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Homogenization of periodic semilinear hypoelliptic PDEs [*]

Alassane Diédhiou [1], Étienne Pardoux [2]

Abstract. — We establish homogenization results for both linear and semilinear partial differential equations of parabolic type, when the linear second order PDE operator satisfies a hypoellipticity asumption, rather than the usual ellipticity condition. Our method of proof is essentially probabilistic.

Résumé. — Nous établissons des résultats d'homogénéisation d'équations aux dérivées partielles paraboliques linéaires et semi–linéaires, sous une hypothèse d'hypoellipticité de l'opérateur aux dérivées partielles du second ordre, au lieu de l'hypothèse usuelle d'ellipticité. Notre méthode de démonstration est essentiellement probabiliste.

1. Introduction

Our aim is to homogenize two classes of periodic semilinear parabolic PDEs, namely

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon u^\varepsilon(t,x) + \dfrac{1}{\varepsilon} e(\dfrac{x}{\varepsilon}, u^\varepsilon(t,x)) + f(\frac{x}{\varepsilon}, u^\varepsilon(t,x)), \\ u^\varepsilon(0,x) = g(x), \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon u^\varepsilon(t,x) + f(\frac{x}{\varepsilon}, u^\varepsilon(t,x), \nabla u^\varepsilon(t,x)\sigma(\dfrac{x}{\varepsilon})) \\ u^\varepsilon(0,x) = g(x), x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

where L_ε is a second order PDE operator (see (1.5)). The novelty of our result lies mainly in the fact that the matrix of second order coefficients of

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L_ε is not assumed to be elliptic, but instead we formulate a hypoellipticity condition of Hörmander type, see the end of this section.

There is by now quite a vast literature concerning the homogenization of second order elliptic and parabolic PDEs with a possibly degenerating matrix of second order coefficients a, see among others [1], [2], [3], [6], [15]. But, as far as we know, in these works, either the coefficient a is allowed to degenerate on sets of measure zero only, or else the equation is linear.

Our method of proof will be mainly probabilistic. We consider the SDE, for $\varepsilon \geqslant 0, x \in \mathbb{R}^d$,

$$X_t^\varepsilon = x + \int_0^t c(\frac{X_s^\varepsilon}{\varepsilon})ds + \frac{1}{\varepsilon}\int_0^t b(\frac{X_s^\varepsilon}{\varepsilon})ds + \sum_{j=1}^d \int_0^t \sigma_j(\frac{X_s^\varepsilon}{\varepsilon})dW_s^j \qquad (1.3)$$

where $\{W_t^j, j = 1, \ldots, d; t \geqslant 0\}$ is a standard d-dimensional Brownian motion. The functions c, b and $\sigma_j, j = 1, \ldots, d$ belong to $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and are periodic with period 1 in each direction.

Under the above conditions, there exists a unique solution $\{X_t^\varepsilon, t \geqslant 0\}$ of (1.3). Setting $\tilde{X}_t^\varepsilon = \frac{1}{\varepsilon}X_{\varepsilon^2 t}^\varepsilon$, then we get with a new d–dimensional standard Brownian motion $\{W_t,\, t \geqslant 0\}$, which in fact depends on ε :

$$\tilde{X}_t^\varepsilon = \frac{x}{\varepsilon} + \varepsilon\int_0^t c(\tilde{X}_s^\varepsilon)ds + \int_0^t b(\tilde{X}_s^\varepsilon)ds + \sum_{j=1}^d \int_0^t \sigma_j(\tilde{X}_s^\varepsilon)dW_s^j. \qquad (1.4)$$

We shall assume that the matrix $\sigma(x)$ of columns vectors $\sigma_j(x)$ satisfies the strong Hörmander condition, given by the

Definition 1.1. — *Let $H(n, x)$ be the set of Lie brackets of $(\sigma_j(x))_{1\leqslant j\leqslant d}$ of order lower than n at the point $x \in \mathbb{R}^d$.*

We say that the matrix σ satisfies the strong Hörmander condition (called SHC) if for all $x \in \mathbb{R}^d$, there exists $n_x \in \mathbb{N}$ such that $H(n_x, x)$ generates $\mathbb{R}^d$.

Now we are going to study the ergodic properties of the processes $\{\tilde{X}_t^\varepsilon,\, t \geqslant 0\}$ like in É. Pardoux [13] under the above strong Hörmander condition.

