Realizability with stateful computations for nonstandard analysis

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Nonstandard analysis

Reaching the inaccessible

Once upon the 17th century...

Leibniz

1684.pdf

$$y = x^2$$

$$y + dy = (x + dx)^2$$

$$y + dy = x^2 + 2x\,dx + dx^2$$

$$\frac{dy}{dx} = 2x + dx = 2x$$

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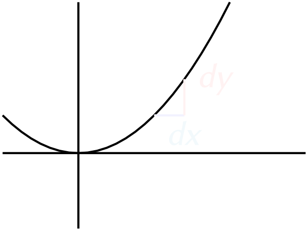

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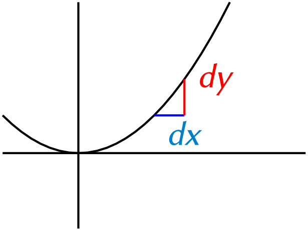

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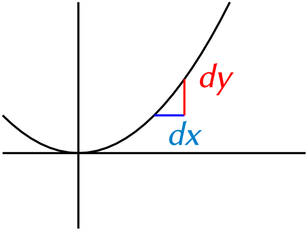

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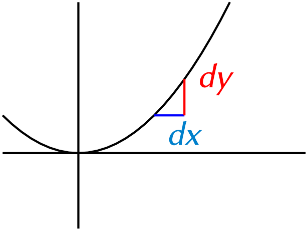

Intuitive explanation

Mismatch between **theoretical objects**:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^k$: infinite
- $\mathbb{R}$, $\mathcal{F}(\mathbb{R}, \mathbb{R})$, $\mathcal{P}(\mathbb{N})$: (very) infinite

and objects that are **concretely accessible**:

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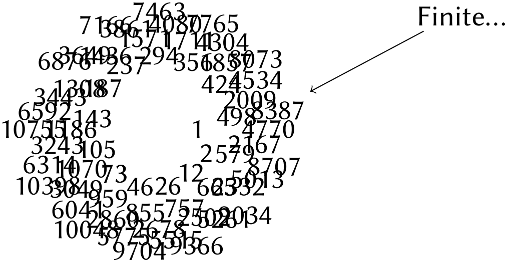

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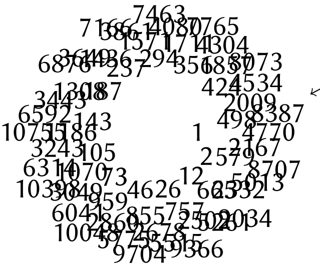

Finite...

... hence there is a number that is greater than all accessible numbers?

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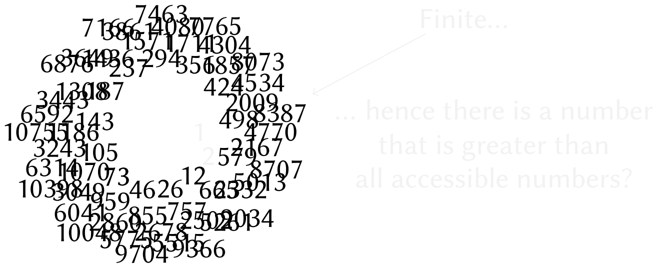

Intuition :

Accessible / non-accessible elements

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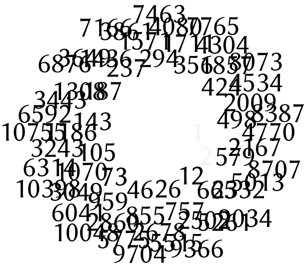

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Robinson (1961):

Standard / non-standard elements

Internal set theory

Nelson IST: syntactical extension

$$\mathcal{L} \quad + \quad \mathrm{st}(x)$$

internal formulas / external formulas

$\mathcal{L}$ — $\mathcal{L} + st(x)$

- **Transfer:** conservativity over standard world

  $\forall^{st} x.A(x) \Rightarrow \forall x.A(x)$ *(A internal)*

- **Idealization:** saturation property

  $\forall^{st}(n \in \mathbb{N}).\exists^{st} x.\forall^{st} y \le n.R(x,y) \Rightarrow \exists x.\forall^{st} y.R(x,y)$ *(R internal)*

