# Revisiting the duality of computation

*An algebraic analysis of classical realizability models*

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# Classical Curry-Howard

Classical logic = Intuitionistic logic + $A \vee \neg A$

**1990**: Griffin discovered that `call/cc` can be typed by Peirce's law
(well-known fact: Peirce's law $\Rightarrow A \vee \neg A$)

**Classical Curry-Howard**:

$\lambda$-calculus + `call/cc`

With side effects come new reasoning principles:

- quote instruction $\sim$ dependent choice
- monotonic memory $\sim$ Cohen's forcing
- ...

# The duality of computation

**Curien-Herbelin (2000):**

## ABSTRACT

We present the $\overline{\lambda}\mu\tilde{\mu}$-calculus, a syntax for $\lambda$-calculus + control operators exhibiting symmetries such as program/context and call-by-name/call-by-value. This calculus is derived from implicational Gentzen's sequent calculus $LK$, a key classical logical system in proof theory. Under the Curry-Howard correspondence between proofs and programs, we can see $LK$, or more precisely a formulation called $LK_{\mu\tilde{\mu}}$, as a syntax-directed system of simple types for $\overline{\lambda}\mu\tilde{\mu}$-calculus.

# Krivine realizability as a tool

**Many applications:**

- Specification problem

*t realizes* $\forall X.X \to X$ *iff* $t \star u \cdot \pi \succ u \star \pi$

- Normalization proofs

*If* $\vdash t : A$ *then* $t$ *normalizes.*

- Soundness proofs

*My calculus doesn't allow any proof term of* $\bot$.

- Witness extraction

*If* $\vdash t : \exists x^{\mathbb{N}}.f(x) = 0$ *then we can use* $t$ *to compute* $n$ *s.t.* $f(n) = 0$.

# Krivine realizability as a model

**Krivine realizability:**

$$A \mapsto \{t : t \Vdash A\}$$

*(intuition: programs that share a common computational behavior given by A)*

**Tarski**

$$A \mapsto |A| \in \mathbb{B}$$

*(intuition: level of truthness)*

**Great news**

Classical realizability semantics gives surprisingly new models!

*(in particular, provides us with a direct construction of* $\mathcal{M} \vDash ZF_\varepsilon + \neg CH + \neg AC$*)*

# This talk

**Question #1**

What is the algebraic structure of Krivine realizability models?

**Question #2**

Are call-by-name and call-by-value inducing the same models?

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# Outline

1. **Krivine classical realizability**
   *Realizability models, AKS, $^{\mathcal{K}}$OCA*
2. **Implicative algebras**
   *Implicative struct., $\lambda_c$-calculus, models & completeness properties*
3. **Disjunctive algebras**
   *Disjunctive struct., $L^{\parr}$ & connection with implicative struct.*
4. **Conjunctive algebras**
   *Conjunctive struct., $L^{\otimes}$ & connection with disjunctive struct.*

*(Bonus:*  *)*

# Krivine classical realizability

*(from an algebraic perspective)*

# Krivine realizability, a 3-steps recipe

1. an operational semantics
2. a logical language
3. formulas interpretation

# Krivine realizability, a 3-steps recipe

1. an operational semantics (*a.k.a. the abstract Krivine machine*)

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\begin{array}{lrcl}
\textsc{Push}: & (t)u \star \pi & \succ_1 & t \star u \cdot \pi \\
\textsc{Grab}: & \lambda x.t \star u \cdot \pi & \succ_1 & t\{x := u\} \star \pi \\
\textsc{Save}: & \mathbf{cc} \star t \cdot \pi & \succ_1 & t \star \mathbf{k}_\pi \cdot \pi \\
\textsc{Restore}: & \mathbf{k}_\pi \star t \cdot \rho & \succ_1 & t \star \pi
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2. a logical language (*a.k.a. a type system*)

| | | | |
|---|---|---|---|
| **1st-order terms** | $e$ | $::=$ | $x \mid f(e_1, \ldots, e_k)$ |
| **Formulas** | $A, B$ | $::=$ | $X(e_1, \ldots, e_k) \mid A \Rightarrow B \mid \forall x.A \mid \forall X.A$ |

3. formulas interpretation

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2. a logical language (*a.k.a. a type system*)
3. formulas interpretation

- pole $\perp\!\!\!\perp$: processes, referee
- falsity value $\|A\|$: stacks, opponent to $A$
- truth value $|A|$ : terms, player of $A$

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- pole $⫫$:

$$t \star \pi \succ p_0 \succ \cdots \succ p_n \in ⫫ ?$$

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**Adequacy**

Typed terms are realizers.

# Realizability models

Given the previous ingredients:

1. a calculus
2. its type system
3. an adequate interpretation of formula

one defines a model $\mathcal{M}_{⫫}$ by:

**Realizability model**

$$\mathcal{M}_{⫫} \vDash A \quad \textit{iff} \quad |A| \cap \mathbf{PL} \neq \emptyset$$

(where **PL** is the set of *proof-like* terms)

In other words:

$A$ is satisfied $\triangleq$ "*there exists a proof-like realizer of A*"

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# Streicher's *Abstract Krivine Structures*

Krivine's classical realisability from (...)
*Thomas Streicher* [2013]

## Abstract Krivine Structures

An AKS is given by $(\Lambda, \Pi, \text{app}, \text{push}, \text{k}_{\_}, \boldsymbol{k}, \boldsymbol{s}, \boldsymbol{cc}, \mathbf{PL}, \perp\!\!\!\perp)$ where:

1. $\Lambda$ and $\Pi$ are non-empty sets (*terms* and *stacks*)
2. app : $t, u \mapsto tu$ is from $\Lambda \times \Lambda$ to $\Lambda$ (*application*)
3. push : $t, \pi \mapsto t \cdot \pi$ is from $\Lambda \times \Pi$ to $\Pi$ (*push*)
4. $\text{k}_{\_}$ : $\pi \mapsto \text{k}_\pi$ is from $\Pi$ to $\Lambda$ (*continuation*)
5. $\boldsymbol{k}$, $\boldsymbol{s}$ and $\boldsymbol{cc}$ are distinguished terms of $\Lambda$;
6. $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ is a relation s.t.: (*pole*)

$$\begin{array}{rcl} t \star u \cdot \pi \in \perp\!\!\!\perp & \Rightarrow & tu \star \pi \in \perp\!\!\!\perp \\ t \star \pi \in \perp\!\!\!\perp & \Rightarrow & \boldsymbol{k} \star t \cdot u \cdot \pi \in \perp\!\!\!\perp \\ tv(uv) \star \pi \in \perp\!\!\!\perp & \Rightarrow & \boldsymbol{s} \star t \cdot u \cdot v \cdot \pi \in \perp\!\!\!\perp \end{array}$$

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**Definitions:**

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↬ *you know the rest!*

# Ordered combinatory algebras

Ordered combinatory algebras and realizability
*Ferrer et al.* [2017]

The Uruguayan approach (similar to PCA for Kleene realizability)

An OCA is given by $(\mathcal{A}, \leq, \mathrm{app}, \boldsymbol{k}, \boldsymbol{s})$ where:

- $(\mathcal{A}, \leq)$ is a poset
- $\mathrm{app} : (a, b) \mapsto ab$ is monotonic
- $\boldsymbol{k}ab \leq a$
- $\boldsymbol{s}abc \leq ac(bc)$

If $\mathcal{A}$ is an OCA, a *filter* over $\mathcal{A}$ is a subset $\Phi \subseteq \mathcal{A}$ s.t.:

- $\boldsymbol{k} \in \Phi$ and $\boldsymbol{s} \in \Phi$
- $\Phi$ is closed under application

Krivine Ordered Combinatory Algebra

A $^{\mathcal{K}}$OCA is given by $(\mathcal{A}, \leq, \mathrm{app}, \mathtt{imp}, \boldsymbol{k}, \boldsymbol{s}, \boldsymbol{e}, \boldsymbol{c}, \Phi)$ where:

- $(\mathcal{A}, \leq, \Phi)$ is a filtered OCA
- $\boldsymbol{e}, \boldsymbol{c} \in \Phi$
- $\mathtt{imp} : (a, b) \mapsto a \to b$ is monotonic from $\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$
- $\boldsymbol{c} \leq ((a \to b) \to a) \to a$
- $a \leq b \to c \quad \Rightarrow \quad ab \leq c \quad$ and $\quad ab \leq c \quad \Rightarrow \quad \boldsymbol{e}a \leq b \to c$