Let us consider the infinitesimal generator of $\{\tilde{X}_t^\varepsilon, t \geqslant 0\}$:

$$\begin{aligned} L_\varepsilon &= \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i x_j} + \sum_{i=1}^d (b_i(x) + \varepsilon c_i(x))\frac{\partial}{\partial x_i} \\ &= \frac{1}{2}\sum_{i,j=1}^d \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^d (\tilde{b}_i(x) + \varepsilon c_i(x))\frac{\partial}{\partial x_i} \end{aligned} \tag{1.5}$$

where $\tilde{b}_i(x) = b_i(x) - \frac{1}{2}\sum_{j=1}^d \frac{\partial}{\partial x_j} a_{ij}(x)$. We will denote by L the operator L_0.

In all the rest of the paper, we assume that the condition

$$\text{The matrix } \sigma \text{ satisfies the SHC} \tag{A}$$

is satisfied.

The paper is organized as follows. Section 2 contains several preliminary results, which are our tools for the homogenization, and which we extend from the classical elliptic case to our hypoelliptic setting. Section 3 applies the above preliminaries to the homogenization of a linear parabolic equation. Finally section 4 studies the homogenization of equation (1.1), and section 5 that of equation (1.2).

2. Preliminaries

2.1. Invariant measure

We rewrite the equation (1.4) in the form:

$$\tilde{X}_t^\varepsilon = \frac{x}{\varepsilon} + \int_0^t b^\varepsilon(\tilde{X}_s^\varepsilon)ds + \int_0^t \sigma(\tilde{X}_s^\varepsilon)dW_s^\varepsilon \tag{2.1}$$

where $b^\varepsilon(\tilde{X}_s^\varepsilon) = \varepsilon c(\tilde{X}_s^\varepsilon) + b(\tilde{X}_s^\varepsilon)$, or in Stratonovich form

$$d\tilde{X}_t^\varepsilon = \sigma_0(\tilde{X}_t^\varepsilon)dt + \sum_{j=1}^d \int_0^t \sigma_j(\tilde{X}_t^\varepsilon) \circ dW_t^j,$$

where $\sigma_0^i = b_i^\varepsilon - \frac{1}{2}\sum_{k=1}^m \sum_{j=1}^d \partial_k \sigma_j^i \sigma_j^k$. From now on, we consider the process $\{\tilde{X}_t^\varepsilon\}$ as taking its values in the d–dimensional torus $\mathbf{T^d} := \mathbb{R}^d/Z^d$.

Define $H = L^2([0,T], \mathbb{R}^{\mathrm{d}})$, let $h \in H$ and $\Phi(h)$ be the solution of:

$$\Phi(h)_t = \frac{x}{\varepsilon} + \int_0^t \sigma_0(\Phi(h)_s)ds + \sum_{j=1}^{d} \int_0^t \sigma_j(\Phi(h)_s)h_s^j ds$$

PROPOSITION 2.1. — *We have*

$$\begin{aligned} \textit{support}(\tilde{X}_t^\varepsilon) &= \overline{\{\Phi(h)_t(x); h \in H\}} \\ &= \mathbf{T^d}. \end{aligned}$$

Proof. — The first equality follows from the Stroock-Varadhan support theorem see [16], and the second is a well known consequence of condition (A), see e.g. V. Jurdjevic [9]. □

Let $\mu_t^\varepsilon(x, dy)$ denote the law of $\tilde{X}_t^\varepsilon$ and $p_t^\varepsilon(x, y)$ its density.

THEOREM 2.2. — *The density $p_t^\varepsilon(x, y)$ is strictly positive for all $(t, x, y) \in \mathbb{R}_+^* \times \mathbf{T^d} \times \mathbf{T^d}$.*

Proof. — This is again a consequence of condition (A), see Michel, Pardoux [11] theorem 3.3.6.1. □

We have the

LEMMA 2.3. — *For all $\varepsilon \geqslant 0$, the $\mathbf{T^d}$-valued diffusion process $\{\tilde{X}_t^\varepsilon, t \geqslant 0\}$ of generator L_ε, has a unique invariant probability μ_ε.*

Proof. — Since $\{\tilde{X}_t^\varepsilon, t \geqslant 0\}$ is a homogeneous Feller process with values in a compact set, μ_ε exists. The proof of the uniqueness is the same as in É. Pardoux [13] by using the fact that, since the transition density $p_t^\varepsilon(x, y)$ is strictly positive, any invariant measure has a strictly positive density.

LEMMA 2.4. — *For any fixed $t > 0$ the function*

$$\begin{aligned} [0,1] \times \mathbf{T^d} \times \mathbf{T^d} &\longrightarrow \mathbb{R}_+ \\ (\varepsilon, x, y) &\longrightarrow p_t^\varepsilon(x, y) \end{aligned}$$

is continuous.

