ON THE HYPERBOLICITY OF SURFACES OF GENERAL TYPE WITH SMALL c_1^2

XAVIER ROULLEAU, ERWAN ROUSSEAU

ABSTRACT. Surfaces of general type with positive second Segre number $s_2 := c_1^2 - c_2 > 0$ are known by results of Bogomolov to be algebraically quasi-hyperbolic i.e. with finitely many rational and elliptic curves. These results were extended by McQuillan in his proof of the Green-Griffiths conjecture for entire curves on such surfaces. In this work, we study hyperbolic properties of minimal surfaces of general type with minimal c_1^2, known as Horikawa surfaces. In principle these surfaces should be the most difficult case for the above conjecture as illustrate the quintic surfaces in $\mathbb{P}^3$. Using orbifold techniques, we exhibit infinitely many irreducible components of the moduli of Horikawa surfaces whose very generic member has no rational curves or even is algebraically hyperbolic. Moreover, we construct explicit examples of algebraically hyperbolic and quasi-hyperbolic orbifold Horikawa surfaces.

1. INTRODUCTION

Our motivations are the following conjectures of Green-Griffiths and Lang:

Conjecture 1 (Lang [23]). *Let X be a complex variety of general type. Then a proper Zariski closed subset Z of X contains all its subvarieties not of general type. In particular, X has only a finite number of codimension-one subvarieties not of general type.*

Even in the case of surfaces, this conjecture is still open, except in some specific cases such as surfaces of general type with irregularity at least two [25]. It has attracted a lot of attention because of its important conjectural links with arithmetic: according to the Bombieri-Lang conjecture, the rational points on a variety of general type defined over a number field should not be Zariski dense.

A surface satisfying Conjecture 1, i.e. with finitely many rational and elliptic curves, is said to be *algebraically quasi-hyperbolic*. Following the terminology of Demailly [12], a projective variety $X \subset \mathbb{P}^n$ is called *algebraically hyperbolic* if there exists a positive real number ε such that

$$2g(C) - 2 \geq \varepsilon \deg C$$

for each reduced irreducible curve $C \subset X$, where $g(C)$ and $\deg C$ are the geometric genus and the degree of the curve $C \hookrightarrow X$ respectively.

The analytic version of Conjecture 1 is:

Conjecture 2 (Green-Griffiths [17], Lang [23]). *Let X be a variety of general type. Then a proper Zariski closed subset of X contains all images of entire curves $f : \mathbb{C} \to X$.*

A projective variety satisfying Conjecture 2 is said to be *quasi-hyperbolic* and a variety that contains no non-constant entire curve is said to be *hyperbolic*.

Let us recall that minimal surfaces of general type X are classified according to their Chern numbers. These Chern numbers c_1^2, c_2 satisfy the two famous inequalities of Noether and Bogomolov-Miyaoka-Yau (see for example [1]):

$$\frac{1}{5}(c_2 - 36) \le c_1^2 \le 3c_2.$$

If $c_1^2 = 3c_2$ then, by Yau [41], X is a quotient of the unit ball. Therefore X is hyperbolic.

A striking overhang toward Conjecture 1 is its proof by Bogomolov for surfaces whose Segre number $s_2 = c_1^2 - c_2$ is positive [3]. The key point in Bogomolov's proof is that for surfaces with positive Segre number, the sufficiently high symmetric differentials have global sections.

An extension of Bogomolov's result to its analytic version was obtained by Lu and Yau [26] proving conjecture 2 for surfaces with $c_1^2 > 2c_2$.

Then McQuillan [28] proved Conjecture 2 for surfaces with positive second Segre number.

For surfaces with $s_2 \le 0$ there is no such a good result. Demailly and El Goul [13] proved Conjecture 2 for some surfaces with $13c_1^2 > 9c_2$.

In the extreme case, a surface that reaches the equality $c_2 = 5c_1^2 + 36$ if c_1^2 is even and $c_2 = 5c_1^2 + 30$ otherwise is called a Horikawa surface. Thus the Horikawa surfaces are the one for which the Segre number $s_2 = c_1^2 - c_2$ is the most negative as possible in the geography of (minimal) surfaces of general type.

As the previous list of results suggests, one is naturally lead to believe that it is for these surfaces that the above conjectures will be the most difficult to establish.

Even for hypersurfaces in $\mathbb{P}^3$, (which have $s_2 < 0$), we are far from complete results. The case of quintics ($c_1^2 = 5, c_2 = 55$) is particularly difficult to treat and this may be explained by the fact that they are Horikawa surfaces, thus with minimal second Segre number. Indeed, a conjecture by Kobayashi predicts that generic quintics should be hyperbolic. We know by Geng Xu [40] that a very generic surface of degree $d \ge 5$ in $\mathbb{P}^3$ contains no curve of geometric genus 0 or 1. But we still do not know that they are hyperbolic, even algebraically hyperbolic.

Moreover, we have very few examples of hyperbolic surfaces of low degree. A striking fact is that no example of hyperbolic surface is known for $d = 5$.

In this paper we investigate the hyperbolic properties of Horikawa surfaces using their very interesting geometric properties. Indeed, we know that most Horikawa surfaces realizing the equality $c_2 = 5c_1^2 + 36$ appear as ramified coverings [19]. Therefore it is very natural to associate to such surface an orbifold. Our philosophy is to use this orbifold to study the geometry of curves in Horikawa surfaces.

We will see that "orbifold" techniques as systematically introduced by Campana ([7], [8], see also [33]) are useful in this context.

Let $N \geq 0$ be an integer and let $\mathbb{F}_N$ be the Nth Hirzebruch surface. The surface $\mathbb{F}_N$ has a natural fibration $\mathbb{F}_N \to \mathbb{P}^1$. We denote by F a fiber, and by T the section of this fibration such that $T^2 = N$. Any divisor D on $\mathbb{F}_N$ is numerically equivalent to $aT + bF$ for a, b integers. We denote by (a, b) the equivalence class of D. The holomorphic Euler characteristic of a surface is $\chi := \frac{c_1^2 + c_2}{12}$. Horikawa obtained the following classification [19]: a Horikawa surface with even c_1^2 is either

(1) a double covering of $\mathbb{P}^2$ branched along an octic ($\chi = 4$),
(2) a double covering of $\mathbb{P}^2$ branched along a curve of degree 10 ($\chi = 7$),
(3) a double covering of $\mathbb{F}_N$ branched along a curve of type $(6, 2a)$, $(2a \geq -N)$ ($\chi = 3N + 2a - 1$).

Here the branch curve has at most ADE singularities. We show:

Theorem 3.   (1) *Let X be a very generic Horikawa surface of type* (1). *Then X has no rational curves.*

(2) *Let X be a very generic Horikawa surface of type* (2). *Then X is algebraically hyperbolic, in particular X has no rational or elliptic curves.*

(3) *Let X be a very generic Horikawa surface of type* (3) *with $a = 3$ and $\chi = 3N + 5$. Then X has no rational curves.*

Here, the terminology "very generic" is used to indicate that the property is satisfied outside a countable union of algebraic subsets in the moduli space. This result is obtained as a consequence of more general results on the algebraic hyperbolicity of branched covers which may be of independent interest.

One of the most natural examples of surfaces with $s_2 < 0$ are hypersurfaces in $\mathbb{P}^3$. In [5], Bogomolov and De Oliveira studied the hyperbolicity of hypersurfaces of degree $d > 5$ with sufficiently many nodes. Translated into the langue of orbifold, their key observation is that for such a surface X, the natural structure of orbifold $\mathcal{X}$ has positive orbifold Segre number $s_2(\mathcal{X}) > 0$. We investigate hyperbolic properties of the quintic surfaces that are degree 5 cyclic covers of $\mathbb{P}^2$ branched over 5 lines in general position. In this study, we develop the theory of jet differentials in the orbifold setting and, as an application, prove

Theorem 4. *Let X be a quintic surface that is a degree 5 cyclic cover of $\mathbb{P}^2$ branched over 5 lines in general position. Then any entire orbifold curve $f : \mathbb{C} \to X$ satisfies a differential equation of order 2.*

Using Nevanlinna theory and constructions of Persson [32] of Horikawa surfaces with maximal Picard number, we exhibit some explicit examples of quasi-hyperbolic Horikawa orbifold surfaces.

Theorem 5. *For χ equal to 4 and $2k - 1$ (for any integer $k > 2$), there exists quasi-hyperbolic orbifold Horikawa surfaces whose minimal resolutions have Euler characteristic χ.*

To finish this introduction, let us remark that in characteristic $p > 0$ the analog of the Lang conjecture is false. In [24], unirational Horikawa surfaces are constructed over fields of positive characteristic, these unirational surfaces have maximal Picard number.

The paper is structured as follows. In section 2 we recall the definition of orbifolds and the classical results we will use. In section 3 we prove our main Theorem 3 and some further results on surfaces that are cyclic ramified covers of the plane. In section 4 we prove Theorem 4. Section 5 is devoted to the construction of quasi-hyperbolic Horikawa orbifold surfaces.

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2. Orbifold set-up

2.1. **Orbifolds.** As in [16], we define orbifolds as a particular type of log pairs. The data (X, Δ) is a log pair if X is a normal algebraic variety (or a normal complex space) and $\Delta = \sum_i d_i D_i$ is an effective $\mathbb{Q}$-divisor where the D_i are distinct, irreducible divisors and $d_i \in \mathbb{Q}$.

For orbifolds, we need to consider only pairs (X, Δ) such that Δ has the form

$$\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) D_i,$$

where the D_i are prime divisors and $m_i \in \mathbb{N}$.

Definition 6. An *orbifold chart* on X compatible with Δ is a Galois covering $\varphi : U \to \phi(U) \subset X$ such that
   (1) U is a domain in $\mathbb{C}^n$ and $\varphi(U)$ is open in X,
   (2) the branch locus of φ is $\lceil \Delta \rceil \cap \varphi(U)$,
   (3) for any $x \in U'' := U \setminus \varphi^{-1}(X_{sing} \cup \Delta_{sing})$ such that $\varphi(x) \in D_i$, the ramification order of φ at x verifies $ord_\varphi(x) = m_i$.

