A probabilistic formula for a Poisson equation with Neumann boundary condition

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Abstract

In this work we extend Brosamler's formula (see [2]) and give a probabilistic solution of a non degenerate Poisson type equation with Neumann boundary condition in a bounded domain of the Euclidean space.

1 Introduction

Let D be a bounded domain in R^d of class $\mathcal{C}^{2,\alpha}$, $0 < \alpha < 1$, and consider the following Poisson equation in D with Neumann boundary condition

$$\begin{cases} \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2 u(x) + \sum_{i=1}^{d} b_i(x)\partial_i u(x) = -f(x), \quad x \in D; \\ \partial_{n_a} u(\alpha) = g(\alpha), \quad \alpha \in \partial D. \end{cases} \tag{1}$$

where $\partial_{n_a}(.) = \frac{1}{2}a\nabla(.).n$ stands for conormal derivative (n being the outward pointing normal), the symmteric matrix $a = [a_{ij}]$ is in $\mathcal{C}^{1,\alpha}(\overline{D})$ and satisfies the following boundedness and uniform ellipticity condition : there exist two positive constants a_0 and a_1 s.t. $\forall \xi \in R^d$

$$a_0 \left\|\xi\right\|^2 \leq a(x)\xi.\xi \leq a_1 \left\|\xi\right\|^2, \tag{2}$$

the vector b is in $\mathcal{C}^{0,\alpha}(\overline{D})$ and f and g are functions respectively in $\mathcal{C}^{0,\alpha}(\overline{D})$ and $\mathcal{C}^{1,\alpha}(\partial D)$. The elliptic operator in (1) will be denoted by A. Setting $\widetilde{b}_i = b_i - \frac{1}{2}\sum_{j=1}^{d}\partial_j a_{ij}$ we can write $A = \frac{1}{2}\nabla.a\nabla + \widetilde{b}.\nabla$. It is well known, see for example [5], that the solution of this problem (unique modulo an additive constant) exists whenever the following compatibility (or centering) condition is verified

$$\int_D f(x)p(x)dx + \int_{\partial D} g(\alpha)p(\alpha)d\alpha = 0, \tag{3}$$

where $p(x)$ is the solution of the adjoint elliptic problem, i.e.

$$\begin{cases} A^* p(x) = 0, & x \in D; \\ \partial n_a p(\alpha) - (\widetilde{b}, n)p(\alpha) = 0, & \alpha \in \partial D. \end{cases} \tag{4}$$

which satisfies $p(x) \geq 0$, $x \in D$, and $\int_D p(x)dx = 1$, in other words p is the invariant probability density of the associated reflecting diffusion $X(t)$ which will be defined in the next section. We say that equation (1) admits a weak solution $u \in H^1(D)$ if $\forall v \in H^1(D)$ we have

$$\int_D \frac{1}{2} a\nabla u.\nabla v dx - \int_D (\widetilde{b}.\nabla u)v dx = \int_D f v dx + \int_{\partial D} g v d\alpha.$$

Such a solution exists and is unique (modulo additive constants) under the conditions (2), (3) and our conditions on the coefficients above (see [6]). Suppose that by some way or another, we can find a weak solution u of (1), then using standard PDE regularity theory (see for example theorem 15.1 of [6]) one can show that this weak solution is indeed a classical solution.

In [2] Brosamler, using probabilistic potential theory and the Shur-Meyer representation theorem for addtitive functionals, establishes the following probabilistic representation of u when $A = \frac{1}{2}\Delta$ and $f = 0$

$$u(x) = \lim_{T\to\infty} E^x \int_0^T g(X_t)dL(t),$$

where $L(t)$ is the boundary local time for the process $X(t)$, see below. It is our aim in this paper to extend this result to much more general operators than the Laplacian and consider a Poisson equation with a non trivial interior function f.

In what follows, the letters x, y, ... are reserved for interior variables and α, β, ... for boundary variables on ∂D. The volume of D will be designated by $|D|$. The obvious notations $(.,.)$, $(.,.)_{2,D}$ etc ... are scalar products. Various unimportant constants are noted c , c' ... and they may vary from line to line while proofs are in process.

