Survival of a single mutant in one dimension

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February 8, 2010

Abstract

We study a one dimensional two-type contact process and give necessary and sufficient conditions on the initial configuration for both types to survive forever. These results are proved under the assumption that the rates of propagation (and death) of the two types are equal.

AMS 60J25, 92D5
Keywords : two–type contact process, survival.

1 Introduction

The aim of this paper is to study the probability that the progeny of a single mutant in an infinite population of residents will survive. We consider this problem in the framework of the one dimensional two–type contact process.

We will prove that if the mutant has no selective advantage nor disadvantage, compared with the individuals of the resident population, then, provided we are in the supercritical case (which means that a single individual's progeny may survive for ever), a single mutant with an empty half–line in front of him, and all sites behind him occupied by resident individuals,

*Supported in part by National Science Foundation grants DMS-0201173 and DMS-0818727
†Supported in part by National Science Foundation grants DMS-0201173 and DMS-0818727

has a progeny which survives forever with positive probability, while any finite number of mutants, with infinitely many residents on both sides, have a progeny which goes extinct a. s. Note that we define the progeny at time t of a given ancestor at time 0 as the set of individuals alive at time t, who are the descendants of that ancestor at time 0.

Let us now explain what we mean by the contact process. Note that this process is often presented in the language of infection. We shall rather consider it here as a model of the spread of a population. Consider first the usual one–type contact process with birth parameter $\lambda > 0$. This process $\{\xi_t,\ t \geq 0\}$ is a $\{0,1\}^{\mathbb{Z}}$–valued Markov process, hence ξ_t is a random mapping which to each $x \in \mathbb{Z}$ associates $\xi_t(x) \in \{0,1\}$. The statement $\xi_t(x) = 1$ means that the site x is occupied at time t, while $\xi_t(x) = 0$ means that site x is empty at time t. The process evolves as follows. Let x be such that $\xi_0(x) = 1$. We wait a random exponential time with parameter $1 + 2\lambda$. At that time, with probability $1/(1 + 2\lambda)$, the individual at site x dies; with probability $\lambda/(1+2\lambda)$, the individual, while continuing its own life at site x, gives birth to another individual; the newborn occupies site $x + 1$ if it is empty, and dies instataneously otherwise; and with probability $\lambda/(1+2\lambda)$, it gives birth to a newborn who occupies site $x - 1$ if it is empty, and dies instataneously otherwise. Then the same operation repeats itself until site x becomes empty, independently of what happened so far. The same happens at any occupied site, and the exponential clocks at various sites are mutually independent. We will use the same notation ξ_t to denote the random element of $\{0,1\}^{\mathbb{Z}}$ defined above, and the random subset of $\mathbb{Z}$ consisting of all sites $x \in \mathbb{Z}$ where $\xi_t(x) = 1$.

The two–type contact process $\{\eta_t,\ t \geq 0\}$ is a $\{0,1,2\}^{\mathbb{Z}}$–valued Markov process which starts from an initial condition (A, B), where A and B are two nonintersecting subsets of $\mathbb{Z}$, A denoting the set of sites which are occupied by type 1 individuals and B the set of sites which are occupied by type 2 individuals at time $t = 0$. In other words,

$$\eta_0(x) = \begin{cases} 0, & \text{if } x \notin A \cup B; \\ 1, & \text{if } x \in A; \\ 2, & \text{if } x \in B. \end{cases}$$

The two–type contact process with equal birth rates λ evolves exactly like the one–type process, with each individual possibly giving birth to individuals of the same type. We shall consider in section 4 the case where the birth rate of the mutants (i. e. type 2 individuals) differs from that of the residents (i. e. type 1 individuals).

The (one-type) contact process has been extensively studied and plays a central role in the theory of interacting particle systems (see (Liggett95), (Liggett99) and references therein) but there are very few papers on the two-type contact process (see (Cox,Schinazi) and (Neuhauser)).

Let us now present a useful construction of the contact process, called the graphical representation, which is valid in both the one–type and the two–type cases (at least in the case of equal birth rates). The important feature of this construction is that processes corresponding to different initial conditions are coupled through it. Indeed, $\{\xi_t,\ t\geq 0\}$ (resp. $\{\eta_t,\ t\geq 0\}$) is a fixed function of both the initial condition, and the set of Poisson point processes, which code all the randomness, which we now introduce.

Consider a collection $\{P^x_t, P^{x,+}_t, P^{x,-}_t,\ t\geq 0;\ x\in\mathbb{Z}\}$ of mutually independent Poisson point processes, such that the P^x's have intensity 1 while both the $P^{x,+}$'s and the $P^{x,-}$'s have intensity λ, all defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. On the set $\mathbb{Z}\times[0,\infty)$ we place a δ on the point (x,t) whenever t belongs to the Poisson process P^x. On that set we also place an arrow from (x,t) to $(x+1,t)$ whenever t belongs to the Poisson process $P^{x,+}$ and an arrow from (x,t) to $(x-1,t)$ whenever t belongs to the Poisson process $P^{x,-}$.

The process $\{\xi^A_t, :\ t\geq 0\}$ is defined as follows. An open path in $\mathbb{Z}\times[0,+\infty)$ is a connected oriented path which moves along the time lines in the increasing t direction without passing through a δ symbol, and along birth arrows, in the direction of the arrow. Now

$$\{y;\ \xi^A_t(y)=1\}=\{y\in\mathbb{Z};\ \exists x\in A \text{ with an open path from } (x,0) \text{ to } (y,t)\}.$$

To construct the two–type contact process, we call line of descendance an open path starting from an occupied site at time 0, and such that any arrow belonging to this path points to an unoccupied site. Note that unlike open paths, lines of descendance depend on the initial configuration of the process. For A, B two disjoint subsets of $\mathbb{Z}$, we define $\{\eta^{A,B}_t,\ t\geq 0\}$ as the $\{0,1,2\}^{\mathbb{Z}}$–valued process whose value at time t is given by

$$\{y;\ \eta^{A,B}_t(y)=1\}=\{y\in\mathbb{Z};\ \exists x\in A \text{ and a line of descendance from } (x,0) \text{ to } (y,t)\}$$
$$\{y;\ \eta^{A,B}_t(y)=2\}=\{y\in\mathbb{Z};\ \exists x\in B \text{ and a line of descendance from } (x,0) \text{ to } (y,t)\}$$

Let $\{\xi^A_t,\ t\geq 0\}$ denote the one–type contact process starting from the configuration whose set of occupied sites is A. We will write ξ^x_t for $\xi^{\{x\}}_t$. We shall use the notation

$$\rho = \mathbb{P}(\xi_t^0 \neq \emptyset,\ \forall t > 0) = \lim_{s\to\infty} \mathbb{P}(\xi_s^0 \neq \emptyset). \tag{1.1}$$

It follows from well–known results on the contact process, see e. g. Liggett (Liggett95), that there exists $\lambda_c < \infty$ such that $\rho > 0$ whenever $\lambda > \lambda_c$.

Given a finite subset $B \subset \mathbb{Z}$, write $B^+ = \{x \in \mathbb{Z},\ x > y, \forall y \in B\}$ and $B^- = \{x \in \mathbb{Z},\ x < y, \forall y \in B\}$.

The aim of this paper is to prove

Theorem 1.1. *Suppose that $\lambda > \lambda_c$ and $0 < |B| < \infty$. Then*

$$\mathbb{P}\left(\{x,\ \eta_t^{A,B}(x) = 2\} \neq \emptyset, \forall t > 0\right) > 0$$

if and only if at least one of the two sets $A \cap B^+$ and $A \cap B^-$ is finite.

From the results needed to prove Theorem 1.1 we can also deduce:

Theorem 1.2. *Suppose that $\lambda > \lambda_c$, $0 < |A| < \infty$ and $0 < |B| < \infty$. Then*

$$\mathbb{P}\left(\{x,\ \eta_t^{A,B}(x) = 1\} \neq \emptyset \text{ and } \{x,\ \eta_t^{A,B}(x) = 2\} \neq \emptyset, \forall t > 0\right) > 0.$$

We conjecture that Theorem 1.2 holds for the two-type contact process on $\mathbb{Z}^d$ for all $d \geq 1$. In (Neuhauser) it is proved that for $d \leq 2$ and all initial configurations $\lim_{t\to\infty} \mathbb{P}(\eta_t(x) = 1, \eta_t(y) = 2) = 0$ for all x, y, while for $d \geq 3$ the process admits invariant measures μ such that for all $x \neq y$, $\mu(\{\eta : \eta(x) = 1, \eta(y) = 2\}) > 0$. Although this last result may be seen as evidence favoring our conjecture (when $d \geq 3$) it does not imply it nor is it implied by it.