Definition 7. An orbifold $\mathcal{X}$ is a log pair (X, Δ) such that X is covered by orbifold charts compatible with Δ.

Remark 8.   (1) In the language of stacks, we have a smooth Deligne-Mumford stack $\pi : \mathcal{X} \to X$, with coarse moduli space X.
   (2) More generally, Campana introduces *geometric orbifolds* in [7] as pairs of this type but which are not necessarily locally uniformizable.

Example 9. Let X be a complex manifold and $\Delta = \sum_i (1 - \frac{1}{m_i}) D_i$ with a support $\lceil \Delta \rceil$ which is a normal crossing divisor, i.e. for any point $x \in X$ there is a holomorphic coordinate system $(V, z_1, \ldots, z_n)$ such that Δ has equation

$$z_1^{(1 - \frac{1}{m_1})} \ldots z_n^{(1 - \frac{1}{m_n})} = 0.$$

Then (X, Δ) is an orbifold. Indeed, fix a coordinate system as above. Set

$$\varphi : U \to V, \ \phi(x_1, \ldots, x_n) = (x_1^{m_1}, \ldots, x_n^{m_n}).$$

Then (U, ϕ) is an orbifold chart on X compatible with Δ.

In the case of surfaces we have more examples of orbifolds looking at singularities that naturally appear in the logarithmic Mori program.

Definition 10. Let (X, Δ), $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) C_i$, be a pair where X is a normal surface and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : \tilde{X} \to X$ be a resolution of the singularities of (X, Δ), so that the exceptional divisors, E_i and the components of $\tilde{\Delta}$, the strict transform of Δ, have normal crossings and

$$K_{\tilde{X}} + \tilde{\Delta} + \sum_i E_i = \pi^*(K_X + \Delta) + \sum_i a_i E_i.$$

We say that (X, Δ) is *klt* (Kawamata log terminal) if $m_i < \infty$ and $a_i > 0$ for every exceptional curve E_i.

Thanks to a result of [30] (see also the appendix of [9]), we have

Example 11. Let (X, Δ), $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) C_i$, be a klt pair with X a surface, then (X, Δ) is an orbifold.

2.2. **Chern classes.** Let $\pi : \mathcal{X} \to (X, \Delta)$ be a two dimensional orbifold. Let S be the singular points of X and of the divisor $\lceil \Delta \rceil$, $\Delta = \sum (1 - \frac{1}{m_i}) D_i$. The orbifold canonical line bundle is: $K_{\mathcal{X}} = \pi^*(K_X + \Delta)$, therefore:

$$c_1(\mathcal{X}) = -\pi^*(K_X + \sum (1 - \frac{1}{m_i}) D_i).$$

To each point p of S, there is a well defined integer $\beta(p)$, the order of the isotropy group, such that by the orbifold Gauss-Bonnet formula [37]:

$$c_2(\mathcal{X}) = e(X) - \sum (1 - \frac{1}{m_i}) e(D_i \setminus S) - \sum_{p \in S} (1 - \frac{1}{\beta(p)}).$$

Let $p \in S$ be a smooth point of X.

Lemma 12. *Suppose that p is an ADE singularity of $\lceil \Delta \rceil$. We have :*

Type	$\beta(p)$
A_1	$m_i m_j$
A_{2n}	$\frac{2}{2n+1}(\frac{1}{m_i} + \frac{1}{2n+1} - \frac{1}{2})^{-2}$
A_{2n-1}	$\frac{4}{n}(\frac{1}{m_i} + \frac{1}{m_j} + \frac{1}{n} - 1)^{-2}$, $n \geq 2$
D_{2n+2}	$\frac{4}{n}(\frac{1}{m_i} + \frac{1}{m_j} + \frac{1}{nm_k} - 1)^{-2}$
D_{2n+3}	$2(2n+1)m_i^2$, $n \geq 1$
E_7	96

where the branches are as follows:

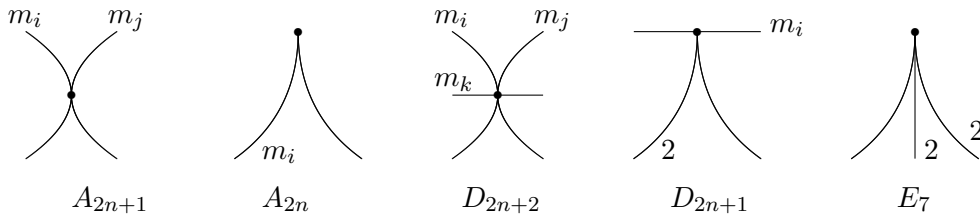A_{2n+1} A_{2n} D_{2n+2} D_{2n+1} E_7

FIGURE 2.1. ADE singularities and conditions on the multiplicities

Proof. See [39] Table 2.3, [38], [9] and [21]. □

Let $C_1, \ldots$ be disjoint reduced divisors on X whose irreducible components are (-2)-curves. There exist a surface X' and map $X \to X'$ such that the C_i's are contracted onto ADE-singularities and that is an isomorphism outside. Let a_n (resp. d_n, e_n) be the number of A_n (resp. D_n, E_n) singularities on X'.

Proposition 13. ([27], 1.8). *The surface X' has a natural structure of orbifold $\mathcal{X}$ and its Chern numbers are $c_1^2(\mathcal{X}) = c_1^2(X)$ and:*

(2.1)
$$c_2(\mathcal{X}) = c_2(X) - \sum (n+1)(a_n + d_n + e_n) + \sum \frac{a_n}{n+1} + \frac{d_n}{4(n-2)} + \frac{e_6}{24} + \frac{e_7}{48} + \frac{e_8}{120}.$$

The denominators $4(n-2)$, 24, 48, 120 are the order of the binary dihedral $BD_{4(n-2)}$, the binary tetrahedral, the binary octahedral, and the binary icosahedral group respectively.

2.3. **Orbifold Riemann-Roch.** Let L be an orbifold line bundle on the orbifold $\mathcal{X}$ of dimension n. We will use Kawazaki's orbifold Riemann-Roch theorem [20] or Toën's for Deligne-Mumford stacks [37] using intersection theory on stacks.

Theorem 14 ([37]). *Let $\mathcal{X}$ be a Deligne-Mumford stack with quasi-projective coarse moduli space and which has the resolution property (i.e every coherent sheaf is a quotient of a vector bundle). Let E be a coherent sheaf on $\mathcal{X}$ then*

$$\chi(\mathcal{X}, E) = \int_{\mathcal{X}} \widetilde{ch}(E) \widetilde{Td}(T_{\mathcal{X}}).$$

From this formula, we obtain the asymptotic:

$$\chi(\mathcal{X}, L^k) = \frac{c_1(L)^n}{n!} k^n + O(k^{n-1}),$$

using orbifold Chern classes.

It should be remarked that we use here the fact that we work only with effective orbifolds: the stabilizers are generically trivial therefore the correction terms of the orbifold Riemann-Roch do not affect the leading term.

We will apply this result to the case of orbifold surfaces $\mathcal{X}$ of general type with big orbifold cotangent bundle $\Omega_{\mathcal{X}}$. Indeed, let $T_{\mathcal{X}}$ be the orbifold tangent bundle. Then $\mathbb{P}(T_{\mathcal{X}})$ is naturally an orbifold and we can apply the previous Riemann-Roch formula to the tautological line bundle $\mathcal{O}_{\mathbb{P}(T_{\mathcal{X}})}(1)$. Orbifold Serre duality gives

$$h^2(\mathcal{X}, S^m \Omega_{\mathcal{X}}) = h^0(\mathcal{X}, S^m \Omega_{\mathcal{X}} \otimes K_{\mathcal{X}}^{1-m}) \hookrightarrow h^0(\mathcal{X}, S^m \Omega_{\mathcal{X}}).$$

As a corollary, if $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$ then

$$H^0(S^m \Omega_{\mathcal{X}}) \geq c.m^3$$

for a positive constant c.

2.4. **Orbifold hyperbolicity.** In the context of manifolds, complex hyperbolicity is concerned with the geometry of rational, elliptic curves and more generally entire curves i.e. holomorphic maps from $\mathbb{C}$. For orbifold hyperbolicity, algebraic curves are replaced by orbifold Riemann surfaces $\mathcal{C}$ i.e. the data of a Riemann surface C of genus $g \geq 0$ and r points $p_1, \ldots, p_r \in X$ marked by orders of stabilizer groups $m_1, \ldots, m_r \geq 2$ corresponding to the $\mathbb{Q}$-divisor $\Delta = \sum_i \left(1 - \frac{1}{m_i}\right) p_i$.

Definition 15. An orbifold Riemann surface $\mathcal{C}$ is *rational* (resp. *elliptic*) if $\deg(K_\mathcal{C}) < 0$ i.e. $2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right) < 0$ (resp. $\deg(K_\mathcal{C}) = 0$).

Example 16. A case-by-case check gives that (C, Δ) is elliptic if it is isomorphic to one of the following orbifold curves:

- $(E, \emptyset)$ where E is an elliptic curve.
- $\left(\mathbb{P}^1, \left(1 - \frac{1}{m_1}\right)\{0\} + \left(1 - \frac{1}{m_2}\right)\{1\} + \left(1 - \frac{1}{m_3}\right)\{\infty\}\right)$ where (m_1, m_2, m_3) is either $(2, 3, 6), (2, 4, 4), (3, 3, 3)$.
- $\left(\mathbb{P}^1, \left(1 - \frac{1}{2}\right)\{0\} + \left(1 - \frac{1}{2}\right)\{1\} + \left(1 - \frac{1}{2}\right)\{p_3\} + \left(1 - \frac{1}{2}\right)\{\infty\}\right)$ with $p_3 \in \mathbb{C} \setminus \{0, 1\}$.