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# Connecting the dots

## From AKS to $^{\mathcal{K}}$OCA

If $(\Lambda, \Pi, \text{app}, \text{push}, \text{k}_{\_}, \boldsymbol{k}, \boldsymbol{s}, \boldsymbol{cc}, \textbf{PL}, \perp\!\!\!\perp)$ is an AKS, then $(\mathcal{P}_{\perp\!\!\!\perp}(\Pi), \leq, \text{app}', \texttt{imp}', \{\boldsymbol{k}\}^{\perp\!\!\!\perp}, \{\boldsymbol{s}\}^{\perp\!\!\!\perp}, \{\boldsymbol{cc}\}^{\perp\!\!\!\perp}, \{\boldsymbol{e}\}^{\perp\!\!\!\perp}, \Phi)$ is a $^{\mathcal{K}}$OCA, with:

- $X \leq Y \quad \triangleq \quad X \supseteq Y$;
- $X \to Y \triangleq \{t \cdot \pi \in \Pi : t \in X^{\perp\!\!\!\perp} \wedge \pi \in Y\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$;
- $\Phi \triangleq \{X \in \mathcal{P}_{\perp\!\!\!\perp} : \exists t \in \textbf{PL}. t \perp\!\!\!\perp X\}$

## From $^{\mathcal{K}}$OCA to AKS

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# Observations

- Everything lays in the order:

$$t \perp\!\!\!\perp A \triangleq t \leq A \qquad \text{(AKS to }^{\mathcal{K}}\text{OCA)}$$

- In particular, $t \perp\!\!\!\perp A \rightarrow B \triangleq t \leq A \rightarrow B$ implies that $tA \leq B$ but the converse implication requires *e*.

- Closure required when defining the AKS:

$$\mathcal{P}_{\perp\!\!\!\perp}(X) \triangleq \{Y \subset X : Y = Y^{\perp\!\!\!\perp\perp\!\!\!\perp}\}$$

- From a filtered OCA, one can define a tripos

$$\mathcal{T} : \begin{cases} \mathbf{Set}^{op} & \rightarrow & \mathbf{HA} \\ X & \mapsto & \mathcal{A}^X \end{cases}$$

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$$t \perp\!\!\!\perp A \triangleq t \leq A \qquad \text{(AKS to } {}^{\mathcal{K}}\text{OCA)}$$

- In particular, $t \perp\!\!\!\perp A \to B \triangleq t \leq A \to B$ implies that $tA \leq B$ but the converse implication requires $\boldsymbol{e}$.

- Closure required when defining the AKS:

$$\mathcal{P}_{\perp\!\!\!\perp}(X) \triangleq \{Y \subset X : Y = Y^{\perp\!\!\!\perp\perp\!\!\!\perp}\}$$

- From a filtered OCA, one can define a tripos

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$$\varphi \vdash \psi \quad \triangleq \quad |\varphi \to \psi| \cap \mathbf{PL} \neq \emptyset$$

# Implicative algebras

# Underlying lattice structures

**Subtyping relation:**

$$\frac{\Gamma \vdash p : T \qquad T <: U}{\Gamma \vdash p : U}\ (\textsc{Sub}) \qquad\qquad \frac{U_1 <: T_1 \qquad T_2 <: U_2}{T_1 \to T_2 <: U_1 \to U_2}\ (\textsc{S-Arr})$$

**Classical realizability:**

if $A <: B$, for any pole, if $t \Vdash A$ then $t \Vdash B$.

In terms of truth values:

**Subtyping** $\qquad A \leq_{\perp\!\!\perp} B \triangleq \|B\| \subseteq \|A\|$

Induces a structure of complete lattice, where $\curlywedge = \bigcup$, as in:

$$\|\forall x.A\|_\rho \triangleq \bigcup_{n\in\mathbb{N}} \|A\{x := n\}\| = \curlywedge \{\|A\{x := n\}\| : n \in \mathbb{N}\}$$

**Realizability:** $\quad \forall = \curlywedge \quad \wedge = \times \quad \exists = \curlyvee \quad \vee = +$

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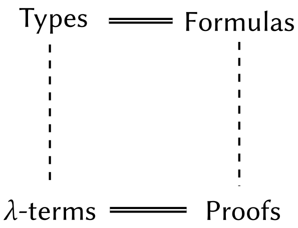
**Boolean algebras:**

quantifiers and connectives both interpreted by meets and joins:

$$\|\forall x.A\| = \|A(0) \wedge A(1) \wedge \ldots \wedge A(n) \wedge \ldots\| = \bigcurlywedge_{n\in\mathbb{N}} \|A(n)\|$$

**Forcing:** $\forall = \wedge = \curlywedge \qquad \exists = \vee = \curlyvee$

# Curry-Howard, one step further




In particular, $a \preccurlyeq b$ reads:

- $a$ is a *subtype* of $b$
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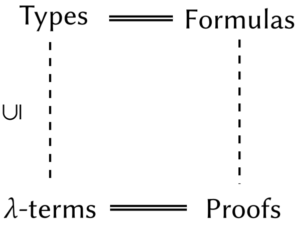


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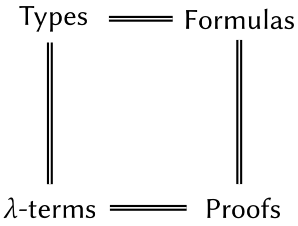


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# Implicative Structures

Implicative algebras: a new (...)
*Alexandre Miquel* [2018]

**Definition:**

Complete meet-semilattice $(\mathcal{A}, \preccurlyeq, \rightarrow)$ s.t.:

- if $a_0 \preccurlyeq a$ and $b \preccurlyeq b_0$ then $(a \rightarrow b) \preccurlyeq (a_0 \rightarrow b_0)$ *(Variance)*
- $\curlywedge_{b \in B}(a \rightarrow b) = a \rightarrow \curlywedge_{b \in B} b$ *(Distributivity)*

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*Examples:*

- complete Heyting/Boolean algebras

*If $\mathcal{H}$ is complete, $a \mapsto b = \curlyvee\{x \in \mathcal{H} : a \wedge x \preccurlyeq b\}$.*

- Ordered Combinatory Algebras

*Complete lattice $\mathcal{P}(\mathcal{A})$ equipped with $A \mapsto B \triangleq \{r \in \mathcal{A} : \forall a \in A. ra \in B\}$.*

- Abstract Krivine Structures

*Complete lattice $\mathcal{P}(\Pi)$, equipped with:*

$$a \preccurlyeq b \triangleq a \supseteq b \qquad a \mapsto b \triangleq a^{\perp\!\!\perp} \cdot b = \{t \cdot \pi : t \in a^{\perp\!\!\perp}, \pi \in b\}$$

# Interpretation of $\lambda$-terms

**Application:**

$$a@b \triangleq \bigwedge\{c \in \mathcal{A} : a \preccurlyeq b \mapsto c\}$$

**Abstraction:**

$$\lambda f \triangleq \bigwedge_{a \in \mathcal{A}} (a \mapsto f(a))$$

Properties

1. If $t \to_\beta u$, then $t^{\mathcal{A}} \preccurlyeq u^{\mathcal{A}}$. ($\beta$-reduction)
2. If $t \to_\eta u$, then $u^{\mathcal{A}} \preccurlyeq t^{\mathcal{A}}$. ($\eta$-expansion)
3. $a@b \preccurlyeq c \quad \Leftrightarrow \quad a \preccurlyeq b \mapsto c$ (Adjunction)

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# Interpretation of formulas

Formulas with parameters:

$$A, B ::= a \mid X \mid A \Rightarrow B \mid \forall X.A \qquad (a \in \mathcal{A})$$

Embedding of closed formulas with parameters:

$$\begin{array}{rcll} a^{\mathcal{A}} & \triangleq & a & (\text{if } a \in \mathcal{A}) \\ (A \Rightarrow B)^{\mathcal{A}} & \triangleq & A^{\mathcal{A}} \to B^{\mathcal{A}} & \\ (\forall X.A)^{\mathcal{A}} & \triangleq & \curlywedge_{a \in \mathcal{A}} (A\{X := a\})^{\mathcal{A}} & \end{array}$$

**Adequacy:** If $\vdash t : A$ then $t^{\mathcal{A}} \preccurlyeq A^{\mathcal{A}}$

In particular:

$$\begin{array}{rcl} \mathbf{K}^{\mathcal{A}} & = & \curlywedge_{a,b \in \mathcal{A}} (a \to b \to a) \\ \mathbf{S}^{\mathcal{A}} & = & \curlywedge_{a,b,c \in \mathcal{A}} ((a \to b \to c) \to (a \to b) \to a \to c) \\ \mathbf{cc} & \triangleq & \curlywedge_{a,b \in \mathcal{A}} (((a \to b) \to a) \to a) \end{array}$$

