A WEAK CONVERGENCE THEOREM FOR PARTICLE MOTION IN A STOCHASTIC FIELD

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1. Introduction

Consider the two–dimensional diffusion process indexed by $n \geq 1$, solution of the SDE

$$\begin{cases} \dfrac{dU_t^n}{dt} = nV_t^n, \ U_0 = u, \\ dV_t^n = \sin(U_t^n)dW_t, \ V_0 = v, \end{cases} \tag{1.1}$$

where $(u, v) \notin \{(k\pi, 0), \ k \in \mathbb{Z}\}$. The aim of this note is to prove the

Theorem 1.1. *As $n \to \infty$,*

$$V^n \Rightarrow v + \frac{1}{\sqrt{2}} \times B,$$

where $\{B_t, \ t \geq 0\}$ is a standard one–dimensional Brownian motion, and the convergence is in the sense of convergence in law in $C(\mathbb{R}_+, \mathbb{R})$.

2. A change of time scale

Note that for any $n \geq 1$, the law of $\{(U_t^n, V_t^n), \ t \geq 0\}$, the solution of (1.1), is characterized by the statement

$$\begin{cases} \dfrac{dU_t^n}{dt} = nV_t^n, \ U_0 = u, \\ V^n \text{ is a martingale}, \ \dfrac{d < V^n >_t}{dt} = \sin^2(U_t^n), \ V_0^n = v. \end{cases}$$

Now define

$$X_t = U^n_{n^{-2/3}t}, \quad Y_t = n^{1/3} V^n_{n^{-2/3}t}.$$

We first note that $X_0 = u$, $Y_0 = n^{1/3}v$, Y is a martingale, and

$$\begin{cases} \dfrac{dX_t}{dt} = n^{-2/3}\dfrac{dU^n}{dt}(n^{-2/3}t) = n^{1/3}V^n_{n^{-2/3}t} = Y_t, \\ < Y >_t = n^{2/3} < V^n >_{n^{-2/3}t}, \ \dfrac{d < Y >_t}{dt} = \sin^2(X_t). \end{cases}$$

If we use a well–known martingale representation theorem, we can pretend that there exists a standard Brownian motion $\{B_t,\ t \geq 0\}$ such that

$$\begin{cases} \dfrac{dX_t}{dt} = Y_t,\ X_0 = u, \\ dY_t = \sin(X_t)dB_t,\ Y_0 = n^{1/3}v. \end{cases} \tag{2.1}$$

Note that the process $\{(X_t, Y_t),\ t \geq 0\}$ still depends upon n, but only through the value of Y_0.

On the other hand, $V_t^n = n^{-1/3}Y_{n^{2/3}t}$. Hence

$$V_t^n = v + n^{-1/3}\int_0^{n^{2/3}t} \sin(X_s)dB_s,$$

in other words V^n is a martingale such that $V_0^n = y$ and

$$< V^n >_t = n^{-2/3}\int_0^{n^{2/3}t} \sin^2(X_s^n)ds.$$

Here we recall the fact that the process X depends upon n (through the initial condition of Y), unless $v = 0$. Consequently

$$\lim_{n\to\infty} < V^n >_t = t \times \lim_{n\to\infty} \frac{1}{n^{2/3}t}\int_0^{n^{2/3}t} \sin^2(X_s^n)ds. \tag{2.2}$$

3. Qualitative properties of the solution of (2.1)

We now consider the two–dimensional diffusion process

$$\begin{cases} \dfrac{dX_t}{dt} = Y_t,\ X_0 = x, \\ dY_t = \sin(X_t)dW_t,\ Y_0 = y, \end{cases} \tag{3.1}$$

with values in the state–space $E = [0, 2\pi) \times \mathbb{R}\backslash\{(0,0),(\pi,0)\}$, where 2π is identified with 0. We first prove that the process $\{(X_t, Y_t),\ t \geq 0\}$ is a conservative E–valued diffusion. Indeed,

