Concurrent Realizability on Conjunctive Structures

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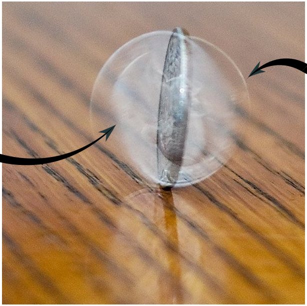
Aix*Marseille université Socialement engagée

Realizability, two sides of the same coin

provides **models** for theories



a **tool** to analyse programs behavior

Realizability, two sides of the same coin

$$t \quad \Vdash \quad A$$

$\nearrow$ $\uparrow$ $\nwarrow$

program *"realises"* *formula*

Intuitively

$$t \Vdash A \quad = \quad \text{"}t \text{ computes (soundly) w.r.t. } A\text{"}$$

Algebrization of realizability

Allows to:

- abstract from implementation details
- focus on their structures
- reason collectively on interpretations

Algebrization of realizability

Implicative algebra **[Miquel 20]**

complete lattice	$(\mathcal{A}, \preccurlyeq, \curlywedge)$	+	$\cdot \to \cdot \in \mathcal{A}^{\mathcal{A}\times\mathcal{A}}$	*"implication"*
		+	$\mathcal{S} \subseteq \mathcal{A}$	*separator*

Application $\quad a@b \triangleq \curlywedge\{c \in \mathcal{A} : a \preccurlyeq b \to c\}$

Abstraction $\quad \lambda f \triangleq \curlywedge_{a\in\mathcal{A}}(a \to f(a))$

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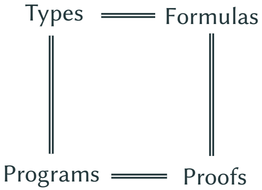

Order relation $\cdot \preccurlyeq \cdot$:

- $A \preccurlyeq B$ — A subtype of B
- $t \preccurlyeq A$ — t realizes A
- $t \preccurlyeq u$ — t is more defined than u

Soundness

1. If $\vdash t : A$ then $t^{\mathcal{A}} \preccurlyeq A^{\mathcal{A}}$ *(w.r.t. typing)*
2. If $t \to_\beta u$ then $t^{\mathcal{A}} \preccurlyeq u^{\mathcal{A}}$. *(w.r.t. computation)*

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Types ═ Formulas ‖ Programs ═ Proofs

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This work

Concurrent calculi:

- many syntactic presentations (CCS, π-calculi, etc...)
- many type systems, none as tight and universal as for λ-calculi
- many works seeking for simpler or more general theories.

Our take

an algebraic presentation of concurrent realizability might provide us with a satisfying framework!

↬ *We follow Beffara's previous work on the matter*

The specification:

- study processes and their types via an ordered algebraic structure
- avoid imposing a priori restrictions on processes.

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Processes with global fusions

Processes

Processes Π

$$P, Q ::= \underbrace{\mathbb{1}}_{\textit{unit}} \mid \underbrace{(P \mid Q)}_{\textit{parallel composition}} \mid \underbrace{u(\vec{x}).P}_{\textit{reception}} \mid \underbrace{\bar{u}\,\langle\vec{v}\rangle}_{\textit{emission}} \mid \underbrace{(\nu y)P}_{\textit{hiding}}$$

↬ *names are taken in* $\mathbb{N}$, $u(\vec{x}).\cdot$ *and* $\nu y.\cdot$ *are binders*

Substitution: $\sigma : \mathbb{N} \to \mathbb{N}$

One-step reduction: $\bar{u}\,\langle\vec{v}\rangle \mid u(\vec{x}).P \longrightarrow_1 P\{\vec{x} := \vec{v}\}$

NB - this is just a parameter of the construction, which applies to other processes calculi

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$$\bar{a}\langle b\rangle$$

$$\uparrow$$

$$u(x).\bar{x}\langle b\rangle \mid \bar{u}\langle a\rangle$$

$$\uparrow$$

$$u(x).\bar{x}\langle b\rangle \mid \bar{y}\langle a\rangle \mid y(x).\bar{u}\langle x\rangle$$

$$\uparrow$$

$$\bar{u}\langle y\rangle \mid u(x).\bar{x}\langle b\rangle \mid u(x).\bar{x}\langle a\rangle \mid y(x).\bar{u}\langle x\rangle$$

$$\downarrow$$

$$\mid u(x).\bar{x}\langle a\rangle \mid \quad .\bar{u}\langle x\rangle$$

$$\downarrow$$

$$.\bar{x}\langle a\rangle \mid$$

$$\downarrow$$

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Global fusions

Intuition

$u \leftrightarrow v$ allows actions on u to synchronize with actions on v.