Recall that an *orbifold map* between orbifolds $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a map between the underlying spaces X_1 and X_2 which lifts to an equivariant map in orbifold charts.

Definition 17. An orbifold map $p : \mathcal{X}_1 \to \mathcal{X}_2$ is an *orbifold covering* if every $x \in X_2$ is in some $U \subset X_2$ such that for every component V of $p^{-1}(U)$ the corresponding orbifold chart $\phi_1 : \tilde{V} \to V$ verifies that $p \circ \phi_1 : \tilde{V} \to U$ is an orbifold chart for U.

Definition 18. Let $\mathcal{X}$ be an orbifold. An orbifold rational (resp. elliptic) curve in $\mathcal{X}$ is the image of an orbifold map $f : \mathcal{C} \to \mathcal{X}$ where $\mathcal{C}$ is rational (resp. elliptic).

A hyperbolic orbifold should not have rational and elliptic curves and more generally:

Definition 19. An orbifold $\mathcal{X}$ is *(Brody-)hyperbolic* if there is no non-constant orbifold map $f : \mathbb{C} \to \mathcal{X}$. An orbifold $\mathcal{X}$ is *quasi-hyperbolic* if the Zariski closure of the union of the images of orbifold maps $f : \mathbb{C} \to \mathcal{X}$ is a proper sub-variety.

Example 20. From the uniformization theorem available in the orbifold setting (see for example [15]), we have that an orbifold curve $\mathcal{C}$ is hyperbolic if and only if $\deg(K_\mathcal{C}) > 0$.

In the algebraic setting, we can generalize the notion of algebraic hyperbolicity to orbifolds:

Definition 21. A compact orbifold $\mathcal{X}$, with ω an orbifold hermitian metric, is *algebraically hyperbolic* if there exists $\epsilon > 0$ such that for any orbifold morphism $f : \mathcal{C} \to \mathcal{X}$ with $\mathcal{C}$ compact,

$$\deg(K_\mathcal{C}) \geq \epsilon \int_C f^* \omega.$$

We suggest the following generalization of Green-Griffiths-Lang conjecture in the orbifold context

Conjecture 22 (Orbifold Green-Griffiths-Lang conjecture)**.** *Let $\mathcal{X}$ be an orbifold of general type. Then there exists a proper sub-orbifold $\mathcal{Z} \subset \mathcal{X}$ which contains the image of all holomorphic maps $f : \mathbb{C} \to \mathcal{X}$.*

Even in the setting of manifolds, this conjecture is widely open. An important result was obtained by McQuillan [28] with the confirmation of the conjecture for surfaces of general type with $c_1^2 - c_2 > 0$. The positivity of the second Segre number ensures by a Riemann-Roch computation as already explained above that $H^0(S^m\Omega_X) > cm^3$. McQuillan has extended his result to 2-dimensional Deligne-Mumford stacks with projective moduli [29] (see also [33] and [34]).

Since orbifolds are examples of DM stacks, a consequence of his result that we shall use is

Theorem 23. *Let $\mathcal{X}$ be an orbifold surface of general type with positive orbifold second Segre number*

$$c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0.$$

Then there exists a proper sub-orbifold $\mathcal{Z} \subset \mathcal{X}$ which contains the image of all orbifold maps $f : \mathbb{C} \to \mathcal{X}$, in other terms: the orbifold $\mathcal{X}$ is quasi-hyperbolic.

Let $\mathcal{C} \to \mathcal{X}$ be an orbifold rational or elliptic curve. The subjacent space to $\mathcal{C}$ is $\mathbb{P}^1$ or a smooth elliptic curve, thus such a curve gives rise to an entire orbifold curve $\mathbb{C} \to \mathcal{X}$. Therefore a quasi-hyperbolic orbifold is algebraically quasi-hyperbolic in the sense that it has only finitely many rational and elliptic orbifold curves.

Example 24. Following [5] (see also [33]), we see that nodal surfaces in $\mathbb{P}^3$ of degree d with l nodes provide examples of orbifolds with positive orbifold second Segre number if

$$l > \frac{8}{3}\left(d^2 - \frac{5}{2}d\right).$$

It should be remarked that this numerical condition can be satisfied only for $d \geq 6$. Moreover, in [5], the authors claim that a subspace $H^0(\mathcal{X}, S^m\Omega_{\mathcal{X}})$ of symmetric orbifold differentials can be extended to the resolutions of such surfaces but it seems to us that there is a gap in the argument of [5] Lemma 2.2.

2.5. **Orbifold jet differentials.** Since Bloch [2], jet differentials have turned out to be a useful tool to study hyperbolic properties of complex manifolds. We would like to use these technics in the orbifold setting extending the construction of Demailly [12] for manifolds to this situation.

Start with an orbifold $\mathcal{X}$ and an orbifold subbundle $\mathcal{V} \subset T_{\mathcal{X}}$. Then we consider the orbifold $\tilde{\mathcal{X}} := \mathbb{P}(\mathcal{V})$ with its natural orbifold line bundle

$$\mathcal{O}_{\tilde{\mathcal{X}}}(-1).$$

We have an orbifold map $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$. We define

$$\tilde{\mathcal{V}} := \pi_*^{-1}\mathcal{O}_{\tilde{\mathcal{X}}}(-1) \subset T_{\tilde{\mathcal{X}}},$$

here π_* is the differential map between $T_{\tilde{\mathcal{X}}}$ and $\pi^*T_{\mathcal{X}}$ (see [14]).

Therefore, as in the case of manifolds, we have an inductive process which gives the orbifold Demailly-Semple jet bundles:

$$(\mathcal{X}_0, \mathcal{V}_0) = (\mathcal{X}, \mathcal{V}), (\mathcal{X}_k, \mathcal{V}_k) = (\tilde{\mathcal{X}}_{k-1}, \tilde{\mathcal{V}}_{k-1}).$$

Taking $\mathcal{V}_0 := T_{\mathcal{X}_0}$, one can define jet differentials $E_{k,m}$ as the direct image sheaf

$$E_{k,m} := (\pi_{0,k})_* \mathcal{O}_{\mathcal{X}_k}(m),$$

where $\pi_{0,k} := \mathcal{X}_k \to \mathcal{X}$ is the natural projection map.

Remark 25. Alternatively, following [12] one could construct Green-Griffiths jet differentials $E_{k,m}^{GG}$ on orbifolds and define $E_{k,m}$ as the subbundle of orbifold jet differentials which are invariant under the reparametrization of germs of orbifold curves.

One can use the orbifold Riemann-Roch to compute the Euler characteristic. In the case of orbifold surfaces, one obtains

$$\chi(\mathcal{X}, E_{k,m}) = m^{k+2}(\alpha_k c_1^2 - \beta_k c_2) + O(m^{k+1}),$$

where c_1 and c_2 denote the orbifold Chern classes and α_k, β_k are just real numbers depending on k.

Using the semi-stability of $T_{\mathcal{X}}$ with respect to $K_{\mathcal{X}}$, which is also true for orbifolds ([21] and [36]), one can control the higher cohomology as in [17]. In particular, one obtains for an orbifold surface of general type [12]

(2.2) $$h^0(\mathcal{X}, E_{2,m}) \geq \frac{m^4}{648}(13c_1^2 - 9c_2) + O(m^3).$$

In the case of smooth hypersurfaces of degree d of $\mathbb{P}^3$, one obtains that

$$h^0(X, E_{2,m}) \geq C.m^4$$

for $d \geq 15$ (and $C > 0$). In general, the bigger c_1^2/c_2 is, the easier it is to obtain global jet differentials. This is another illustration that Horikawa surfaces are the most difficult cases to deal with. Therefore, for smooth surfaces in $\mathbb{P}^3$ of degree $d = 5$, one has to take k much larger. In contrast, we will provide below an example of an orbifold surface of $\mathbb{P}^3$ of degree 5 where $h^0(\mathcal{X}, E_{2,m}) \geq C.m^4$ (for $C > 0$).

As in the case of manifolds, one obtains as a consequence of Ahlfors-Schwarz lemma:

Theorem 26 ([29], [34]). *Let $\pi : \mathcal{X} \to X$ be a projective orbifold and an ample line bundle A on X such that $H^0(\mathcal{X}_k, \mathcal{O}_{\tilde{\mathcal{X}}_k}(m) \otimes \pi_{0,k}^* A^{-1}) \simeq H^0(\mathcal{X}, E_{k,m} \otimes A^{-1})$ has a non-zero section P. Then every orbifold entire curve $f : \mathbb{C} \to \mathcal{X}$ must satisfy the algebraic differential equation $P(f) = 0$.*

The generalization of jet differentials to orbifolds can be applied to classical problems which are not yet solved by Nevanlinna theory. Among these problems, we suggest the following one. Consider $f : \mathbb{C} \to \mathbb{P}^n$ a holomorphic curve which ramifies with multiplicity divisible by m_i over $H_1, \ldots, H_q$, hypersurfaces of degrees d_i of $\mathbb{P}^n$ in general position. Then it defines an orbifold entire curve $f : \mathbb{C} \to (\mathbb{P}^n, \Delta)$ where $\Delta := \sum_i \left(1 - \frac{1}{m_i}\right) H_i$. The condition

$$K_{\mathbb{P}^n} + \Delta > 0,$$

is equivalent to

$$\sum_i \left(1 - \frac{1}{m_i}\right) d_i > n + 1.$$

In this setting, Conjecture 22 takes the following form

Conjecture 27. *Let $H_1, \ldots, H_q$ be hypersurfaces of degrees d_i of $\mathbb{P}^n$ in general position. Assume that $f : \mathbb{C} \to \mathbb{P}^n$ is a holomorphic curve which ramifies over H_i with multiplicity divisible by m_i and that*

$$\sum_i \left(1 - \frac{1}{m_i}\right) d_i > n + 1,$$

then f is algebraically degenerate i.e. its image is contained in a proper algebraic hypersurface.