Proof. — In order to prove this Lemma it suffices to prove that the function $(\varepsilon, x) \longrightarrow p_t^\varepsilon(x, .)$ from $[0,1] \times \mathbf{T^d}$ into $C(\mathbf{T^d})$ is continuous. Consider the map

$$\begin{aligned} (\varepsilon, x) &\longrightarrow \mu_t^\varepsilon(x, dy) \\ [0,1] \times \mathbf{T^d} &\longrightarrow E, \end{aligned}$$

where E denotes the set of probability measures on $\mathbf{T^d}$ equiped with the topology of weak convergence. This map is continuous, since the map $(\varepsilon, x) \longrightarrow \tilde{X}_t^\varepsilon$ with values in $L^2(\Omega)$ is continuous. It now suffices to show that the densities are equicontinuous, in order to deduce from Ascoli's theorem the wished continuity. We know that

$$||p^\varepsilon_\cdot(x, \cdot)||_{L^1([t-\alpha,t+\alpha]\times\mathbf{T^d})} = 2\alpha,$$

so for some large enough n (whose value depends on d),

$$||p^\varepsilon_\cdot(x, \cdot)||_{H^{-n}([t-\alpha,t+\alpha]\times\mathbf{T^d})} \leqslant C(\alpha),$$

then by the hypoellipticity of $\frac{\partial}{\partial t} - L^*_\varepsilon$ (cf. e.g. Lemma 5.2 p.122 of [17]) for all $m > 0$, there exists $C(m) > 0$ such that

$$||p^\varepsilon_\cdot(x, .)||_{H^m([t-\alpha,t+\alpha]\times\mathbf{T^d})} \leqslant C(m),$$

and from the Sobolev embedding we deduce that

$$\sup_{\varepsilon\in[0,1],x\in\mathbf{T^d}} ||p^\varepsilon_t(x, .)||_{C^1(\mathbf{T^d})} \leqslant C, \tag{2.2}$$

which establishes the wished equicontinuity. □.

Since $p^\varepsilon_t(x, y) > 0,\ \forall(\varepsilon, x, y) \in [0, 1] \times \mathbf{T^d} \times \mathbf{T^d}$ and p_t is a continuous function of (ε, x, y) on this compact set, for each $t > 0$, there exists $(\varepsilon_0, x_0, y_0) \in [0, 1] \times \mathbf{T^d} \times \mathbf{T^d}$ such that

$$c_t := \inf_{\varepsilon,x,y} p^\varepsilon_t(x, y) = p^{\varepsilon_0}_t(x_0, y_0) > 0.$$

We now prove the

Lemma 2.5. — *For any $t > 0$, and $x, x' \in \mathbf{T^d}$, we have*

$$||p^\varepsilon_t(x, .) - p^\varepsilon_t(x', .)||_{L^1(\mathbf{T^d})} \leqslant 2(1 - c_1)^{[t]}.$$

Proof. — For any coupling of X^x_t and $X^{x'}_t$ we have

$$||p^\varepsilon_t(x, .) - p^\varepsilon_t(x', .)||_{L^1(\mathbf{T^d})} \leqslant 2\mathbb{P}(X^x_t \neq X^{x'}_t).$$

We first define a coupling of $(X^x_n, X^{x'}_n,\ n = 0, 1, 2, \dots,\ [t])$. Let us consider a map $F_\varepsilon : \mathbf{T^d} \times [0, 1] \longrightarrow \mathbf{T^d}$, such that if the random variable η has the uniform distribution on $[0, 1]$, the random variable $F_\varepsilon(x, \eta)$ has the probability density $\frac{p^\varepsilon_1(x,y)-c_1}{1-c_1}$.

Let $(U_1,\xi_1,\eta_1,\eta'_1,\ldots,U_n,\xi_n,\eta_n,\eta'_n,\ldots)$ be independent random variables such that the random variables $\xi_n,\eta_n,\ \eta'_n$ are uniformly distributed over $[0,1]$, and

$$U_n=\begin{cases}1 & \text{with probability } c_1\\ 0 & \text{with probability } 1-c_1.\end{cases}$$

We now define recursively the sequences $X_n^x,\ X_n^{x'},\ n\geqslant 1$. For each $n\geqslant 0$, if $X_n^x=X_n^{x'}$, then we set

$$X_{n+1}^x=X_{n+1}^{x'}=U_{n+1}\xi_{n+1}+(1-U_{n+1})F(X_n^x,\eta_{n+1}),$$

if not then we set

$$\begin{cases}X_{n+1}^x &= U_{n+1}\xi_{n+1}+(1-U_{n+1})F(X_n^x,\eta_{n+1})\\ X_{n+1}^{x'} &= U_{n+1}\xi_{n+1}+(1-U_{n+1})F(X_n^{x'},\eta'_{n+1}).\end{cases}$$

So we have

$$\mathbb{P}(X_n^x\neq X_n^{x'})=\mathbb{P}(U_1=0,U_2=0,\ldots,U_n=0)=(1-c_1)^n.$$