- **Standardization:** comprehension scheme

  $\forall^{st} B.\exists^{st} A.\forall^{st} z.(z \in A \Leftrightarrow z \in B \wedge C(z))$ *(any C)*

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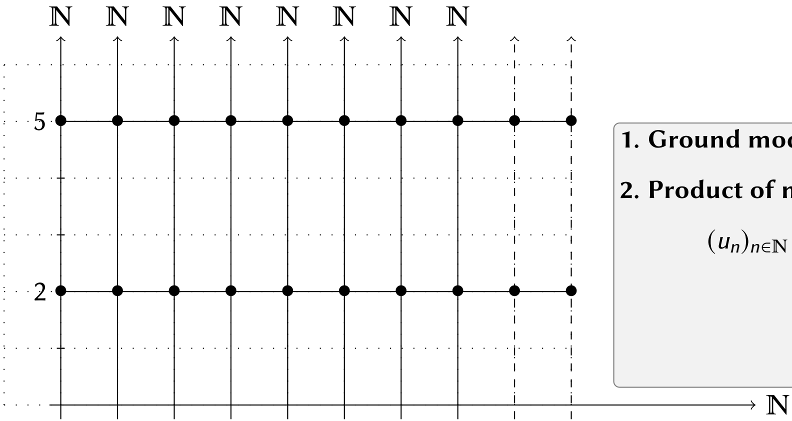
Nonstandard models

1. Ground model

3. Quotient (*w.r.t.* $\mathcal{U}$)

$[u_n]$

Nonstandard models

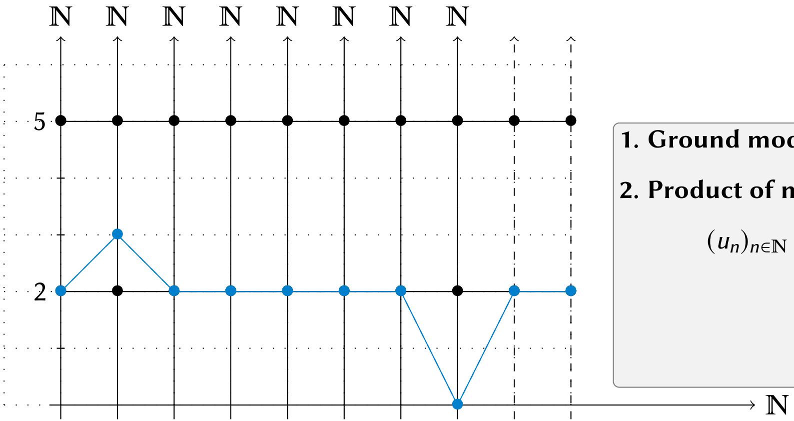


1. Ground model

2. Product of models

$$(u_n)_{n\in\mathbb{N}}$$

Nonstandard models



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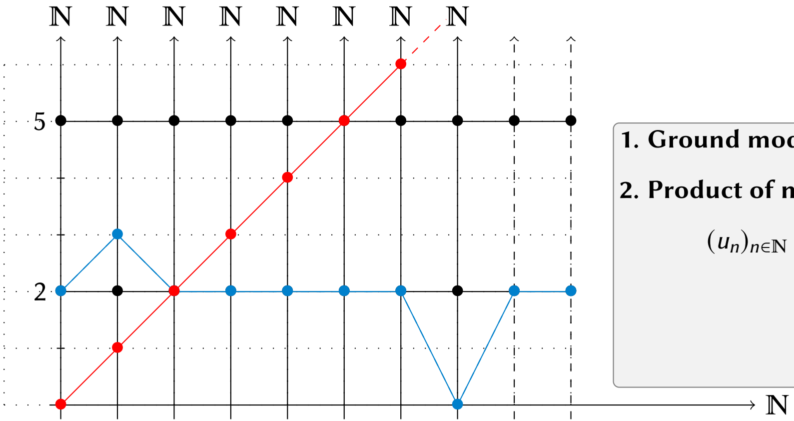