2.6. **Horikawa orbifolds.** Let us recall that for minimal surfaces of general type, we have the inequalities $5c_1^2 + 36 \geq c_2$ if c_1^2 is even, $5c_1^2 + 30 \geq c_2$ if c_1^2 is odd. Surfaces realizing the equalities are called Horikawa surfaces. Horikawa gave a classification of these surfaces [19]:

Theorem 28. *Let Z be a Horikawa surface with c_1^2 even, then there is a birational map $Z \to X'$ where X' is a surface branched along a curve with ADE singularities, more precisely X' is either*

(1) *a double covering of $\mathbb{P}^2$ branched along an octic ($\chi = 4$),*
(2) *a double covering of $\mathbb{P}^2$ branched along a curve of degree 10 ($\chi = 7$),*
(3) *a double covering of $\mathbb{F}_N$ branched along a curve of type $(6, 2a)$, $(2a \geq -N)$ ($\chi = 3N + 2a - 1$).*

In the sequel, we will say that a Horikawa surface is of type (i) according to the above classification.

It is therefore natural to associate orbifolds to Horikawa surfaces.

Definition 29. We say that an orbifold $\mathcal{X} = (X, \Delta)$ is Horikawa if there is a Horikawa surface Z with a birational map $Z \to X'$ where $X' \to X$ is a branched covering with ramification divisor Δ.

Example 30. From Horikawa's theorem $(\mathbb{P}^2, (1 - \frac{1}{2})C)$ where C is a curve of degree 8 or 10 with at most ADE singularities is a Horikawa orbifold.

It is possible to construct Horikawa surfaces with $c_1^2 = 5$ as the (resolutions of) 5-fold covers of $\mathbb{P}^2$ branched along five lines. More generally, examples of Horikawa orbifolds are provided by $(\mathbb{P}^2, (1 - \frac{1}{5})C)$ where C is a curve of degree 5 with at most nodes.

3. Algebraic hyperbolicity

3.1. **Horikawa surfaces with even first Chern number.** Having in mind Lang's conjecture claiming that there are only a finite number of curves of genus 0 or 1 on surfaces of general type, the aim of this section is to provide a lower bound on the genus of a curve on some explicit surfaces.

To study curves in Horikawa surfaces, one strategy is to study orbifold curves in Horikawa orbifolds. Using this philosophy, we will prove

Theorem 31.   (1) *Let X be a very generic Horikawa surface of type (1). Then X has no rational curves.*
   (2) *Let X be a very generic Horikawa surface of type (2). Then X is algebraically hyperbolic, in particular X has no rational or elliptic curves.*

(3) *Let X be a very generic Horikawa surface of type* (3) *with $a = 3$ and $\chi = 3N + 5$. Then X has no rational curves.*

This theorem will be a consequence of more general results which we establish now.

3.2. **Covers of the plane.** Let $D = \cup D_k \subset \mathbb{P}^2$, where D_k is a very general curve of degree d_k. For $n > 1$ dividing $d = \sum d_k$, let us consider $p : X \to \mathbb{P}^2$ the n-cyclic covering branched along D (for the construction of cyclic cover see [1]).

The singularities of D are the intersection points of the D_k's. For such a singularity, there exist local coordinates z_1, z_2 such that Δ has equation $z_1 z_2 = 0$; the singularity on the cover is therefore $\{t^n = z_1 z_2\}$ it is a A_{n-1}. The map p is an orbifold covering between X and $(\mathbb{P}^2, \Delta)$ where $\Delta = \left(1 - \frac{1}{n}\right) \sum D_i$.

Theorem 32. *Let $f : C \to X$ be an orbifold compact curve in X not contained in the branch locus D. Then*

$$\deg(K_C) \geq (d - \frac{d}{n} - 4) \deg C,$$

where $\deg C$ is the degree of C computed as $\deg p(f(C))$.
For $d > 4$ and $(d, n) \neq (5, 5), (6, 2), (6, 3), (8, 2)$, the surface X is algebraically hyperbolic modulo the rational and elliptic curves in the branch locus of $p : X \to \mathbb{P}^2$. For $(d, n) = (5, 5), (6, 3)$ or $(8, 2)$, X has no rational curves except the rational curves in the branch locus.

Remark 33. Of course, if X is smooth, every curve $f : C \to X$ is an orbifold curve.

Remark 34. If there are no rational or elliptic curves in $\lceil \Delta \rceil$, then there are no rational or elliptic curves in the branch locus.

Remark 35. The Chern numbers of the desingularisation of X are:

$$\begin{array}{rl} c_1^2 = & n(-3 + (1 - \frac{1}{n})d)^2 \\ c_2 = & 3n + (n - 1)(d^2 - 3d). \end{array}$$

For $(d, n) = (6, 2)$, the surface X is $K3$: such a surface contains an infinite number of elliptic curves [17]. The three other cases $(d, n) = (5, 5), (6, 3)$ or $(8, 2)$ are Horikawa surfaces. The case $(5, 5)$ is a quintic surface. For $(d, n) = (6, 3)$, we have $c_1^2 = 3$ and $5c_1^2 + 30 = c_2 = 45$. The case $(d, n) = (8, 2)$ is a Horikawa surface of type (1).

For the proof of Theorem 32, we use the main Theorem of Xi Chen in [10] (see also [31]):

Theorem 36. *Let $D = \cup D_k \subset \mathbb{P}^2$ be as above. For all reduced curve $C \subset \mathbb{P}^2$, we have:*

(3.1) $$2g(C) - 2 + i(C, D) \geq (d - 4) \deg C,$$

where $d = \sum d_k$ and $i(C, D)$ is the number of distinct points of $\nu^* D$ if $\nu : C' \to C$ is the normalization.

Proof. (Of Theorem 32). Let $f : C \to X$ be an orbifold map. We have the following commuting diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & X \\
\Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle p} \\
(C'_1, \Delta') & \xrightarrow[\ g\ ]{} & (\mathbb{P}^2, \Delta)
\end{array}
$$

where $C_1 = p(f(C))$, C'_1 is the desingularisation of C_1, and g is an orbifold morphism. Let

$$
\begin{aligned}
g^*(D_j) &= \sum_{i=1}^{i(C_1,D)} t_{i,j} p_i, \\
g^*(D) &= \sum_{i=1}^{i(C_1,D)} t_i p_i.
\end{aligned}
$$

Now, let $\tilde{\Delta} = \sum_{i=1}^{i(C_1,D)} \left(1 - \frac{1}{\tilde{m}_i}\right) p_i$ be the minimal orbifold structure on C'_1 such that $g : (C'_1, \tilde{\Delta}) \to (\mathbb{P}^2, \Delta)$ is an orbifold morphism. The conditions for g to be an orbifold morphism are $n | \tilde{m}_i . t_{i,j}$ for all j. Therefore the minimal orbifold structure is given by

$$
\tilde{m}_i = \mathrm{lcm}_j \left(\frac{n}{\gcd(n, t_{i,j})} \right).
$$

As $t_i = \sum_j t_{ij}$, we have $\gcd(n, t_{ij}) \leq t_i$ for all i, j, thus:

$$
\frac{n}{t_i} \leq \frac{n}{\gcd(n, t_{i,j})} \leq \mathrm{lcm}_j \left(\frac{n}{\gcd(n, t_{i,j})} \right) = \tilde{m}_i.
$$

So, we have:

$$
\sum_{i=1}^{i(C_1,D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \sum_{i=1}^{i(C_1,D)} \left(1 - \frac{t_i}{n}\right) = i(C_1, D) - \frac{d}{n} \deg C_1.
$$

Now, inequality 3.1 of [10] gives:

$$
2g(C'_1) - 2 + i(C_1, D) \geq (d - 4) \deg C_1,
$$

therefore:

$$
2g(C'_1) - 2 + \sum_{i=1}^{i(C,D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \left(d - \frac{d}{n} - 4\right) \deg C_1.
$$

To conclude, we observe that

$$
\deg(K_C) \geq 2g(C'_1) - 2 + \sum_{i=1}^{i(C,D)} \left(1 - \frac{1}{\tilde{m}_i}\right)
$$

because C is a uniformisation of C'_1. □

This implies (1) and (2) of Theorem 31.

When $n = d$, the surface X is the degree d hypersurface in $\mathbb{P}^3$ defined by:

$$
X = \{x_4^d - G(x_1, x_2, x_3) = 0\},
$$

where G is a homogeneous polynomial in x_1, x_2, x_3 of degree d defining the curve $C \subset \mathbb{P}^2$. We can apply the above results to X:

Corollary 37. *If C is very generic and $d \geq 6$, then the smooth algebraic surface X is algebraically hyperbolic. If C is very generic and $d = 5$, then X contains no curves of geometric genus 0.*

This result should be compared to results of Clemens, Ein, Pacienza, Voisin establishing the algebraic hyperbolicity of very general hypersurfaces of $\mathbb{P}^n$ of large degree (see [14] for a survey).

Moreover let us mention that, even if here we are interested in surfaces, since inequality 3.1 of Chen generalizes to higher dimension (see [10] and [31]), the proof above immediately generalizes to

Theorem 38. *If $D \subset \mathbb{P}^n$ is a very generic hypersurface of degree d and $d \geq 2n+2$, then the degree d branched cover $X \subset \mathbb{P}^{n+1}$ ramified over D is algebraically hyperbolic.*

3.3. **Covers of Hirzebruch surfaces $\mathbb{F}_N$, Algebraically hyperbolic Horikawa surfaces.** In order to prove Theorem 31 (3), we study covers of Hirzebruch surfaces.

For $N \geq 0$, let $\mathbb{F}_N := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(N))$ be the Hirzebruch surface and let T and F the divisors on $\mathbb{F}_N$ generating $Pic(\mathbb{F}_N)$ with $T^2 = N$, $T.F = 1$ and $F^2 = 0$.
Let $D = \cup D_k \subset \mathbb{F}_N$ where each D_k is a very general member of a base point free complete linear series. And let $D \sim aT + bF$ with $a(N-1) + b \geq 0$.