Similarly we define $F_{t-[t]}(x,.)$ such that $F_{t-[t]}(x,\eta)$ possesses the probability density $p^\varepsilon_{t-[t]}(x,y)$ and if $X_{[t]}^x=X_{[t]}^{x'}$, then we set
$X_t^x=X_t^{x'}=F_{t-[t]}(X_{[t]}^x,\eta_{[t]+1})$, else

$$\begin{cases}X_t^x &= F_{t-[t]}(X_{[t]}^x,\eta_{[t]+1})\\ X_t^{x'} &= F_{t-[t]}(X_{[t]}^{x'},\eta'_{[t]+1}).\end{cases}$$

Hence we get $\mathbb{P}(X_t^x\neq X_t^{x'})=\mathbb{P}(X_{[t]}^x\neq X_{[t]}^{x'})=(1-c_1)^{[t]}$. Since the densities of $X_t^x,\ X_t^{x'}$ are respectively $p_t^\varepsilon(x,y)$ and $p_t^\varepsilon(x',y)$, we have that:

$$\begin{aligned}||\mu_t^\varepsilon(x,.)-\mu_\varepsilon||_{TV} &= \int|p_t^\varepsilon(x,y)-p^\varepsilon(y)|dy\\ &= \int|p_t^\varepsilon(x,y)-\int\mu_\varepsilon(dx')p_t^\varepsilon(x',y)|dy\\ &= \int|\int\mu_\varepsilon(dx')(p_t^\varepsilon(x,y)-p_t^\varepsilon(x',y))|dy\\ &\leqslant \int\int\mu_\varepsilon(dx')|p_t^\varepsilon(x,y)-p_t^\varepsilon(x',y)|dy\\ &\leqslant 2(1-c_1)^{[t]}.\end{aligned}$$

Hence we have the

Lemma 2.6. — *There exists a constant $\rho > 0$ such that for any $\varepsilon \geqslant 0$ and $f \in L^\infty(\mathbf{T^d})$,*

$$|\mathbb{E}(f(\tilde{X}_t^\varepsilon)) - \int f(x)\mu_\varepsilon(dx)| \leqslant ||f||_{L^\infty(\mathbf{T^d})} e^{-\rho[t]}$$

If f is centered with respect to μ_ε, i.e. $\int_{\mathbf{T^d}} f(x)\mu_\varepsilon(dx) = 0$, then we get

$$|\mathbb{E}(f(\tilde{X}_t^\varepsilon))| \leqslant ||f||_{L^\infty(\mathbf{T^d})} e^{-\rho[t]},\ t > 0$$

We shall need the following result

Lemma 2.7. —

$$\mu_\varepsilon \Rightarrow \mu$$

(in the sense of weak convergence of probability measures), as $\varepsilon \to 0$.

Proof. — The collection $\{\mu_\varepsilon,\ \varepsilon > 0\}$ is tight, since these are measures on the compact set $\mathbf{T^d}$. For each $\varepsilon > 0$, $t > 0$, $f \in C(\mathbf{T^d})$,

$$\int_{\mathbf{T^d}} f(x)\mu_\varepsilon(dx) = \int_{\mathbf{T^d}} \mathbb{E}_x f(\tilde{X}_t^\varepsilon)\mu_\varepsilon(dx). \tag{2.3}$$

But as $\varepsilon \to 0$, clearly $\tilde{X}_t^\varepsilon \to \tilde{X}_t$ in $L^2(\Omega)$, uniformly with respect to the starting point x. Hence taking the limit in (2.3) along a subsequence $\{\varepsilon_k\}$ along which μ_{ε_k} converges weakly to ν, we deduce that

$$\int_{\mathbf{T^d}} f(x)\nu(dx) = \int_{\mathbf{T^d}} \mathbb{E}_x f(\tilde{X}_t)\nu(dx).$$

This is true for all $t > 0$ and all $f \in C(\mathbf{T^d})$. Hence all accumulation points of the collection $\{\mu_\varepsilon\}$, as $\varepsilon \to 0$, equal the invariant measure μ, and $\mu_\varepsilon \Rightarrow \mu$, as $\varepsilon \to 0$.

2.2. Ergodic theorem

From Lemma 2.6 and Lemma 2.7, we deduce the

Proposition 2.8. — *If $f \in L^\infty(\mathbf{T^d})$, then for any $t > 0$,*

$$\int_0^t f\left(\frac{X_s^\varepsilon}{\varepsilon}\right) ds \longrightarrow t \int_{\mathbf{T^d}} f(x)\mu(dx)$$

in probability, as $\varepsilon \longrightarrow 0$.