Fusions - $\mathcal{E}$

A *fusion* $e \in \mathcal{E}$ is an equivalence relation $\cdot \underset{e}{\sim} \cdot$ over $\mathbb{N}$.

Fusion vs. substitution

We can define

- $\varepsilon_\tau \triangleq \bigvee_{x \in \mathbb{N}} (x \leftrightarrow \tau(x))$, the fusion induced by τ;
- $\sigma_e^\bullet : x \mapsto x_e^\bullet$, the substitution induced by e;

  with $x_e^\bullet$ a canonical representative of $[x]_e$

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Processes with fusions

Specification: extend a process calculus *without* affecting its theory
↬ *technically made possible by only having fusions at top-level*

Processes with global fusions - $\bar{\Pi}$

$$\bar{\Pi} \triangleq \Pi \times \mathcal{E} = \{(P, e) \mid P \in \Pi, e \in \mathcal{E}\}$$

↬ *we extend syntax, substitution, α-equivalence, ≡, reduction, etc., everything works.*

Syntax

$$(P, e) \mid (Q, f) \triangleq (P \mid Q, ef), \quad \text{etc...}$$

Reduction

$$(P, e) \longrightarrow_1 (Q, f) \triangleq \; e = f \;\; \wedge \;\; P^{\sigma_e^\bullet} \longrightarrow_1 Q^{\sigma_f^\bullet}$$

which entails:

$$(u(x).P \mid \bar{v}\langle y \rangle, u{\leftrightarrow}v) \longrightarrow_1 (P\{x := y\}, u{\leftrightarrow}v)$$

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Realizability interpretation

Concurrent realizability [Beffara'06]

Main insight from Krivine realizability:

interpretation parameterized by a *pole* $\perp\!\!\!\perp$, specifying "correct" interactions

Pole

Any $\perp\!\!\!\perp \subseteq \{p \in \bar{\Pi} \mid \mathrm{FN}(p) = \emptyset\}$ that is closed under $\equiv$.

↪ *characterize the soundness of interaction wrt. the renaming mechanism*

This induces a map $(\cdot)^\perp$ on $\mathbb{P}$:

$$A^\perp = \{p \in \bar{\Pi} | \forall q \in A, \bar{\nu}(p \mid q) \in \perp\!\!\!\perp\}$$

Behaviors - $\mathbb{B} \triangleq \{A \in \mathbb{P} : A^{\perp\perp} = A\}$

- allows to define truth values
- induces a complete lattice.

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Interpreting MLL formulas

Main idea

$\cdot \otimes \cdot$ amounts to parallel composition without communication.

Technically: *injections $\iota_1, \iota_2 : \mathbb{N} \longrightarrow \mathbb{N}$ with disjoint codomains*

For any PGFs p, q, we let : *(writing $p^i \triangleq p^{\iota_i}$, $p^{-i} \triangleq p^{\iota_i^{-1}}$)*

- $p \bullet q \triangleq p^1 \mid q^2$ *(tensor)*
- $p * q \triangleq ((\nu \mathbb{N}^1)(p \mid q^1))^{-2}$ *(application)*

On behaviors $A, B \in \mathbb{B}$, we define:

$$
\begin{aligned}
1 &:= \{(\mathbb{1}, \Delta_{\mathbb{N}})\}^{\perp\perp} \\
A \bullet B &:= \{p \bullet q \mid p \in A,\ q \in B\} \\
A \mid B &:= \{p \mid q \mid p \in A,\ q \in B\}
\end{aligned}
$$

$$
\begin{aligned}
A \otimes B &:= (A \bullet B)^{\perp\perp} \\
A \parr B &:= (A^\perp \otimes B^\perp)^\perp \\
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Easy example

Proposition

For any $A \in \mathbb{P}$, we have that $I \triangleq \bigvee_{n \in \mathbb{N}} (n.1 \leftrightarrow n.2) \Vdash A \multimap A$.