Theorem 39. *(Chen, [11] Corollary 1.12). We have:*

$$(3.2) \qquad 2g(C) - 2 + i(C, D) \geq \min(a - 3, b - 2) \deg C$$

for all reduced irreducible curves $C \subset \mathbb{F}_N$ with $C \not\subset D$ where $\deg C = (T+F).C$.

As before, for n dividing a and b, we consider $p : X \to \mathbb{F}_N$ a n-cyclic covering branched along D. Then we have

Theorem 40. *Let $f : C \to X$ be an orbifold compact curve in X not contained in the branch locus. Let c, d be the integers such that $p(f(C)) = C_1 \sim cT + dF$, then:*

$$\deg(K_C) \geq \min(a - 3, b - 2)(c(N+1) + d) - \frac{1}{n}(bc + ad + acN).$$

Proof. We follow the notations and proof of theorem 32 : C_1' is the desingularisation of C_1. First we remark that

$$\deg C_1 = c(1 + N) + d, \ C_1.D = bc + ad + acN.$$

Then, using the inequality 3.2, we obtain:

$$2g(C_1') - 2 + \sum_{i=1}^{i(C,D)} \left(1 - \frac{1}{\tilde{m}_i}\right) \geq \min\left(a - 3, b - 2\right)(\deg C_1) - \frac{C_1.D}{n}.$$

$\square$

We wish to apply Theorem 40 to the Horikawa surfaces (for which $a = 6$, $n = 2$). To obtain a non-trivial result, we need $b \geq 3$ if $N \geq 1$ and $b \geq 6$ if $N = 0$ (recall that $a(N-1) + b \geq 0$). Recall moreover that b must be divisible by 2. The case $b = 4$ gives :

$$\deg(K_C) \geq c(N+1) + d - \frac{1}{2}(bc + 6d + 6cN) = c(-1 - 3N) - 2d,$$

and therefore is not interesting. If $b \geq 6$, we have:

$$\deg(K_C) \geq 3(c(N+1) + d) - \frac{1}{2}(bc + 6d + 6cN) = c(3 - \frac{b}{2}),$$

thus we obtain $\deg(K_C) \geq 0$ for $a = 6$, $b = 6$.

Let us suppose that $D \sim 6T + 6F$ on $\mathbb{F}_N$ ($N \geq 0$) is a very general member of a base point free complete linear series. The double cover of $\mathbb{F}_N$ ramified over D is a Horikawa surface X with $\chi = 3N + 5$. That implies (3) of Theorem 31.

The proof of Theorem 31 is now complete.

We would like to emphasize here the fact that this result is valid for a very generic surface in some irreducible component of the moduli space of Horikawa surfaces.

3.4. **Quintic surfaces.** Our starting point was the fact that we do not know quasi-hyperbolic quintics. Consider 5 lines $L_1, \ldots, L_5$ in general position on $\mathbb{P}^2$. The degree 5 ramified cover of $\mathbb{P}^2$ branched over the L_i's is a quintic surface X in $\mathbb{P}^3$. If $\ell_i = \ell_i(x_1, x_2, x_3)$ is an equation of L_i, an equation of X is :

$$X = \{x_4^5 - \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 = 0\}.$$

As the five lines are in general position, their union is a degree 5 curve D with 10 nodal singularities. Over such a singularity p, there exists local coordinates x, y such that the equation of D is $xy = 0$. The equation of the singularity of X over p is then $z^5 - xy = 0$ and the surface X contains therefore 10 A_4. We consider this surface X as an orbifold. A corollary of Theorem 32 is

Corollary 41. *X has no orbifold rational curves except over the 5 lines of the ramification locus.*

As X is an orbifold of general type, a natural question is

Problem 42. Show that X has only finitely many (orbifold) elliptic curves.

A first result toward this is

Theorem 43. *The image in $\mathbb{P}^2$ of any orbifold elliptic curve $f : C \to X$ is not rational. More precisely, it is mapped in $\mathbb{P}^2$ to a singular genus one curve which intersects every line L_i with multiplicity divisible by 5.*

Proof. As above, we have the following diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & X \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle p} \\
(C_1', \Delta') & \xrightarrow{\ g\ } & (\mathbb{P}^2, \Delta)
\end{array}
$$

where $C_1 = p(f(C))$, C_1' is the desingularisation of C_1, and g is an orbifold morphism. Therefore $(C_1', \Delta' = \sum \left(1 - \frac{1}{m_i}\right) p_i)$ is an orbifold elliptic curve i.e. $K_{C_1'} + \Delta' = 0$. A case-by-case check gives that (C_1', Δ') is isomorphic to one of the following orbifold curve:

   □ $(E, \emptyset)$ where E is an elliptic curve

□ $\left(\mathbb{P}^1, \left(1 - \frac{1}{m_1}\right)\{0\} + \left(1 - \frac{1}{m_2}\right)\{1\} + \left(1 - \frac{1}{m_3}\right)\{\infty\}\right.$ where (m_1, m_2, m_3) is either $(2,3,6), (2,4,4), (3,3,3)$

□ $\left(\mathbb{P}^1, \left(1 - \frac{1}{2}\right)\{0\} + \left(1 - \frac{1}{2}\right)\{1\} + \left(1 - \frac{1}{2}\right)\{p_3\} + \left(1 - \frac{1}{2}\right)\{\infty\}\right.$ with $p_3 \in \mathbb{C} \setminus \{0,1\}$.

Let

$$g^*(L_j) = \sum_{i=1}^{i(C_1, L)} t_{i,j} p_i.$$
$$g^*(L) = \sum_{i=1}^{i(C_1, L)} t_i p_i.$$

The conditions for g to be an orbifold morphism are $5 | m_i.t_{i,j}$ for all j. As $\gcd(5, m_i) = 1$ by the preceding case-by-case analysis, we conclude that $5 | t_{i,j}$. Therefore $g : C_1' \to (\mathbb{P}^2, \Delta)$ is an orbifold morphism. From theorem 32, we deduce that this excludes the case $C_1' = \mathbb{P}^1$. Therefore in the above list, only the first case one can occur. So C_1 is a genus one curve. The condition $5 | t_{i,j}$ implies that $\deg(C_1) = \sum_j t_{ij}$ is divisible by 5 and C_1 cannot be a smooth elliptic curve.   □

Therefore we have reduced the problem of counting orbifold elliptic curves in these quintics to counting genus one curves in $\mathbb{P}^2$ which intersects every line L_i with multiplicity divisible by 5.

Remark 44. The following computation shows that it is natural to believe that there are finitely many of these curves.

The Severi variety $V^{d,\delta}$ of plane curves of degree d and δ nodes has dimension $3d + g - 1$. Let $C \in V^{d,\delta}$ be a curve which intersects L_i in $\sum_j \beta_j$ points with multiplicity $5j$. The total number of conditions is

$$D := \sum_{i=1}^{5} \sum_{j=1}^{l_i} \beta_j (5j - 1),$$

with $\sum_j 5j\beta_j = d$. Then $D = \sum_i (d - \sum_j \beta_j)$, et $\sum \beta_j \leq \frac{d}{5}$. Therefore $D \geq \sum_i d(1 - \frac{1}{5}) = 4d$. Thus for $g \leq 1$ the number of conditions is greater than the number of parameters.

4. ANALYTIC HYPERBOLICITY

4.1. **Nevanlinna theory.** Nevanlinna theory can be used to study entire curves in ramified covers. We will briefly recall in this context the truncated defect relation of Cartan following the notations of [22] to which we refer for details.

Let $f : \mathbb{C} \to \mathbb{P}^n$ be a linearly-non degenerate entire curve and $D_1, \ldots, D_q$ be q hyperplanes in general position. Let us denote as usual $N^{[n]}(f, r, D_i)$ and $T(f, r, D_i)$, the truncated counting function and the characteristic function.

The defect is defined by

$$\delta^{[n]}(f, D_i) := \liminf_{r \to \infty} \left(1 - \frac{N^{[n]}(f, r, D_i)}{T(f, r, D_i)}\right).$$

Then we have the truncated defect relation of Cartan:

Theorem 45. $\sum \delta^{[n]}(f, D_i) \leq n + 1$.

We can apply this to ramified covers of the plane. Let $X \subset \mathbb{P}^3$ be the degree d cover of $\mathbb{P}^2$ ramified over d lines D_i.

Theorem 46. *If $d \geq 6$ then X is a quasi-hyperbolic orbifold.*

Proof. Composing with the projection, we obtain an orbifold map $g : \mathbb{C} \to (\mathbb{P}^2, \Delta)$. If g is linearly degenerate then from Theorem 32, its image lies in the branch locus. Let us suppose that it is linearly non-degenerate. By the First Main Theorem of Nevanlinna theory, we have

$$\frac{d}{2} N^{[2]}(f, r, D_i) \leq N(f, r, D_i) \leq T(f, r, D_i).$$

Therefore

$$\delta^{[2]}(f, D_i) \geq 1 - \frac{2}{d},$$

which implies

$$\sum \delta^{[2]}(f, D_i) \geq d - 2,$$

contradicting Cartan's relation if $d \geq 6$. $\qquad\square$

Unfortunately we see in the previous proof that we cannot say anything for $d = 5$, the case of the quintics we studied. An interesting question is to prove that we have again algebraic degeneracy of entire curves in this case. Note that, according to Conjecture 27, this should be true since $K_{\mathbb{P}^2} + \Delta = \mathcal{O}(1)$. Surprisingly, it seems that this problem is still open.

Problem 47. *Prove that any entire curve $f : \mathbb{C} \to \mathbb{P}^2$ intersecting 5 lines in general position with multiplicity at least 5 is algebraically degenerate.*

Nevertheless, we can use Nevanlinna theory to construct some hyperbolic orbifold Horikawa surfaces.