The paper is organized as follows. In section 2, we recall and prove several results on the one–type contact process which are needed in further sections. In section 3, we study the case of a single or a finite number of mutants confronted with an infinite number of residents, in the case of equal birth rates. Theorems 1.1 and 1.2 are proved in subsections 3.3 and 3.4 respectively. Finally, in section 4, we conclude with some remarks on the case of unequal birth rates (i. e. when one of the two species has a selective advantage). We formulate one result and two conjectures.

In all of this paper, we assume that $\lambda > \lambda_c$.

2 Some results on the one–type contact process

Let $\mathbb{Z}^-$ be the set of integers smaller than or equal to 0 and let $\mathbb{Z}^+$ be the set of integers greater than or equal to 0.

Let $r_t = \sup\left\{x : \xi_t^{\mathbb{Z}^-}(x) = 1\right\}$ and let $\ell_t = \inf\left\{x : \xi_t^{\mathbb{Z}^+}(x) = 1\right\}$.

It is known that since $\lambda > \lambda_c$, there exits $v = v(\lambda) > 0$ such that

$$\lim_{t\to\infty} \frac{r_t}{t} = -\lim_{t\to\infty} \frac{\ell_t}{t} = v \text{ a.s. and in } L^1(\Omega, \mathcal{F}, \mathbb{P}).$$

For a proof of these results the reader is referred to Theorems VI.2.19 and VI.2.24 in (Liggett95).

Let $R_t = \sup_{s\le t} r_s$.

Lemma 2.1. *$\mathbb{P}(r_t \ge a) \ge \frac{\rho}{2}\mathbb{P}(R_t \ge a)$, $\forall t, a$.*

PROOF: Let $\tau_a = \inf\{s : \ r_s \ge a\}$. Then

$$\mathbb{P}(r_t \ge a | R_t \ge a) = \mathbb{P}(r_t \ge a | \tau_a \le t).$$

By the strong Markov property this is bounded below by

$$\inf_{s\ge 0} \mathbb{P}(\xi_s^0 \cap [0,\infty) \ne \emptyset),$$

which by symmetry is at least

$$\inf_{s\ge 0} \frac{1}{2}\mathbb{P}(\xi_s^0 \ne \emptyset) = \frac{\rho}{2}.$$

□

Lemma 2.2. *$\lim_{t\to\infty} \frac{R_t}{t} = v$ a.s. and in L^1.*

PROOF: The a. s. convergence follows from the a.s convergence of $\frac{r_t}{t}$ and the fact that $v > 0$. For the L^1 convergence note first that since $\frac{R_t}{t} \ge \frac{r_t}{t}$ and $\frac{r_t}{t}$ converges to v in L^1, it suffices to show that

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{R_t}{t} - v\right]^+ = 0.$$

To do so fix $\varepsilon > 0$ and let $c = \frac{2}{\rho}$. Then write

$$\begin{aligned}\lim_{t\to\infty} \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{R_t}{t} - v \geq \varepsilon n\right) &\leq \lim_{t\to\infty} c \sum_{n=1}^{\infty} \mathbb{P}\left(\frac{r_t}{t} - v \geq \varepsilon n\right) \\ &\leq \lim_{t\to\infty} \frac{c}{\varepsilon} \mathbb{E}\left[\frac{r_t}{t} - v\right]^+ \\ &= 0,\end{aligned}$$

where we have used Lemma 2.1 for the first inequality, and the L^1 convergence of r_t/t for the equality. Hence

$$\limsup_{t\to\infty} \mathbb{E}\left[\frac{R_t}{t} - v\right]^+ \leq \limsup_{t\to\infty} \varepsilon \sum_{n=0}^{\infty} \mathbb{P}\left(\frac{R_t}{t} - v \geq \varepsilon n\right) \leq \varepsilon.$$

Since ε is arbitrary the lemma is proved. □

Although the following lemma is well known, we did not find it in previous publications and we include it here for the sake of completness.

Lemma 2.3. *Suppose* $\lambda > \lambda_c$, *let*

$r'_t = \sup\{x \in \mathbb{Z} : x \in \xi_t^0$ *and there is an infinite open path starting from* $(x,t)\}$

and let $\tau^0 = \inf\{s : \xi_s^0 = \emptyset\}$ *Then*

$$\mathbb{P}\left(\lim_t \frac{r'_t}{t} = v(\lambda) | \tau^0 = \infty\right) = 1.$$

Proof: Let $0 < \epsilon < v$. Then write:

$$\begin{aligned}&\mathbb{P}\left(|\xi_n^0 \cap [(v(\lambda) - 2\epsilon)n, (v(\lambda) - \epsilon)n]| \leq \frac{\epsilon\rho n}{2}, |\tau^0 = \infty\right) \\ &\leq \mathbb{P}\left(|\xi_n^0 \cap [(v(\lambda) - 2\epsilon)n, (v(\lambda) - \epsilon)n]| \leq \frac{\epsilon\rho n}{2}, [(v(\lambda) - 2\epsilon)n, (v(\lambda) - \epsilon)n] \subset [\ell_n, r_n] | \tau^0 = \infty\right) \\ &\qquad + \mathbb{P}\left(\ell_n > (v(\lambda) - 2\epsilon)n | \tau^0 = \infty)\right) + \mathbb{P}\left(r_n < (v(\lambda) - \epsilon)n | \tau^0 = \infty)\right) \\ &\leq \mathbb{P}\left(|\xi_n^{\mathbb{Z}} \cap [(v(\lambda) - 2\epsilon)n, (v(\lambda) - \epsilon)n]| \leq \frac{\epsilon\rho n}{2}, |\tau^0 = \infty\right) \\ &\qquad + \mathbb{P}\left(\ell_n > (v(\lambda) - 2\epsilon)n | \tau^0 = \infty)\right) + \mathbb{P}\left(r_n < (v(\lambda) - \epsilon)n | \tau^0 = \infty)\right),\end{aligned}$$

where the last inequality is due to the fact that $\xi_n^0(x) = \xi_n^{\mathbb{Z}}(x)$ for any $x \in [\ell_n, r_n]$.

We now show that the sum on n of each of the three terms of the right hand side above converges: For the first of these terms, the convegence is a consequece of the fact that for any n the distribution of $\xi_n^{\mathbb{Z}}$ is stochastically above the upper invariant measure of the contact process and of Theorem 1 of (Durrett,Schonmann) .For the third term the convergence follows from Corollary 3.22 in Chapter VI of (Liggett95). For the second term it follows by that same corollary applied to ℓ_n and our choice of ϵ. We have thus proved that

$$\sum_n \mathbb{P}\left(|\xi_n^0 \cap [(v(\lambda)-2\epsilon)n,(v(\lambda)-\epsilon)n]| \leq \frac{\epsilon\rho n}{2}, |\tau^0=\infty\right) < \infty.$$

This, the Markov property and Theorem 3.29 in Chapter VI of (Liggett95) imply that

$$\sum_n \mathbb{P}(r'_n < (v(\lambda)-2\epsilon)n|\tau^0=\infty) < \infty.$$

Since ϵ is arbitrary and $r'_n \leq r_n$ we get: $\mathbb{P}(\lim_n \frac{r'_n}{n} = v(\lambda)|\tau^0=\infty) = 1$ and the lemma follows from the fact that $\sup_{0\leq s\leq t\leq 1} r'_{n+t} - r'_{n+s}$ is bounded above by a Poisson r.v. of parameter λ. $\square$

It now follows:

Corollary 2.4. *Suppose $\lambda > \lambda_c$ and let A be an infinite subset of $\mathbb{Z}_+$. Then for any $v' < v$, there exists an infinite open path starting from $A\times\{0\}$, which lies on the right of the line $\{(v't,t),\ t\geq 0\}$.*

PROOF: There exists a strictly increasing sequence $\{x_k,\ k\geq 1\}\subset A$ such that there is an infinite open path starting from each x_k. Now for each $n\geq 0$ and some $R\in\mathbb{N}$ define

$$r'_{t,n} = \sup\{x\in\mathbb{Z} : x\in\xi_t^n \text{ and there is an infinite open path starting from } (x,t)\}$$

and $A_n = \{r'_{t,n} > v't - R + n, \forall t\geq 0\}$

It follows from the last Lemma that for R large enough, $\mathbb{P}(A_n) = \mathbb{P}(A_0) > 0$. From now on such an R is fixed. From the ergodic theorem,

$$\frac{1}{n}\sum_{j=0}^{n-1} \mathbf{1}_{A_j} \to \mathbb{P}(A_0) > 0,$$

hence a. s. infinitely many A_n occur. So almost surely, one A_n with $n \geq R$ occurs. Now choose k large enough such that $x_k \geq n$. Clearly there exists an infinite open path starting from $(x_k, 0)$ which lies on the right of the line $\{(v't, t),\ t \geq 0\}$. □

Corollary 2.5. *The critical values of λ for the contact processes on $\mathbb{N}$ and $\mathbb{Z}$ are equal.*

Let μ^+ denote the upper invariant measure for the contact process on $\mathbb{N}$. This is defined as follows. Denote by $\{\chi_t,\ t \geq 0\}$ the one–type contact process on $\mathbb{N}$. This process takes its values in $\{0,1\}^{\mathbb{N}}$. In accordance with the above conventions, for $A \subset \mathbb{N}$, we write χ_t^A for the contact process on $\mathbb{N}$ starting with the initial condition $\chi_0^A(x) = 1$ iff $x \in A$. Then μ^+ is the weak limit, as $t \to \infty$, of the law of $\chi_t^{\mathbb{N}}$.