Proof :

By definition, $A \multimap A = (A \bullet A^\perp)^\perp$.

Consider $p \in A$ and $q \in A^\perp$, we have to prove that $I \perp\!\!\!\perp p^1 \mid q^2$. Using properties of fusions, we get

$$\bar{\nu}(p^1 \mid q^2 \mid I) \equiv_\alpha (\nu_2)(\nu_1)(p^1 \mid q^2 \mid I) \equiv_\alpha (\nu_2)(p^2 \mid q^2) \equiv_\alpha (\overline{\nu})(p \mid q)$$

We conclude by observing that $(\overline{\nu})(p \mid q) \in \perp\!\!\!\perp$.

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Proof :

By definition, $A \multimap A = (A \bullet A^{\perp})^{\perp}$.

Consider $p \in A$ and $q \in A^{\perp}$, we have to prove that $I \perp\!\!\!\perp p^1 \mid q^2$. Using properties of fusions, we get

$$\bar{v}(p^1 \mid q^2 \mid I) \equiv_\alpha (v_2)(v_1)(p^1 \mid q^2 \mid I) \equiv_\alpha (v_2)(p^2 \mid q^2) \equiv_\alpha (\overline{v})(p \mid q)$$

We conclude by observing that $(\overline{v})(p \mid q) \in \perp\!\!\!\perp$.

Easy example

Proposition

For any $A \in \mathbb{P}$, we have that $I \triangleq \bigvee_{n \in \mathbb{N}} (n.1 \leftrightarrow n.2) \Vdash A \multimap A$.

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We conclude by observing that $(\overline{v})(p \mid q) \in \perp\!\!\!\perp$.

Conjunctive structures

Conjunctive involutive structures

Conjunctive structures **[M. 2020]**

A CS is a complete lattice $(\mathbb{C}, \preccurlyeq)$ equipped with $\otimes$ / $(\cdot)^\perp$ s.t.:

- $\otimes$ is monotone / $(\cdot)^\perp$ is antimonotone;
- $\curlyvee_{b \in \mathfrak{B}}(a \otimes b) = a \otimes \left(\curlyvee_{b \in \mathfrak{B}} b\right)$ and $\curlyvee_{b \in \mathfrak{B}}(b \otimes a) = \left(\curlyvee_{b \in \mathfrak{B}} b\right) \otimes a$;
- $\left(\curlyvee_{b \in \mathfrak{B}} b\right)^\perp = \curlywedge_{b \in \mathfrak{B}} b^\perp$ (De Morgan's law).

A CS is *involutive* (CIS) if $(\cdot)^\perp$ is, *unitary* with $1 \in \mathbb{C}$.

Examples

- any Boolean algebra $(\mathbb{B}, \preccurlyeq, \wedge, \vee, \neg)$,
- the call-by-value λ-calculus (not involutive though)
- Girard's phase space induces a CIS.

Separators & internal logic

Any CIS induces an interpretation of MLL formulas:

$$
\begin{array}{rcl|rcl}
a \parr b & \triangleq & (a^\perp \otimes b^\perp)^\perp & \exists F & \triangleq & \bigcurlyvee_{a \in \mathbb{C}} F(a) \\
a \multimap b & \triangleq & (a \otimes b^\perp)^\perp & \forall F & \triangleq & \bigcurlywedge_{a \in \mathbb{C}} F(a)
\end{array}
$$

Combinators

- $S_3 \triangleq \bigcurlywedge_{a,b \in \mathbb{C}} (a \otimes b) \multimap (b \otimes a)$ *commutativity*
- $S_4 \triangleq \bigcurlywedge_{a,b,c \in \mathbb{C}} (a \multimap b) \multimap (a \otimes c) \multimap (b \otimes c)$ *compat.* $\multimap$
- $S_5 \triangleq \bigcurlywedge_{a,b,c \in \mathbb{C}} ((a \otimes b) \otimes c) \multimap (a \otimes (b \otimes c))$ *associativity*
- $S_6 \triangleq \bigcurlywedge_{a \in \mathbb{C}} a \multimap (1 \otimes a)$ *unit*
- $S_7 \triangleq \bigcurlywedge_{a \in \mathbb{C}} (1 \otimes a) \multimap a$ *unit*
- $S_8 \triangleq \bigcurlywedge_{a,b \in \mathbb{C}} (a \multimap b) \multimap (b^\perp \multimap a^\perp)$ *contrapositive*

Separator

Any upwards closed set $\mathcal{S} \subseteq \mathbb{C}$ s.t.:

- $\mathcal{S}$ contains the MLL-combinators $S_3, \ldots, S_8$.