Theorem 48. *Let $a \geq 3$ be an integer. Then there exists a quasi-hyperbolic orbifold Horikawa surface whose minimal resolution has $\chi = 2a - 1$.*

Proof. Let $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Consider the divisor

$$\Delta = \left(1 - \frac{1}{2}\right) \sum_{i=1}^{6} G_i + \left(1 - \frac{1}{2}\right) \sum_{i=1}^{2a} H_i,$$

where the G_i are fibers of the first projection p_1, and the H_i are fibers of the second projection p_2. Then $(\mathbb{F}_0, \Delta)$ is an Horikawa orbifold. The corresponding Horikawa surface has $\chi = 2a - 1$.

Let $f : \mathbb{C} \to (\mathbb{F}_0, \Delta)$ be an orbifold entire curve. Then $h := p_1 \circ f : \mathbb{C} \to \mathbb{P}^1$ is an entire curve with multiplicity at least 2 over the 6 points $a_i := p_1(G_i)$. Then

$$\sum \delta^{[1]}(h, a_i) \geq 6 \left(1 - \frac{1}{2}\right).$$

The truncated defect relation (i.e. Nevanlinna Second Main Theorem in this case), which gives the upper bound 2 for the sum of the defects, implies that $p_1 \circ f$ is constant since . Therefore f has its image contained in a fiber of p_2 and has multiplicity at least 2 over $2a$ points of this fiber. This implies again that f is constant since $2a \geq 6$. $\qquad\square$

We can apply this kind of construction to numerical quintics. A quintic surface has $c_1^2 = 5$ and $c_2 = 55$. Horikawa [18] proved that a surface with $c_1^2 = 5$ and $c_2 = 55$, called a numerical quintic, is either a quintic or a double cover.

Let us consider surfaces of the second type with their attached natural geometric orbifolds (X, Δ).

Theorem 49. *There exists a numerical quintic whose geometric orbifold is quasi-hyperbolic.*

Proof. Let $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Consider the divisor

$$\Delta = \left(1 - \frac{1}{2}\right) \sum_{i=1}^{5} G_i + \left(1 - \frac{1}{2}\right)(H_1 + H_2 + C + D_1 + D_2),$$

where the G_i are fibers of the first projection p_1, the H_i are fibers of the second projection p_2, C is a curve of type $(2, 1)$, and the D_i are curves of type $(1, 1)$ such that:

(1) G_1, H_1, C and D_1 meet in a point a.
(2) G_1, H_2, C and D_2 meet in a point b.

Then $\lceil \Delta \rceil$ is a curve of type $(6, 8)$ with two quadruple points lying on a single fiber of $\mathbb{F}_0$.

$(\mathbb{F}_0, \Delta)$ is a numerical quintic orbifold.

Let $f : \mathbb{C} \to (\mathbb{F}_0, \Delta)$ be an orbifold entire curve. Then $p_1 \circ f : \mathbb{C} \to \mathbb{P}^1$ is an entire curve with multiplicity at least 2 over 5 points. The truncated defect relation (i.e. Nevanlinna Second Main Theorem in this case) implies that $p_1 \circ f$ is constant. Therefore f has its image contained in a fiber of p_2. For a generic fiber i.e. not G_1, f has multiplicity at least 2 over at least 5 points of this fiber. This implies again that f is constant. Therefore the image of f is contained in G_1. $\square$

One can remark that Cartan's theorem unfortunately says nothing in the case of Horikawa surfaces which are degree 2 covers of the plane $\mathbb{P}^2$.

4.2. **Jet differentials.** As we have just seen, applications of Nevanlinna theory are quite limited. This is one motivation to develop the tool of orbifold jet differentials.

Let us illustrate it in the case of quintics X that are $\mathbb{Z}/5\mathbb{Z}$-covers of the plane ramified over 5 lines in general position. First one could think of using jets of order 1 to apply Theorem 23.

As already mentioned in paragraph 3.4, the surface X has a natural structure of orbifold $\mathcal{X} = (X, 0)$.

Proposition 50. *The Chern numbers of $\mathcal{X}$ are:*

$$c_1^2(\mathcal{X}) = 5$$
$$c_2(\mathcal{X}) = 7,$$

Proof. By paragraph 3.4, X contains $10A_4$ singularities. Let us denote by $X' \to X$ the minimal resolution of these singularities. The Chern numbers of X' are equals to the Chern numbers of a smooth quintic (Brieskorn resolution Theorem [6]) i.e. $K_{X'}^2 = 5$, $e(X') = 55$. We have $K_{\mathcal{X}}^2 = K_X^2 = 5$ and $c_2(\mathcal{X}) = e(X') - 10 \cdot (5 - \frac{1}{5}) = 7$ (see Proposition 13). $\square$

Therefore the orbifold second Segre number is negative and we cannot apply Theorem 23.

Turning to jet differentials of order 2, one sees that

$$13c_1^2(\mathcal{X}) - 9c_2(\mathcal{X}) = 2 > 0.$$

Therefore by inequality (2.2):

Corollary 51. *We have:*

$$h^0(X, E_{2,m} \otimes L^{-1}) \geq 2.\frac{m^4}{648} + O(m^3),$$

for L an ample line bundle on X. As a consequence (see (2.2) and Theorem 26), any entire orbifold curve $f : \mathbb{C} \to X$ satisfies a differential equation of order 2.

There are many known quintic surfaces with quotient singularities (see e.g. [35]), but as far as we know, the surfaces that are $\mathbb{Z}/5\mathbb{Z}$-covers of the plane ramified over 5 lines in general position are the only one such that the associated orbifold satisfies $13c_1^2(\mathcal{X}) - 9c_2(\mathcal{X}) > 0$.

This result can also be seen as a first step toward problem 47.

5. PERSSON-HORIKAWA SURFACES

Let us recall some notations. For N a positive integer, the Nth Hirzebruch surface is $\mathbb{F}_N = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(N))$. We denote by F denote a fiber of the $\mathbb{P}^1$-bundle $\mathbb{F}_N \to \mathbb{P}^1$ and by T a section s.t. $T^2 = N$. We denote by (a, b) a divisor equivalent to $aT + bF$ in the Néron-Severi group.

Recall (see Theorem 28) that a Horikawa surface is the double cover of the plane branched along an octic (case $\chi = 4$), or a curve of degree 10 (case $\chi = 7$) or a double cover of $\mathbb{F}_N$ branched over a curve D of type $(6, 2a)$, with $2a \geq -N$. In this section we will exhibit quasi-hyperbolic Horikawa orbifolds of each type.

5.1. **Horikawa surfaces with $\chi = 2k-1$.** Let us prove the following Proposition:

Proposition 52. *For any integer $k > 2$, there exists a quasi-hyperbolic Horikawa orbifold $\mathcal{X}$ whose minimal resolution has Euler characteristic equals to $\chi = 2k-1$.*

Proof. The divisor $-2(F + T)$ is the canonical divisor on Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Following Persson's construction in [32] Lemma 4.5, there exists on $\mathbb{F}_0$ a $\mathbb{Q}$-divisor:

$$\Delta = (1 - \frac{1}{k})(F_1 + F_2) + (1 - \frac{1}{2})(C_1 + C_2 + E_0 + E_1 + E_2 + E_3)$$

such that the orbifold $(\mathbb{F}_0, \Delta)$ is uniformizable for all $k \geq 2$ and the desingularisation of the uniformization is a Horikawa surface with Chern invariants $\chi = 2k - 1$ and $c_1^2 = 2\chi - 6$. The curves C_1, C_2 are $(1,1)$-curves meeting with multiplicity 2 into one point. The curves F_1, F_2 are fibers of the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, the curves E_i are sections of this fibration. The singularities of $\lceil \Delta \rceil$ are $6A_1 + 4D_4 + D_6$.

Let us use the results of subsection 2.2. Two singularities of type A_1 are on branches with multiplicities m_i equal to 2 (thus $\beta(p) = 4$), $4A_1$ are on branches with multiplicities 2 and k (with $\beta(p) = 2k$), the branches of the $4D_4$ have the same multiplicities : $2, 2, k$ (giving $\beta(p) = 4k^2$) and the branches of the D_6 point have multiplicities 2, thus $\beta(p) = 32$ by Lemma 12.
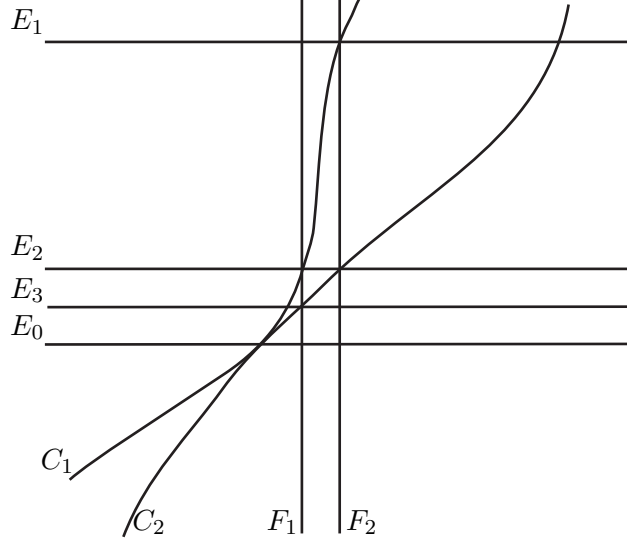
Let us compute the orbifold Chern numbers of the associated orbifold $\mathcal{X}$. We have:

$$(K + \Delta)^2 = (T + (1 - \tfrac{1}{k})F)^2 = T^2 + 2(1 - \tfrac{1}{k})TF + (1 - \tfrac{1}{k})^2 F^2 = 2(1 - \tfrac{2}{k})$$

because $F^2 = 0$, $FT = 1$, $T^2 = N = 0$ (see section 3.3). For the second Chern number of $\mathcal{X}$, (2.1) gives:

$$c_2(\mathcal{X}) = 4 - 2(1 - \tfrac{1}{k})(2 - 4) - \tfrac{1}{2}(3(2 - 3) + (2 - 2) + 2(2 - 4))$$
$$- (2(1 - \tfrac{1}{4}) + 4(1 - \tfrac{1}{2k}) + 4(1 - \tfrac{1}{4k^2}) + (1 - \tfrac{1}{32}))$$

and

$$c_2(\mathcal{X}) = \frac{33}{32} - \frac{2}{k} + \frac{1}{k^2}.$$

Therefore :

