

AX-SCHANUEL FOR SHIMURA VARIETIES

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ABSTRACT. We prove the Ax-Schanuel theorem for a general (pure) Shimura variety.

1. INTRODUCTION

Let Ω be a hermitian bounded symmetric domain corresponding to a semisimple group \mathbf{G} , and let $\Gamma \subset \mathbf{G}(\mathbb{Z})$ be a finite index subgroup. Then $X = \Gamma \backslash \Omega$ has the structure of a quasi-projective algebraic variety. A variety X arising in this way is called a (*connected, pure*) *Shimura variety*. We refer to [4, 5] or [12] for a detailed introduction to Shimura varieties. A Shimura variety X is endowed with a collection of *weakly special subvarieties*. There is a smaller collection of *special subvarieties*, which are precisely the weakly special subvarieties that contain a *special point*. For a description of these see e.g. [10].

Let $q : \Omega \rightarrow X$ be the natural projection map, and let $D \subset \Omega \times X$ be the graph of q . Recall that Ω sits naturally as an open subset in its *compact dual* $\widehat{\Omega}$, which has the structure of a projective variety. By an *algebraic subvariety* $W \subset \Omega \times X$ we mean a (complex analytically) irreducible component of $\widehat{W} \cap (\Omega \times X)$ for some algebraic subvariety $\widehat{W} \subset \widehat{\Omega} \times X$. In the sequel, $\dim U$ denotes the complex dimension of a complex analytic set. Though at some points we will refer implicitly to sets in real Euclidean space, any reference to real dimension will be specifically noted.

Our result is the following.

Theorem 1.1. *With notation as above, let $W \subset \Omega \times X$ be an algebraic subvariety. Let U be an irreducible component of $W \cap D$ whose dimension is larger than expected, that is,*

$$(*) \quad \text{codim } U < \text{codim } W + \text{codim } D.$$

the codimensions being in $\Omega \times X$ or, equivalently,

$$(**) \quad \dim W < \dim U + \dim X.$$

Then the projection of U to X is contained in a proper weakly special subvariety of X .

If one takes $q : \Omega \rightarrow X$ to be the map $\exp : \mathbb{C}^n \rightarrow (\mathbb{C}^\times)^n$, namely the cartesian power of the complex exponential, then the statement is an equivalent form of the Ax-Schanuel theorem of Ax [2]. In this form it is

given a new proof in [25]. Note however that $(\mathbb{C}^\times)^n$ is a “mixed” Shimura variety but not a “pure” one, so this case is not formally covered by the above theorem.

One expects equality in $(*)$ and $(**)$ above, on dimensional considerations, and such a component U always has dimension *at least* $\dim W - \dim X$ (see e.g. Łojasiewicz [11], III.4.6). Thus the theorem asserts that all components of such intersections which are atypical in dimension are accounted for by weakly special subvarieties.

Since weakly special varieties are “bi-algebraic” in the sense of [10], they do indeed give rise to atypical intersections. For example, in the extreme case that $W = \Omega_1 \times X_1$ where X_1 is a weakly special subvariety of X and Ω_1 is a connected component of the preimage of X_1 in Ω under the uniformization map, we get $\dim W = \dim U + \dim X_1 < \dim U + \dim X$.

As in earlier papers [25, 10, 22], the proof combines arguments from complex geometry (Hwang-To), the geometry/group theory underlying Shimura varieties, o-minimality, and monodromy (Deligne-André). The ingredients from o-minimality include the counting theorem of Pila-Wilkie, and results of Peterzil-Starchenko giving powerful “definable” versions of the classical theorems of Remmert-Stein, and Chow. Throughout the paper, we use ‘definable’ to mean ‘definable in the o-minimal structure $\mathbb{R}_{\text{an},\exp}$ '; see [7] and [6], where the o-minimality of $\mathbb{R}_{\text{an},\exp}$ is established, building on [26].

The crucial new ingredient in this paper is the observation of additional algebraicity properties, and for these we make further essential use of the above mentioned results of Peterzil-Starchenko. However, there is also a purely complex analytic approach, which we allude to below.

We expect that Theorem 1.1, which is sometimes called the “hyperbolic Ax-Schanuel conjecture” [3], will have applications to problems within the Zilber-Pink conjecture, where it can play a role analogous to that of the “Ax-Lindemann theorem”, which it generalizes, in proving cases of the André-Oort conjecture, see e.g. [21]. One such application is in [3].

2. PRELIMINARIES

We gather some preliminary remarks, definitions, and results.