Separators & internal logic

Any CIS induces an interpretation of MLL formulas:

$$\begin{array}{rcl|rcl} a \parr b & \triangleq & (a^\perp \otimes b^\perp)^\perp & \exists F & \triangleq & \curlyvee_{a\in\mathbb{C}} F(a) \\ a \multimap b & \triangleq & (a \otimes b^\perp)^\perp & \forall F & \triangleq & \curlywedge_{a\in\mathbb{C}} F(a) \end{array}$$

A model?

How to discriminate *valid* formulas?

Combinators

- $S_3 \triangleq \curlywedge_{a,b\in\mathbb{C}}(a\otimes b) \multimap (b\otimes a)$ *commutativity*
- $S_4 \triangleq \curlywedge_{a,b,c\in\mathbb{C}}(a \multimap b) \multimap (a\otimes c) \multimap (b\otimes c)$ *compat.* $\multimap$
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Separators & internal logic

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$$\begin{array}{rcl|rcl} a \parr b & \triangleq & (a^\perp \otimes b^\perp)^\perp & \exists F & \triangleq & \Upsilon_{a\in\mathbb{C}} F(a) \\ a \multimap b & \triangleq & (a \otimes b^\perp)^\perp & \forall F & \triangleq & \curlywedge_{a\in\mathbb{C}} F(a) \end{array}$$

Combinators

- $S_3 \triangleq \curlywedge_{a,b\in\mathbb{C}} (a \otimes b) \multimap (b \otimes a)$ *commutativity*
- $\cdots$

Separator

Any upwards closed set $\mathcal{S} \subseteq \mathbb{C}$ s.t.:

- $\mathcal{S}$ contains the MLL-combinators $S_3, \ldots, S_8$.
- If $a \multimap b \in \mathcal{S}$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$.

Conjunctive Involutive Monoidal Algebra

unitary CIS + separator.

Interpretation of MLL

Semantic judgement

A judgment $\vdash a : \Gamma$, where $a \in \mathbb{C}$ and Γ has parameters in $\mathbb{C}$, is *sound* if $a \preccurlyeq [\![\Gamma]\!]$.

Soundness

There exist $\mathbf{I}, \mathbf{c}, \mathbf{t}, \mathbf{ex}(\sigma) \in \mathcal{S}$ (for any *CIMA*) s.t. the following are sound:

$$\frac{}{\vdash 1 : \mathbb{1}}\,(\mathbb{1}) \qquad \frac{}{\vdash \mathbf{I} : A^{\perp}, A}\,(\textsf{Ax}) \qquad \frac{\vdash a : A_1, \ldots, A_k}{\vdash \mathbf{ex}(\sigma) * a : A_{\sigma(1)}, \ldots, A_{\sigma(k)}}\,(\textsf{Ex})$$

$$\frac{\vdash a : \Gamma, A \quad \vdash b : B, \Delta}{\vdash \mathbf{t} * a * b : \Gamma, A \otimes B, \Delta}\,(\otimes) \qquad \frac{\vdash a : \Gamma, A \quad \vdash b : A^{\perp}, \Delta}{\vdash \mathbf{c} * a * b : \Gamma, \Delta}\,(\textsc{Cut})$$

$$\frac{\vdash a : \Gamma, A\{X := B\}}{\vdash a : \Gamma, \exists X.A}\,(\exists) \qquad \frac{\vdash a : \Gamma, A \quad X \notin \mathsf{FV}(\Gamma)}{\vdash a : \Gamma, \forall X.A}\,(\forall)$$

Concurrent realizability as a CIMA

Conjunctive structure

The tuple $(\mathbb{B}, \subseteq, \otimes, (\cdot)^{\perp})$ is a conjunctive involutive structure.