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{31}{32} - \frac{2}{k} - \frac{1}{k^2},$$

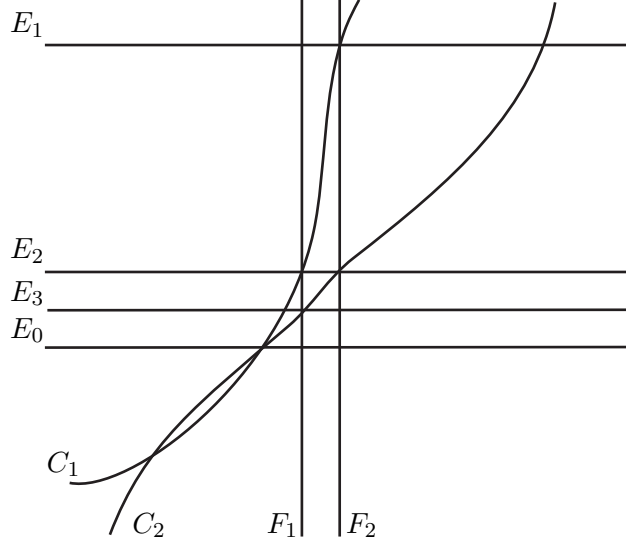and the orbifold $\mathcal{X}$ is quasi-hyperbolic for $k > 2$ by Theorem 23. □

Remark 53. Let $X \to \mathbb{F}_0$ be the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$-cover of F_0 branched over Δ. The desingularisation of X is a Horikawa surface. The surface X has only ADE singularities and has a natural structure of orbifold $\mathcal{X}'$. The map $\mathcal{X}' \to \mathcal{X}$ is an orbifold covering, in particular $c_1(\mathcal{X}')^2 = 2kc_1(\mathcal{X})^2$ and $c_2(\mathcal{X}') = 2kc_2(\mathcal{X})$.

5.1.1. *Generalization : a family of quasi-hyperbolic Horikawa orbifold with* $\chi = 2k - 1$, $k \geq 4$. Let us consider the configuration of curves on $\mathbb{F}_0$ in figure 5.1.

The two $(1, 1)$-curves C_1, C_2 in $\mathbb{F}_0$ are in general position. The singularities of Δ are $7A_1 + 5D_4$. One D_4 with multiplicities $(2, 2, 2)$, $4D_4$ with multiplicities $(2, 2, k)$, $4A_1$ with $(2, k)$ and $3A_1$ with $(2, 2)$.

Let $\mathcal{X}$ be the associated orbifold. By subsection 2.2, its Chern numbers are: $c_1(\mathcal{X})^2 = 2(1 - \tfrac{2}{k})$ and

$$c_2(\mathcal{X}) = 4 - 2(1 - \tfrac{1}{k})(2 - 4) - \tfrac{1}{2}(3(2 - 3) + (2 - 2) + 2(2 - 5))$$
$$- (3(1 - \tfrac{1}{4}) + 4(1 - \tfrac{1}{2k}) + 4(1 - \tfrac{1}{4k^2}) + (1 - \tfrac{1}{16}))$$

FIGURE 5.1. The divisor Δ on $\mathbb{F}_0$.

thus $c_2(\mathcal{X}) = \frac{21}{16} - \frac{2}{k} + \frac{2}{k^2}$ and

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{11}{16} - \frac{2}{k} - \frac{2}{k^2}.$$

It is positive if $k \geq 4$ and then the orbifold $\mathcal{X}$ is quasi-hyperbolic by Theorem 23.

5.2. **Horikawa surfaces with $\chi = 4k - 1$.** Let us give another construction of quasi-hyperbolic Horikawa surfaces:

Proposition 54. *For every integer $k > 1$, there exists a quasi-hyperbolic orbifold Horikawa surface $\mathcal{X}$ with*

$$\begin{array}{ll} c_1^2(\mathcal{X}) = & 4(1 - \frac{1}{k}) \\ c_2(\mathcal{X}) = & \frac{17}{12} - \frac{2}{k} - \frac{1}{k^2}. \end{array}$$

The desingularisation of the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$-cover of the subjacent space to $\mathcal{X}$ is a Horikawa surface with $\chi = 4k - 1$.

Proof. The canonical divisor on $\mathbb{F}_2$ is $K_{\mathbb{F}_2} = -2T$. Let $k > 1$ be an integer. Persson ([32] Proposition 4.7) constructed on $\mathbb{F}_2$ a divisor :

$$\Delta = (1 - \frac{1}{k})(F_1 + F_2) + (1 - \frac{1}{2})(A + B + C)$$

whose irreducible components F_i, A, B, C have the following configuration : F_1 and F_2 are fibers, C has type $(1, -2)$, A has type $(2, 0)$, B has type $(3, 0)$.

We have $e(A) = e(B) = 0$, $CA = CB = 0$ and $F_iA = 2$, $F_iB = 3$. The common points to A and B are $2A_{11}$ singularities of $\lceil \Delta \rceil$.

The singularities on Δ are $2A_1 + 2A_{11}$ with multiplicities $(2, 2)$, $4A_1$ with multiplicities $(2, k)$, and $4D_4$ with multiplicities $(2, 2, k)$ (see figure 5.2).
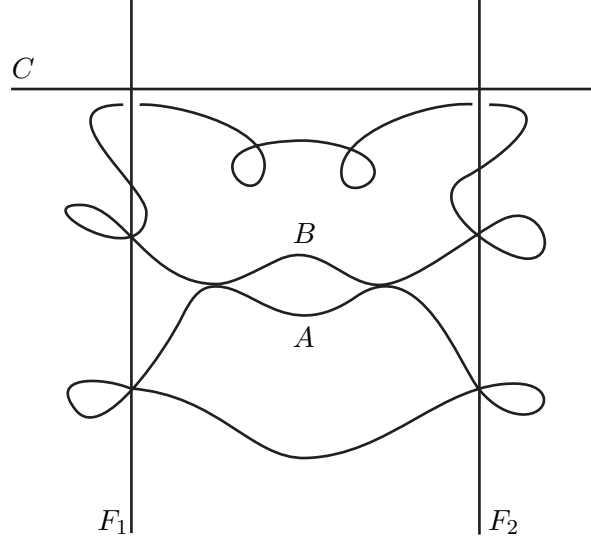
FIGURE 5.2. The divisor Δ on $\mathbb{F}_0$.

Let us compute the orbifold Chern numbers of $\mathcal{X} = (\mathbb{F}_2, \Delta)$:

$$c_1^2(\mathcal{X}) = (K_{\mathbb{F}_2} + \Delta)^2 = \left((1 - \frac{1}{k})(0,2) + (1 - \frac{1}{2})(6,-2) + (-2,0) \right)^2 = 4(1 - \frac{1}{k}),$$

and

$$c_2(\mathcal{X}) = 4 - 2(1 - \tfrac{1}{k})(2 - 4) - \tfrac{1}{2}((0 - 4) + (0 - 8) + 0)$$
$$-(2(1 - \tfrac{1}{4}) + 2(1 - \tfrac{1}{24}) + 4(1 - \tfrac{1}{2k}) + 4(1 - \tfrac{1}{4k^2})).$$

Therefore : $c_2(\mathcal{X}) = \frac{31}{12} - \frac{2}{k} + \frac{1}{k^2}$ and

$$c_1(\mathcal{X})^2 - c_2(\mathcal{X}) = \frac{17}{12} - \frac{2}{k} - \frac{1}{k^2}$$

is positive for $k > 1$ and by Theorem 23, $\mathcal{X}$ is quasi-hyperbolic. $\square$

5.3. **Examples of quasi-hyperbolic orbifold double octic.**

5.3.1. *First construction.* The Horikawa surfaces with $\chi = 4$ are double cover of the plane ramified over an octic with at most ADE singularities.

Proposition 55. *There exists a quasi-hyperbolic orbifold Horikawa surface whose minimal resolution has $\chi = 4$.*

Proof. Let $\mathcal{X}$ be the orbifold whose subjacent variety is $\mathbb{P}^2$ and with $\Delta = \frac{1}{2}(Q + L_1 + L_2 + L_3 + L_4)$ where Q is the Steiner quartic (the unique quartic curve with 3 cusps, see [32]), L_1, L_2, L_3 are the tangents to the 3 cusps and L_4 is the unique bitangent of Q (see figure 5.3).