2.1. Shimura varieties

According to the definition given, a Shimura variety X may not be smooth, and the covering $q : \Omega \rightarrow X$ may be ramified, if Γ contains elliptic elements. For example, $j : \mathbb{H} \rightarrow \mathbb{C}$ is ramified at $\text{SL}_2(\mathbb{Z})i$ and $\text{SL}_2(\mathbb{Z})\rho$, where $\rho = \exp(2\pi i/3)$, but in this case \mathbb{C} is smooth.

By passing to a finite index subgroup we may always assume that the uniformization is unramified and the Shimura variety is smooth, and hence a complex manifold. This does not affect the validity of 1.1. Hence we may and do assume throughout that X is smooth and that $q : \Omega \rightarrow X$ is unramified.

2.2 Definability

The definition and basic results on o-minimal structures over a real closed field may be found in [18]. As already mentioned, ‘definable’ will mean ‘definable in the o-minimal structure $\mathbb{R}_{an,exp}$ ’.

Let \mathcal{F} be the classical Siegel domain for the action of Γ on Ω . Then the uniformization $q : \Omega \rightarrow X$ restricted to \mathcal{F} is *definable*. For a general Shimura variety this result is due to Klingler-Ullmo-Yafaev [10], generalizing results of Peterzil-Starchenko for moduli spaces of abelian varieties [19].

We shall need the following results, which can be seen as definable generalizations of GAGA-type theorems.

Theorem 2.1. [*“Definable Remmert-Stein”*, [18], Theorem 65.3] *Let M be a definable complex manifold and E a definable complex analytic subset of M . If A is a definable, complex analytic subset of $M - E$ then its closure \overline{A} is a complex analytic subset of M .* \square

The following is a slight generalization of a theorem stated by Peterzil-Starchenko [18], Theorem 4.5, which may be proved by combining their statement with “Definable Remmert-Stein” above. This strengthening has also been observed by Scanlon [24], Theorem 2.11, and, in a slightly less general form, in [20].

Theorem 2.2. [*“Definable Chow”*] *Let Y be a quasiprojective algebraic variety, and let $A \subset Y$ be definable, complex analytic and closed in Y . Then A is algebraic.*

Proof. We follow the proof in [20]. By taking an affine open set in Y , it suffices to consider the case where Y is an affine subset of projective space. Then Y is a definable, complex analytic subset of $M \setminus E$ where M is a projective variety and E is a closed algebraic subset of M . Then, by “Definable Remmert-Stein”, above, the closure of A in M is a definable, complex analytic subset of M , hence complex analytic in the ambient projective space. Thus A must be algebraic by Chow’s theorem, or by the Peterzil-Starchenko version [18], Theorem 4.5. \square

3. SOME ALGEBRAICITY RESULTS

We have the uniformization

$$q : \Omega \rightarrow X$$

in which X is a quasi-projective variety, the map q is complex analytic and surjective. It is further invariant under the action of some discrete group Γ of holomorphic automorphisms of Ω , and as noted above the restriction of q to a suitable fundamental domain \mathcal{F} for this action is definable.

Suppose that $V \subset X$ is a relatively closed algebraic subvariety. Then $q^{-1}(V) \subset \Omega$ is a closed complex analytic set which is Γ -invariant, and definable on a fundamental domain \mathcal{F} . The same statement holds for the uniformization

$$q \times \text{id} : \Omega \times X \rightarrow X \times X$$

and $V \subset X \times X$, which is invariant under $\Gamma \times \{\text{id}\}$, where now $q \times \text{id}$ is definable on $\mathcal{F} \times X$. We observe that the converse holds.

Theorem 3.1. *Let $A \subset \Omega \times X$ be a closed, complex analytic set which is $\Gamma \times \{\text{id}\}$ -invariant, and such that $A \cap \mathcal{F} \times X$ is definable. Then $(q \times \text{id})(A) \subset X \times X$ is a closed algebraic subset.*

Proof. The image $(q \times \text{id})(A)$ is closed and complex analytic in $X \times X$. Since $(q \times \text{id})(A) = (q \times \text{id})(A \cap \mathcal{F} \times X)$ it is also definable, and so it is algebraic by ‘‘Definable Chow’’ (2.2). \square

3.1. Descending Hilbert scheme loci. Now we fix some algebraic subvariety $W \subset \Omega \times X$, with $\widehat{W} \subset \widehat{\Omega} \times X$ its Zariski closure, and U an irreducible component of $W \cap D$. We make no assumptions here on the dimension of U . By the *Hilbert polynomial* $P_W(\nu)$ of W we mean the Hilbert polynomial of \widehat{W} .