Combinators

The following behaviors are realized by pure fusions:

1. $\bigcap_{A \in \mathbb{B}} A \multimap A$.
2. $\bigcap_{A,B \in \mathbb{B}} (A \otimes B) \multimap (B \otimes A)$.
3. $\bigcap_{A,B,C \in \mathbb{B}} (A \multimap B) \multimap (B \multimap C) \multimap A \multimap C$.
4. $\bigcap_{A,B,C \in \mathbb{B}} ((A \otimes B) \otimes C) \multimap (A \otimes (B \otimes C))$.
5. $\bigcap_{A \in \mathbb{B}} A \multimap (1 \otimes A)$.
6. $\bigcap_{A \in \mathbb{B}} (1 \otimes A) \multimap A$.
7. $\bigcap_{A,B \in \mathbb{B}} (A \multimap B) \multimap (B^{\perp} \multimap A^{\perp})$.

↬ *Essentially computations with the right fusions.*

Separator

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5. $\bigcap_{A \in \mathbb{B}} A \multimap (1 \otimes A)$.
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7. $\bigcap_{A,B \in \mathbb{B}} (A \multimap B) \multimap (B^{\perp} \multimap A^{\perp})$.

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Separator

The set of non-empty behaviors $\mathcal{S}_{\mathbb{B}} \triangleq \mathbb{B} \setminus \emptyset$ defines a separator.

Theorem

Any concurrent realizability interpretation induces a CIMA.

Concurrent realizability as a CIMA

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Theorem

Any concurrent realizability interpretation induces a CIMA.

Parallel composition

What about concurrency?

So far, we essentially described a linear tensorial calculus.

What about concurrency?

Composition: increasing and $\curlyvee$-continuous function $\diamond : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Compositional structure induced by $\diamond$

Quotient $\mathbb{C}/{\equiv_\diamond}$ where $\equiv_\diamond$ is the minimum equivalence relation s.t.

$$\frac{}{a \diamond \mathbb{1} \equiv_\diamond a} \qquad \frac{}{\mathbb{1} \diamond a \equiv_\diamond a} \qquad \frac{}{a \diamond b \equiv_\diamond b \diamond a} \qquad \frac{}{a \diamond (b \diamond c) \equiv_\diamond (a \diamond b) \diamond c}$$

$$\frac{a \equiv_\diamond a' \quad b \equiv_\diamond b'}{a \diamond b \equiv_\diamond a' \diamond b'} \qquad \frac{a \equiv_\diamond a' \quad b \equiv_\diamond b'}{a \otimes b \equiv_\diamond a' \otimes b'} \qquad \frac{a \equiv_\diamond a'}{a^\perp \equiv_\diamond a'^\perp}$$

$$\frac{a \equiv_\diamond a' \quad b \equiv_\diamond b' \quad a \preccurlyeq b}{a' \preccurlyeq b'}$$

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PGF composition

The operation $\|$ is a composition over $\mathbb{B}$, the equivalence $\equiv_\|$ is equality.

↬ *We even have a term Φ s.t. $\Phi * t * u = t \| u$.*

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This extra structure is necessary in the sense that:

Parallel composition cannot be derived

There exists a CIMA in which no term induces a composition which equivalence relation is already valid.

Embedding of the π-calculus

We know how to define all this from a π-calculus. What about the converse?

Embedding of the π-calculus

Honda & Yoshida defined a set of combinators complete wrt. π-calculus.