For $\eta \in \{0,1\}^{\mathbb{N}}$, let $Y(\eta) = \inf\{x > 0 :\ \eta(x) = 1\}$.

Lemma 2.6. *a) There exist constants $K, c > 0$ such that $\mu^+(Y > n) \leq Ke^{-cn}$ for all $n \geq 0$.*

b) $\alpha := \mathbb{E}_{\mu^+}(Y) < \infty$.

For the proof of this result, we will need the following

Lemma 2.7. *Denoting again $r_t = \sup\{x,\ \xi_t^{\mathbb{Z}^-}(x) = 1\}$, we have*

$$\mu^+(Y > n) = \mathbb{P}(\inf_{t>0} r_t \leq -n).$$

Proof: We first exploit the well–known self–duality of the contact process. Since there is a one to one correspondance between the open paths from some $(y, 0)$, $y \in \mathbb{N}$, to some (x, t), $x \in (0, n]$ and the open paths from some $(x, 0)$, $x \in (0, n]$ to some (y, t), $y \in \mathbb{N}$ obtained by reversing the directions of the arrows,

$$\mathbb{P}(\exists x \in (0, n] :\ \chi_t^{\mathbb{N}}(x) = 1) = \mathbb{P}(\exists x \in (0, n] :\ \chi_t^x \neq \emptyset).$$

Letting $t \to \infty$ in the above identity yields

$$\mu^+(Y \leq n) = \mathbb{P}(\exists x \in (0, n],\ \chi_t^x \neq \emptyset,\ \forall t > 0).$$

The last right hand side is the probability that there is an infinite open path starting from some $(x,0)$, $x \in (0,n]$, which visits only points located at the right of the vertical line $\{1\} \times \mathbb{R}_+$. This has the same probability as the event that there is in $(-n,\infty) \times \mathbb{R}_+$ an infinite open path starting in $(-n,0] \times \{0\}$, i. e. it equals $\mathbb{P}(\inf_{t>0} r_t > -n)$. The result follows. $\square$

Proof of Lemma 2.6

Part b) follows from part a) and in view of Lemma 2.7, to prove part a) it suffices to show that for the contact process on $\mathbb{Z}$ there exist constants $K, c > 0$ such that

$$\mathbb{P}(\inf_{t>0} r_t \leq -n) \leq K e^{-cn} \quad \forall\, n \geq 1.$$

It follows from Corollary VI.3.22 in (Liggett95) that for some $K_1, c > 0$ we have:

$$\mathbb{P}\left(r_t \leq \frac{v}{2} t\right) \leq K_1 e^{-ct} \quad \forall\, t \geq 1. \tag{2.1}$$

From now on let $[t]$ be the integer part of t. From (2.1) we get:

$$\begin{aligned}
\mathbb{P}\left(\inf_{n \geq [t],\ n \in \mathbb{N}} r_n \leq \frac{v}{2} t\right) &\leq \mathbb{P}\left(\bigcup_{n \geq [t],\ n \in \mathbb{N}} \left\{r_n \leq \frac{v}{2} n\right\}\right) \\
&\leq \sum_{n \geq [t]} \mathbb{P}\left(r_n \leq \frac{v}{2} n\right) \\
&\leq K_2 e^{-ct},
\end{aligned}$$

for some $K_2 > 0$. Next define $\tau_n = \inf\{n < s \leq n+1,\ r_s \leq \frac{v}{2} t\}$ (with the convention that $\tau_n = n+1$ on the set $\{\inf_{n<s\leq n+1} r_s > \frac{v}{2} t\}$). Now note that

$$\left\{\inf_{n<s\leq n+1} r_s \leq \frac{v}{2} t\right\} \cap \{r_{n+1} - r_{\tau_n} \leq 0\} \subset \left\{r_{n+1} \leq \frac{v}{2} t\right\}$$

and that

$$\mathbb{P}\left(r_{n+1} - r_{\tau_n} \leq 0 \,\middle|\, \inf_{n<s\leq n+1} r_s \leq \frac{v}{2} t\right) \geq \mathbb{P}(X = 0),$$

where a X is Poisson r.v. of parameter λ. Hence,

$$\mathbb{P}\left(\inf_{n<s\leq n+1} r_s \leq \frac{v}{2} t\right) \leq [\mathbb{P}(X=0)]^{-1} \mathbb{P}\left(r_{n+1} \leq \frac{v}{2} t\right) = e^{\lambda} \mathbb{P}\left(r_{n+1} \leq \frac{v}{2} t\right).$$

Therefore, using (2.1) we get

$$\begin{aligned}\mathbb{P}\left(\inf_{s\geq t} r_s \leq \frac{v}{2}t\right) &\leq \sum_{n\geq[t]} \mathbb{P}\left(\inf_{n<s\leq n+1} r_s \leq \frac{v}{2}t\right)\\ &\leq e^{\lambda}\sum_{n\geq[t]} \mathbb{P}\left(r_{n+1} \leq \frac{v}{2}t\right)\\ &\leq K_3 e^{-ct},\end{aligned}$$

for some constant K_3. We have shown in particular that

$$\mathbb{P}\left(\inf_{s\geq t} r_s \leq 0\right) \leq Ke^{-ct}. \tag{2.2}$$

Fix $\beta > 0$ such that $2\lambda\beta < 1 + v\beta$. Now, write

$$\mathbb{P}\left(\inf_{t\geq 0} r_t \leq -n\right) \leq \mathbb{P}\left(\inf_{0\leq t\leq \beta n} r_t \leq -n\right) + \mathbb{P}\left(\inf_{t\geq \beta n} r_t \leq 0\right).$$

It follows from (2.2) that the second term of the right hand side decays exponentially in n. Hence, the lemma will be proved if we show that the first term also decays exponentially in n. To do so, let $\sigma = \inf\{t : r_t \leq -n\}$ and let Y_n be a Poisson random variable of parameter $2\lambda\beta n$. It now follows from the Strong Markov property applied at the stopping time σ that:

$$\mathbb{P}(r_{2\beta n} \leq v\beta n) \geq \mathbb{P}(\sigma \leq \beta n)\mathbb{P}(Y_n \leq (1 + v\beta)n).$$

Since, given our choice of β, $\lim_{n\to\infty} \mathbb{P}(Y_n \leq (1 + v\beta)n) = 1$ and by (2.1), $\mathbb{P}(r_{2\beta n} \leq v\beta n)$ decays exponentially in n, the same happens to $\mathbb{P}(\sigma \leq \beta n) = \mathbb{P}(\inf_{0\leq t\leq \beta n} r_t \leq -n)$. □

Let T^{-1} be the operator on the set of probability measures on $\{0,1\}^{\mathbb{N}}$ defined by

$$T^{-1}(\nu)\left(\eta(x_1) = \gamma_1, ..., \eta(x_n) = \gamma_n\right) = \nu\left(\eta(x_1 + 1) = \gamma_1, ..., \eta(x_n + 1) = \gamma_n\right),$$

for any $n \geq 1$, $\gamma_1, \ldots, \gamma_n \in \{0,1\}$.

The natural partial order on $\{0,1\}^{\mathbb{N}}$ induces a partial order on the set of probability measures on $\{0,1\}^{\mathbb{N}}$ which we denote by $\leq$. Recalling that μ^+ is the upper invariant measure for the contact process on $\mathbb{N}$, we have

Lemma 2.8. $T^{-1}(\mu^+) \geq \mu^+$.