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# Implicative algebras

**Separator $\mathcal{S}$:**

1. $\kappa^{\mathcal{A}} \in \mathcal{S}$, $\mathrm{s}^{\mathcal{A}} \in \mathcal{S}$,($\boldsymbol{cc} \in \mathcal{S}$) *(Combinators)*
2. If $a \in \mathcal{S}$ and $a \preccurlyeq b$, then $b \in \mathcal{S}$. *(Upwards closure)*
3. If $(a \to b) \in \mathcal{S}$ and $a \in \mathcal{S}$, then $b \in \mathcal{S}$. *(Modus ponens)*

**Implicative algebras:**

$$(\mathcal{A}, \preccurlyeq, \to) \quad + \quad \text{separator } \mathcal{S}$$

**Examples:**

- Complete Boolean algebras
- Abstract Krivine structures

*The set $\mathcal{S} = \{a \in \mathcal{P}(\Pi) : a^{\perp\!\!\!\perp} \cap PL \neq \emptyset\}$ is a separator.*

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# Internal logic

**Entailment:** $a \vdash_{\mathcal{S}} b \quad \triangleq \quad a \to b \in \mathcal{S}$

## Properties

1. $\vdash_{\mathcal{S}}$ is a preorder
2. if $a \preccurlyeq b$ then $a \vdash_{\mathcal{S}} b$ (Subtyping)
3. if $a \vdash_{\mathcal{S}} b$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$ (Closure under $\vdash_{\mathcal{S}}$)

## Adjunction

$a \vdash_{\mathcal{S}} b \to c$ if and only if $a \times b \vdash_{\mathcal{S}} c$

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**Quantifiers:**

$$\forall_{i\in I} a_i \triangleq \bigwedge_{i\in I} a_i \qquad \exists_{i\in I} a_i \triangleq \bigwedge_{c\in A}\Big(\bigwedge_{i\in I}(a_i \to c) \to c\Big)$$

**Semantic rules:**

$$\frac{\Gamma \vdash t : a_i \quad \text{for all } i \in I}{\Gamma \vdash t : \forall_{i\in I} a_i} \qquad \frac{\Gamma \vdash t : \forall_{i\in I} a_i \quad i_0 \in I}{\Gamma \vdash t : a_{i_0}}$$

$$\frac{\Gamma \vdash t : a_{i_0} \quad i_0 \in I}{\Gamma \vdash \lambda x.xt : \exists_{i\in I} a_i} \qquad \frac{\Gamma \vdash t : \exists_{i\in I} a_i \quad \Gamma, x : a_i \vdash u : c \quad (\text{for all } i \in I)}{\Gamma \vdash t(\lambda x.u) : c}$$

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$$a \times b \triangleq \bigcurlywedge_{c \in \mathcal{A}} ((a \to b \to c) \to c) \qquad a + b \triangleq \bigcurlywedge_{c \in \mathcal{A}} ((a \to c) \to (b \to c) \to c)$$

**Semantic rules:**

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$$\frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\pi_1 : a} \qquad \frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\pi_2 : b} \qquad \frac{\Gamma \vdash t : a}{\Gamma \vdash \lambda lr.lt : a + b} \qquad \frac{\Gamma \vdash t : b}{\Gamma \vdash \lambda lr.rt : a + b}$$

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$a \vdash_{\mathcal{S}} b \to c$ if and only if $a \times b \vdash_{\mathcal{S}} c$

# Advertisement

Up to this point, everything you saw has been formalized in Coq.

# Advertisement

**Implicative Structures:**

Complete meet-semilattice $(\mathcal{A}, \preccurlyeq, \to)$ s.t.:

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- $\curlywedge_{b\in B}(a \to b) = a \to \curlywedge_{b \in B} b$ *(Distributivity)*

```
Class ImplicativeStructure `{CL:CompleteLattice} := {
   ↦ : X → X → X;
   arrow_mon_l : ∀ a a' b, a ≼ a' → a' ↦ b ≼ a ↦ b;
   arrow_mon_r : ∀ a b b', b ≼ b' → a ↦ b ≼ a ↦ b';
   arrow_meet : ∀ a B, ⋏_{b∈B}(a ↦ b) = a ↦ ⋏_{b∈B} b
 }.
```

# Advertisement

**Application:**

$$a@b \triangleq \bigwedge\{c \in \mathcal{A} : a \preccurlyeq b \mapsto c\}$$

```
Definition app a b := ⋏ (fun c ⇒ a ≼ b ↦ c).
```

**Abstraction:**

$$\lambda f \triangleq \bigwedge_{a \in \mathcal{A}}(a \mapsto f(a))$$

```
Definition abs f := ⋏ (fun x ⇒ ∃ a, x = a ↦ f a).
```

# Advertisement

**Adequacy:**

$$\vdash t : T \quad \Rightarrow \quad \mathsf{t}^{\mathcal{A}} = \mathsf{x} \quad \Rightarrow \quad \mathsf{T}^{\mathcal{A}} = \mathsf{a} \quad \Rightarrow \quad \mathsf{x} \preccurlyeq \mathsf{a}$$

```
Theorem adequacy_empty:
 ∀ t T x a, fv_typ T = {} → typing_trm empty t T→
              translated t x → translated_typ T a → x ≼ a.
```

# Advertisement

**Try it!**

`https://gitlab.com/emiquey/ImplicativeAlgebras/`

# A incredibly nice framework

**Adjunction**

$$a \vdash_{\mathcal{S}} b \mapsto c \qquad \text{if and only if} \qquad a \times b \vdash_{\mathcal{S}} c.$$

*Proof.* (⇒) *Assume that* $t := a \mapsto b \mapsto c \in \mathcal{S}$. *We shall find* $u \in \mathcal{S}$ *s.t.:*

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$$u \preccurlyeq \Big( \bigcurlywedge_{d \in \mathcal{A}} (a \mapsto b \mapsto d) \mapsto d \Big) \mapsto c$$

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$\Leftarrow \quad \lambda y.y(a \mapsto b \mapsto c) \preccurlyeq (\curlywedge_{d \in \mathcal{A}} (a \mapsto b \mapsto d) \mapsto d) \mapsto c$ *($\beta$-reduction)*

$\Leftrightarrow \quad (\lambda y.y(a \mapsto b \mapsto c))@(\curlywedge_{d \in \mathcal{A}} (a \mapsto b \mapsto d) \mapsto d) \preccurlyeq c$ *(adjunction)*

$\Leftarrow \quad (\curlywedge_{d \in \mathcal{A}} (a \mapsto b \mapsto d) \mapsto d)@(a \mapsto b \mapsto c) \preccurlyeq c$ *($\beta$-reduction)*

$\Leftrightarrow \quad (\curlywedge_{d \in \mathcal{A}} (a \mapsto b \mapsto d) \mapsto d) \preccurlyeq (a \mapsto b \mapsto c) \mapsto c$ *(adjunction)*

$\Leftarrow \quad (a \mapsto b \mapsto c) \mapsto c \preccurlyeq (a \mapsto b \mapsto c) \mapsto c$ *(meet def.)*

□

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($\Leftarrow$) *Assume that* $(a \times b) \mapsto c \in \mathcal{S}$. *It suffices to prove that:*

$$\lambda fab.f(\lambda z.zab) \preccurlyeq ((a \times b) \mapsto c) \mapsto (a \mapsto b \mapsto c)$$

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Within Coq:

```
Proof. intros a b c; split;intro H.
– ...   . realizer (λ+ λ+ [$0] $1).
– ...   . realizer (λ+ λ+ λ+ [$2] (λ+ ([[$0] $2] $1))).
Qed.
```


# Implicative tripos

**Adjunction**

$$a \vdash_{\mathcal{S}} b \to c \qquad \text{if and only if} \qquad a \times b \vdash_{\mathcal{S}} c$$

($\looparrowright$ $(\mathcal{A}/\mathcal{S}, \vdash_{\mathcal{S}}, \times, +, \to)$ *is a Heyting algebra*)

**Tripos:**

$$\mathcal{T} : \begin{cases} \mathbf{Set}^{op} & \to & \mathbf{HA} \\ I & \mapsto & \mathcal{A}^I/\mathcal{S}[I] \end{cases}$$

**Collapse criteria**

The following are equivalent:

1. $\mathcal{T}$ is isomorphic to a forcing tripos
2. $\mathcal{S} \subseteq \mathcal{A}$ is a principal filter of $\mathcal{A}$.
3. $\mathcal{S} \subseteq \mathcal{A}$ is finitely generated and $⋔ \in \mathcal{S}$.