Proof. — Set $\tilde{f}_\varepsilon(x) = f(x) - \int_{\mathbf{T^d}} f(x)\mu_\varepsilon(dx)$. It follows from Lemma 2.7 that

$$\lim_{\varepsilon \longrightarrow 0} \int_{\mathbf{T^d}} f(x)\mu_\varepsilon(dx) = \int_{\mathbf{T^d}} f(x)\mu(dx),$$

at least for $f \in C(\mathbf{T^d})$. However, an argument very similar to that used to prove (2.2) above yields that the density p_ε of the invariant measure μ_ε satisfies

$$\|p_\varepsilon\|_{L^\infty(\mathbf{T^d})} \leqslant C, \quad \forall \varepsilon \geqslant 0.$$

This allows us to extend the above convergence to $f \in L^\infty(\mathbf{T^d})$. Hence it suffices to show that $\int_0^t \tilde{f}_\varepsilon(\frac{X_s^\varepsilon}{\varepsilon})ds \to 0$, as $\varepsilon \to 0$. We have $X_s^\varepsilon = \varepsilon \tilde{X}^\varepsilon_{\frac{s}{\varepsilon^2}}$, from which we deduce that $\int_0^t \tilde{f}_\varepsilon(\frac{X_s^\varepsilon}{\varepsilon})ds = \varepsilon^2 \int_0^{\frac{t}{\varepsilon^2}} \tilde{f}_\varepsilon(\tilde{X}_u^\varepsilon)du$.

From the Markov property of the process $\tilde{X}^\varepsilon$, and Lemma 2.6, we get:

$$\begin{aligned}
\mathbb{E}[(\int_0^t \tilde{f}_\varepsilon(\tilde{X}_u^\varepsilon)du)^2] &= 2\mathbb{E}[\int_0^t \int_0^s \tilde{f}_\varepsilon(\tilde{X}_s^\varepsilon)\tilde{f}_\varepsilon(\tilde{X}_u^\varepsilon)]dsdu \\
&\leqslant 2C||\tilde{f}_\varepsilon||^2_{L^\infty(\mathbf{T^d})} \int_0^t \int_0^s e^{-\rho[s-u]}dsdu \\
&\leqslant 2Ce^\rho||\tilde{f}_\varepsilon||^2_{L^\infty(\mathbf{T^d})} \int_0^t \int_0^s e^{-\rho[s-u]}dsdu \\
&= 2Ce^\rho\rho^{-2}(-1 + \rho t + e^{-\rho t})||\tilde{f}_\varepsilon||^2_{L^\infty(\mathbf{T^d})},
\end{aligned}$$

hence

$$\mathbb{E}\left[\left(\varepsilon^2 \int_0^{\frac{t}{\varepsilon^2}} \tilde{f}_\varepsilon(\tilde{X}_u^\varepsilon)du\right)^2\right] \leqslant 2Ce^\rho\rho^{-2}\left(-\varepsilon^4 + \rho\varepsilon^2 t + \varepsilon^4 e^{-\rho\frac{t}{\varepsilon^2}}\right)||\tilde{f}_\varepsilon||^2_{L^\infty(\mathbf{T^d})},$$

from which the Proposition follows.

2.3. The Poisson equation

We have the

Theorem 2.9. — *If $f \in C^\infty(\mathbf{T^d})$ is such that $\int_{\mathbf{T^d}} f(x)\mu(dx) = 0$ then the PDE*

$$L\hat{f}(x) + f(x) = 0,\ x \in \mathbf{T^d}$$

has a solution $\hat{f} \in C^\infty(\mathbf{T^d})$, which is given by the probabilistic formula $\hat{f}(x) = \int_0^{+\infty} \mathbb{E}_x(f(\tilde{X}_t))dt$.

Proof. — Let us consider the parabolic PDE:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - Lu(t,x) = 0, \ t > 0, \ x \in \mathbf{T^d} \\ u(0,x) = f(x), \ x \in \mathbf{T^d} \end{cases}$$

This equation has a solution in $C^\infty(\mathbb{R}_+ \times \mathbf{T^d})$, by the hypoellipticity of the operator $\frac{\partial}{\partial t} - L$, which is given by the Feynman-Fac formula: $u(t,x) = \mathbb{E}_x[f(\tilde{X}_t)]$.