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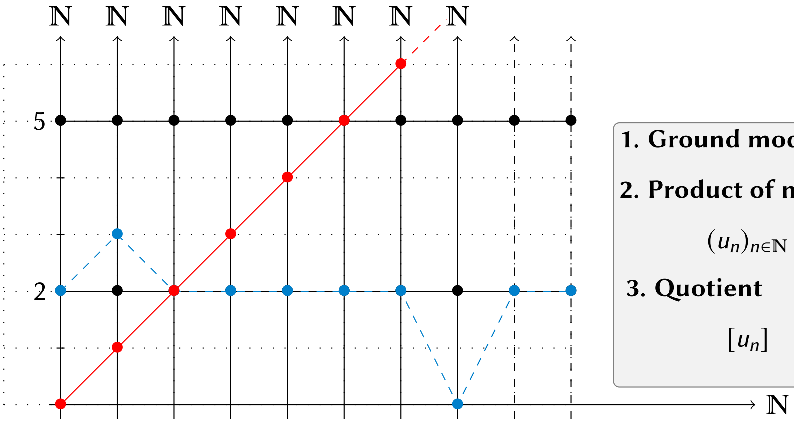

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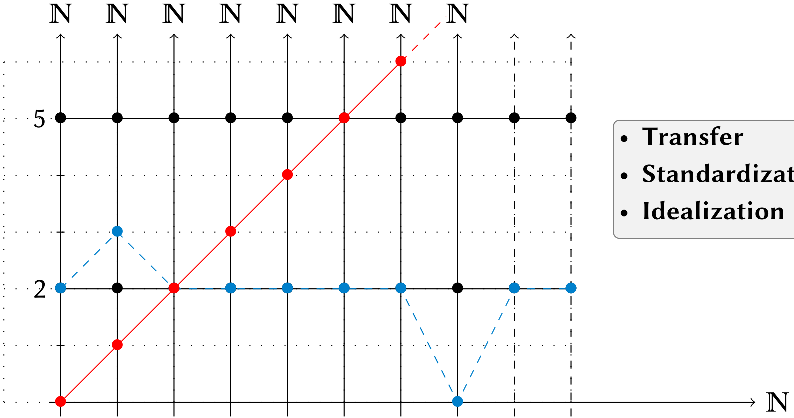

- **Transfer**
- **Standardization**
- **Idealization**

Constructive interpretations

State of the art:

- pioneer works by Moerdijk, Palmgren and Avigad
- variants of Kreisel's modified realizability
  Van den Berg, Briseid & Safarik, Ferreira & Gaspar, D. & Gaspar

$\looparrowright$ all inspired by Nelson's syntactical approach
translations of formulas inducing conservative extensions of Heyting arithmetic

This talk

Computational interpretation relying on stateful computations

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Realizability

Unveiling the computational contents of proofs

Realizability defines models

Realizability:

$$A \mapsto \{t : t \Vdash A\}$$

(intuition: programs that share a common computational behavior given by A)

Tarski

$$A \mapsto |A| \in \mathbb{B}$$

(intuition: level of truthness)

Great news

Realizability semantics gives surprisingly new models!

(e.g., Krivine realizability provides a direct construction of $\mathcal{M} \vDash ZF_\varepsilon + \neg CH + \neg AC$)

Realizability, a 3-steps recipe

1. formulas
2. an operational semantics
3. formulas interpretation

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1. formulas (*a.k.a. types*)

   ↬ *simple types*, 2^{nd} − *order logic* , *ZF*, ...

2. an operational semantics (*a.k.a. your favorite calculus*)

   ↬ *some* λ − *calculus* , *a combinators algebra, PCF, etc.*

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   ↬ $|A| = \{t \in \Lambda : t \Vdash A\}$

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Adequacy

$$\text{If} \quad \mathfrak{p} : (\Gamma \vdash A) \quad \text{then} \quad \mathfrak{p}^* \in |A|.$$

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If $\quad \Gamma \vdash t : A \quad$ then $\quad t \in |A|$.