composition				
(I).	$u^*x.(P,\ Q)$	$\stackrel{\text{def}}{=}$	$c_1c_2 \blacktriangleright (\mathcal{D}(uc_1c_2),\ c_1^*x.P,\ c_2^*x.Q)$	c_1, c_2 fresh.
(II).	$u^*x.c \blacktriangleright P$	$\stackrel{\text{def}}{=}$	$c' \blacktriangleright u^*x.P\{c'/c\}$	c' fresh.
(III).	$u^*x.\Lambda$	$\stackrel{\text{def}}{=}$	$\mathcal{K}(u)$	
synchronization				
(IV).	$u^*x.\mathcal{C}(v^+\tilde{w})$	$\stackrel{\text{def}}{=}$	$c \blacktriangleright (\mathcal{S}(ucv),\ \mathcal{C}(c^+\tilde{w}))$	$x \notin \{v\tilde{w}\}$, c fresh.
(V).	$u^*x.\mathcal{C}(v^-\tilde{w})$	$\stackrel{\text{def}}{=}$	$c \blacktriangleright (\mathcal{S}(uvc),\ \mathcal{C}(c^-\tilde{w}))$	$x \notin \{v\tilde{w}\}$, c fresh.
binding-I				
(VI).	$u^*x.\mathcal{M}(vx)$	$\stackrel{\text{def}}{=}$	$\mathcal{FW}(uv)$	$x \neq v$
(VII).	$u^*x.\mathcal{FW}(xv)$	$\stackrel{\text{def}}{=}$	$\mathcal{B}_l(uv)$	$x \neq v$
(VIII).	$u^*x.\mathcal{FW}(vx)$	$\stackrel{\text{def}}{=}$	$\mathcal{B}_r(uv)$	$x \neq v$
binding-II				
(IX).	$u^*x.\mathcal{C}(\tilde{v_1}x^+\tilde{v_2})$	$\stackrel{\text{def}}{=}$	$c \blacktriangleright u^*x.(\mathcal{FW}(cx),\ \mathcal{C}(\tilde{v_1}c^+\tilde{v_2}))$	$x \notin \{\tilde{v_1}\}$, c fresh.
binding-III				
(X).	$u^*x.\mathcal{C}(x^-\tilde{v})$	$\stackrel{\text{def}}{=}$	$c \blacktriangleright u^*x.(\mathcal{FW}(xc),\ \mathcal{C}(c^-\tilde{v}))$	c fresh.
(XI).	$u^*x.\mathcal{B}_r(vx^-)$	$\stackrel{\text{def}}{=}$	$c_1c_2c_3 \blacktriangleright u^*x.(\mathcal{D}(vc_1c_2),\ \mathcal{S}(c_1xc_3),\ \mathcal{B}_r(c_2c_3))$	$x \neq v$ $\quad c_1, c_2, c_3$ fresh.
(XII).	$u^*x.\mathcal{S}(vx^-w)$	$\stackrel{\text{def}}{=}$	$c_1c_2 \blacktriangleright u^*x.(\mathcal{S}(vc_1c_2),\ \mathcal{M}(c_1x),\ \mathcal{B}_l(c_2w))$	$x \neq v$ $\quad c_1, c_2$ fresh.

Figure 2: Name Abstraction

Embedding of the π-calculus

Essentially, one needs to have reduction rules for these combinators:

$$
\begin{array}{rcl}
\mathcal{D}(uww'),\ \mathcal{M}(uv) & \longrightarrow & \mathcal{M}(wv),\ \mathcal{M}(w'v) \\
\mathcal{FW}(uw),\ \mathcal{M}(uv) & \longrightarrow & \mathcal{M}(wv) \\
\mathcal{K}(u),\ \mathcal{M}(uv) & \longrightarrow & \Lambda
\end{array}
\qquad
\begin{array}{rcl}
\mathcal{B}_l(uw),\ \mathcal{M}(uv) & \longrightarrow & \mathcal{FW}(vw) \\
\mathcal{B}_r(uw),\ \mathcal{M}(uv) & \longrightarrow & \mathcal{FW}(wv) \\
\mathcal{S}(uww'),\ \mathcal{M}(uv) & \longrightarrow & \mathcal{FW}(ww')
\end{array}
$$

Figure 1: Reduction Rules for Atomic Agents

Embedding of the π-calculus

Essentially, one needs to have reduction rules for these combinators:

$\mathcal{D}(uww'), \mathcal{M}(uv)$	$\longrightarrow$	$\mathcal{M}(wv), \mathcal{M}(w'v)$	$\mathcal{B}_l(uw), \mathcal{M}(uv)$	$\longrightarrow$	$\mathcal{FW}(vw)$
$\mathcal{FW}(uw), \mathcal{M}(uv)$	$\longrightarrow$	$\mathcal{M}(wv)$	$\mathcal{B}_r(uw), \mathcal{M}(uv)$	$\longrightarrow$	$\mathcal{FW}(wv)$
$\mathcal{K}(u), \mathcal{M}(uv)$	$\longrightarrow$	Λ	$\mathcal{S}(uww'), \mathcal{M}(uv)$	$\longrightarrow$	$\mathcal{FW}(ww')$