PROOF: Consider the contact process $\{\chi_t,\ t \geq 0\}$ this time on $\mathbb{N} \cup \{0\}$, starting again from $\chi_0 \equiv 1$. Let now $\{\overline{\chi}_t,\ t \geq 0\}$ denote the same process, with the same initial condition and the same realization of the graphical representation, except that we delete all arrows between states 0 and 1. The restriction to $\mathbb{N}$ of the asymptotic (as $t \to \infty$) law of $\overline{\chi}_t$ coincides with μ^+, while the same law associated with χ_t coincides with $T^{-1}(\mu^+)$.The result follows from the fact that for all $t > 0$, $x \geq 1$, $\mathbb{P}(\chi_t(x) \geq \overline{\chi}_t(x)) = 1$. $\square$

Our next proposition is taken from (vdBerg,Haggstrom,Kahn) (See Theorem 2 in that reference). Although there the result is stated and proved for the contact process on $\mathbb{Z}$, their proof also holds for the contact process on $\mathbb{N}$.

Proposition 2.9. *Let $\{\xi_t(x),\ x \in \mathbb{N},\ t \geq 0\}$ denote the one–type contact process starting at time $t = 0$ from a deterministic configuration.Then, for each $t > 0$, conditioned on the event $\{\xi_t(x) = 1\}$, the collections of random variables $\{1-\xi_t(y),\ 0 < y < x\}$ and $\{\xi_t(y),\ y > x\}$ are positively associated.*

Lemma 2.10. *Let f be a continuous increasing real valued function on $\{0,1\}^{\mathbb{N}}$ which depends only upon coordinates which are greater than or equal to $x+1$ (for some $x \in \mathbb{N}$). Then*

$$\int f d\mu^+(\cdot|\eta(1) = 0, \ldots, \eta(x-1) = 0, \eta(x) = 1) \geq \int f d\mu^+.$$

PROOF: Consider the contact process $\{\chi_t,\ t \geq 0\}$ on $\mathbb{N}$, starting from $\chi_0 \equiv 1$. We deduce from Proposition 2.9:

$$\mathbb{E}(f(\chi_t) \mid \chi_t(1) = 0, \ldots, \chi_t(x-1) = 0, \chi_t(x) = 1) \geq \mathbb{E}(f(\chi_t) \mid \chi_t(x) = 1).$$

It then follows from Lemma 2.11 below that

$$\mathbb{E}(f(\chi_t) \mid \chi_t(1) = 0, \ldots, \chi_t(x-1) = 0, \chi_t(x) = 1) \geq \mathbb{E}(f(\chi_t)).$$

It remains to let $t \to \infty$. $\square$

Lemma 2.11. *Let $\{\chi_t,\ t \geq 0\}$ denote the contact process on $\mathbb{N}$, starting from any deterministic initial condition. For any $t > 0$, the law of χ_t has positive correlations.*

Proof: For the contact process on $[1, \cdots, n]$, the result follows from Theorem 2.14 on page 80 of Liggett (Liggett95). Our result then follows by letting $n \to \infty$. $\square$

Note that Lemma 2.11 applies as well to the contact process $\{\xi_t,\ t \geq 0\}$ on $\mathbb{Z}$.

Let $S^{x,y} = \{\xi_t^x \neq \emptyset,\ \forall t > 0;\ \xi_t^y \neq \emptyset,\ \forall t > 0\}$. Recall that both processes $\{\xi_t^x,\ t \geq 0\}$ and $\{\xi_t^y,\ t \geq 0\}$ are constructed with the same set of Poisson processes $\{P_t^x, P_t^{x,+}, P_t^{x,-},\ x \in \mathbb{Z}\}$ as explained above. Note that on the event $S^{x,y}$ the process starting from $\{x\}$ survives but this does not mean that if we start from $\{x, y\}$ the progeny of (say) x lives forever. We now show that (recall the definition of ρ in (1.1))

Lemma 2.12. *For all $x, y \in \mathbb{Z}$*

$$\mathbb{P}(S^{x,y}) \geq \rho^2.$$

Proof: Denoting by μ the upper invariant measure of the contact process $\{\xi_t,\ t \geq 0\}$ on $\mathbb{Z}$, i. e. μ is the limit as $t \to \infty$ of the law of $\xi_t^{\mathbb{Z}}$, we have by the same duality argument already used in the proof of Lemma 2.7 the identities

$$\begin{aligned}
\mathbb{P}(\xi_t^x \neq \emptyset, \forall t > 0) &= \mu(\eta(x) = 1),\\
\mathbb{P}(\xi_t^y \neq \emptyset, \forall t > 0) &= \mu(\eta(y) = 1),\\
\mathbb{P}(S^{x,y}) &= \mu(\eta(x) = 1, \eta(y) = 1).
\end{aligned}$$

Letting $t \to \infty$ in the result of Lemma 2.11 applied to the contact process on $\mathbb{Z}$ implies that μ has positive correlations. Hence

$$\mu(\eta(x) = 1, \eta(y) = 1) \geq \mu(\eta(x) = 1) \times \mu(\eta(y) = 1).$$

The result follows from this inequality and the three above identities. $\square$

We now fix some $\lambda > \lambda_c$ and let $v = v(\lambda)$. We pick

$$0 < \varepsilon < \frac{v}{2} \wedge \frac{\rho^2}{4}.$$

From now on t_0 will be a large enough multiple of $\frac{2}{v}$ so that the following holds :

$$\mathbb{P}\left(B(t_0,\varepsilon)\right) \geq 1-\varepsilon, \tag{2.3}$$

where

$$B(t_0,\varepsilon) = \left\{ v-\varepsilon \leq \frac{r_{t_0}}{t_0} \leq v+\varepsilon, v-\varepsilon \leq -\frac{\ell_{t_0}}{t_0} \leq v+\varepsilon \right\}.$$

Let us define new processes. For any $z \in \mathbb{Z}$, we write

$$\begin{aligned} r_t^z &= \sup\left\{x : \xi_t^z(x) = 1\right\} - z, \\ \ell_t^z &= \inf\left\{x : \xi_t^z(x) = 1\right\} - z, \end{aligned}$$

where as usual the sup (resp. the inf) over an empty set is $-\infty$ (resp. $+\infty$).

Now we define the event

$$\begin{aligned} C(t_0,\varepsilon) = &\left\{ v-\varepsilon \leq \frac{r^0_{t_0}}{t_0} \leq v+\varepsilon, v-\varepsilon \leq -\frac{\ell^0_{t_0}}{t_0} \leq v+\varepsilon \right\} \\ &\bigcap \left\{ v-\varepsilon \leq \frac{r^{vt_0}_{t_0}}{t_0} \leq v+\varepsilon,\ v-\varepsilon \leq -\frac{\ell^{vt_0}_{t_0}}{t_0} \leq v+\varepsilon \right\}, \end{aligned}$$

and prove:

Lemma 2.13. *Let ε be as above. Then, for any large enough t_0, we have:*

$$\mathbb{P}\left(C(t_0,\varepsilon)\right) \geq \rho^2 - 2\varepsilon.$$

Proof: First note that on the event $\{\xi_t^0 \neq \emptyset, \forall t>0\}$ we have: $r_t^0 = r_t$ and $\ell_t^0 = \ell_t$ and a similar result holds for r^{vt_0} and ℓ^{vt_0}. Hence the result follows from translation invariance, Lemma 2.12 and (2.3). □

From now on, t_0 will be a large enough multiple of $\frac{2}{v}$ such that both the inequality (2.3) and the conclusion of Lemma 2.13 hold.

3 The two–type contact process with equal birth rates

Let η_t denote the contact process with two types. For $A, B \subset \mathbb{Z}$ with $A \cap B = \emptyset$, $\{\eta_t^{A,B},\ t \geq 0\}$ now denotes the contact process where at time zero A is the set of sites occupied by individuals of type 1, and B is the set of sites occupied by individuals of type 2. The dynamics is the same as before, using the same construction with the same collection of Poisson processes, except that now an individual of type $\alpha \in \{1,2\}$ located at site z gives birth at time t to an individual of the same type at site $z+1$ (resp. at site $z-1$), if t is a point of the Poisson process $P^{x,+}$ (resp. $P^{x,-}$) and the site $z+1$ (resp. $z-1$) is not occupied at time t.

3.1 A single mutant in front of an infinite number of residents may survive

In this subsection, we consider the process $\{\eta_t^{A,B},\ t \geq 0\}$ only in the case where $A < B$, meaning that all points in A are located on the left of each point of B. In other words, the initial configurations belong to the set:

$$\mathcal{L} := \{\eta : \ \eta(x) = 1, \eta(y) = 2 \ \Rightarrow \ x < y\}.$$

Given the nearest neighbor character of our process, whenever it starts in $\mathcal{L}$, it remains in $\mathcal{L}$ with probability 1.

