cdot D \cdot H \leq (3c_2 - c_1^2)\cdot H + 3(K_X + D)\cdot D \cdot H - D^2 \cdot H.$$

Finally, thanks to the adjunction formula, we get $K_X \cdot D \cdot H \leq (3c_2 - c_1^2) \cdot H + 2K_D \cdot H|_D$. The inequality (8.1.1) now follows from the assumption that $-K_D$ is pseudoeffective. □

Proof of Theorem 1.5. Let H be an ample divisor in X. The divisor K_X is big so we can find a positive integer m such that $(m \cdot K_X - H)$ is linearly equivalent to an effective divisor E.

Let us first prove that the family of polarized varieties $(D, H|_D)$ is bounded. We note that as each D has only rational singularities the theorem of Kollár and Matsusaka [KM83] applies, that is to bound the family $(D, H|_D)$, it suffices to bound the intersection numbers

$$H^2 \cdot D \qquad \text{and} \qquad H \cdot K_D = H \cdot (K_X + D) \cdot D.$$

For $H^2 \cdot D$, we note that, as long as D is not a component of E we can use the inequality (8.1.1), to get

$$0 \leq H^2 \cdot D \leq mH \cdot (3c_2 - c_1^2).$$

For the second term $K_D \cdot H$, we use Theorem 1.2 to find

$$\begin{aligned} 0 &\leq 3c_2(\Omega_X^1(\log D)) \cdot H - c_1^2(\Omega_X^1(\log D)) \cdot H \\ &= (3c_2 - c_1^2) \cdot H + 2(K_X + D) \cdot D \cdot H - K_X \cdot D \cdot H. \end{aligned}$$

We immediately deduce that

$$-\frac{1}{2}(3c_2 - c_1^2) \cdot H \leq H \cdot (K_X + D) \cdot D = H \cdot K_D \leq 0.$$

Therefore, the family of polarized varieties $(D, H|_D)$ is bounded.

It now remains to show the finiteness of the family $(D, D|_H)$. Aiming for a contradiction assume that the family is not finite. As the family is bounded, after going to a smooth model of X, we are reduced to the case of a fibration. The additivity of the Kodaira dimension ("easy additivity") shows that, as $-K_D$ is pseudoeffective, X cannot be of general type; a contradiction. □

Remark 8.2. In [LM97, Thm. 4], in the setting where X is non-uniruled and smooth and D is reduced, the Miyaoka-Yau inequality 7.2 is claimed to be valid. As a consequence a stronger version of Theorem 1.5 is obtained. Unfortunately we have been unable to verify the details of the proof of [LM97, Thm. 4]. The main point of difficulty is that within the proof of this theorem, in [LM97, Subsect. 3.1], the authors claim that given a smooth projective, threefold X of general type with an ample divisor H, for sufficiently large m, there is a general member $S \in |m \cdot H|$ for which the following conditions hold.

(8.2.1) The restriction $(\Omega_X \log(D))|_S$ is semipositive with respect to $(P_\sigma(K_X + D))|_S$, where P_σ is the positive part of the divisorial Zariski decomposition of $K_X + D$.

(8.2.2) The restriction $(P_\sigma(K_X + D))|_S$ of the positive part of $K_X + D$ verifies the equality $P_\sigma(K_X + D)|_S \cdot N((K_X + D)|_S) = 0$, where $N(K_X + D|_S)$ is the negative part of the Zariski decomposition of the pseudoeffective divisor $(K_X + D)|_S$.

Although Item (8.2.1) in the conditions above can most likely be recovered by [CP15, Thm. 2.1] and the arguments in Sections 3 and 4 in the current paper, the second condition (8.2.2) is more problematic as the underlying assumption is that Zariski decomposition is functorial; a condition that in general does not hold.

Remark 8.3. Starting with a general type variety X and a divisor D such that (X, D) is lc, thanks to [BCHM10], it is certainly possible to establish a Miyaoka-Yau inequality using a minimal model of (X, D). More precisely, let $\pi : (X, D) \dashrightarrow (X', D')$ be a LMMP map resulting in the minimal model (X', D'). Let $\widetilde{\pi} : \widetilde{X} \to X'$ be a desingularization of π factoring through $\mu : \widetilde{X} \to X$. Now, one can use the original arguments of Miyaoka, together with those of Megyesi, to show that the inequality

$$\big(3c_2(\Omega_{X'} \log(D') - (K'_X + D')^2)\big) \cdot H^{n-2} \geq 0$$

holds for any ample divisor $H \subset X'$. Furthermore, we can use known results on the behaviour of Chern classes under birational morphisms to show that

$$\big(3c_2(\Omega_{\widetilde{X}} \log(\widetilde{D})) - (K_{\widetilde{X}} + \widetilde{D})^2)\big) \cdot \widetilde{\pi}^*(H)^{n-2} \geq 0. \tag{8.3.1}$$

But the inequality (8.3.1) is hardly independent of the divisor D. In fact in the inequality (8.3.1) even the polarization $(\pi^* H)$ depends on D. Therefore, the inequality (8.3.1) is far from being useful in the context of Lang-Vojta's conjecture.

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