# Implicative tripos

## Adjunction

$$a \vdash_{\mathcal{S}} b \to c \qquad \text{if and only if} \qquad a \times b \vdash_{\mathcal{S}} c$$

($\looparrowright$ $(\mathcal{A}/\mathcal{S}, \vdash_{\mathcal{S}}, \times, +, \to)$ *is a Heyting algebra*)

**Tripos:**

$$\mathcal{T} : \begin{cases} \mathbf{Set}^{op} & \to & \mathbf{HA} \\ I & \mapsto & \mathcal{A}^I/\mathcal{S}[I] \end{cases}$$

## Collapse criteria

The following are equivalent:

1. $\mathcal{T}$ is isomorphic to a forcing tripos
2. $\mathcal{S} \subseteq \mathcal{A}$ is a principal filter of $\mathcal{A}$.
3. $\mathcal{S} \subseteq \mathcal{A}$ is finitely generated and $\pitchfork \in \mathcal{S}$.

# Completeness of implicative triposes

**Theorem [Miquel 18]**

Each **Set**-based tripos is (isomorphic to) an implicative tripos.

The proof is based on several observations:

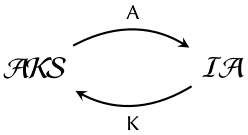
- *generic predicate*: there exists $\Sigma$ and $\mathtt{tr} \in \mathcal{T}(\Sigma)$ s.t.

$$[\![-]\!]_X : \left\{ \begin{array}{ccc} \Sigma^X & \rightarrow & \mathcal{T}(X) \\ \sigma & \mapsto & \mathcal{T}(\sigma)(\mathtt{tr}) \end{array} \right. \qquad \text{is surjective}$$

  ↬ *each predicate on $X$ has a* **code** *in* $\Sigma^X$
- we can define codes $\dot{\wedge}, \dot{\vee}, \dot{\Rightarrow}$ for connectives
  $\dot{\forall}, \dot{\exists}$ for quantifiers
- this *almost* endows $\Sigma$ with a structure of complete HA
- it "leads" to an implicative algebra
  ↬ *the corresponding tripos is* ***isomorphic*** *to the original one*

# Categorifying a bit more

We have:




**Questions:**

1. Can we define categories for $\mathcal{IA}$ / $\mathcal{AKS}$?
2. Does this diagram have a categorical meaning?

# The category of Implicative Algebras

The category of Implicative Algebras and Realizability
*W. Ferrer, O. Malherbe* [2018]

Assume two IAs $\mathcal{A}$ and $\mathcal{B}$

## Applicative morphism

$f : \mathcal{A} \to \mathcal{B}$ with $r \in \mathcal{S}_{\mathcal{B}}$ such that:

1. $f(\mathcal{S}_{\mathcal{A}}) \subseteq \mathcal{S}_{\mathcal{B}}$
2. If $a \vdash a'$, then $r \preccurlyeq f(a \to a') \to f(a) \to f(a')$ $\quad (\forall a, a' \in \mathcal{A})$
3. $f(\curlywedge P) = \curlywedge\{f(x) : x \in P\}$ $\quad (\forall P \subseteq \mathcal{A})$

## Computationally dense morphism

$f : \mathcal{A} \to \mathcal{B}$ applicative with $h : \mathcal{S}_{\mathcal{B}} \to \mathcal{S}_{\mathcal{A}}$ monotonic, $t \in \mathcal{S}_{\mathcal{B}}$ s.t.:

$$t \preccurlyeq f(h(b)) \to b \qquad (\forall b \in \mathcal{S}_{\mathcal{B}})$$

## Proposition

The two notions give rise to categories **IA** / **IAc**.

# The category of Implicative Algebras

The category of Implicative Algebras and Realizability
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**Good news:**

- The two notions also give rise to categories **AKS** / **AKSc**.
- The maps $A : \mathcal{AKS} \to \mathcal{IA}$ and $K : \mathcal{IA} \to \mathcal{AKS}$ extend to functors:


**Theorem**

These functors form an adjoint pair.

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- The maps $A : \mathcal{AKS} \to \mathcal{IA}$ and $K : \mathcal{IA} \to \mathcal{AKS}$ extend to functors:

$$\mathbf{AKS} \underset{K}{\overset{A}{\rightleftarrows}} \mathbf{IA} \quad (\bot) \qquad\qquad \mathbf{AKSc} \underset{K}{\overset{A}{\rightleftarrows}} \mathbf{IAc} \quad (\bot)$$

**Theorem**

These functors form an adjoint pair.

# The end?

**Implicative structures:**

- simple algebraic structures
- adequate embedding of types and terms

**Implicative algebras:**

- encompass usual approaches to realizability
- generalize Boolean algebras and forcing
- complete w.r.t. **Set**-based triposes

*Does the quest of algebraic foundations for classical realizability stop here?*

**Questions:**

- logic = $\forall, \rightarrow$ ?
- call-by-name $\lambda_c$-calculus ?

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# Disjunctive algebras

*(fast-tracked)*

# Decomposing the arrow

**Logic:**

$$A \longrightarrow B \quad \triangleq \quad \neg A \vee B$$

Different axiomatic:

$$\begin{array}{lcl} S1 & : & (A \vee A) \longrightarrow A \\ S2 & : & A \longrightarrow (A \vee B) \end{array} \qquad \begin{array}{lcl} S3 & : & (A \vee B) \longrightarrow (B \vee A) \\ S4 & : & (A \longrightarrow B) \longrightarrow ((C \vee A) \longrightarrow (C \vee B)) \end{array}$$

**$\lambda$-calculus:**

$$\lambda x.t \triangleq \tilde{\mu}([x], \beta).\langle t \,\|\, \beta \rangle : \neg A \parr B$$

- $L^{\parr}$ fragment of Munch-Maccagnoni's system L
- embedding of the *call-by-name* $\lambda$-calculus

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# Focusing on disjunction: $L^{\parr}$

Focalisation and Classical Realisability
*Guillaume Munch-Maccagnoni* [2009]

**Syntax:**

| | | | |
|---|---|---|---|
| **Contexts** | $e$ | $::=$ | $\alpha \mid (e_1, e_2) \mid [t] \mid \mu x.c$ |
| **Terms** | $t$ | $::=$ | $x \mid \mu(\alpha_1,\alpha_2).c \mid \mu[x].c \mid \mu\alpha.c$ |
| **Commands** | $c$ | $::=$ | $\langle t \,\|\, e\rangle$ |

**Types** $\quad A, B ::= X \mid A \parr B \mid \neg A \mid \forall X.A$

**Type system:**

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \,\|\, e\rangle : \Gamma \vdash \Delta}\ (\textsc{Cut})$$

$$\frac{\Gamma \mid e_1 : A \vdash \Delta \qquad \Gamma \mid e_2 : B \vdash \Delta}{\Gamma \mid (e_1, e_2) : A \parr B \vdash \Delta}\ (\parr\vdash) \qquad \frac{c : \Gamma \vdash \Delta, \alpha_1 : A, \alpha_2 : B}{\Gamma \vdash \mu(\alpha_1,\alpha_2).c : A \parr B \mid \Delta}\ (\vdash\parr)$$

$$\frac{\Gamma \vdash t : A \mid \Delta}{\Gamma \mid [t] : \neg A \vdash \Delta}\ (\neg\vdash) \qquad \frac{c : \Gamma, x : A \vdash \Delta}{\Gamma \vdash \mu[x].c : \neg A \mid \Delta}\ (\vdash\neg)$$

# Disjunctive structures

Complete meet-semilattice $(\mathcal{A}, \preccurlyeq, \parr, \neg)$:

1. $\neg$ is anti-monotonic
2. $\parr$ is monotonic
3. $\bigcurlywedge_{b \in B}(a \parr b) = a \parr (\bigcurlywedge_{b \in B} b)$ and $\bigcurlywedge_{b \in B}(b \parr a) = (\bigcurlywedge_{b \in B} b) \parr a$
4. $\neg \bigcurlywedge_{a \in A} a = \bigcurlyvee_{a \in A} \neg a$

**Examples:**

- complete Boolean algebras
- classical realizability in $L^{\parr}$

Induced implication

$(\mathcal{A}, \preccurlyeq, \overset{\parr}{\mapsto})$ with $a \overset{\parr}{\mapsto} b \triangleq \neg a \parr b$ is an implicative structure