The 2 points where L_4 is tangent to Q are A_3 singularities of $\lceil \Delta \rceil$. The singularity of $\lceil \Delta \rceil$ at the cusp Q have type E_7. The three lines $L_1, \ldots, L_3$ meet at
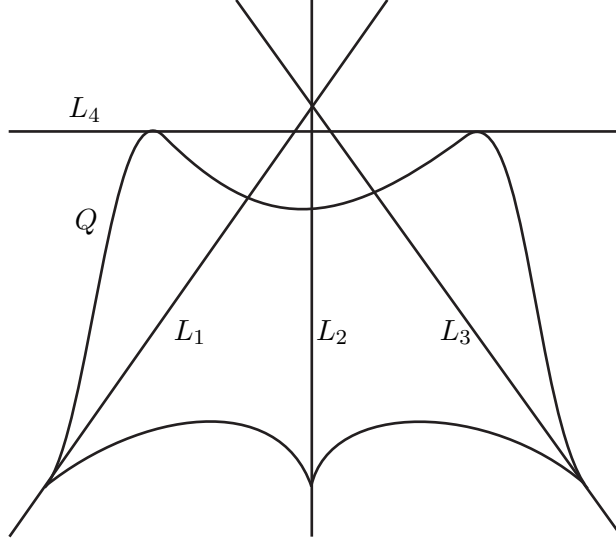
FIGURE 5.3. The Steiner quartic, its bitangent and the 3 tangents to its cusps.

a unique point, giving a D_4 singularity. The singularities of the curve $\lceil \Delta \rceil$ are $6A_1 + 2A_3 + D_4 + 3E_7$. We have:

$$c_1^2(\mathcal{X}) = (-3 + (1 - \frac{1}{2})8)^2 = 1.$$

Moreover:

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(e(Q \setminus S) + \sum_i e(L_i \setminus S)) - \sum_{p \in S} 1 - \frac{1}{\beta(p)},$$

where S is the set of singularities of $\lceil \Delta \rceil$. Therefore:

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(2 - 8 + 3(2 - 4) + (2 - 5)) - 12 + \frac{6}{4} + \frac{2}{8} + \frac{1}{16} + \frac{3}{96} = \frac{11}{32},$$

and $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) = \frac{21}{32} > 0$. By Theorem 23, $\mathcal{X}$ is quasi-hyperbolic. The desingularisation of the degree 2 cover ramified over $\lceil \Delta \rceil$ is a Horikawa surface with $\chi = 4$. $\qquad\square$

5.3.2. *Second construction: a pencil of quasi-hyperbolic Horikawa surfaces.*

Proposition 56. *There exists a pencil of quasi-hyperbolic orbifold Horikawa surfaces whose minimal resolutions have $\chi = 4$.*

Proof. Let us consider the curve in Proposition 55, and replace the line L_3 by a generic line L_3' going through the intersection point of L_1 and L_2.

That gives a degree 8 curve Δ with singularities

$$9A_1 + 2A_3 + D_4 + 2E_7$$

For which $c_2(\mathcal{X}) = \frac{3}{4}$ and $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$. By Theorem 23, $\mathcal{X}$ is quasi-hyperbolic. $\qquad\square$

5.4. **A family of quasi-hyperbolic double covers branched over a degree 10 curve.** Let us prove:

Proposition 57. *There exists a 4 dimensional family of quasi-hyperbolic Horikawa orbifolds whose minimal resolutions have $\chi = 7$.*

Proof. Let C be the degree 10 curve that is the union of the degree 8 curve in Proposition 55 and two lines in general position. It has singularities:

$$23A_1 + 2A_3 + d_4 + 3E_7.$$

The orbifold Chern classes of $\mathcal{X} = (\mathbb{P}^2, C)$ are :

$$c_1^2(\mathcal{X}) = (-3 + \frac{10}{2})^2 = 4$$

and

$$c_2(\mathcal{X}) = 3 - \frac{1}{2}(2 - 16 + 3(2-6) + (2-7) + 2(2-9)) - 29 + \frac{23}{4} + \frac{2}{8} + \frac{1}{16} + \frac{3}{96} = \frac{83}{32},$$

thus : $c_1^2(\mathcal{X}) - c_2(\mathcal{X}) > 0$ and by Theorem 23, $\mathcal{X}$ is quasi-hyperbolic. The moduli of 2 lines in $\mathbb{P}^2$ is 4 dimensional. $\qquad\square$

REFERENCES

[1] Barth W., Hulek K., Peters C., Van De Ven A., *Compact complex surfaces*, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete, 2nd edition, Springer-Verlag, Berlin, (2004).

[2] Bloch A., *Sur les systèmes de fonctions uniformes satisfaisant à l'équation d'une variété algébrique dont l'irrégularité dépasse la dimension,* J. de Math., 5 (1926), 19–66.

[3] Bogomolov F., *Families of curves on a surface of general type*, Soviet Math. Dokl. 18 (1977), 1294–1297.

[4] Bogomolov F., De Oliveira B., *Symmetric tensors and geometry of $\mathbb{P}^N$ subvarieties*, Geom. Funct. Anal. 18 (2008), no. 3, 637–656

[5] Bogomolov F., De Oliveira B., *Hyperbolicity of nodal hypersurfaces*, J. Reine Angew. Math. 596 (2006), 89–101.

[6] Brieskorn, E., *Singular elements of semi-simple algebraic groups*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 279–284. Gauthier-Villars, Paris, 1971.

[7] Campana F., *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier 54 (2004), 499–665.

[8] Campana F., *Orbifoldes spéciales et classification biméromorphe des variétés kählériennes compactes*, arXiv:0705.0737.

[9] Campana F., Claudon B., *Abelianity conjecture for special compact Kahler threefolds*, preprint arXiv:1107.0168.

[10] Chen Xi, *On Algebraic Hyperbolicity of Log Varieties*, Commun. Contemp. Math. 6 (2004), no. 4, 513–559. Also available as preprint math.AG/0111051.

[11] Chen Xi, *On Algebraic Hyperbolicity of Log surfaces*, preprint math.AG/0103084.

[12] Demailly J.-P., *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*, Proc. Sympos. Pure Math., vol.62, Amer. Math.Soc., Providence, RI, 1997, 285–360.

[13] Demailly, J.-P., El Goul, J., *Hyperbolicity of generic surfaces of high degree in projective 3-space*, Amer. J. Math. 122 (2000), no. 3, 515–546.

[14] Diverio S., Rousseau E., *A survey on hyperbolicity of projective hypersurfaces*, Publicacoes Matematicas do IMPA, 2011.

[15] Farkas H. M. , Kra I., *Riemann Surfaces*, Springer-Verlag, New-York, 1980, second edition.

[16] Ghigi A., Kollár J., *Kähler-Einstein metrics on orbifolds and Einstein metrics on spheres*, Comment. Math. Helv. 82 (2007), 877–902.

[17] Green M., Griffiths P., *Two applications of algebraic geometry to entire holomorphic mappings*, The Chern Symposium 1979, Proc. Inter. Sympos. Berkeley, CA, 1979, Springer-Verlag, New-York, 1980, 41–74.

[18] Horikawa E., *On deformations of quintic surfaces*, Invent. math. 31, 43–85 (1975).

[19] Horikawa E., *Algebraic surfaces of general type with small c_1^2*, Ann. of Math. 104(2) (1976), 357–387.

[20] Kawasaki T., *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math., 16(1):151–159, 1979.

[21] Kobayashi R., *Uniformization of complex surfaces*, Kähler metric and moduli spaces, 313–394, Adv. Stud. Pure Math., 18-II, Academic Press, Boston, MA, 1990

[22] Kobayashi S., *Hyperbolic complex spaces*, Springer-Verlag, Berlin, 1998.

[23] Lang, S., *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. (N.S.) 14 (1986), no. 2, 159–205.

[24] Liedtke C., Schütt M., *Unirational Surfaces on the Noether Line*, Pacific J. Math. 239 (2009), no. 2, 343–356.

[25] Lu S.S.Y., *On surfaces of general type with maximal Albanese dimension*, J. Reine Angew. Math. 641 (2010), 163–175.

[26] Lu S.S.Y., Yau S.T., *Holomorphic curves in surfaces of general type.* Proc. Nat. Acad. Sci. USA, 87 (January 1990), 80–82.

[27] Megyesi G., "Generalisation of the Bogomolov-Miyaoka-Yau inequality to singular surfaces", Proc. London Math. Soc. (3) 78 (1999), no. 2, 241–282.

[28] McQuillan M., *Diophantine approximations and foliations*, Publ. IHES 87 (1998), 121–174.

[29] McQuillan M., *Bloch hyperbolicity*, preprint IHES (2001).

[30] Nakamura S., *Classification and uniformisation of log-cannonical singularities in the presence of branch loci (in Japanese)*, Ph. D. thesis, Saitama University, Japan, 1989.

[31] Pacienza G., Rousseau E., *On the logarithmic Kobayashi conjecture*, J. Reine Angew. Math. 611 (2007), 221–235.

[32] Persson U., *Horikawa surfaces with maximal Picard numbers*, Math. Ann. 259 (1982), no. 3, 287–312.

[33] Rousseau E., *Hyperbolicity of geometric orbifolds*, Trans. Amer. Math. Soc. 362 (2010), no. 7, 3799–3826.

[34] Rousseau E., *Degeneracy of holomorphic maps via orbifolds*, to appear in Bull. Soc. Math. France.

[35] Schütt M., *Quintic Surfaces with maximum and other Picard numbers*, to appear in Journal Math. Soc. Japan, arXiv 0812.3519v5.

[36] Tian G., Yau S.-T., *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, Adv. Ser. Math. Phys. 1 1 (1987), p. 574–628, Mathematical aspects of string theory (San Diego, Calif., 1986).

[37] Töen B., *Théorèmes de Riemann-Roch pour les champs de Deligne Mumford*, K-Theory, 18(1) (1999) 33–76.

[38] Uludag A.M., *Covering relations between ball-quotient orbifolds* Math. Ann. 328 (2004), no. 3, 503–523.

[39] Uludag A.M., *Galois coverings of the plane by K3 surfaces*, Kyushu J. Math. 59 (2005), no. 2, 393–419

[40] Xu G., *Subvarieties of general hypersurfaces in projective space*, J. Differential Geom. 39 (1994), no. 1, 139–172.

[41] Yau, S.T., *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA 74 (1977), 1798–1799.

Xavier Roulleau,
roulleau@math.ist.utl.pt
Département de Mathematica,
Instituto Superior Técnico,
Avenida Rovisco Pais
1049-001 Lisboa
Portugal

Erwan Rousseau,
erwan.rousseau@cmi.univ-mrs.fr
Laboratoire d'Analyse, Topologie, Probabilités
Université d'Aix-Marseille et CNRS
39, rue Frédéric Joliot-Curie
13453 Marseille Cedex 13
France