Let M be the Hilbert scheme of all subvarieties of $\widehat{\Omega} \times X$ with Hilbert polynomial P . Then M also has the structure of an algebraic variety. Corresponding to $y \in M$ we have the subvariety $W_y \subset \Omega \times X$, and we have the incidence variety (universal family)

$$B = \{(z, x, y) \in \Omega \times X \times M : (z, x) \in W_y\},$$

and the family of the intersections of its fibres over M with D , namely

$$A = \{(z, x, y) \in \Omega \times X \times M : (z, x) \in W_y \cap D\}.$$

Then A is a closed complex analytic subset of $\Omega \times X \times M$. It has natural projection $\theta : A \rightarrow M$, with $(z, x, y) \mapsto y$. Then, for each natural number k , the set

$$A(k) = \{(z, x, y) \in A : \dim_{(z, x)} \theta^{-1}\theta(z, x, y) \geq k\},$$

the dimension being the dimension at (z, x) of the fibre of the projection in A , is closed and complex analytic see e.g. the proof of [16], Lemma 8.2, and references there.

Now we have the projection $\psi : \Omega \times X \times M \rightarrow \Omega \times X$, and consider

$$Z = Z(k) = \psi(A(k)).$$

Then as M is compact, ψ is proper and so Z is closed in $\Omega \times X$. Note that Z is Γ -invariant and $Z \cap (\mathcal{F} \times X)$ is definable.

Lemma 3.2. *Let $T = (q \times \text{id})(Z)$. Then $T \subset X \times X$ is closed and algebraic.*

Proof. Since Z is Γ -invariant and $Z \cap \mathcal{F} \times X$ is definable, this follows as in Theorem 3.1. \square

Remark. One may also prove Lemma 3.2 by more geometric methods along the lines of the argument in [13], which uses the method of compactification of complete Kähler manifolds of finite volume of [14] based on L^2 -estimates of $\overline{\partial}$.

4. PROOF OF THEOREM 1.1

Proof. We argue by induction, in the first instance (upward) on $\dim \Omega$. For a given $\dim \Omega$, we argue (upward) on $\dim W - \dim U$. Finally, we argue by induction (downward) on $\dim U$.

Suppose then for purposes of contradiction that the projection of U is not contained in any proper weakly special subvariety but that we have $\dim U > \dim W - \dim X$.

We carry out the constructions of §3.1 with $k = \dim U$ and keep the notation there. We let $A(k)' \subset A(k)$ be the irreducible component which contains $U \times [W]$, and $Z' = \psi(A'(k)) \subset Z$ be the corresponding irreducible component of Z , and $V = (q \times \text{id})(Z')$ the irreducible component of T , which is therefore algebraic by Lemma 3.2. Now, by assumption V contains $(q \times \text{id})(U)$, and so it is not contained in any proper weakly special of the diagonal Δ_X , and thus has Zariski-dense monodromy by André-Deligne [1].

Consider the family F_0 of algebraic varieties corresponding to $A(k)'$. Let $\Gamma_0 \subset \Gamma$ be the subgroup of elements γ such that any element of F_0 is invariant under γ . For any $\mu \in \Gamma$ define $E_\mu \subset F_0$ to be the subset corresponding to algebraic subvarieties invariant under μ . Then, for $\mu \in \Gamma - \Gamma_0$, $E_\mu \subsetneq F_0$ is an algebraic subvariety. Hence, a very general element¹ W' of F_0 is invariant under exactly the subset Γ_0 of Γ . Let Θ be the connected component of the Zariski closure of Γ_0 in $\mathbf{G}(\mathbb{R})$.

Lemma 4.1. Θ is a normal subgroup of $\mathbf{G}(\mathbb{R})$.

Proof. Note that there is an action of Γ on $A(k)$ given by $\gamma \cdot (z, x, [W]) = (\gamma z, x, [\gamma W])$, and the map $A(k) \rightarrow Z$ is equivariant with respect to this action. Since $A(k) \rightarrow Z$ is proper and the action of Γ is discrete, it follows that $\Gamma \backslash A(k) \rightarrow \Gamma \backslash Z \cong T$ is a proper map of analytic varieties.

Let $\Gamma \backslash A(k)'$ denote the image of $A(k)'$ in $\Gamma \backslash A(k)$. Then $\phi : \Gamma \backslash A(k)' \rightarrow V$ is a proper map of analytic varieties, and thus the fibers of ϕ have finitely many components. Let Γ_1 be the image of $\pi_1(\Gamma \backslash A(k)') \rightarrow \pi_1(V) \rightarrow \Gamma$. Then $\pi_1(A(k)')$ has finite index image in the monodromy group of V , and thus Γ_1 is Zariski-dense in $\mathbf{G}(\mathbb{R})$ by André-Deligne [1].