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- complete Boolean algebras

$$a \parr b \triangleq a \vee b \qquad\qquad \neg a \triangleq \neg a$$

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**Examples:**

- complete Boolean algebras
- classical realizability in $L^{\parr}$

- $\mathcal{A} \triangleq \mathcal{P}(\Pi)$
- $a \preccurlyeq b \triangleq a \supseteq b$
- $a \parr b \triangleq (a, b)$
- $\neg a \triangleq [a^{\perp\!\!\!\perp}]$

**Induced implication**

$(\mathcal{A}, \preccurlyeq, \xrightarrow{\parr})$ with $a \xrightarrow{\parr} b \triangleq \neg a \parr b$ is an implicative structure

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$(\mathcal{A}, \preccurlyeq, \overset{\parr}{\mapsto})$ with $a \overset{\parr}{\mapsto} b \triangleq \neg a \parr b$ is an implicative structure

# Interpreting $L^{\parr}$

$$\perp\!\!\!\perp \triangleq ?$$

Intuition:

$$t \Vdash A \quad \text{if} \quad t^{\mathcal{A}} \preccurlyeq A^{\mathcal{A}} \qquad \text{and} \qquad t \perp\!\!\!\perp \pi \quad \text{if} \quad t^{\mathcal{A}} \preccurlyeq \pi^{\mathcal{A}}$$

**Terms:**

- $\mu^{-}.c \triangleq \bigcurlywedge_{a\in\mathcal{A}}\{a : c(a) \in \perp\!\!\!\perp\}$
- $\mu^{()}.c \triangleq \bigcurlywedge_{a,b\in\mathcal{A}}\{a \parr b : c(a,b) \in \perp\!\!\!\perp\}$
- $\mu^{[]}.c \triangleq \bigcurlywedge_{a\in\mathcal{A}}\{\neg a : c(a) \in \perp\!\!\!\perp\}$

**Contexts:**

- $(a,b) \triangleq a \parr b$
- $[a] \triangleq \neg a$
- $\mu^{+}.c \triangleq \bigcurlyvee_{a\in\mathcal{A}}\{a : c(a) \in \perp\!\!\!\perp\}$

**Properties:**

# Interpreting $L^{⅋}$

$$\bot\!\!\!\bot \triangleq \{(t,e) : t \preccurlyeq e\}$$

**Terms:**

- $\mu^{-}.c \triangleq \bigcurlywedge_{a\in\mathcal{A}}\{a : c(a) \in \bot\!\!\!\bot\}$
- $\mu^{()}.c \triangleq \bigcurlywedge_{a,b\in\mathcal{A}}\{a ⅋ b : c(a,b) \in \bot\!\!\!\bot\}$
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**Properties:**

# Interpreting $L^{⅋}$

$$⫫ \triangleq \{(t,e) : t \preccurlyeq e\}$$

**Terms:**

- $\mu^{-}.c \triangleq \curlywedge_{a\in\mathcal{A}}\{a : c(a) \in ⫫\}$
- $\mu^{()}.c \triangleq \curlywedge_{a,b\in\mathcal{A}}\{a ⅋ b : c(a,b) \in ⫫\}$
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**Contexts:**

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- $[a] \triangleq \neg a$
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**Properties:**

If $c_1 \rightarrow_\beta c_2$ then $c_1^{\mathcal{A}} \trianglelefteq c_2^{\mathcal{A}}$. *(β-reduction)*

**Adequacy**

1. for any term $t$, if $\Gamma \vdash t : A \mid \Delta$, then $(t[\sigma])^{\mathcal{A}} \preccurlyeq A[\sigma]^{\mathcal{A}}$;
2. for any context $e$, if $\Gamma \mid e : A \vdash \Delta$, then $(e[\sigma])^{\mathcal{A}} \succcurlyeq A[\sigma]^{\mathcal{A}}$;
3. for any command $c$, if $c : (\Gamma \vdash \Delta)$, then $(c[\sigma])^{\mathcal{A}} \in ⫫$.

# Interpreting $L^{\parr}$

$$\perp\!\!\!\perp \triangleq \{(t,e) : t \preccurlyeq e\}$$

**Terms:**

- $\mu^{-}.c \triangleq \curlywedge_{a\in\mathcal{A}}\{a : c(a) \in \perp\!\!\!\perp\}$
- $\mu^{()}.c \triangleq \curlywedge_{a,b\in\mathcal{A}}\{a \parr b : c(a,b) \in \perp\!\!\!\perp\}$
- $\mu^{[]}.c \triangleq \curlywedge_{a\in\mathcal{A}}\{\neg a : c(a) \in \perp\!\!\!\perp\}$

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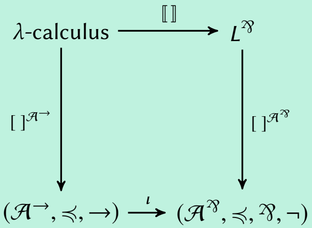


*(Soundness)*

# Disjunctive algebras

Russel's axioms:

$$
\begin{array}{lcll}
s_1^{\parr} & \triangleq & \curlywedge_{a\in\mathcal{A}} & [(a \parr a) \mapsto a] \\
s_2^{\parr} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} & [a \mapsto (a \parr b)] \\
s_3^{\parr} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} & [(a \parr b) \mapsto (b \parr a)] \\
s_4^{\parr} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} & [(a \mapsto b) \mapsto (c \parr a) \mapsto (c \parr b)] \\
s_5^{\parr} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} & [(a \parr (b \parr c)) \mapsto ((a \parr b) \parr c)]
\end{array}
$$

**Separator $\mathcal{S}$:**

1. If $a \in \mathcal{S}$ and $a \preccurlyeq b$ then $b \in \mathcal{S}$ (upward closure)
2. $s_1, s_2, s_3, s_4$ and $s_5$ are in $\mathcal{S}$ (combinators)
3. If $a \overset{\parr}{\mapsto} b \in \mathcal{S}$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$ (modus ponens)

# Disjunctive algebras

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**Examples:**

- Complete Boolean algebras

*All combinators verify $(s_i)^{\mathcal{B}} = \top$, thus $\top$ or any filter define separators.*

- realizability models in $L^{!\parr}$

# Disjunctive algebras

$$\begin{array}{lcl} s_1^{\parr} & \triangleq & \bigcurlywedge_{a\in\mathcal{A}} [(a \parr a) \mapsto a] \\ s_2^{\parr} & \triangleq & \bigcurlywedge_{a,b\in\mathcal{A}} [a \mapsto (a \parr b)] \\ s_3^{\parr} & \triangleq & \bigcurlywedge_{a,b\in\mathcal{A}} [(a \parr b) \mapsto (b \parr a)] \\ s_4^{\parr} & \triangleq & \bigcurlywedge_{a,b,c\in\mathcal{A}} [(a \mapsto b) \mapsto (c \parr a) \mapsto (c \parr b)] \\ s_5^{\parr} & \triangleq & \bigcurlywedge_{a,b,c\in\mathcal{A}} [(a \parr (b \parr c)) \mapsto ((a \parr b) \parr c)] \end{array}$$

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**Examples:**

- Complete Boolean algebras
- realizability models in $L^{!\parr}$

*The set of realized falsity values is again a separator.*

# Internal logic

Recall:

$$a \vdash_{\mathcal{S}} b \;\triangleq\; a \overset{\parr}{\mapsto} b \in \mathcal{S}$$

**Sum type:**

- $a \parr b \vdash_{\mathcal{S}} a + b$
- $a + b \vdash_{\mathcal{S}} a \parr b$

**Negation:**

- $\neg a \vdash_{\mathcal{S}} a \overset{\parr}{\mapsto} \bot$
- $a \overset{\parr}{\mapsto} \bot \vdash_{\mathcal{S}} \neg a$
- $a \vdash_{\mathcal{S}} \neg\neg a$
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In an implicative structure, we don't have in general:

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**Combinators:**

$$\mathbf{K}, \mathbf{S}, \mathbf{cc} \in \mathcal{S}$$

**Theorem**

$\mathcal{S}$ is an implicative separator

# Internal logic

Recall:

$$a \vdash_{\mathcal{S}} b \;\triangleq\; a \overset{⅋}{\mapsto} b \in \mathcal{S}$$

**Sum type:**

- $a ⅋ b \vdash_{\mathcal{S}} a + b$
- $a + b \vdash_{\mathcal{S}} a ⅋ b$

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In an implicative structure, we don't have in general:

$$\bigwedge_{b\in B}(a+b) \overset{?}{=} a + (\bigwedge_{b\in B} b) \qquad \bigwedge_{b\in B}(b+a) \overset{?}{=} (\bigwedge_{b\in B} b) + a \qquad (\bigwedge_{a\in A} a) \to \bot \overset{?}{=} \bigvee_{a\in A}(a \to \bot)$$

↬ *Implicative algebras are more general !*

# Recap

**Disjunctive structures:**

- induced by classical realizability
- allow to adequately embed $\mathrm{L}^{\parr}$
- are implicative structures

**Disjunctive algebras:**

- are intrinsically classical
- are implicative algebras
- do not necessarily collapse to a forcing situation

**Conclusion**

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**Conclusion**

Implicative algebras are more general.