Proposition 3.1. *Whenever the initial condition (x, y) belongs to E,*

$$\inf\{t > 0,\ (X_t, Y_t) \in \{(0,0),(\pi,0)\}\} = +\infty \quad a.\ s.$$

Proof: We define the stopping time

$$\tau = \inf\{t,\ (X_t, Y_t) = (0,0)\}.$$

Let $R_t = X_t^2 + Y_t^2$, $Z_t = \log R_t$, $t \geq 0$. A priori, Z_t takes its values in $[-\infty, +\infty)$. Itô calculus on the interval $[0, \tau)$ yields

$$\begin{aligned} dX_t^2 &= 2X_tY_tdt, \\ dY_t^2 &= 2\sin(X_t)Y_tdW_t + \sin^2(X_t)dt, \\ dZ_t &= \frac{dR_t}{R_t} - \frac{d < R >_t}{2R_t^2} \\ &= \frac{2Y_tX_t + \sin^2(X_t)}{R_t}dt - 2\frac{\sin^2(X_t)Y_t^2}{R_t^2}dt \\ &\quad + 2\frac{Y_t\sin(X_t)}{R_t}dW_t. \end{aligned}$$

Now clearly $|\sin(x)| \leq |x|$, $\sin^2(x) \leq x^2$, and it follows from the above and standard inequalities that on the time interval $[0, \tau)$,

$$Z_t \geq Z_0 - 2t + \int_0^t \varphi_s dW_s,$$

where $|\varphi_s| \leq 1$. Hence the process $\{Z_t,\ t \geq 0\}$ is bounded from below on any finite time interval, which implies that $\tau = +\infty$ a. s., since $\tau = \inf\{t, Z_t = -\infty\}$. A similar argument shows that $\tau' = +\infty$ a. s., where

$$\tau' = \inf\{t,\ (X_t, Y_t) \in \{(0,0), (\pi, 0)\}\}.$$

□

We next prove the (here and below $\mathcal{B}_E$ stands for the σ–algebra of Borel subsets of E)

Proposition 3.2. *The collection of transition probabilities*

$$\{p((x,y); t, A) := \mathbb{P}((X_t, Y_t) \in A),\ (x,y) \in E,\ t > 0,\ A \in \mathcal{B}_E\}$$

has a smooth density $p((x,y); t, (x', y'))$ with respect to Lebesgue's measure $dx'dy'$ on E.

Proof: Consider the Lie algebra of vector fields on E generated by $X_1 = \sin(x)\frac{\partial}{\partial y}$, $X_2 = [X_0, X_1]$ and $X_3 = [[X_0, X_1], X_0]$, where $X_0 = y\frac{\partial}{\partial x}$. This Lie algebra has rank 2 at each point of E. The result is now a standard consequence of the well–known Malliavin calculus, see e. g. Nualart [4]. □

Proposition 3.3. *The E–valued diffusion process $\{(X_t, Y_t),\ t \geq 0\}$ is topologically irreducible, in the sense that for all $(x, y) \in E$, $t > 0$, $A \in \mathcal{B}_E$ with non empty interior,*

$$\mathbb{P}_{x,y}((X_t, Y_t) \in A) > 0.$$

PROOF: From Stroock–Varadhan's support theorem, see e. g. Ikeda–Watanabe [2], the support of the law of (X_t, Y_t) starting from $(X_0, Y_0) = (x, y)$ is the closure of the set of points which the following controlled ode can reach at time t by varying the control function $\{u(s),\ 0 \leq s \leq t\}$:

$$\begin{cases} \dfrac{dx}{ds}(s) = y(s), \quad x(0) = x; \\ \dfrac{dy}{ds}(s) = \sin(x(s))u(s), \quad y(0) = y. \end{cases} \tag{3.2}$$

It is not hard to show that the set of accessible points at time $t > 0$ by the solution of (3.2) is dense in E. The result now follows from the fact that the transition probability is absolutely continuous with respect to Lebesgue's measure, see Proposition 3.2. □

We next prove the

Lemma 3.4.

$$\mathbb{P}\left(|Y_t| \to \infty, \text{ as } t \to \infty\right) = 0.$$

PROOF: The Lemma follows readily from the fact that

$$Y_t = W\left(\int_0^t \sin^2(X_s)ds\right),$$

where $\{W(t),\ t \geq 0\}$ is a scalar Brownian motion. □

Hence the topologically irreducible E–valued Feller process $\{(X_t, Y_t),\ t \geq 0\}$ is recurrent. Its unique (up to a multiplicative constant) invariant measure is the Lebesgue measure on E, in particular the process is null–recurrent. It then follows from (ii) in Theorem 20.21 from Kallenberg [3]