It is clear that F_0 is invariant under Γ_1 .

Now, letting $\text{stab}(W')$ denote the stabilizer of W' , we have $\text{stab}(\gamma \cdot W') = \gamma \cdot \text{stab}(W') \cdot \gamma^{-1}$. It follows that Γ_0 , and hence also Θ is invariant under conjugation by Γ_1 , and hence under its Zariski closure, which is all of $\mathbf{G}(\mathbb{R})$. This completes the proof. \square

¹in the sense of being the complement of countably many proper subvarieties

Lemma 4.2. Θ is the identity subgroup.

Proof. We argue by contradiction. Without loss of generality we may assume that W is a very general member of F_0 , and hence is invariant by exactly Θ . Since Θ is a \mathbb{Q} -group by construction, it follows that \mathbf{G} is isogenous to $\Theta \times \Theta'$ and we have a splitting $\Omega = \Omega_\Theta \times \Omega_{\Theta'}$ of hermitian symmetric domains. Replacing Γ by a finite index subgroup we also have a splitting $X \cong X_\Theta \times X_{\Theta'}$.

As W is invariant under Θ it is of the form $\Omega_\Theta \times W_1$ where $W_1 \subset \Omega_{\Theta'} \times X_{\Theta'} \times X_\Theta$. Moreover, D splits as $D_\Theta \times D_{\Theta'}$. Let U_1 be the projection of U to W_1 . Since the map from D_Θ to X_Θ has discrete pre-images, it is easy to see that $\dim U = \dim U_1$.

Now let W' be the projection of W_1 to $\Omega_{\Theta'} \times X_{\Theta'}$. Then letting U' be a component of $W' \cap D_{\Theta'}$ we easily see that U' is the projection of U to $\Omega_{\Theta'} \times X_{\Theta'}$. Now let W'' be the Zariski closure of U' . It follows by induction that

$$\dim U' + \dim X_{\Theta'} \leq \dim W''.$$

Now for the projection map $\psi : W_1 \rightarrow W'$, the generic fiber dimension of W'' is the same as the generic fiber dimension over U' , and thus

$$\dim U_1 + \dim X_{\Theta'} \leq \dim \psi^{-1}(W'') \leq \dim W_1,$$

from which it follows that

$$\dim U + \dim X \leq \dim W$$

as desired. \square

It follows that W is not invariant by any infinite subgroup of Γ . The following lemma thus reaches a contradiction, and completes the proof.

Lemma 4.3. W is invariant by an infinite subgroup of Γ .

Proof. As before, let \mathcal{F} be a definable fundamental domain for π , and consider the definable set

$$I = \{\gamma \in \mathbf{G}(\mathbb{R}) \mid \dim_{\mathbb{R}} ((\gamma \cdot W) \cap D \cap (\mathcal{F} \times X)) = \dim_{\mathbb{R}} U\}.$$

Clearly, I contains $\gamma \in \Gamma$ whenever U intersects $\gamma \mathcal{F} \times X$. Thus, as in [22] by the work of Hwang–To [9, Theorem 2] and the volume estimate of Klingler–Ullmo–Yafaev [10, Lemma 5.3] I contains a polynomial number of integer matrices². It follows by the Pila–Wilkie theorem [23] that I contains irreducible real algebraic curves C containing arbitrarily many integer points – in particular, containing at least 2 integer points.

If W_c is constant in c , then W is stable under $C \cdot C^{-1}$. Since C contains at least 2 integer points, it follows that W is stabilized by a non-identity integer point, completing the proof (since Γ is torsion free). So we assume that W_c varies with $c \in C$. Note that since C contains an integer point that

² Ordered by height, there are at least T^δ integer points of height at most T for some fixed $\delta > 0$ and arbitrarily large T .

$q(W_c \cap D)$ is not contained in a weakly special subvariety for at least one $c \in C$, and thus for all but a countable subset of C (since there are only countably many families of weakly special subvarieties).

We now have 2 cases to consider. First, suppose that $U \subset W_c$ for $c \in C$. Then we may replace W by $W_c \cap W_{c'}$ for a generic $c, c' \in C$ and lower $\dim W$, contradicting our induction hypothesis on $\dim W - \dim U$.