# Conjunctive algebras

*(entering call-by-value)*

# Conjunctive algebras

**Logic:**

$$A \to B \quad \triangleq \quad \neg(A \wedge \neg B)$$

**$\lambda$-calculus:**

$$\lambda x.t \triangleq [\mu(x,[\alpha]).\langle t \parallel \alpha\rangle] : \neg(A \otimes \neg B)$$

↬ *$L^{\otimes}$ fragment*

↬ *call-by-value $\lambda$-calculus*

**Same process:**

1. Conjunctive structures $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$
2. Adequate embedding of $L^{\otimes}$
3. Conjunctive algebras

# Conjunctive structures

Complete meet-semilattice $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$:

1. $\neg$ is anti-monotonic
2. $\otimes$ is monotonic
3. $\curlyvee_{b\in B}(a \otimes b) = a \otimes (\curlyvee_{b\in B} b)$ and $\curlyvee_{b\in B}(b \otimes a) = (\curlyvee_{b\in B} b) \otimes a$
4. $\neg \curlyvee_{a\in A} a = \curlywedge_{a\in A} \neg a$

**Examples:**

- complete Boolean algebras
- classical realizability in $L^{\otimes}$

**Duality**

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, \parr, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \parr, \neg)$ conjunctive str.

# Conjunctive structures

Complete meet-semilattice $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$:

1. $\neg$ is anti-monotonic
2. $\otimes$ is monotonic
3. $\curlyvee_{b\in B}(a \otimes b) = a \otimes (\curlyvee_{b\in B} b)$ and $\curlyvee_{b\in B}(b \otimes a) = (\curlyvee_{b\in B} b) \otimes a$
4. $\neg \curlyvee_{a\in A} a = \curlywedge_{a\in A} \neg a$

**Examples:**

- complete Boolean algebras

$$a \otimes b \triangleq a \wedge b \qquad\qquad \neg a \triangleq \neg a$$

- classical realizability in $L^{\otimes}$

**Duality**

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, \invamp, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \invamp, \neg)$ conjunctive str.

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**Examples:**

- complete Boolean algebras
- classical realizability in $L^{\otimes}$

- $\mathcal{A} \triangleq \mathcal{P}(\mathcal{V})$
- $a \preccurlyeq b \triangleq a \subseteq b$
- $a \otimes b \triangleq (a, b)$
- $\neg a \triangleq [a^{\perp\!\!\perp}]$

Duality

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.

# Conjunctive structures

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**Examples:**

- complete Boolean algebras
- classical realizability in $L^{\otimes}$

## Duality

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, \parr, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \parr, \neg)$ conjunctive str.

# Conjunctive algebras ?

Russel's axioms:

$$\begin{aligned}
\mathbf{s}_1^{\otimes} &\triangleq \textstyle\curlywedge_{a\in\mathcal{A}} [a \overset{\otimes}{\mapsto} (a \otimes a)] \\
\mathbf{s}_2^{\otimes} &\triangleq \textstyle\curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\mapsto} a] \\
\mathbf{s}_3^{\otimes} &\triangleq \textstyle\curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\mapsto} (b \otimes a)] \\
\mathbf{s}_4^{\otimes} &\triangleq \textstyle\curlywedge_{a,b,c\in\mathcal{A}} [(a \overset{\otimes}{\mapsto} b) \overset{\otimes}{\mapsto} (c \otimes a) \overset{\otimes}{\mapsto} (c \otimes b)] \\
\mathbf{s}_5^{\otimes} &\triangleq \textstyle\curlywedge_{a,b,c\in\mathcal{A}} [(a \otimes (b \otimes c)) \overset{\otimes}{\mapsto} ((a \otimes b) \otimes c))]
\end{aligned}$$

**Separator $\mathcal{S}$:**

(1) If $a \in \mathcal{S}$ and $a \preccurlyeq b$ then $b \in \mathcal{S}$ (upward closure)

(2) $\mathbf{s}_1^{\otimes}, \mathbf{s}_2^{\otimes}, \mathbf{s}_3^{\otimes}, \mathbf{s}_4^{\otimes}$ and $\mathbf{s}_5^{\otimes}$ are in $\mathcal{S}$ (combinators)

(3) If $a \overset{\otimes}{\mapsto} b \in \mathcal{S}$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$ (modus ponens)

**Reversed disjunctive algebra**

If $(\mathcal{A}, \preccurlyeq, \parr, \neg, \mathcal{S})$ is a disjunctive algebra, then $(\mathcal{A}, \succcurlyeq, \parr, \neg, \neg^{-1}(\mathcal{S}))$ is a conjunctive algebra.

# Conjunctive algebras ?

Russel's axioms:

$$\begin{array}{lcl} s_1^{\otimes} & \triangleq & \curlywedge_{a\in\mathcal{A}} [a \overset{\otimes}{\mapsto} (a \otimes a)] \\ s_2^{\otimes} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\mapsto} a] \\ s_3^{\otimes} & \triangleq & \curlywedge_{a,b\in\mathcal{A}} [(a \otimes b) \overset{\otimes}{\mapsto} (b \otimes a)] \\ s_4^{\otimes} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} [(a \overset{\otimes}{\mapsto} b) \overset{\otimes}{\mapsto} (c \otimes a) \overset{\otimes}{\mapsto} (c \otimes b)] \\ s_5^{\otimes} & \triangleq & \curlywedge_{a,b,c\in\mathcal{A}} [(a \otimes (b \otimes c)) \overset{\otimes}{\mapsto} ((a \otimes b) \otimes c))] \end{array}$$

**Separator $\mathcal{S}$:**

(1) If $a \in \mathcal{S}$ and $a \preccurlyeq b$ then $b \in \mathcal{S}$ (upward closure)

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(3) If $a \overset{\otimes}{\mapsto} b \in \mathcal{S}$ and $a \in \mathcal{S}$ then $b \in \mathcal{S}$ (modus ponens)

**Reversed disjunctive algebra**

If $(\mathcal{A}, \preccurlyeq, \parr, \neg, \mathcal{S})$ is a disjunctive algebra, then $(\mathcal{A}, \succcurlyeq, \parr, \neg, \neg^{-1}(\mathcal{S}))$ is a conjunctive algebra.

# Conjunctive algebras ✗

## Reversed disjunctive algebra

If $(\mathcal{A}, \preccurlyeq, ⅋, \neg, \mathcal{S})$ is a disjunctive algebra, then $(\mathcal{A}, \succcurlyeq, ⅋, \neg, \neg^{-1}(\mathcal{S}))$ is a conjunctive algebra.

## C'était trop beau pour être vrai

The converse seems impossible to prove...