For a configuration $\eta \in \mathcal{L}$, we define

$$\begin{aligned} br(\eta) &= \sup\{x : \eta(x) = 1\} \text{ and} \\ b\ell(\eta) &= \inf\{x : \eta(x) = 2\}. \end{aligned}$$

We now have the following consequence of Lemma 2.13 (here $\mathbb{P}^{A,B}$ denotes the law of $\{\eta_t^{A,B},\ t \geq 0\}$) :

Corollary 3.1. *For t_0 large enough, we have*

$$\mathbb{P}^{(-\infty,0],\{vt_0\}}\left(\left\{br(\eta_{t_0}) \leq \frac{vt_0}{2}\right\} \cap C(t_0,\varepsilon)\right) \geq \frac{\rho^2}{2} - \varepsilon.$$

Proof:

By Lemma 2.13 and symmetry arguments we have:

$$\mathbb{P}^{\{0\}\{vt_0\}}\left(\left\{br(\eta_{t_0}) \leq \frac{vt_0}{2}\right\} \cap C(t_0,\varepsilon)\right) \geq \frac{\rho^2}{2} - \varepsilon$$

On $C(t_0,\varepsilon)$ there is an open path from $(0,0)$ to some point in $[-(v+\varepsilon)t_0, -(v-\varepsilon)t_0] \times \{t_0\}$. Any open path starting from $(vt_0, 0)$ remains strictly to the right of the previous path, since otherwise there would be an open path from $(vt_0, 0)$ to $[-(v+\varepsilon)t_0, -(v-\varepsilon)t_0] \times \{t_0\}$, which cannot occur on the event $C(t_0, \varepsilon)$. Consequently for the initial configuration $\{0\}\{vt_0\}$ the first of these paths is always occupied by a type 1 particle. Therefore, adding to the initial configuration extra 1-type particles to the left of the origin does not alter the process to the right of that open path. Hence

$$\begin{aligned}
\mathbb{P}^{(-\infty,0],\{vt_0\}} & \left(\left\{ br(\eta_{t_0}) \leq \frac{vt_0}{2} \right\} \cap C(t_0, \varepsilon) \right) \\
&= \mathbb{P}^{\{0\}\{vt_0\}} \left(\left\{ br(\eta_{t_0}) \leq \frac{vt_0}{2} \right\} \cap C(t_0, \varepsilon) \right) \\
&\geq \frac{\rho^2}{2} - \varepsilon.
\end{aligned}$$

$\square$

To show that a similar result holds for the two type contact process on $(-\infty, \frac{3}{2}vt_0]$, we start with another lemma concerning the two type contact process on $\mathbb{Z}$:

Lemma 3.2. *As $t_0 \to \infty$,*

$$\mathbb{P}^{\{0\}\{vt_0\}} \left(\left\{ br(\eta_{t_0}) \leq \frac{vt_0}{2} \right\} \cap C(t_0, \varepsilon) \right) -$$
$$\mathbb{P}^{\{0\}\{vt_0\}} \left(\left\{ \exists\ x \in \left(\frac{vt_0}{2}, \frac{3vt_0}{4} \right] ;\ \eta_{t_0}(x) = 2 \right\} \cap \left\{ br(\eta_{t_0}) \leq \frac{vt_0}{2} \right\} \cap C(t_0, \varepsilon) \right)$$

converges to 0.

Proof: It suffices to show that

$$\mathbb{P}^{\{0\}\{vt_0\}} \left(\left\{ \forall\ x \in \left(\frac{vt_0}{2}, \frac{3vt_0}{4} \right] ;\ \eta_{t_0}(x) \neq 2 \right\} \cap \left\{ br(\eta_{t_0}) \leq \frac{vt_0}{2} \right\} \cap C(t_0, \varepsilon) \right),$$

converges to 0 as t_0 goes to infinity. But on the event $\{br(\eta_{t_0}) \leq \frac{vt_0}{2}\}$ there are no 1's at time t_0 on the interval $[\frac{vt_0}{2} + 1, \frac{3vt_0}{4}]$, hence we only need to prove that for the one type contact process

$$\mathbb{P}^{\{vt_0\}} \left(\left\{ \forall\ x \in \left(\frac{vt_0}{2}, \frac{3vt_0}{4} \right] ;\ \eta_{t_0}(x) = 0 \right\} \cap C(t_0, \varepsilon) \right)$$

converges to 0 as t_0 goes to infinity. But on the event $C(t_0,\varepsilon)$ the set of occupied points in the interval $\left(\frac{vt_0}{2},\frac{3vt_0}{4}\right]$ is the same whether the initial condition of the process is $\mathbb{Z}$ or $\{vt_0\}$. Since starting from $\mathbb{Z}$ we have more occupied points than under the upper invariant measure the result follows from the fact that under the upper invariant measure the probability of having an empty interval of length n tends to 0 as n tends to infinity. □

From now on we shall use $\{{}_a\zeta_t,\ t\geq 0\}$ to denote the two–type contact process on $(-\infty,a]$. Now we can prove:

Corollary 3.3. *Provided t_0 is large enough, we have*

$$\mathbb{P}^{(-\infty,0],\{vt_0\}}\left(\left\{br({}_{\frac{3vt_0}{2}}\zeta_{t_0})\leq\frac{vt_0}{2}\right\}\cap\left\{\exists x:\ \frac{vt_0}{2}<x<\frac{3vt_0}{2},\ {}_{\frac{3vt_0}{2}}\zeta_{t_0}(x)=2\right\}\right)\geq\frac{\rho^2}{2}-2\varepsilon.$$

Proof: In this proof we will consider the two type contact process on both $\mathbb{Z}$ and $(-\infty,\frac{3}{2}vt_0]$. These two processes are constructed on the same probability space with the same Poisson processes. For the second of these pocesses $\{P_t^x;\ x>\frac{3}{2}vt_0\}$, $\{P_t^{x,-};\ x>\frac{3}{2}vt_0]\}$ and $\{P_t^{x,+};\ x\geq\frac{3}{2}vt_0\}$ play no role. These processes η_t and ${}_{\frac{3vt_0}{2}}\zeta_t$ are assumed to start both from the configuration $\left((-\infty,0],\{\frac{vt_0}{2}\}\right)$.

On the set

$$\left\{\exists x\in\left(\frac{vt_0}{2},\frac{3vt_0}{4}\right];\ \eta_{t_0}(x)=2\right\},$$

there is an open path from $(vt_0,0)$ to $(\frac{vt_0}{2},\frac{3vt_0}{4}]\times\{t_0\}$. We now show that the probability that this path ever reaches the vertical line $\{x=\frac{3vt_0}{2}\}$ between time 0 and time t_0 converges to 0 as t_0 goes to infinity. Indeed, if that happened, there would be either an open path from $(vt_0,0)$ to $\{\frac{3}{2}vt_0\}\times[0,\frac{3}{8}t_0]$ or an open path from $\{\frac{3}{2}vt_0\}\times[\frac{3}{8}t_0,t_0]$ to $[\frac{1}{2}vt_0,\frac{3}{4}vt_0]\times\{t_0\}$. The existence of the first of these paths has a probability which converges to 0 as t_0 goes to infinity by Lemma 2.2. By reversing the arrows and using symmetry and again Lemma 2.2, we see that the same happens to the second path.

Hence, if we define

$$\begin{aligned}G=&\left\{\exists\text{ an open path from }(vt_0,0)\text{ to }\left(\frac{vt_0}{2},\frac{3vt_0}{4}\right]\times\{t_0\}\right\}\\&\cap\left\{\nexists\text{ an open path from }(vt_0,0)\text{ to }\left(\frac{vt_0}{2},\frac{3vt_0}{4}\right]\times\{t_0\}\text{ which touches the line }x=\frac{3vt_0}{2}\right\},\end{aligned}$$

we deduce from Lemma 3.2 that

$$\mathbb{P}^{\{0\}\{vt_0\}}\left(C(t_0,\varepsilon)\cap\left\{br(\eta_{t_0})\le\frac{vt_0}{2}\right\}\right)-\mathbb{P}^{\{0\}\{vt_0\}}\left(G\cap C(t_0,\varepsilon)\cap\left\{br(\eta_{t_0})\le\frac{vt_0}{2}\right\}\right)$$

converges to 0 as t_0 goes to infinity . The result follows from Corollary 3.1 and the following claim: starting both η_t and ${}_{\frac{3vt_0}{2}}\zeta_t$ from $(\{0\}\{vt_0\})$ we have:

$$\begin{aligned}&G\cap C(t_0,\varepsilon)\cap\{br(\eta_{t_0})\le\frac{vt_0}{2}\}\\&\quad\subset\left\{br({}_{\frac{3vt_0}{2}}\zeta_{t_0})\le\frac{vt_0}{2}\right\}\cap\left\{\exists\ \frac{vt_0}{2}<x<\frac{3vt_0}{2},\ {}_{\frac{3vt_0}{2}}\zeta_{t_0}(x)=2\right\}.\end{aligned}$$

To justify this claim note first that on the event G there exists a rightmost open path from $(vt_0,0)$ to $\left(\frac{vt_0}{2},\frac{3vt_0}{4}\right]$, which remains to the left of the line $x=\frac{3vt_0}{2}$. Now on the event G, the processes η_t and ${}_{\frac{3vt_0}{2}}\zeta_t$ must coincide up to time t_0 on any point to the left of or on that open path. $\square$

We now introduce the following partial order on $\{0,1,2\}^{\mathbb{Z}}$:
$\eta_1\succeq\eta_2$ whenever both

$$\{x:\eta_1(x)=2\}\subset\{x:\eta_2(x)=2\}\ \text{ and }\ \{x:\eta_2(x)=1\}\subset\{x:\eta_1(x)=1\}\,. \tag{3.1}$$

Intuitively $\succeq$ means "more 1's" and "fewer 2's".