By Lemma 2.6 with $\varepsilon = 0$, $u(t,.) \longrightarrow 0$ in $L^\infty(\mathbf{T^d})$ at exponential speed as $t \to \infty$, and if we set $v(t,x) = \int_0^t u(s,x)ds$, we have

$$||v(t,.)||_\infty = \sup_{x \in \mathbf{T^d}} |v(t,x)| \leqslant C,$$

and

$$v(t,x) \longrightarrow v(x) = \int_0^{+\infty} \mathbb{E}_x[f(\tilde{X}_t)]dt, \quad \text{as } t \longrightarrow \infty.$$

By Alaoglu's theorem (see e.g. A. Friedman[7] p.169) there exists a sequence $t_n \longrightarrow \infty$, such that $v(t_n,.) \longrightarrow v$ in $L^\infty(\mathbf{T^d})$ for the weak star topology. Since $u(t,x) = Lv(t,x) + f(x)$, we have

$$\forall \varphi \in C^\infty(\mathbf{T^d}), (u(t_n), \varphi) = (v(t_n), L^*\varphi) + (f, \varphi)$$

and and letting n tend to infinity, we get

$$(v, L^*\varphi) + (f, \varphi) = 0, \forall \varphi \in C^\infty(\mathbf{T^d}),$$

i.e. v solves the PDE

$$Lv + f = 0$$

in the sense of distributions. Then by the hypoellipticity of L we have that $v \in C^\infty(\mathbf{T^d})$.

3. Homogenization of a linear parabolic equation

The functions a, b, c satisfy the conditions of section 1. Let us consider the functions e belonging to $C^\infty(\mathbb{R}^{\mathrm{d}}, \mathbb{R})$, and $f : \mathbb{R}^{\mathrm{d}} \longrightarrow \mathbb{R}$ measurable and bounded, and both are periodic with period 1 in each direction, and $g \in C(\mathbb{R}^{\mathrm{d}})$ with at most polynomial growth at infinity. For $\varepsilon > 0$, we consider the linear PDE:

$$\begin{cases} \dfrac{\partial u^\varepsilon(t,x)}{\partial t} = L_\varepsilon u^\varepsilon(t,x) + (\frac{1}{\varepsilon} e(\frac{x}{\varepsilon}) + f(\frac{x}{\varepsilon}))u^\varepsilon(t,x), \ t > 0, \ x \in \mathbb{R}^{\mathrm{d}}, \\ u^\varepsilon(0,x) = g(x), x \in \mathbb{R}^{\mathrm{d}}, \end{cases} \tag{3.1}$$

where

$$L_\varepsilon = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\frac{x}{\varepsilon})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}(\frac{1}{\varepsilon}b_i(\frac{x}{\varepsilon}) + c_i(\frac{x}{\varepsilon}))\frac{\partial}{\partial x_i}.$$

We assume that

$$\int_{\mathbf{T^d}} b_i(x)\mu(dx) = 0,\ i = 1,\dots,d,\ \int_{\mathbf{T^d}} e(x)\mu(dx) = 0,$$

where μ is the invariant probability of the process $\tilde{X}_t = \tilde{X}_t^0, t \geqslant 0$. If we set

$$Y_t^\varepsilon = \int_0^t (\frac{1}{\varepsilon}e(\frac{X_s^\varepsilon}{\varepsilon}) + f(\frac{X_s^\varepsilon}{\varepsilon}))ds, t \geqslant 0,$$

then the solution of (3.1) is given by

$$u^\varepsilon(t,x) = \mathbb{E}_x[g(X_t^\varepsilon)\exp(Y_t^\varepsilon)]$$

where X_t^ε is the solution of (1.1).

Let $\hat{e}(x) = \int_0^\infty \mathbb{E}_x[e(\tilde{X}_t)]dt$, and

$$\hat{b}_i(x) = \int_0^\infty \mathbb{E}_x[b_i(\tilde{X}_t)]dt, i = 1,\dots,d,\ x \in \mathbf{T^d}$$

be solutions of the Poisson equations

$$L\hat{e}(x) + e(x) = 0,\ L\hat{b}_i(x) + b_i(x) = 0, i = 1,\dots,d.$$

Let us define

$$\begin{aligned} A &= \int_{\mathbf{T^d}} (I + \nabla\hat{b})a(I + \nabla\hat{b})^*(x)\mu(dx); \\ C &= \int_{\mathbf{T^d}} (I + \nabla\hat{b})(c + a\nabla\hat{e})(x)\mu(dx); \\ D &= \int_{\mathbf{T^d}} (\frac{1}{2}\nabla\hat{e}^*.a\nabla\hat{e} + f + \nabla\hat{e}c)(x)\mu(dx). \end{aligned}$$

Then $u(t,x) = \mathbb{E}[g(x + Ct + A^{\frac{1}{2}}W_t)]e^{Dt}$ is the solution of

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} C_i \frac{\partial u(t,x)}{\partial x_i} + Du(t,x) \\ u(0,x) = g(x),\ x \in \mathbb{R}^{\mathrm{d}}, \end{cases} \tag{3.2}$$

THEOREM 3.1. — *For any $t \geqslant 0, x \in \mathbb{R}^{\mathrm{d}}$ we have*

$$u^\varepsilon(t,x) \longrightarrow u(t,x)$$

when $\varepsilon \longrightarrow 0$.