Technically, we are missing the adjunction:

$$a \preccurlyeq b \overset{\otimes}{\mapsto} c \quad \not\Leftrightarrow \quad a @^{\otimes} b \preccurlyeq c$$

Indeed we have:

$$\bigcurlywedge (a \overset{\otimes}{\mapsto} b) \overset{?}{\neq} a \overset{\otimes}{\mapsto} \bigcurlywedge b \qquad \bigcurlywedge (a \overset{\otimes}{\mapsto} b) = (\bigcurlyvee a) \overset{\otimes}{\mapsto} b$$

**Intuitions:**

- conjunctive ⇝ positive ⇝ joins $\curlyvee$, yet axioms with $\forall / \curlywedge$
- values are not closed by application (MP flawed)

# Inspecting L$^\otimes$ deduction system

$$\frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \mid e : A \vdash \Delta}{\langle t \| e \rangle : \Gamma \vdash \Delta} \ (\textsc{Cut})$$

$$\frac{(\alpha : A) \in \Delta}{\Gamma \mid \alpha : A \vdash \Delta}(ax\vdash) \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A \mid \Delta}(\vdash ax)$$

$$\frac{c : \Gamma \vdash \Delta, x : A}{\Gamma \mid \mu x.c : A \vdash \Delta}(\mu\vdash) \qquad \frac{c : \Gamma, \alpha : A \vdash \Delta}{\Gamma \vdash \mu\alpha.c : A \mid \Delta}(\vdash\mu)$$

$$\frac{c : (\Gamma, x : A, x' : B \vdash \Delta)}{\Gamma \mid \mu(x, x').c : A \otimes B \vdash \Delta}(\otimes\vdash) \qquad \frac{\Gamma \vdash t : A \mid \Delta \quad \Gamma \vdash u : B \mid \Delta}{\Gamma \vdash (t, u) : A \otimes B \mid \Delta}(\vdash\otimes)$$

$$\frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \mu[\alpha].c : \neg A \vdash \Delta}(\neg\vdash) \qquad \frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \vdash [e] : \neg A \vdash \Delta}(\vdash\neg)$$

# Inspecting $L^{\otimes}$ deduction system

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (\textsc{Cut})$$

$$\frac{A \in \Delta}{\Gamma, A \vdash \Delta}(ax\vdash) \qquad \frac{A \in \Gamma}{\Gamma \vdash A, \Delta}(\vdash ax)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}(\otimes\vdash) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \otimes B, \Delta}(\vdash\otimes)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}(\neg\vdash) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A \vdash \Delta}(\vdash\neg)$$

# Inspecting $L^{\otimes}$ deduction system

**One-sided sequents, inlining cuts and contexts**

$$\frac{}{\Gamma, A \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \otimes B}$$

$$\frac{\Gamma, A, B \vdash C \quad \Gamma, A, B \vdash \neg C}{\Gamma \vdash \neg(A \otimes B)}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, A \vdash \neg C}{\Gamma \vdash \neg A}$$

# Inspecting $L^{\otimes}$ deduction system

**One-sided sequents, inlining cuts and contexts**

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$$\frac{\Gamma, A \vdash C \quad \Gamma, A \vdash \neg C}{\Gamma \vdash \neg A}$$

↬ *this is a calculus of* ***contradiction***

# Conjunctive algebras

Axioms through contradictions:

$$\begin{array}{lcl}
\mathbf{s}_1^{\otimes} & \triangleq & \bigcurlywedge_{a\in\mathcal{A}} \neg[\neg(a\otimes a)\otimes a] \\
\mathbf{s}_2^{\otimes} & \triangleq & \bigcurlywedge_{a,b\in\mathcal{A}} \neg[\neg a\otimes(a\otimes b)] \\
\mathbf{s}_3^{\otimes} & \triangleq & \bigcurlywedge_{a,b\in\mathcal{A}} \neg[\neg(a\otimes b)\otimes(b\otimes a)] \\
\mathbf{s}_4^{\otimes} & \triangleq & \bigcurlywedge_{a,b,c\in\mathcal{A}} \neg[\neg(\neg a\otimes b)\otimes(\neg(c\otimes a)\otimes(c\otimes b))] \\
\mathbf{s}_5^{\otimes} & \triangleq & \bigcurlywedge_{a,b,c\in\mathcal{A}} \neg[\neg(a\otimes(b\otimes c))\otimes((a\otimes b)\otimes c)]
\end{array}$$

**Separator $\mathcal{S}$:**

1. If $a\in\mathcal{S}$ and $a\preccurlyeq b$ then $b\in\mathcal{S}$. *(upward closure)*
2. $\mathbf{s}_1^{\otimes},\mathbf{s}_2^{\otimes},\mathbf{s}_3^{\otimes},\mathbf{s}_4^{\otimes}$ and $\mathbf{s}_5^{\otimes}$ are in $\mathcal{S}$. *(combinators)*
3. If $\neg(a\otimes b)\in\mathcal{S}$ and $a\in\mathcal{S}$ then $\neg b\in\mathcal{S}$. *(deduction)*
4. If $a\in\mathcal{S}$ and $b\in\mathcal{S}$ then $a\otimes b\in\mathcal{S}$. *(pairs)*

***Classical:*** If $\neg\neg a\in\mathcal{S}$ then $a\in\mathcal{S}$.

# Conjunctive algebras

$$
\begin{array}{lcl}
\mathbf{s}_1^{\otimes} & \triangleq & \curlywedge_{a\in\mathcal{A}} \neg[\neg(a\otimes a)\otimes a] \\
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**Examples:**

- Complete Boolean algebras
- realizability models in $L^{\otimes}$

# Conjunctive algebras

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**Examples:**

- Complete Boolean algebras
- realizability models in $L^{\otimes}$

# Internal logic

**Remark:**

- in general, we only have: $\dfrac{a \vdash_{\mathcal{S}} b \quad a \in \mathcal{S}}{\neg\neg b \in \mathcal{S}}$
- $a \vdash_{\mathcal{S}} b$ can be composed:

$$a \vdash_{\mathcal{S}} b \quad \textit{and} \quad b \vdash_{\mathcal{S}} c \quad \textit{implies} \quad a \vdash_{\mathcal{S}} c$$

**Negation:**

- $\neg a \vdash_{\mathcal{S}} a \overset{\otimes}{\mapsto} \bot$
- $a \overset{\otimes}{\mapsto} \bot \vdash_{\mathcal{S}} \neg a$
- $a \vdash_{\mathcal{S}} \neg\neg a$
- $\neg\neg a \vdash_{\mathcal{S}} a$

**Heyting Algebra** $a \times b \triangleq a \otimes b$ and $a + b \triangleq \neg(\neg a \otimes \neg b)$

1. $a \times b \vdash_{\mathcal{S}} a$
2. $a \times b \vdash_{\mathcal{S}} b$
3. $a \vdash_{\mathcal{S}} a + b$
4. $b \vdash_{\mathcal{S}} a + b$
5. $a \vdash_{\mathcal{S}} b \overset{\otimes}{\mapsto} c$ iff $a \times b \vdash_{\mathcal{S}} c$

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# $\lambda$-calculus

## $\lambda$-calculus

$$\lambda f \triangleq \bigcurlywedge_{a \in \mathcal{A}} (a \overset{\otimes}{\mapsto} f(a)) \qquad\qquad ab \triangleq \bigcurlywedge \{\neg\neg c : a \preccurlyeq b \overset{\otimes}{\mapsto} c\}$$

## Properties

- If $a \in \mathcal{S}$ and $b \in \mathcal{S}$ then $ab \in \mathcal{S}$.
- $(\lambda f)a \preccurlyeq \neg\neg f(a)$

## Combinators

1. $\mathbf{s} \in \mathcal{S}$
2. $\mathbf{k} \in \mathcal{S}$

**$\lambda$-calculus**

If $\mathcal{S}$ is classical and $t$ is a closed $\lambda$-term, then $t^{\mathcal{A}} \in \mathcal{S}$.

# $\lambda$-calculus

## $\lambda$-calculus

$$\lambda f \triangleq \bigcurlywedge_{a \in \mathcal{A}} (a \overset{\otimes}{\mapsto} f(a)) \qquad\qquad ab \triangleq \bigcurlywedge \{\neg\neg c : a \preccurlyeq b \overset{\otimes}{\mapsto} c\}$$

## Properties

- If $a \in \mathcal{S}$ and $b \in \mathcal{S}$ then $ab \in \mathcal{S}$.
- $(\lambda f)a \preccurlyeq \neg\neg f(a)$

## Combinators

1. $\mathbf{s} \in \mathcal{S}$
2. $\mathbf{\kappa} \in \mathcal{S}$

**$\lambda$-calculus**

If $\mathcal{S}$ is classical and $t$ is a closed $\lambda$-term, then $t^{\mathcal{A}} \in \mathcal{S}$.

# Conjunctive tripos

Great news:

**Theorem**

If $(\mathcal{A}, \preccurlyeq, \rightarrow, \mathcal{S})$ is a classical conjunctive algebra, the functor:

$$\mathcal{T} : I \mapsto \mathcal{A}^I/\mathcal{S}[I] \qquad \mathcal{T}(f) : \begin{cases} \mathcal{A}^I/\mathcal{S}[I] & \rightarrow \quad \mathcal{A}^J/\mathcal{S}[J] \\ [(a_i)_{i\in I}] & \mapsto \quad [(a_{f(j)})_{j\in J}] \end{cases}$$

(where $f : J \rightarrow I$) defines a tripos.

# Duality, the come-back

## Structures

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, ⅋, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, ⅋, \neg)$ conjunctive str.