This partial order extends to probability measures on the set of configurations: $\mu_1\succeq\mu_2$ means that there exists a probability measure ν on $(\{0,1,2\}^{\mathbb{Z}})^2$ with marginals μ_1 and μ_2 such that $\nu(\{(\eta,\zeta):\eta\succeq\zeta\})=1$. The same notation will be used below for measures on $\{0,1,2\}^A$, for some $A\subset\mathbb{Z}$. We now state the

Definition 3.4. *Let η_1, η_2 be two random configurations, μ_1 and μ_2 their respective probability distributions. We shall say that $\eta_1\succeq\eta_2$* **a. s.** *whenever (3.1) holds a. s., and that $\eta_1\succeq\eta_2$* **in distribution** *whenever $\mu_1\succeq\mu_2$.*

Remark 3.5. *The reader might think that a more natural definition of the inequality in distribution would be to say that $\mu_1\succeq\mu_2$ whenever $\mu_1(f)\ge\mu_2(f)$ for all $f:\{0,1,2\}^{\mathbb{Z}}\to\mathbb{R}$ which are increasing in the sense that $\eta_1\succeq\eta_2$ implies $f(\eta_1)\ge f(\eta_2)$. Theorem II.2.4 in (Liggett95) says that for the standard partial order on $\{0,1\}^{\mathbb{Z}}$ the two definitions are equivalent. It is clear that this theorem can be extended to our partial order, but we shall not need this result here.*

Note that $\eta_1 \succeq \eta_2$ implies $br(\eta_1) \geq br(\eta_2)$ and that if $\gamma \succeq \zeta$, the coupling between the contact processes starting form different initial conditions deduced from the graphical representation produces the property

$$\mathbb{P}(\eta_t^\gamma \succeq \eta_t^\zeta \ \forall t \geq 0) = 1. \tag{3.2}$$

In the sequel for any probability measure μ on $\{0,1,2\}^{\mathbb{Z}}$ and any $i \in \mathbb{N}$, $T^i(\mu)$ will denote the measure μ translated by i. That is the measure such that for all $n \in \mathbb{N}$, all $x_1 < x_2 < \cdots < x_n$ and all possible values of $a_1, \ldots, a_n$ we have:

$$T^i(\mu)(\{\eta : \eta(x_1) = a_1, \ldots, \eta(x_n) = a_n\}) =$$
$$\mu(\{\eta : \eta(x_1 - i) = a_1, \ldots, \eta(x_n - i) = a_n\}) \quad (*).$$

Moreover, if μ is a measure on $A^{[n,\infty)}$ where A is any non-empty subset of $\{0,1,2\}$, then $T^i(\mu)$ will be the measure on $A^{[n+i,\infty)}$ satisfying (*).

As before μ^+ denotes the upper invariant mesure for the contact process on $\mathbb{N}$ and μ_2^+ will be the measure obtained from μ^+ by means of the map: $F : \{0,1\}^{\mathbb{N}} \to \{0,2\}^{\mathbb{N}}$ given by $F(\eta)(x) = 2\eta(x)$. With a slight abuse of notation the measures μ^+ and μ_2^+ will also be seen as measures on $\{0,1,2\}^{\mathbb{N}}$ and a similar abuse of notation will be applied to the translates of these measures.

We start the process $\{\eta_t,\ t \geq 0\}$ from the initial distribution $\overline{\mu}$ determined by

- (i) The projection of $\overline{\mu}$ on $\{0,1,2\}^{(-\infty,vt_0]}$ is the point mass on the configuration
$$\eta(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } 0 < x < vt_0, \\ 2, & \text{if } x = vt_0, \end{cases}$$
- (ii) the projection of $\overline{\mu}$ on $\{0,1,2\}^{[vt_0+1,\infty)}$ is $T^{vt_0}(\mu_2^+)$.

In the sequel η^0 will denote a random initial configuration distributed according to $\overline{\mu}$. In other words, we assume that $\eta_0 = \eta^0$.

We now proceed as follows. We partition the probability space into a countable number of events: $H, J_0, J_1, \ldots$ and let the process run on a time interval of length t_0. Then we show that the distribution of η_{t_0} conditioned on any event of the partition is $\preceq$ than a convex combination of translations of $\bar{\mu}$. Hence the unconditioned distribution of η_{t_0} is also $\preceq$ such a convex combination. Then we replace η_{t_0} by a random configuration η^1 whose distribution is this convex combination and let the process run on another time interval of length t_0 and so on.

For each $n \in \{\frac{3vt_0}{2}\} \cup \{2vt_0, 2vt_0+1, \ldots\}$ we define two new processes: ${}_n\zeta_s$ on $\{0,1,2\}^{(-\infty,n]}$ and ${}_n\xi_s$ on $\{0,2\}^{[n+1,\infty)}$. These evolve like the process η_t and are constructed with the same Poisson processes $P_t^{x,-}, P_t^{x,+}$ and P_t^x. For the first of these processes the Poisson processes $\{P_t^x : x > n\}$, $\{P_t^{x,+} : x \geq n\}$ and $\{P_t^{x,-} : x > n\}$ play no role. A similar statement holds for the second process. The initial distribution of these processes are the projections of $\overline{\mu}$ on $\{0,1,2\}^{(-\infty,n]}$ and $\{0,1,2\}^{[n+1,\infty)}$ respectively. Since we only consider cases where $n \geq \frac{3vt_0}{2}$, the second of these projections concentrates on $\{0,2\}^{[n+1,\infty)}$.

Our partition of the probability space is given by :

$$\begin{aligned} H &= \left\{ br\left({}_{\frac{3vt_0}{2}}\zeta_{t_0}\right) \leq \frac{vt_0}{2},\ \exists\ \frac{vt_0}{2} < x < \frac{3vt_0}{2} :\ {}_{\frac{3vt_0}{2}}\zeta_{t_0}(x) = 2 \right\}, \\ J_m &= \{Q_{t_0} = vt_0 + m\} \cap H^c \quad \text{for} \quad m = 0, 1, \ldots, \end{aligned}$$

where $Q_{t_0} = \max\{R_{t_0}, vt_0\}$ (recall that $R_t = \sup_{s \leq t} r_s$).

Since the initial distribution considered here is $\preceq$ than the initial distribution of Corollary 3.3, we have

$$\mathbb{P}(H) \geq \frac{\rho^2}{2} - 2\varepsilon > 0. \tag{3.3}$$

Note that on H

1. The set $\{x : \eta_{t_0}(x) = 1\}$ is contained in $(-\infty, \frac{vt_0}{2}]$ (indeed since $\{x,\ {}_{\frac{3vt_0}{2}}\zeta_{t_0}(x) = 2\} \neq \emptyset$, $\{x,\ \eta_{t_0}(x) = 1\} = \{x,\ {}_{\frac{3vt_0}{2}}\zeta_{t_0}(x) = 1\}$).

2. The set $\{x : \eta_{t_0}(x) = 2\}$ contains $\{x :\ {}_{\frac{3vt_0}{2}}\xi_{t_0}(x) = 2\}$.

We also claim that conditioned on H, the distribution of ${}_{\frac{3vt_0}{2}}\xi_{t_0}$ is $\geq T^{\frac{3vt_0}{2}}\mu_2^+$ (this follows from Lemma 2.8 and the fact that the process ${}_{\frac{3vt_0}{2}}\xi_t$ is independent of H).

Therefore, the distribution of η_{t_0} conditioned on H is $\preceq \nu$ where ν is determined by:

1. The projection of ν on $\{0,1,2\}^{(-\infty,\frac{3vt_0}{2}]}$ is the point mass on the configuration
$$\eta(x) = \begin{cases} 1, & \text{if } x \le \frac{vt_0}{2}, \\ 0, & \text{if } \frac{vt_0}{2} < x \le \frac{3vt_0}{2} \end{cases}$$
and

2. the projection of ν on $\{0,1,2\}^{[\frac{3vt_0}{2}+1,\infty)}$ is $T^{\frac{3vt_0}{2}}(\mu_2^+)$.