A simple realizability interpretation

Types & terms: (excerpt)

1st-order exp.	e	$::=$	$x \mid 0 \mid S(e) \mid f(e_1, \ldots, e_n)$
Formulas	A, B	$::=$	$\mathrm{Nat}(e) \mid X(e_1, \ldots, e_n) \mid A \to B \mid \ldots$
			$\mid \forall x.A \mid \exists x.A \mid \forall X.A \mid \exists X.A$
Terms	t, u	$::=$	$x \mid 0 \mid \mathbf{succ} \mid \mathbf{rec} \mid \lambda x.t \mid t\,u \mid \ldots$

where $f : \mathbb{N}^n \to \mathbb{N}$ is any arithmetical function.

Typing rules:

$$\frac{}{\Gamma \vdash 0 : \mathrm{Nat}(0)} \qquad \frac{}{\Gamma \vdash \mathbf{rec} : \forall Z.Z(0) \to (\forall^{\mathbb{N}} y.(Z(y) \to Z(S(y)))) \to \forall^{\mathbb{N}} x.Z(x)}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x\,.\,t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\,(\to_E)$$

$$\frac{\Gamma \vdash t : A[x := n]}{\Gamma \vdash t : \exists x.A} \qquad \frac{\Gamma \vdash t : A[X(x_1, \ldots, x_n) := B]}{\Gamma \vdash t : \exists X.A} \qquad \ldots$$

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Typing rules:

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Reductions:

$$\overline{(\lambda x.t)u \triangleright_\beta t[u/x]} \qquad \overline{\mathbf{rec}\ u_0\ u_1\,(\mathbf{succ}\ t) \triangleright_\beta u_1\ t\,(\mathbf{rec}\ u_0\ u_1\ t)} \qquad \cdots$$

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Key ideas:

- realizers compute

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Value restriction

Realizability is **flexible**.

Value restriction

Scenario: *you want to change the calculus* *(here: call-by-value)*

$$V \quad ::= \quad 0 \mid \mathbf{succ}\ V \mid \lambda x.t \mid \cdots \qquad\qquad \overline{(\lambda x.t)V \triangleright_{\mathrm{v}} t[V/x]}$$

What about the interpretation ?

↬ *same recipe!*

1. formulas:
2. terms:
3. interpretation:

$$|\{A\} \mapsto B|_\rho \quad \triangleq \quad \{t \in \Lambda : \forall V \in |A|_\rho.(t\ V \in |B|_\rho)\}$$

Quick check:

✓ $|\{A\} \mapsto B|_\rho$ saturated ✓ adequacy preserved

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Scenario: *you want to change the calculus* *(here: call-by-value)*

$$V \quad ::= \quad 0 \mid \mathbf{succ}\ V \mid \lambda x.t \mid \cdots \qquad\qquad \overline{(\lambda x.t)V \triangleright_{\mathrm{v}} t[V/x]}$$

What about the interpretation ?

↬ *same recipe!*

1. formulas: $A, B \quad ::= \quad \ldots \mid \{A\} \mapsto B$
2. terms:

$$\frac{\Gamma \vdash t : A \to B}{\Gamma \vdash t : \{A\} \mapsto B} \qquad \frac{\Gamma \vdash t : \{A\} \mapsto B \quad \Gamma \vdash V : A}{\Gamma \vdash t\, V : B}$$

3. interpretation:

$$|\{A\} \mapsto B|_\rho \quad \triangleq \quad \{t \in \Lambda : \forall V \in |A|_\rho.(t\ V \in |B|_\rho)\}$$

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Value restriction

In the sequel, **stateful computations**: evaluation strategy *matters*!