## Algebras

1. If $(\mathcal{A}^{⅋}, \mathcal{S}^{⅋})$ is a ⅋-algebra, then $\neg^{-1}(\mathcal{S}^{⅋}) = \{a : \neg a \in \mathcal{S}^{⅋}\}$ is a valid separator for the dual $\otimes$-structure.
2. If $(\mathcal{A}^{\otimes}, \mathcal{S}^{\otimes})$ is a $\otimes$-algebra, then $\neg^{-1}(\mathcal{S}^{\otimes}) = \{a : \neg a \in \mathcal{S}^{\otimes}\}$ is a valid separator for the dual ⅋-structure.

## Triposes

Let $(\mathcal{A}, \mathcal{S})$ be a ⅋-algebra and $(\bar{\mathcal{A}}, \bar{\mathcal{S}})$ its dual $\otimes$-algebra. The family:

$$\varphi_I : \begin{cases} \bar{\mathcal{A}}/\bar{\mathcal{S}}[I] & \to & \mathcal{A}/\mathcal{S}[I] \\ [a_i] & \mapsto & [\neg a_i] \end{cases}$$

defines a tripos isomorphism.

# Duality, the come-back

## Structures

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, \parr, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \parr, \neg)$ conjunctive str.

## Algebras

1. If $(\mathcal{A}^{\parr}, \mathcal{S}^{\parr})$ is a $\parr$-algebra, then $\neg^{-1}(\mathcal{S}^{\parr}) = \{a : \neg a \in \mathcal{S}^{\parr}\}$ is a valid separator for the dual $\otimes$-structure.
2. If $(\mathcal{A}^{\otimes}, \mathcal{S}^{\otimes})$ is a $\otimes$-algebra, then $\neg^{-1}(\mathcal{S}^{\otimes}) = \{a : \neg a \in \mathcal{S}^{\otimes}\}$ is a valid separator for the dual $\parr$-structure.

## Triposes

Let $(\mathcal{A}, \mathcal{S})$ be a $\parr$-algebra and $(\bar{\mathcal{A}}, \bar{\mathcal{S}})$ its dual $\otimes$-algebra. The family:

$$\varphi_I : \left\{ \begin{array}{ccc} \bar{\mathcal{A}}/\bar{\mathcal{S}}[I] & \rightarrow & \mathcal{A}/\mathcal{S}[I] \\ [a_i] & \mapsto & [\neg a_i] \end{array} \right.$$

defines a tripos isomorphism.

# Duality, the come-back

## Structures

1. $(\mathcal{A}, \preccurlyeq, \otimes, \neg)$ conjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \otimes, \neg)$ disjunctive str.
2. $(\mathcal{A}, \preccurlyeq, \parr, \neg)$ disjunctive str. $\Rightarrow$ $(\mathcal{A}, \succcurlyeq, \parr, \neg)$ conjunctive str.

## Algebras

1. If $(\mathcal{A}^{\parr}, \mathcal{S}^{\parr})$ is a $\parr$-algebra, then $\neg^{-1}(\mathcal{S}^{\parr}) = \{a : \neg a \in \mathcal{S}^{\parr}\}$ is a valid separator for the dual $\otimes$-structure.
2. If $(\mathcal{A}^{\otimes}, \mathcal{S}^{\otimes})$ is a $\otimes$-algebra, then $\neg^{-1}(\mathcal{S}^{\otimes}) = \{a : \neg a \in \mathcal{S}^{\otimes}\}$ is a valid separator for the dual $\parr$-structure.

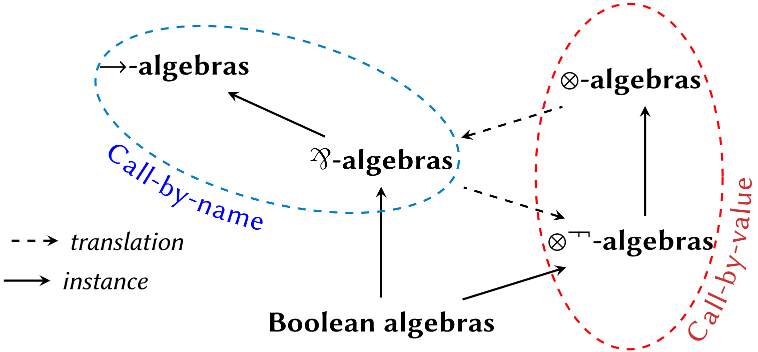
## Triposes

Let $(\mathcal{A}, \mathcal{S})$ be a $\parr$-algebra and $(\bar{\mathcal{A}}, \bar{\mathcal{S}})$ its dual $\otimes$-algebra. The family:

$$\varphi_I : \left\{ \begin{array}{ccc} \bar{\mathcal{A}}/\bar{\mathcal{S}}[I] & \rightarrow & \mathcal{A}/\mathcal{S}[I] \\ [a_i] & \mapsto & [\neg a_i] \end{array} \right.$$

defines a tripos isomorphism.

# Final picture



# Fun fact

## Oliva & Streicher (2008)

Krivine = Kleene ∘ Friedman

**Definition 8.1** (Definition of the negative translation)**.** The formula $A^\perp$ is defined by induction on $A$ by the equations

$$(X(e_1,\ldots,e_k))^\perp \equiv X(e_1,\ldots,e_k) \qquad (\mathrm{null}(e))^\perp \equiv \mathrm{null}(\mathrm{neg}(e))$$
$$(A \Rightarrow B)^\perp \equiv A^{\neg\neg} \wedge B^\perp \qquad (\forall x\, A)^\perp \equiv \exists x\, A^\perp$$
$$(\{e\} \Rightarrow B)^\perp \equiv \mathrm{nat}(e) \wedge B^\perp \qquad (\forall X\, A)^\perp \equiv \exists X\, A^\perp$$

(using the unary function 'neg' defined in section 2.1), whereas the formula $A^{\neg\neg}$ is defined as $A^{\neg\neg} \equiv \neg_R A^\perp \equiv A^\perp \Rightarrow R$.

In our setting:

# Fun fact

## Oliva & Streicher (2008)

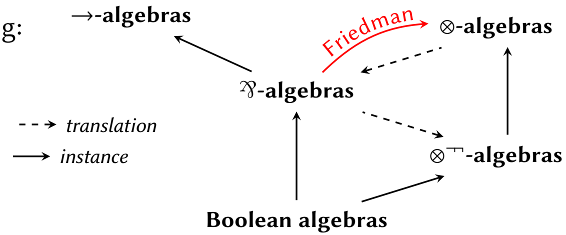
Krivine = Kleene ∘ Friedman

**Definition 8.1** (Definition of the negative translation)**.** The formula $A^\perp$ is defined by induction on $A$ by the equations

$$\begin{aligned}(X(e_1,\ldots,e_k))^\perp &\equiv X(e_1,\ldots,e_k) & (\mathrm{null}(e))^\perp &\equiv \mathrm{null}(\mathrm{neg}(e))\\ (A\Rightarrow B)^\perp &\equiv A^{\neg\neg}\wedge B^\perp & (\forall x\, A)^\perp &\equiv \exists x\, A^\perp\\ (\{e\}\Rightarrow B)^\perp &\equiv \mathrm{nat}(e)\wedge B^\perp & (\forall X\, A)^\perp &\equiv \exists X\, A^\perp\end{aligned}$$

(using the unary function 'neg' defined in section 2.1), whereas the formula $A^{\neg\neg}$ is defined as $A^{\neg\neg} \equiv \neg_R A^\perp \equiv A^\perp \Rightarrow R$.

In our setting:



# Conclusion

What we have:

- **Disjunctive algebras**, particular cases of implicative ones
- **Conjunctive algebras**, harder to handle
- **Duality** between conjunctive and disjunctive algebras

What is left:

1. By-value implicative algebras:
   - Does it exist?
   - Relation to (by-name) implicative algebras?
   - Tripos equivalence ?
2. Combination of disjunctive and conjunctive algebras:
   - Would it collapse to a forcing situation?
   - Any chance to get call-by-push-value algebras?
3. Algebraic counterpart of strategy/side-effects:
   - Lazy algebras?
   - Algebraic counterpart of memory?

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# References

## Implicative algebras

- A. Miquel, *Implicative algebras: a new foundation for realizability and forcing*, 2018, arXiv
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  ↬ *Coq development:* https://gitlab.com/emiquey/ImplicativeAlgebras

## AKS, OCA and categorical stuffs

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## Disjunctive & conjunctive algebras

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