Lemma 3.5. *For all $M > 0$, as $t \to \infty$,*

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{|Y_s| \leq M\}} ds \to 0 \quad a.\ s.$$

4. A path decomposition of the process $\{X_t, Y_t),\ t \geq 0\}$

We first define two sequences of stopping times. Let $T_0 = 0$ and

$$\begin{aligned} \text{for } \ell \text{ odd}, \quad & T_\ell = \inf\{t > T_{\ell-1},\ |Y_t| \geq M+1\}, \\ \text{for } \ell \text{ even}, \quad & T_\ell = \inf\{t > T_{\ell-1},\ |Y_t| \leq M\}. \end{aligned}$$

Let now $\tau_0 = T_1$. We next define recursively $\{\tau_k,\ k \geq 1\}$ as follows. Given τ_{k-1}, we first define

$$L_k = \sup\{\ell \geq 0,\ \tau_{k-1} \geq T_{2\ell+1}\}.$$

Now let

$$\eta_k = \begin{cases} \tau_{k-1}, & \text{if } \tau_{k-1} < T_{2L_k+2}, \\ T_{2L_k+3}, & \text{if } \tau_{k-1} \geq T_{2L_k+2}, \end{cases}$$

We now define

$$\tau_k = \inf\{t > \eta_k,\ |X_t - X_{\eta_k}| = 2\pi\} \wedge \inf\{t > \eta_k,\ |Y_t - Y_{\eta_k}| > 1\}.$$

It follows from the above definitions that

$$\int_0^t \mathbf{1}_{\{|Y_s| \geq M+1\}} \sin^2(X_s) ds \leq \sum_{k=1}^{\infty} \int_{\eta_k \wedge t}^{\tau_k \wedge t} \sin^2(X_s) ds \leq \int_0^t \sin^2(X_s) ds.$$

Define

$$\begin{aligned} K^0 &= \{k \geq 1,\ |Y_{\tau_k} - Y_{\eta_k}| < 1\}, \\ K^1 &= \{k \geq 1,\ |Y_{\tau_k} - Y_{\eta_k}| = 1\}, \\ K_t &= \{k \geq 1,\ \eta_k < t\}, \\ K_t^0 &= K^0 \cap K_t, \\ K_t^1 &= K^1 \cap K_t. \end{aligned}$$

We first prove the

Lemma 4.1.

$$\frac{1}{t} \sum_{k \in K_t^1} (\tau_k - \eta_k) \to 0$$

in $L^1(\Omega)$ as $M \to \infty$, uniformly in $t > 0$.

Proof: We shall use repeatedly the fact that since $|Y_{\eta_k}| \geq M > 2$, $|Y_{\eta_k}| - 1 \geq |Y_{\eta_k}|/2$. We have that (see the Appendix below), since $\tau_k - \eta_k \leq 4\pi/|Y_{\eta_k}|$,

$$\begin{aligned} \mathbb{P}(k \in K^1 | \mathcal{F}_{\eta_k}) &\leq \mathbb{P}(\sup_{\eta_k \leq t \leq \tau_k} |Y_t - Y_{\eta_k}| \geq 1 | \mathcal{F}_{\eta_k}) \\ &\leq 2\exp(-|Y_{\eta_k}|/8\pi). \end{aligned}$$

Consequently, using again the inequality $\tau_k - \eta_k \le 4\pi/|Y_{\eta_k}|$, we deduce that

$$\begin{aligned}\mathbb{E}\left[(\tau_k - \eta_k)\mathbf{1}_{\{k\in K^1\}}|\mathcal{F}_{\eta_k}\right] &\le \frac{8\pi}{|Y_{\eta_k}|}\exp(-|Y_{\eta_k}|/8\pi)\\ &\le \frac{8\pi}{|Y_{\eta_k}|}\exp(-M/8\pi)\end{aligned}$$