Proof. — We know from Theorem 2.9 that the functions $\hat{b}_i$ and $\hat{e}$ belong to $C^\infty(\mathbf{T^d})$. We then can copy the proof of theorem 3.1 in É. Pardoux [13].

4. Homogenization of a semilinear parabolic equation 1

Let us consider the semilinear parabolic equation

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon u^\varepsilon(t,x) + \frac{1}{\varepsilon} e(\frac{x}{\varepsilon}, u^\varepsilon(t,x)) + f(\frac{x}{\varepsilon}, u^\varepsilon(t,x)), \\ u^\varepsilon(0,x) = g(x). \end{cases} \tag{4.1}$$

where

$$L_\varepsilon = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(\frac{x}{\varepsilon})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}(\frac{1}{\varepsilon}b_i(\frac{x}{\varepsilon}) + c_i(\frac{x}{\varepsilon})\frac{\partial}{\partial x_i}.$$

The assumptions on a, b, c are the same as in the previous section.The function g belongs to $C(\mathbb{R}^{\mathrm{d}})$, with at most a polynomial growth at infinity. Again the b_i's verify the condition

$$\int_{\mathbf{T^d}} b_i(x)\mu(dx) = 0, \ i = 1, \ldots, d.$$

We assume that

$$e, f : \mathbb{R}^{\mathrm{d}} \times \mathbb{R} \longrightarrow \mathbb{R}$$

are measurable, periodic with respect to their first variable, with period one in each direction, the function e is C^∞ with respect to x, continuous in y uniformly with respect to x, twice continuously differentiable in y, uniformly with respect to x, and moreover for all $y \in \mathbb{R}$,

$$\int_{\mathbf{T^d}} e(x,y)\mu(dx) = 0,$$

and verifies $e(x,y) = e_0(x,y) + e_1(x)y$. We assume that there exists a constant K such that

$$|e_1(x)| + |e_0(x,y)| + |\frac{\partial e_0}{\partial y}(x,y)| + |\frac{\partial^2 e_0}{\partial y^2}(x,y)| \leqslant K, \ x \in \mathbf{T^d}, y \in \mathbb{R}.$$

We assume also that for some $k \in \mathbb{R}$, all $x \in \mathbb{R}$, $y, y' \in \mathbb{R}$,

$$(f(x,y) - f(x,y'))(y - y') \leqslant k|y - y'|^2,$$

and

$$|f(x,y)| \leqslant C(1+y^2).$$

From the above assumptions on e and similarly as in Theorem 2.9, for each $y \in \mathbb{R}$, there exists a solution of the Poisson equation

$$L\hat{e}(x,y) + e(x,y) = 0,\ x \in \mathbf{T^d},\ y \in \mathbb{R},$$

which is given given by

$$\hat{e}(x,y) = \int_0^\infty \mathbb{E}_x[e(\tilde{X}_t, y)]dt.$$

The function $y \longrightarrow \mathbb{E}_x[e(\tilde{X}_t, y)]$ is twice differentiable with respect to y according to the assumptions on e, and we get

$$|\mathbb{E}_x[\frac{\partial e}{\partial y}(\tilde{X}_t, y)]| \leqslant Ke^{-\rho[t]};\ |\mathbb{E}_x[\frac{\partial^2 e}{\partial y^2}(\tilde{X}_t, y)]| \leqslant Ke^{-\rho[t]}.$$

We now prove that $\hat{e}$ belongs to $C^2(\mathbf{T^d} \times \mathbb{R})$ and the derivatives of order one and two with respect y verify the Poisson equations

$$L\frac{\partial \hat{e}}{\partial y}(x,y) + \frac{\partial e}{\partial y}(x,y) = 0;\ L\frac{\partial^2 \hat{e}}{\partial y^2}(x,y) + \frac{\partial^2 e}{\partial y^2}(x,y) = 0.$$

For $\delta > 0$, we have

$$\begin{aligned} |\hat{e}(x,y+\delta) - \hat{e}(x,y)| &\leqslant \int_0^T |\mathbb{E}_x[e(\tilde{X}_t, y+\delta) - e(\tilde{X}_t, y)]|dt \\ &+ \int_T^\infty |\mathbb{E}_x[e(\tilde{X}_t, y+\delta) - e(\tilde{X}_t, y)]|dt \\ &\leqslant \int_0^T |\mathbb{E}_x[e(\tilde{X}_t, y+\delta) - e(\tilde{X}_t, y)]|dt + Ce^{-\rho T}. \end{aligned}$$

Let us choose T large enough such that $Ce^{-\rho T} < \frac{\varepsilon}{2}$, and using the Lebesgue dominated convergence theorem we have, fo any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\delta < \eta \Longrightarrow |\hat{e}(x,y+\delta) - \hat{e}(x,y)| \leqslant \varepsilon.$$

By the same argment we show that the functions $y \longrightarrow \frac{\partial \hat{e}}{\partial y}(x,y)$ and $y \longrightarrow \frac{\partial^2 \hat{e}}{\partial y^2}(x,y)$ are continuous.