It follows from Lemma 2.10 (applied to μ_2^+ instead of μ^+) that if Y is a $\mathbb{N}$–valued random variable such that
$$\mathbb{P}(Y = n) = \mu_2^+(\{\eta : \eta(x) = 0,\ x = 1, \dots, n-1, \eta(n) = 2\}),$$
then
$$\nu \preceq \sum_{n=1}^{\infty} \mathbb{P}(Y = n) T^{\frac{vt_0}{2}+n}\overline{\mu}.$$
Hence the distribution of η_{t_0} given H is $\preceq \sum_{n=1}^{\infty} \mathbb{P}(Y = n) T^{\frac{vt_0}{2}+n}\overline{\mu}$.

A similar argument shows that the conditional distribution of η_{t_0} given J_m is $\preceq$
$$\sum_{n=1}^{\infty} \mathbb{P}(Y = n) T^{vt_0+n+m}\overline{\mu}\ ,$$
where Y is distributed as above.

It follows from the above arguments that $\eta_{t_0} \preceq \mu^1$ in distribution, where
$$\mu^1 := \mathbb{P}(H) \sum_{n=1}^{\infty} \mathbb{P}(Y = n) T^{\frac{vt_0}{2}+n}\overline{\mu} + \sum_{m=0}^{\infty} \mathbb{P}(J_m) \sum_{n=1}^{\infty} \mathbb{P}(Y = n) T^{vt_0+m+n}\overline{\mu} \quad (3.4)$$

We can now state:

Proposition 3.6. *If t_0 is large enough, there exists a positive integer valued random variable $Z(t_0)$ such that*

a) $\mu^1 = \sum_{n=1}^{\infty} \mathbb{P}(Z(t_0) = n) T^{\frac{vt_0}{2}+n}\overline{\mu}$.

b) $Z(t_0)$ has an exponentially decaying tail.

c) $w := \mathbb{E}(\frac{Z(t_0)}{t_0}) < v$.

PROOF:

Part a) follows from (3.4) and part b) follows from part a) of Lemma 2.6 and the fact that R_{t_0} is bounded by a Poisson random variable of parameter λt_0. To prove part c) write

$$\begin{aligned}
\mathbb{E}(Z(t_0)) = & \sum_{n=1}^{\infty} \mathbb{P}(H)\mathbb{P}(Y=n)\left(\frac{vt_0}{2}+n\right) \\
& + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \mathbb{P}(H^c, Q_{t_0} = vt_0+m)\mathbb{P}(Y=n)(vt_0+n+m) \\
\le & \ \mathbb{P}(H)\left[\frac{vt_0}{2}+\mathbb{E}(Y)\right] + \mathbb{P}(H^c)\mathbb{E}(Y) \\
& + \sum_{m=0}^{\infty} \mathbb{P}(H^c, Q_{t_0} = vt_0+m)(vt_0+m) \\
= & \ \mathbb{P}(H)\frac{vt_0}{2} + \mathbb{E}(Y) + \mathbb{E}(Q_{t_0}) - \mathbb{E}(Q_{t_0}; H) \\
\le & \ \mathbb{P}(H)\frac{vt_0}{2} + \mathbb{E}(Y) + \mathbb{E}(Q_{t_0}) - \mathbb{P}(H)vt_0 \\
= & \ \mathbb{E}(Q_{t_0}) + \mathbb{E}(Y) - \mathbb{P}(H)\frac{vt_0}{2}.
\end{aligned}$$

Hence it follows from Lemma 2.2 that

$$\limsup_{t_0\to\infty} \frac{\mathbb{E}(Z(t_0))}{t_0} \le v\left(1-\frac{\mathbb{P}(H)}{2}\right),$$

and the result follows from (3.3). $\square$

We can now prove:

Proposition 3.7. *Let $\overline{\mu}$ be the initial distribution of the process. Then*

$$\limsup_{t\to\infty} \frac{br(\eta_t)}{t} < v \quad a.s.$$

PROOF: Choose t_0 large enough, such that the conclusion of Proposition 3.6 holds true. It follows from that same Proposition, the Markov property and (3.2) that for all $k \in \mathbb{N}$ the distribution of η_{kt_0} is $\preceq \sum_n \mathbb{P}(U_1 + \cdots + U_k =$

$n)T^n\overline{\mu}$ where the U_i's are i.i.d. random variables distributed as $Z(t_0)$. It then follows that

$$\mathbb{P}\left(\frac{br(\eta_{kt_0})}{kt_0} \geq z\right) \leq \mathbb{P}\left(\frac{U_1 + \cdots + U_k}{kt_0} \geq z\right)$$

for any real z. Using part c) of Proposition 3.6 and standard large deviation estimates we get that for for any $z > \frac{\mathbb{E}(Z(t_0))}{t_0}$ we have:

$$\sum_k \mathbb{P}\left(\frac{br(\eta_{kt_0})}{kt_0} \geq z\right) < \infty.$$

Hence, by the Borel-Cantelli lemma we get:

$$\limsup_k \frac{br(\eta_{kt_0})}{kt_0} \leq w \ a.s.$$

where w is as in Proposition 3.6. Hence the result holds along the sequence kt_0. Finally the gaps are easy to control since for any initial configuration, the process $br(\eta_t)$ makes jumps to the right which are bounded above by a Poisson process of parameter λ. □

It follows readily from this result that

Corollary 3.8.

$$\gamma := \mathbb{P}^{\mathbb{Z}_-,\{1\}}(\textit{the type 2 population survives for ever}) > 0.$$

Proof: First suppose that the initial distribution of the process is $\overline{\mu}$ and call η_0 the initial random configuration. It then follows from the above Corollary and Corollary 2.4 that for some $x > 0$ there is an infinite open path starting at $(x, 0)$ such that for any (y, t) in this path we have $\eta_t(y) = 2$. This conclusion remains true if we suppress all the initial "2's" to the right of x. The corollary then follows from the Markov property and (3.2). □

3.2 A finite number of mutants do not survive in between a double infinity of residents

The aim of this subsection is to prove

Theorem 3.9. *Consider the two type contact process $\{\eta_t^{A,B},\ t \geq 0\}$, where $|B| < \infty$, and the set A contains an infinite number of points located both to the left and to the right of B,*

Then a. s. there exists $t < \infty$ such that

$$\{x;\ \eta_s^{A,B}(x) = 2\} = \emptyset, \quad \forall s \geq t.$$

Let us first prove the following weaker statement. We shall then verify that the Theorem follows from it.

Proposition 3.10. *For any $n, m \in \mathbb{N}$ let $A_{n,m} = \{x \in \mathbb{Z} : x \leq -m \text{ or } x \geq n\}$ and $B = \{0\}$, then a. s. there exists $t < \infty$ such that*

$$\{x;\ \eta_s^{A_{n,m},B}(x) = 2\} = \emptyset, \quad \forall s \geq t.$$

PROOF: By the Markov property and (3.2) it suffices to prove the result for $n = m = 1$. Indeed starting from that configuration, for any n, $m > 1$, with positive probability we find ourselves at time one with the same unique type 2 individual located at $x = 0$, sites $-m+1, \ldots, -1$ empty, sites $1, \ldots, n-1$ empty, and some of the other sites occupied by type 1 individuals.

Let α_t denote the number of descendants at time t of the unique initial type 2 individual (hence α_t denotes also the number of type 2 individuals at time t). On the event that the lineage of the unique type 2 individual survives for ever we have $\alpha_t \to \infty$ as $t \to \infty$ a. s. Hence if that event has positive probability, $\mathbb{E}(\alpha_t) \to \infty$ as $t \to \infty$. Consequently for any $\delta > 0$,

$$T_\delta = \inf\{t > 0,\ \mathbb{E}(\alpha_t) \geq 1 + \delta\} < \infty.$$

Denote by $r'_t(x)$ the supremum of the set of sites occupied by the descendants of the individual $(x, 0)$. Clearly, whatever the initial configuration is $\mathbb{E}[r'_t(x) - x] \leq \mathbb{E}[r_t]$, where as above

$$r_t = \sup\left\{x :\ \xi_t^{\mathbb{Z}^-}(x) = 1\right\}.$$

From the result recalled at the beginning of section 2, there exists T' such that

$$\mathbb{E}\left[\frac{r_t}{t}\right] \leq v + 1, \quad \forall t \geq T'.$$

Recall that in our initial configuration all sites are occupied (who occupies each site is irrelevant to contradict the fact that $T_\delta < \infty$, which we now do).