On the other hand, whenever $k \in K^0$,

$$\tau_k - \eta_k \ge 2\pi/(|Y_{\eta_k}|+1) \ge \pi/|Y_{\eta_k}|.$$

Now, provided $t \ge 4\pi/M$,

$$\begin{aligned}2t &\ge t + \frac{4\pi}{M}\\ &\ge \mathbb{E}\left[\sum_{k\in K_t^0}(\tau_k - \eta_k)\right]\\ &\ge \pi\mathbb{E}\left[\sum_{k\in K_t}\mathbf{1}_{\{k\in K^0\}}\frac{1}{|Y_{\eta_k}|}\right]\\ &\ge \frac{\pi}{2}\mathbb{E}\left[\sum_{k\in K_t}\frac{1}{|Y_{\eta_k}|}\right],\end{aligned}$$

since

$$\begin{aligned}\mathbb{P}(k\in K^0|\mathcal{F}_{\eta_k}) &= 1 - \mathbb{P}(k\in K^1|\mathcal{F}_{\eta_k})\\ &\ge 1 - 2\exp(-M/8\pi)\\ &\ge 1/2,\end{aligned}$$

provided M is large enough. Finally

$$\begin{aligned}\frac{1}{t}\mathbb{E}\left[\sum_{k\in K_t^1}(\tau_k - \eta_k)\right] &\le 32\exp(-M/8\pi)\frac{\mathbb{E}\left[\sum_{k\in K_t}|Y_{\eta_k}|^{-1}\right]}{\mathbb{E}\left[\sum_{k\in K_t}|Y_{\eta_k}|^{-1}\right]}\\ &= 32\exp(-M/8\pi)\\ &\to 0,\end{aligned}$$

as $M \to \infty$, uniformly in t. □

Now, for any $k \in K^0$,

$$\int_{\eta_k}^{\tau_k} \sin^2(X_s)ds = \frac{\tau_k - \eta_k}{2\pi} \int_0^{2\pi} \sin^2(x)dx + \int_{\eta_k}^{\tau_k} \sin^2(X_s) \left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right] ds,$$

and we have

$$\left|\int_{\eta_k}^{\tau_k} \sin^2(X_s) \left[1 - \frac{Y_s(\tau_k - \eta_k)}{2\pi}\right] ds\right| = \left|\int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} \sin^2(X_s) \frac{Y_r - Y_s}{2\pi} drds\right| \leq \frac{1}{2\pi} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| drds.$$

Finally we have the

Lemma 4.2. *Uniformly in $t > 0$,*

$$\frac{\sum_{k \in K_t^0} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| drds}{\sum_{k \in K_t^0} (\tau_k - \eta_k)} \to 0$$

a. s., as $M \to \infty$.

Proof: Since $|Y_t - Y_{\eta_k}| \leq 1$ for $\eta_k \leq t \leq \tau_k$,

$$\begin{aligned} \frac{\sum_{k \in K_t^0} \int_{\eta_k}^{\tau_k} \int_{\eta_k}^{\tau_k} |Y_r - Y_s| drds}{\sum_{k \in K_t^0} (\tau_k - \eta_k)} &\leq 2 \sup_{k \in K_t^0} (\tau_k - \eta_k) \\ &\leq 8\pi/M \\ &\to 0, \end{aligned}$$

as $M \to \infty$, uniformly in t. □

We are now in a position to prove the following ergodic type theorem, from which Theorem 1.1 will follow :

Proposition 4.3. *As $t \to \infty$,*

$$\frac{1}{t} \int_0^t \sin^2(X_s)ds \to \frac{1}{2}$$

in probability.

Proof: We first note that

$$[0, t] = B_t^0 \cup B_t^1 \cup C_t,$$

where

$$\begin{aligned} B^0_t &= [0,t] \cap \left(\cup_{k\in K^0_t}[\eta_k,\tau_k]\right), \\ B^1_t &= [0,t] \cap \left(\cup_{k\in K^1_t}[\eta_k,\tau_k]\right), \\ C_t &= [0,t]\backslash(B^0_t \cup B^1_t). \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{t}\int_0^t \sin^2(X_s)ds = \frac{1}{t}\int_0^t \mathbf{1}_{B^0_t}(s)\sin^2(X_s)ds + \frac{1}{t}\int_0^t \mathbf{1}_{B^1_t}(s)\sin^2(X_s)ds \\ + \frac{1}{t}\int_0^t \mathbf{1}_{C_t}(s)\sin^2(X_s)ds. \end{aligned}$$