2 The reflecting Diffusion

The operator A is the generator of a (free) diffusion $X_0(t)$ in R^d. The reflecting diffusion in the conormal direction $\frac{1}{2}a.n$, starting from $\overline{D}$, solves the SDE $X(t) = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s - \frac{1}{2}\int_0^t an(X_s)dL(s)$, where W is a Brownian motion and $L(t)$ denotes the local time of the process X on the boundary ∂D, see [8]. This is the continuous additive functional given rigorously by the formula

$$L(t) = \lim_{\delta \to 0} \frac{1}{\delta} \int_0^t I_{\{\partial D^\delta\}}(X_s)ds,$$

where $\partial D^\delta = \{x \in \overline{D}/d(x, \partial D) \leq \delta\}$. The Markov process $X(t)$ generates a strongly continuous semigroup of contractions P_t on $\mathcal{C}(\overline{D})$ endowed with the topology of uniform convergence. It is well known that the function $P_t v(x)$, $v \in \mathcal{C}(\overline{D})$, solves the parabolic problem

$$\begin{cases} \partial_t w(t,x) = Aw(t,x), \quad t > 0, \ x \in D; \\ w(0,x) = v(x), \quad x \in \overline{D}; \\ \partial_{n_a} w(t,\alpha) = 0, \quad t > 0, \ \alpha \in \partial D. \end{cases}$$

The generator of the semigroup P_t is noted $\mathcal{A}$. We have $\mathcal{A}v = Av$, $v \in \mathcal{D}(\mathcal{A})$, in the sens of distributions.

2.1 Exponential ergodicity

We are inspired by [9] section 2.

Lemma 1 *The process $X(t)$ has a unique absolutely continuous invariant probability measure with density $p(x)$ in $\mathcal{C}^{2,\alpha}(\overline{D})$. Moreover, there exist two strictly positive constants c, c' s.t. $\forall x \in \overline{D}$ we have $c \leq p(x) \leq c'$.*

Proof. Existence of an invariant probability measure follows from the fact that X_t is a Feller process with values in the compact set $\overline{D}$. Uniqueness follows from the ellipticity of the matrix of the diffusion coefficients. The absolute continuity of that invariant measure also follows from the non degeneracy of the matrix a. Now the density p must solve the stationary Fokker-Planck equation, i. e. equation (4). Moreover from [4] theorem 6.31, p is continuous on $\overline{D}$. On the other hand, it follows from the Harnack inequality that if the function p were to vanish at some point of D it would identically vanish everywhere in D and consequently, by continuity, would identically vanish in $\overline{D}$. This contradiction shows that $p > 0$ in D. Suppose then that there exists an $\alpha_0 \in \partial D$ s.t. $p(\alpha_0) = 0$. The existence of the local minimum α_0 of the trace of p on the boundary implies that $\nabla p.\tau(\alpha_0) = 0$ for any vector field τ on the boundary, hence in particular for the projection of the conormal on the tangent plane at α_0. It then suffices to apply lemma 3.4 p. 34 of [4] to see that we would have $\partial_{n_a} p < 0$ which again contradicts our required boundary behaviour of p. Therefore p has to be strictly positive everywhere on the compact $\overline{D}$. The existence of the constants c, c' then is a consequence of the continuity of p. ■

The following easy technical result shows that many calculations, once based on the measure $p(x)dx$, become simpler.

Lemma 2 *Let $v \in H^1(D)$. We have*

$$\int_D Av.vpdx = -\frac{1}{2}(a\nabla v, \nabla v)_{2,p} + \int_{\partial D} \partial_{n_a} v.vpd\alpha.$$

Proof. As a is symmetric, we have by the Green formula

$$\int_D Av.vpdx = -\frac{1}{2}\int_D (a\nabla v, \nabla v)pdx + \frac{1}{2}\int_D A^*p.v^2dx \\ -\frac{1}{2}\int_{\partial D} B^*p.v^2 d\alpha + \int_{\partial D} \partial_{n_a} v.vpd\alpha.$$

■

We are in the position to establish exponential ergodicity in $L^2(D,p)$.

Lemma 3 *There exists a constant $k > 0$ s. t. for all $\varphi \in \mathcal{C}(\overline{D})$ with $\int_D \varphi(x)p(x)dx = 0$ we have*

$$\|P_t\varphi\|_{2,p} \le \|\varphi\|_{2,p} \exp(-kt). \tag{5}$$

Proof. Let $\varphi \in \mathcal{D}(\mathcal{A})$ and set $v = P_t\varphi$. As $\partial_{n_a} v = 0$, it suffices starting from lemma 2 to repeat the calculations page 502 of [9] using the invariant measure supplied by lemma 1 to arrive at the inequality

$$\frac{d}{dt}\|P_t\varphi\|_{2,p}^2 \le -2k\|P_t\varphi\|_{2,p}^2,$$

for some $k > 0$ and all t, from which it follows immediately that we have

$$\|P_t \varphi\|_{2,p} \leq \|\varphi\|_{2,p} \exp(-kt).$$

It then suffices to use a density argument. ■

As a consequence of this lemma, it makes sense to define the following element of $L^2(D)$

$$\int_0^\infty E^x v(X_t)dt,$$

for any continuous function v on $\overline{D}$ with $\int_D v(x)p(x)dx = 0$.