On the other hand, if it is not true that $U \subset W_c$ for $c \in C$ then $W_c \cap D$ varies with c , and so we may set W' to be the Zariski closure of $C \cdot W$. This increases the dimension of W by 1, but then $\dim W' \cap D = \dim U + 1$ as well, and thus we again contradict our induction hypothesis, this time on $\dim U$. This completes the proof. \square

This contradiction completes the proof of Theorem 1.1. \square

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REFERENCES

- [1] Y. André, Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part, *Compositio* **82** (1992), 1–24.
- [2] J. Ax, On Schanuel’s conjecture, *Ann. of Math.* **93** (1971), 252–268.
- [3] C. Daw and J. Ren, Applications of the hyperbolic Ax-Schanuel conjecture, arXiv:1703.08967v3.
- [4] P. Deligne, Travaux de Shimura, in *Séminaire Bourbaki, 23ème Année* (1970/71), Exp. 389, *Lecture Notes in Math.* **244**, pp 123–165, Springer-Verlag, New York, 1966.
- [5] P. Deligne, Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, *Automorphic forms, representations and L-functions*, Part 2, 247–289, Proc. Symp. Pure Math. XXXiii, AMS, Providence, 1977.
- [6] L. van den Dries, A. Macintyre, D. Marker, The elementary theory of restricted analytic fields with exponentiation, *Annals* **140** (1994), 183–205.
- [7] L. van den Dries and C. Miller, On the real exponential field with restricted analytic functions, *Israel J. Math.* **85** (1994), 19–56.
- [8] L. van den Dries and C. Miller, Geometric categories and o-minimal structures, *Duke Math. J.* **84** (1996), 497–540.
- [9] J. Hwang and W. To, Volumes of complex analytic subvarieties of Hermitian symmetric spaces, *American Journal of Mathematics* **124** (2002), 1221–1246.
- [10] B. Klingler, E. Ullmo, and A. Yafaev, The hyperbolic Ax-Lindemann-Weierstrass conjecture *Publ. Math. IHES* **123** (2016), 333–360.
- [11] S. Łojasiewicz, *Introduction to complex analytic geometry*, Birkhäuser, Basel, 1991.
- [12] J. Milne, *Introduction to Shimura varieties*, available from www.jmilne.org, 2004.
- [13] N. Mok, Zariski closures of images of algebraic subsets under the uniformization map on finite-volume quotients of the complex unit ball. Preprint 2017, available in <http://hkumath.hku.hk/~imr/IMRPreprintSeries/2017/IMR2017-2.pdf>

- [14] N. Mok and J.-Q. Zhong, Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature, *Ann. of Math.* **129** (1989), 427–470.
- [15] Y. Peterzil and S. Starchenko, Uniform definability of the Weierstrass \wp functions and generalized tori of dimension one, *Selecta N. S.* **10** (2004), 525–550.
- [16] Y. Peterzil and S. Starchenko, Complex analytic geometry in a non-standard setting, *Model Theory with Applications to Algebra and Analysis*, pp 117–166, Z. Chatzidakis, A. Pillay, and A. Wilkie, editors, LMS Lecture Note Series **349**, CUP, 2008.
- [17] Y. Peterzil and S. Starchenko, Complex analytic geometry and analytic-geometric categories, *Crelle* **626** (2009), 39–74. [
- [18] Y. Peterzil and S. Starchenko, Tame complex analysis and o-minimality, *Proceedings of the ICM*, Hyderabad 2010.
- [19] Y. Peterzil and S. Starchenko, Definability of restricted theta functions and families of abelian varieties, *Duke Math. J.* **162** (2013), 731–765.
- [20] J. Pila and J. Tsimerman, The André-Oort conjecture for the moduli space of abelian surfaces, *Compositio* **149** (2013), 204–214.
- [21] J. Pila and J. Tsimerman, Ax-Lindemann for \mathcal{A}_g , *Ann. of Math.* **179** (2014), 659–681.
- [22] J. Pila and J. Tsimerman, Ax-Schanuel for the j -function, *Duke Math. J.* **165** (2016), 2587–2605.
- [23] J. Pila and A. J. Wilkie, The rational points of a definable set, *DMJ* **133** (2006), 591–616.
- [24] T. Scanlon, Algebraic differential equations from covering maps, preprint, arXiv:1408.5177.
- [25] J. Tsimerman, Ax-Schanuel and o-minimality, *O-minimality and diophantine geometry*, 216–221, LMS Lecture Note Series **421**, G. Jones and A. J. Wilkie, editors, CUP, 2015.
- [26] A. J. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function, *J. Amer. M. Soc.* **9** (1996), 1051–1094.

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