For n odd, let $Z_t(n)$ be the number of sites which at time t are in a line of descendance starting at time 0 in the interval $[-(n-1)/2, \dots, (n-1)/2]$. Now by stationarity whenever $t \geq T_\delta$,

$$\mathbb{E}(Z_t(n)) \geq n(1+\delta).$$

On the other hand, if $t \geq T'$, by symmetry,

$$\mathbb{E}(Z_t(n)) \leq n + 2t(v+1).$$

Choosing $n > 2t(v+1)/\delta$, the last two inequalities yield a contradiction. $\square$

In order to deduce Theorem 3.9 from Proposition 3.10, we shall need the following Lemma where, as above, the η_t's for various initial conditions are defined with the same unique graphical representation.

Lemma 3.11. *Let $(x_n)_{n\geq 0}$ be a strictly increasing sequence of strictly positive integers and let $(y_m)_{m\geq 0}$ be a strictly decreasing sequence of strictly negative integers. Then,*

$$\mathbb{P}(\exists\ n : \forall\ t > 0, \exists\ x : \eta_t^{\{x_n\},\{x_n-1,x_n-2,\dots\}}(x) = 1) = 1,$$

and

$$\mathbb{P}(\exists\ m : \forall\ t > 0, \exists\ x : \eta_t^{\{y_m\},\{y_m+1,y_m+2,\dots\}}(x) = 1) = 1.$$

Proof: Define for n, $m \geq 0$ the events

$$C_n = \{\forall\ t > 0, \exists\ x : \eta_t^{\{x_n\},\{x_n-1,x_n-2,\dots\}}(x) = 1\},$$
$$D_m = \{\forall\ t > 0, \exists\ x : \eta_t^{\{y_m\},\{y_m+1,y_m+2,\dots\}}(x) = 1\}.$$

From Corollary 3.8, symmetry and translation invariance,

$$\mathbb{P}(C_n) = \mathbb{P}(D_m) = \gamma \quad \forall n, m \geq 0.$$

On the set $\{(x,t) : x \in \mathbb{Z}, t \geq 0\}$ the Poisson processes used in the construction are n-fold mixing with respect to translations on $\mathbb{Z}$ for any $n \in \mathbb{N}$. Since $x_{n+1} \geq x_n + 1$, this implies that for all $k \geq 1$

$$\lim_{N\to\infty} \mathbb{P}\left(\cap_{j=0}^k C_{Nj}^c\right) = (1-\gamma)^k.$$

Consequently

$$\mathbb{P}\left(\cap_{n\geq 0} C_n^c\right) \leq (1-\gamma)^k$$

for all $k \geq 1$. This shows that

$$\mathbb{P}\left(\cup_{n\geq 0} C_n\right) = 1.$$

The result for the D_m's is proved similarly. $\square$

Proof of Theorem 3.9 By the Markov property, it suffices to consider the case where $A = \{y_n : n \in \mathbb{N}\} \cup \{x_n : n \in \mathbb{N}\}$, $B = \{0\}$ and the sequences (x_n) and (y_n) are as in the previous lemma.

For all $n, m \geq 1$, we define

$$E_{n,m} = \left\{\forall t > 0,\ \exists x : \eta_t^{\{x_n\},\{x_n-1,x_n-2,\dots\}}(x) = 1\right\} \bigcap \left\{\forall t > 0,\ \exists x : \eta_t^{\{y_m\},\{y_m+1,y_m+2,\dots\}}(x) = 1\right\}.$$

From the last Lemma we know that $\mathbb{P}(\cup_{n,m} E_{n,m}) = 1$. Hence, it suffices to show that for all $n, m \in \mathbb{N}$, we have:

$$\mathbb{P}(\forall\ t > 0\ \exists x :\ \eta_t^{A,\{0\}}(x) = 2, E_{n,m}) = 0.$$

But on the event $E_{n,m}$ the evolution of "2"'s is not altered by adding "1"'s to the left of y_m or to the right of x_n. Therefore the result follows from Proposition 3.10. $\square$

3.3 Proof of Theorem 1.1

The only if part follows from Theorem 3.9. Let us prove the if part. We consider the case where $|A \cap B^+| < \infty$. The other case is treated similarly.

Recall the definition of the set of configurations

$$\mathcal{L} = \{\eta;\ \text{s. t. } \eta(x) = 2,\ \text{and}\ \ \eta(y) = 1 \text{ imply } y < x\}.$$

We let

$$T = \inf\{t \geq 0,\ \eta_t \in \mathcal{L}\}.$$

Clearly $|A \cap B^+| < \infty$ implies that

$$\mathbb{P}^{A,B}(T < \infty) > 0.$$

Hence from the strong Markov property it remains to show that whenever $A \cap B^+ = \emptyset$,

$$\mathbb{P}^{A,B}(\text{the type 2 population survives for ever}) > 0.$$

This last statement follows from translation invariance, (3.2) and Corollary 3.8.

3.4 Proof of Theorem 1.2

By the Markov property and symmetry it suffices to show that the theorem holds for some A and B. To prove this, let $(x_n)_{n\geq 0}$ and $(y_m)_{m\geq 0}$ be as in the statement of Lemma 3.11 and let C_n and D_m be as in the proof of that lemma. It follows from that same lemma that there exist n and m such that $\mathbb{P}(C_n \cap D_m) > 0$. This implies that

$$\mathbb{P}^{\{y_m\}\{x_n\}}(\forall\ t > 0\ \exists\ x, y: \ \eta_t(x) = 1, \eta_t(y) = 2) > 0.$$

Hence, the theorem holds when $A = \{x_n\}$ and $B = \{y_m\}$.

3.5 Corollary for the one–type contact process

The following is an immediate consequence of the above results.

Corollary 3.12. *Let $\{\xi_t^A,\ t \geq 0\}$ denote the one–type contact process starting from the configuration ξ_0 and let $A = \{x,\ \xi_0(x) = 1\}$. It follows from our results that*

1. *if A contains both a sequence which converges to $+\infty$ and a sequence which converges to $-\infty$, then no individual has a progeny which survives for ever;*
2. *if $|A| = +\infty$ but $\sup A < \infty$, then exactly one individual has a progeny which survives for ever.*

Proof: The first statement is a consequence of Theorem 3.9. For the second statement first note that it follows from (3.2) and Corollary 3.8 that for any initial condition having a rightmost individual, the probability that this individual has a progeny which survives forever is bounded below by $\gamma > 0$. We then define an increasing sequence of stopping times: τ_1 is the smallest time at which the progeny of the rightmost initial individual dies out, τ_2 is the smallest time at which the progeny of the rightmost individual at time τ_1 dies out and so on. It then follows from a repeated application of the Strong Markov Property that $\mathbb{P}(\tau_n < \infty) \leq \gamma^n$. Hence, with probability 1 for some k, $\tau_k = \infty$ which implies that at least one individual has a progeny wchich survives forever. Suppose now that two individuals, say $x < y$, have a progeny which survives for ever with positive probability. Adding infinitely many individuals at time $t = 0$ on the right of y cannot possibly modify the fate of the progeny of x. This would mean that the progeny of x would survive for ever with positive probability, in the presence of infinitely many individuals at time $t = 0$ on both of its sides. This contradicts Theorem 3.9.

4 Remarks about the case of unequal birth rates

Assume that the type 1 individuals have the birth rate μ, and type 2 individuals have the birth rate λ.

It is not hard to deduce from our argument that for $\mu > \lambda_c$ there exists $\varepsilon > 0$ such that the conclusion of 1.1 remains true if $\mu - \varepsilon < \lambda < \mu$. However, we conjecture that this is not the case for all values of λ in the interval (λ_c, μ). Consider now the *right contact process*, where each individual gives birth to offsprings on its right at rate λ, and does not give birth to any offspring on its left. Let now λ_{cc} denote the critical value of the parameter λ, such that whenever $\lambda > \lambda_{cc}$, the one–type right contact process starting from $\{0\}$ has a positive probability of survival. Going back to our two–types contact process, whenever $\lambda > \lambda_{cc}$, whatever the value of μ may be, the progeny of a single type 2 individual with a finite number of type 1 individuals on its right at time 0, has a positive probability of survival.

In the other direction, we conjecture that if the rates favor type 2 individuals (i.e. $\lambda > \mu > \lambda_c$) then a unique type 2 individual has a positive probability of having descendants at all times even when all remaining sites are occupied at time 0 by type 1 individuals.

Acknowledgment We thank T.M. Liggett and J. Steif for useful discussions concerning the proof of Lemma 2.10, and F. Ezanno for his careful reading of an earlier version of this paper.

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