Now $C_t \subset \{s \in [0,t],\ |Y_s| \le M+1\}$, so for each fixed $M > 0$, it follows from Lemma 3.5 that the last term can be made arbitrarily small, by choosing t large enough. The second term goes to zero as $M \to \infty$, uniformly in t, from Lemma 4.1. Finally the first term equals the searched limit, plus an error term which goes to 0 as $M \to \infty$, uniformly in t, see Lemma 4.2 and the following fact, which follows from the combination of Lemma 4.1 and Lemma 3.5 :

$$\frac{1}{t}\sum_{k\in K^0_t}(\tau_k - \eta_k) \to 1$$

in probability, as $n \to \infty$. □

We can finally proceed with the

Proof of Theorem 1.1 All we have to show is that (see (2.2))

$$\lim_{n\to\infty}\frac{1}{n^{2/3}t}\int_0^{n^{2/3}t}\sin^2(X^n_s)ds = \frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)dx = \frac{1}{2}$$

in probability. In the case $v = 0$, the process $\{(X^n_t, Y^n_t)\}$ does not depend upon n, and the result follows precisely from Proposition 4.3. Now suppose that $v \neq 0$. In that case, the result can be reformulated equivalently as follows. For some $x \in \mathbb{R}$, $y \neq 0$, each $t > 0$, define the process $\{(X^t_s, Y^t_s),\ 0 \le s \le t\}$ as the solution of the SDE

$$\begin{cases} \dfrac{dX^t_s}{ds} = Y^t_s,\ X^t_0 = x, \\ dY^t_s = \sin(X^t_s)dW_s,\ Y^t_0 = \sqrt{t}y, \end{cases}$$

We need to show that

$$\frac{1}{t}\int_0^t \sin^2(X^t_s)ds \to \frac{1}{2\pi}\int_0^{2\pi}\sin^2(x)dx$$

in probability, as $t \to \infty$. Note that in time t, the process Y^t starting from $\sqrt{t}y$ can come back near the origin.

It is easily seen, by introducing the Markov time $\tau_M^t = \inf\{s > 0,\ |Y_s^t| \le M\}$ and exploiting the strong Markov property, that

$$\frac{1}{t}\int_0^t \mathbf{1}_{\{|Y_s^t|\le M\}} ds \to 0 \quad \text{a. s.}$$

follows readily from Lemma 3.5. The rest of the argument leading to Proposition 4.3 is based upon limits as $M \to \infty$, uniformly with respect to t. It thus remains to check that the fact that Y_0^t now depends upon t does not spoil this uniformity, which is rather obvious. □

5. Appendix

For the convenience of the reader, we prove the following

Proposition 5.1. *Let η and τ be two stopping times such that $0 \le \eta \le \tau \le \eta + T$ and $M_t = \int_0^t \varphi_s dB_s$, where $\{B_t,\ t \ge 0\}$ is a standard Brownian motion and $\{\varphi_t,\ t \ge 0\}$ is progressively measurable and satisfies $|\varphi_t| \le k$ a. s., for all $t \ge 0$. Then for all $c > 0$,*

$$\mathbb{P}\left(\sup_{\eta\le t\le\tau} |M_t - M_\eta| \ge c\right) \le 2\exp\left(-\frac{c^2}{2k^2T}\right).$$

Proof: From the optional stopping theorem, it suffices to treat the case $\eta = 0$, $\tau = T$. We have

$$\mathbb{P}(\sup_{0\le t\le T} |M_t| \ge c) = \mathbb{P}(\sup_{0\le t\le T} M_t \ge c) + \mathbb{P}(\inf_{0\le t\le T} M_t \le -c).$$

We estimate the first term on the right. The second one is bounded by the same quantity. Define for all $\lambda > 0$

$$\mathcal{M}_t^\lambda = \exp\left(\lambda M_t - \frac{\lambda^2}{2}\int_0^t \varphi_s^2 ds\right).$$

$$\begin{aligned}\mathbb{P}(\sup_{0\le t\le T} M_t \ge c) &\le \mathbb{P}(\sup_{0\le t\le T} \mathcal{M}_t^\lambda \ge \exp(\lambda c - \lambda^2k^2T/2)) \\ &\le \exp(\lambda^2k^2T/2 - \lambda c),\end{aligned}$$

from Doob's inequality, since $\{\mathcal{M}_t^\lambda,\ t \ge 0\}$ is a martingale with mean one. Optimizing the value of λ, we deduce that

$$\mathbb{P}(\sup_{0\le t\le T} M_t \ge c) \le \exp\left(-\frac{c^2}{2k^2T}\right),$$

from which the result follows. □

References

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