Shorthands:

$$\forall^{\{\mathbb{N}\}}x.A \triangleq \forall x.(\{\mathrm{Nat}(x)\} \mapsto A)$$
$$\exists^{\{\mathbb{N}\}}x.A \triangleq \forall X.(\forall^{\{\mathbb{N}\}}x.(A \to X)) \to X$$

We can define **wit** such that:

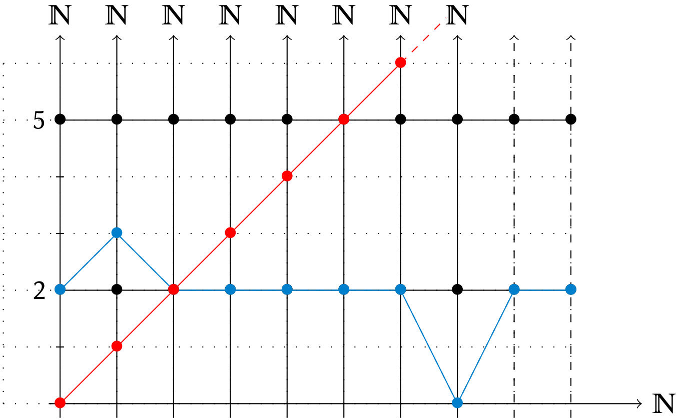
Witness extraction

If $t \in |\exists^{\{\mathbb{N}\}}x.A|_\rho$ then $\exists n \in \mathbb{N}, \exists\ u \in |A[x := n]|_\rho$, t **wit** $\triangleright^* (\overline{n}, u)$.

Stateful realizability interpretation

From states to slices

Stateful computations



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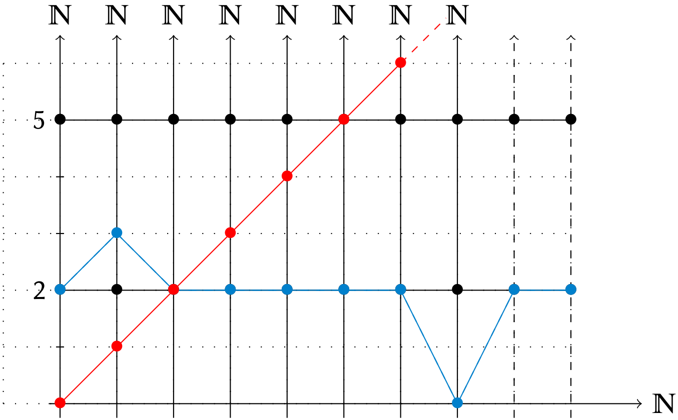

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λ-calculus with states:

Terms $\quad t, u \quad ::= \quad \ldots \mid \mathbf{get} \mid \mathbf{set}$ $\qquad$ States $\quad \mathfrak{S} \triangleq \mathbb{N}$

Reductions:

$$\frac{t \triangleright_\beta t'}{t \triangleright^{\mathfrak{s}}_{\mathfrak{s}} t'} \qquad \frac{}{\mathbf{get} \triangleright^{\mathfrak{s}}_{\mathfrak{s}} \mathfrak{s}} \qquad \frac{\mathfrak{s}'' = \max(\mathfrak{s}, \mathfrak{s}')}{\mathbf{set}\,\bar{\mathfrak{s}}\; t \triangleright^{\mathfrak{s}'}_{\mathfrak{s}''} t} \qquad \frac{t \triangleright^{\mathfrak{s}}_{\mathfrak{s}'} t'}{C[t] \triangleright^{\mathfrak{s}}_{\mathfrak{s}'} C[t']}$$

where $C[\cdot]$ are well-chosen contexts

Realizability interpretation

Individuals
$\mathbb{N}^{\mathfrak{S}}$

Truth values
$\mathcal{P}(\Lambda \times \mathfrak{S})$

Predicates
$\mathbb{N}^k \to \mathcal{P}(\Lambda \times \mathfrak{S})$

Application: $F@(f_1, \ldots, f_k)$ defined slice-wise

$F@(f_1, \ldots, f_k) \triangleq \{(t;\mathfrak{s}) : (t;\mathfrak{s}) \in F(f_1(\mathfrak{s}), \ldots, f_k(\mathfrak{s}))\}$

Adequacy

Typing rules are adequate, except 2[nd]-order comprehension rules .