3 The Neumann problem

Here is our main result.

Theorem 4 *Assume that the boundedness and uniform ellipticity condition* (2) *as well as the centering condition* (3) *hold. Assume also the above regularity of ∂D and of the coefficients a, b, f and g. Then a classical solution of problem* (1) *admits the following probabilistic representation. For all $x \in \overline{D}$ we have*

$$u(x) = \lim_{T \to \infty} \left[\int_0^T E^x f(X_t)dt + E^x \int_0^T g(X_t)dL(t) \right]. \qquad (6)$$

Proof. We first use a well known technical device to work on a homogeneous system. Let $\widetilde{g}(x)$, $x \in \overline{D}$, be a function at least of class $\mathcal{C}^{2,\alpha}(\overline{D})$ s.t. $\partial_{n_a}\widetilde{g} = g$ and $\int_D \widetilde{g}(x)p(x)dx = 0$. In order to verify that such a function exists, it suffices for example to choose

$$\widetilde{g}(x) = \overline{g}(x) - \int_D \overline{g}(y)p(y)dy,$$

where $\overline{g}$ is the solution of the following boundary value problem, for some $\lambda > 0$,

$$\begin{cases} Aw(x) - \lambda w(x) = 0, & x \in D; \\ \partial_{n_a} w(\alpha) = g(\alpha), & \alpha \in \partial D. \end{cases}$$

Note that given our assumptions on the coefficients, $\overline{g}$ is in $\mathcal{C}^{2,\alpha}(\overline{D})$, see for example theorem 6.31 of [4]. Setting $\widetilde{u} = u - \widetilde{g}$ our boundary problem (1) becomes homogeneous, namely $\widetilde{u}$ solves the system

$$\begin{cases} A\widetilde{u}(x) = -\widetilde{f}(x), \quad x \in D; \\ \partial_{n_a}\widetilde{u}(\alpha) = 0, \quad \alpha \in \partial D, \end{cases} \tag{7}$$

where $\widetilde{f} = f + A\widetilde{g}$. Note that $\widetilde{f} \in \mathcal{C}^{0,\alpha}(\overline{D})$. By the Green formula, the centering condition for this last system is nothing but the condition (3).

Let $\{W(t,x),\ t>0,\ x \in D\}$ be the solution of the parabolic equation

$$\begin{cases} \partial_t W(t,x) = AW(t,x) + \widetilde{f}(x), \quad t>0,\ x \in D; \\ W(0,x) = 0,\ x \in \overline{D}; \\ \partial_{n_a} W(t,\alpha) = 0,\ t>0,\ \alpha \in \partial D. \end{cases}$$

By theorem 5.3.2. of [3], $W(t,x)$ is once continuously differentiable in t and twice continuously differentiable in x. From Itô's formula we deduce that

$$W(T,x) = E^x \int_0^T \widetilde{f}(X_s)ds.$$

We also have

$$\begin{aligned} &\frac{1}{2}(a\nabla W(T,.), \nabla W(T,.))_{2,D} \\ &= (\widetilde{b}.\nabla W(T,.), W(T,.))_{2,D} + (\widetilde{f} - \partial_T W(T,.), W(T,.))_{2,D}. \end{aligned} \tag{8}$$

On the other hand, we have by standard inequalities,

$$\begin{aligned} \left|(\widetilde{b}.\nabla W(T,.), W(T,.))_{2,D}\right| &\leq \sup_{\overline{D}} \left\|\widetilde{b}\right\| \left\|\nabla W(T,.)\right\|_{2,D} \left\|W(T,.)\right\|_{2,D} \\ &\leq \frac{1}{4}a_0 \left\|\nabla W(T,.)\right\|_{2,D}^2 + c\left\|W(T,.)\right\|_{2,D}^2, \end{aligned} \tag{9}$$

where c is a positive constant which depends only on our coefficients and on $\overline{D}$. Moreover

$$\left|(\widetilde{f} - \partial_T W(T,.), W(T,.))_{2,D}\right| \leq c(\left\|\widetilde{f}\right\|_{2,D}^2 + \left\|P_T\widetilde{f}\right\|_{2,D}^2 + \left\|W(T,.)\right\|_{2,D}^2). \tag{10}$$

It follows from (8), (9), (10) and (2) that

$$\left\|W(T,.)\right\|_{H^1(D)} \leq C\left(1 + \left\|W(T,.)\right\|_{2,D}\right).$$

