Summary of thesis

ROUSSEAU Erwan, Université de Bretagne Occidentale (Brest).

Title : About the conjecture of Kobayashi and the hyperbolicity of projective hypersurfaces in dimension 2 and 3.

EXAMINING BOARD :

J.P. DEMAILLY (Grenoble)	rapporteur
J. WINKELMANN (Nancy)	rapporteur
T. LEVASSEUR (Brest)	examinateur
J. HUISMAN (Brest)	examinateur
C. SORGER (Nantes)	examinateur
G. DETHLOFF (Brest)	directeur de thèse

My works deal with the hyperbolicity of complex varieties and the study of Demailly-Semple jets. The subject of my thesis is "the conjecture of Kobayashi on the hyperbolicity of projective hypersurfaces", with professor Gerd DETHLOFF as my advisor.

In 1970, S. Kobayashi [Ko.70] asked: Is it true that the complement of a generic hypersurface of degree $d \geq 2n+1$ in $\mathbb{P}^n_{\mathbb{C}}$ is hyperbolic ?

In the case of $\mathbb{P}^2_{\mathbb{C}}$, it is known that the complement of a very generic curve with k components $C_1, ..., C_k$ of degrees $(d_1, ..., d_k)$ is hyperbolic and hyperbolically imbedded in $\mathbb{P}^2_{\mathbb{C}}$ in the following cases:
1) $k \geq 5$ with any degrees [Bab84]
2) $k = 4$ and $\sum d_i \geq 5$ ([Gr74], [DSW92])
3) $k = 3$ and $d_1, d_2, d_3 \geq 2$ ([DSW92],[DSW94])
4) $k = 1$ and $d_1 \geq 15$ ([E.G])

The first part of my thesis consists in the study of the case $k = 2$, using jets developped by J-P. Demailly, J. El Goul, G. Dethloff, S.Lu. ([De95] ; [DL96]).

Main notions : Let X be a complex manifold of dimension n and $f : (\mathbb{C}, 0) \to X$ a germ of holomorphic curve. Following [De95], we introduce the vector bundle $E_{k,m}^{GG}T_X^* \to X$ whose fibers are complex valued polynomials $Q(f', f", ..., f^{(k)})$ on the fibers of J_kX, the bundle of k-jets of germs of parametrized curves in X, of weighted degree m with respect to the $\mathbb{C}^*$ action: $Q(\lambda f', \lambda^2 f'', ..., \lambda^k f^{(k)}) = \lambda^m \ Q(f', f", ..., f^{(k)})$. We define the subbundle $E_{k,m}T_X^* \subset E_{k,m}^{GG}T_X^*$, called the bundle of invariant jet differentials of order k and degree m, i.e : $Q((f \circ \phi)', (f \circ \phi)", ..., (f \circ \phi)^{(k)}) = \phi'(0)^m Q(f', f", ..., f^{(k)})$ for every $\phi \in G_k$ the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$.

The interest of these bundles is underlined by the following theorem formulated by Green and Griffiths [GG80] and finally proved by Siu : Let $P(f', f", ..., f^{(k)}) = P(f)$ a polynomial differential operator, on the first k derivatives of f, globally defined on X, with values in A^{-1}, A ample line bundle. Then for $f : \mathbb{C} \to X$ a non constant holomorphic entire curve in X, $P(f) = 0$ for any such P.

My work on the hyperbolicity of the complements of plane algebraic curves in the two component case is based on the logarithmic generalization of Demailly's jet bundles developped by Dethloff and Lu, and used by El Goul in the case of a single component.

The main result of this study is the following theorem I have obtained:

Theorem:

The complement of a very generic curve with two components in $\mathbb{P}^2$ of degrees $d_1 \leq d_2$ is hyperbolic in the sense of Kobayashi in the following cases:

1) $d_1 \geq 5$

2) $d_1 = 4$ *et* $d_2 \geq 7$

3) $d_1 = 4$ *et* $d_2 = 4$

4) $d_1 = 3$ *et* $d_2 \geq 9$

5) $d_1 = 2$ *et* $d_2 \geq 12$.

This result was published in **Comptes Rendus Mathématiques de l'Académie des Sciences in 2003 (Ser. I 336 (2003) 635-640)**:

" Hyperbolicité du complémentaire d'une courbe : le cas de deux composantes ".

The second part of my thesis is the study of Demailly-Semple jets in dimension 3. My work is based on the representation theory. We define $A_k = \bigoplus_m (E_{k,m}T_X^*)_x$ the algebra of the germs of differential operators at a point $x \in X$. This algebra can be seen as a representation of the general linear group Gl_n. Demailly [De95] has given a characterization of 2-jets of degree m :

$Gr^{\cdot} E_{2,m} T_X^* = \underset{\lambda_1+2\lambda_2=m}{\oplus} \Gamma^{(\lambda_1,\lambda_2,0)} T_X^*$, where Γ is the Schur functor.

My work in dimension 3 has first shown the necessity of considering jets of order 3. Indeed, thanks to Schur complexes I have shown the following result:

Theorem:

Let X a smooth hypersurface in $\mathbb{P}^4$:

Then: $H^0(X, E_{2,m} T_X^*) = 0$, *i.e there are no non zero globally defined 2-jets.*

The algebraic study has led me to a characterization of 3-jets of degree m in dimension 3. First I have obtained the following result:

Theorem:

In dimension 3:

$$A_3 = C[f_i', w_{ij}, w_{ij}^k, W]$$

where $W = \begin{vmatrix} f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \\ f_1''' & f_2''' & f_3''' \end{vmatrix}$, $w_{ij} = f_i' f_j'' - f_i'' f_j'$,

$w_{ij}^k = (f_k')^4 d(\frac{w_{ij}}{(f_k')^3}) = f_k'(f_i' f_j''' - f_i''' f_j') - 3f_k''(f_i' f_j'' - f_i'' f_j')$.

Going back to geometry my main result is :

Theorem:

If X is a complex variety of dimension 3:

Then:

$$Gr^{\cdot} E_{3,m} T_X^* = \underset{0 \le \gamma \le \frac{m}{5}}{\oplus} \Big(\underset{\{\lambda_1+2\lambda_2+3\lambda_3=m-\gamma;\lambda_i-\lambda_j\ge\gamma, i<j\}}{\oplus} \Gamma^{(\lambda_1,\lambda_2,\lambda_3)} T_X^* \Big)$$

where Γ is the Schur functor.

A Riemann-Roch computation has given the

Proposition *For X a smooth hypersurface in $\mathbb{P}^4$* of degree d

$$\chi(X, E_{3,m}T_X^*) = \frac{m^9}{81648 \times 10^6} d(389d^3 - 20739d^2 + 185559d - 358873) + O(m^8)$$

Corollary *For $d \geq 43$, $\chi(X, E_{3,m}T_X^*) \sim \alpha(d)m^9$ with $\alpha(d) > 0$.*

My projects in view are to obtain some results on the hyperbolicity of generic hypersurfaces in $\mathbb{P}^4$ of sufficently high degree.

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