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Glueing theorem

Theorem

The interpretation of internal formulas can be decomposed as the product of its slices.

formally : $(t; \mathfrak{s}) \in |A|_\rho \Leftrightarrow t \in |\overline{A}^{\mathfrak{s}}|^{\mathfrak{s}}_\rho$

Consequence

2nd-order comprehension rules are adequate for **internal formulas**.

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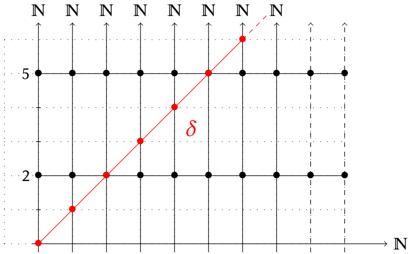

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   realized with **get**
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   elimination only for internal formulas

Transfer

For any *internal* formula $A(x)$ we have:

$$\bigcap_{f\in\mathbb{N}^{\mathfrak{S}}} |A|^{\mathfrak{S}}_{x\mapsto f} = \bigcap_{n\in\mathbb{N}} |A|^{\mathfrak{S}}_{x\mapsto n^*} \qquad \bigcup_{f\in\mathbb{N}^{\mathfrak{S}}} |A|^{\mathfrak{S}}_{x\mapsto f} = \bigcup_{n\in\mathbb{N}} |A|^{\mathfrak{S}}_{x\mapsto n^*}$$

Intuition: internal $\Rightarrow$ glueing (i.e. slice wise)

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The following formulas are realized: *($A(x)$ internal)*

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Nonstandard reasoning principles

External induction

For any formula $A(x)$ s.t. $FV(A(x)) = \{x\}$:

$$\mathbf{rec} \Vdash A(0^*) \rightarrow \forall^{\{st\}} x.(A(x) \rightarrow A(S(x))) \rightarrow \forall^{\{st\}} x.A(x).$$

induction restricted to standard elements

Overspill

For any internal formula A:

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existence of nonstandard elements

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$(R : \mathbb{N}^2 \to \mathbb{N})$

Intuition:

1. from a realizer of $\forall^{\{st\}} n. \exists^{\{st\}} x. \forall^{\{st\}} y. (y \leq n \to R(x,y))$
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$$u_n \Vdash \forall^{\{st\}} y. (y \leq n \to R(\tau_n, y)) \qquad (\textit{in the state } \mathfrak{s} = n)$$

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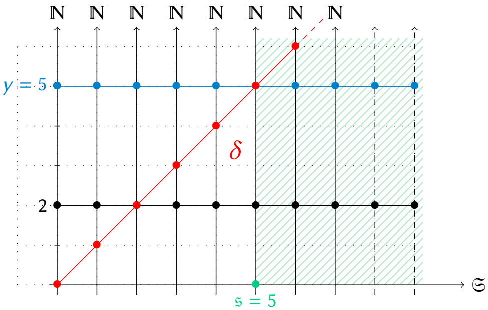

$\forall^{st} y. y \leq \delta$

Conclusion

Ongoing & future work

Towards a quotient

Missing step:

quotient up to some ultrafilter $\mathcal{U}$

Here:

$$|A|^{\mathfrak{S}} \overset{?}{\mapsto} |A|^*$$

A guideline:

Łoś 's theorem

$$t \in |A|^* \quad \text{iff} \quad \{\mathfrak{s} \in \mathfrak{S} : (t;\mathfrak{s}) \in |A|_\rho\} \in \mathcal{U}$$

Here: *only* 1^{st}*-order internal formulas*

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Future work

- **Quotient**: unsatisfactory as such
- **Missing**: LLPO, full standardization
- **Next steps**:
  - consider the classical case
  - compare with existing related works
    (functional interp. / bounded realizability / ...)
  - is this the direct presentation of some logical translation?

Questions?