The exponential bound (5) gives us enough room to extend the definition of the resolvent kernels below the origin, therefore for any positive k' with $k' < k/2$ we have by Cauchy-Schwarz and Fubini

$$\begin{aligned} \|W(T,.)\|_{2,D}^2 &\leq c \int_D dx \int_0^T e^{2k's} \left| P_s \widetilde{f}(x) \right|^2 ds \\ &\leq c' \left\| \widetilde{f} \right\|_{2,D}^2 . \end{aligned}$$

Consequently, $\|W(T,.)\|_{H^1(D)}^2 \leq c \left\| \widetilde{f} \right\|_{2,D}^2$. There then exists a sequence of times $T_n \to \infty$ s.t. $W(T_n,.)$ converges weakly in $H^1(D)$; it is clear that this limit is nothing but $\int_0^\infty E^x \widetilde{f}(X_t)dt$ since the latter is a well defined element of $L^2(D)$ as explained at the end of the previous section. Note that this convergence actually takes place for the whole sequence T by uniqueness of the limit point. For all $v \in H^1(D)$ we have the relation

$$\frac{1}{2}(a\nabla W(T,.), v)_{2,D} - (\widetilde{b}.\nabla W(T,.), v)_{2,D} = (\widetilde{f} - P_T \widetilde{f}, v)_{2,D},$$

and as by lemma 3 we have $\lim_{T\to\infty} (P_T \widetilde{f}, v)_{2,D} = 0$ provided the condition (3) holds, we see that $\widetilde{u}(x) = \int_0^\infty E^x[f(X_t) + A\widetilde{g}(X_t)]dt$ is indeed a weak solution of the homogeneous problem (7).

On the other hand, let v be an arbitrary continuous function on $\overline{D}$. The dominated convergence theorem implies the relation

$$(u, v)_{2,D} = (\widetilde{g}, v)_{2,D} + \lim_{T\to\infty} (E^{\cdot} \int_0^T [f(X_t) + A\widetilde{g}(X_t)]dt, v)_{2,D}.$$

Now applying again the Itô formula and then taking an expectation we can write

$$E^x \widetilde{g}(X_T) = \widetilde{g}(x) + E^x \int_0^T A\widetilde{g}(X_t)dt - E^x \int_0^T g(X_t)dL(t).$$

This equation combined with the following consequence of lemma 3

$$\lim_{T\to\infty} (E^{\cdot} \widetilde{g}(X_T), v)_{2,D} = 0$$

implies our representation formula for weak solutions u since v is arbitrary.
■

Remark 5 *For the sake of completeness, it is worth mentionning that in [1] a similar integral representation in Lipschitz domains is given for the Laplacian (again taking $f = 0$) by means of the transition densities of the reflecting Brownian motion considered as the realisation of a suitable symmetric Dirichlet form. We also mention that, from the point of view of the asymptotics in general PDE theory, the limiting time behaviour of the solutions of more general (nonlinear) parabolic equations is studied in [7], using analytical methods.*

References

[1] R. F. Bass and E. P. Hsu, (1991) Some potential theory for reflecting Brownian motion in Holder and Lipschitz domains, *Ann. Prob.* **19** (2), p. 486–508.

[2] G. A. Brosamler, (1976) A probabilistic solution of the Neumann problem, *Math. Scand.* **38**, p. 137–147.

[3] A. Friedman, (1964) *Partial differential equations of parabolic type*, Prentice-Hall.

[4] D. Gilbarg and N. S. Trudinger, (1998) *Elliptic partial differential equations of second order, second edition*, Springer-Verlag, Grundlehren der mathematischen wissenschaften 0224.

[5] G. Giraud, (1933) Problèmes mixtes et Problèmes sur des variétés closes, relativement aux équations linéaires du type elliptique, *Ann. Soc. Polon. Math.* **12**, p. 35–54.

[6] O. A. Ladyzenskaya and N. N. Ural'ceva, (1968) *Equations aux dérivées partielles de type elliptique*, Dunod.

[7] F. Da Lio, (2008) Large time behavior of solutions to parabolic equations with Neumann boundary conditions, *J. Math. Anal. Appl.* **339** (1), p. 384–398.

[8] P. L. Lions and A. S. Sznitman, (1984) Stochastic differential equations with reflecting boundary conditions, *Comp. Pure Appl. Math. 37*, p. 511–537.

[9] E. Pardoux, (1999) Homogenization of linear and semi-linear second order parabolic PDEs with periodic coefficients : a probabilistic approach, *J. Func. Anal.*, **167**, p. 498–520.