Figure 1: Reduction Rules for Atomic Agents

In our setting:

$$\mathsf{F}(a,b) \mid \mathsf{M}(a,c) \preccurlyeq \mathsf{M}(b,c)$$

Using $\triangleright$ the right adjoint to $\cdot \mid \cdot$, we can define:

$$\mathsf{F}(a,b) \triangleq \bigcurlywedge_{x \in \mathbb{N}} \left(\mathsf{M}(a,x) \triangleright \mathsf{M}(b,x)\right)$$

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Figure 1: Reduction Rules for Atomic Agents

Similarly, we define *Honda-Yoshida combinators* of $\mathbb{C}$ s.t.:

$$\begin{array}{rcl} \mathsf{K}(a)|\mathsf{M}(a,x) & \preccurlyeq & 1 \\ \mathsf{F}(a,b)|\mathsf{M}(a,x) & \preccurlyeq & \mathsf{M}(b,x) \\ \mathsf{D}(a,b,c)|\mathsf{M}(a,x) & \preccurlyeq & \mathsf{M}(b,x)|\mathsf{M}(c,x) \end{array} \quad \Bigg| \quad \begin{array}{rcl} \mathsf{Bl}(a,b)|\mathsf{M}(a,x) & \preccurlyeq & \mathsf{F}(x,b) \\ \mathsf{Br}(a,b)|\mathsf{M}(a,x) & \preccurlyeq & \mathsf{F}(b,x) \\ \mathsf{S}(a,b,c)|\mathsf{M}(a,x) & \preccurlyeq & \mathsf{F}(b,c) \end{array}$$

Honda-Yoshida algebra

CIMA+ $\mathsf{M} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ s.t. all Honda-Yoshida combinators belong to $\mathcal{S}$.

Conclusion

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What we have:

- CIMA, provides an interpretation of MLL
- additional structure for parallel composition
- realizability based on PGF induces a CIMA with parallel composition
- embedding of π-calculus using Honda & Yoshida combinators

Future work

1. Instantiate on different calculi, see if they fit.
2. Could this be a structured framework for comparing calculi?
   ↪ *(for instance, synchrone vs. asynchrone, monadic vs. polyadic)*
3. Add exponentials, additives
   ↪ *following GOI/Duchesne's PhD, Honda-Yoshida?*

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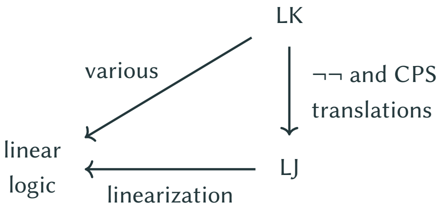
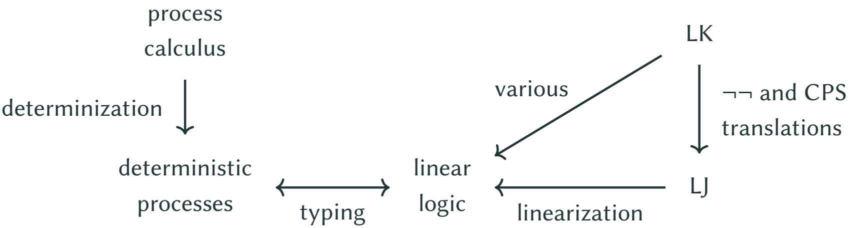

LK
various
¬¬ and CPS
translations
linear
logic
LJ
linearization

Conclusion


process
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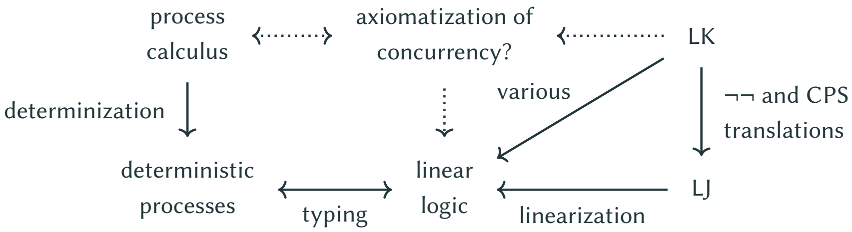
process
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The end

Thank you for your attention!