Let us consider the map :

$$y \longrightarrow (\hat{e}(.,y), \frac{\partial \hat{e}}{\partial y}(.,y), \frac{\partial^2 \hat{e}}{y^2}(.,y))$$

from $\mathbb{R}$ into $(C^2(\mathbf{T^d}))^3$. For any sequence y_n converging to y, we have by the hypoellipticity of L, and the smoothness of e, $\frac{\partial e}{\partial y}$, $\frac{\partial^2 e}{\partial y^2}$ (see e.g. [17])

$$||\hat{e}(.,y_n)||_{C^3(\mathbf{T^d})} + ||\frac{\partial \hat{e}}{\partial y}(.,y_n)||_{C^3(\mathbf{T^d})} + ||\frac{\partial^2 \hat{e}}{\partial y^2}(.,y_n)||_{C^3(\mathbf{T^d})} \leqslant C,$$

so the functions $\hat{e}(.,y_n)$, $\frac{\partial \hat{e}}{\partial y}(.,y_n)$ and $\frac{\partial^2 \hat{e}}{\partial y^2}(.,y_n)$ are equicontinuous, together with their derivatives in x of order one and two. Then from Ascoli's theorem $\hat{e}(.,y_n)$, $\frac{\partial \hat{e}}{\partial y}(.,y_n)$, $\frac{\partial^2 \hat{e}}{\partial y^2}(.,y_n)$ have subsequences converging uniformly in $C^2(\mathbf{T^d})$. Since the sequences $\hat{e}(x,y_n)$, $\frac{\partial \hat{e}}{\partial y}(x,y_n)$, and $\frac{\partial^2 \hat{e}}{\partial y^2}(x,y_n)$ converge respectively to $\hat{e}(x,y)$, $\frac{\partial \hat{e}}{\partial y}(x,y)$, $\frac{\partial^2 \hat{e}}{\partial y^2}(x,y)$ then

$$\begin{array}{rcl} \hat{e}(.,y_n) & \longrightarrow & \hat{e}(.,y) \\ \dfrac{\partial \hat{e}}{\partial y}(.,y_n) & \longrightarrow & \dfrac{\partial \hat{e}}{\partial y}(.,y), \\ \dfrac{\partial^2 \hat{e}}{\partial y^2}(.,y_n) & \longrightarrow & \dfrac{\partial^2 \hat{e}}{\partial y^2}(.,y), \end{array}$$

in $C^2(\mathbf{T^d})$. Hence $\hat{e}$, $\frac{\partial \hat{e}}{\partial y}$ and $\frac{\partial^2 \hat{e}}{\partial y^2}$ are continuous in (x,y) and their partial derivatives with respect to x of order one and two are also continuous. The limiting equation (4.1), can be formulated as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} C_i(u(t,x))\frac{\partial u}{\partial x_i}(t,x) \\ \qquad\qquad +D(u(t,x)), \\ u(0,x) = g(x), \end{cases} \tag{4.2}$$

where

$$\begin{array}{rcl} A & = & \displaystyle\int_{\mathbf{T^d}} (I+\nabla\hat{b})a(I+\nabla\hat{b})^*(x)\mu(dx), \\ C(y) & = & \displaystyle\int_{\mathbf{T^d}} (I+\nabla\hat{b})(c+a\frac{\partial^2 \hat{e}}{\partial x \partial y}(.,y))(x)\mu(dx), \\ D(y) & = & \displaystyle\int_{\mathbf{T^d}} [\langle \frac{\partial \hat{e}}{\partial x}(.,y), c\rangle - \frac{\partial \hat{e}}{\partial y}(.,y)e(.,y) \\ & + & \dfrac{\partial^2 \hat{e}^*}{\partial x \partial y}(.,y)a\dfrac{\partial \hat{e}}{\partial x}(.,y) + f(.,y)](x)\mu(dx). \end{array}$$

Then we have the following

THEOREM 4.1. — *For all* $t \geqslant 0, x \in \mathbb{R}^{\mathrm{d}}$,

$$u^\varepsilon(t,x) \longrightarrow u(t,x), \ \textit{when } \varepsilon \longrightarrow 0,$$

where u^ε *is the solution of the equation (4.1) and* u *the solution of (4.2).*

Proof. — The functions $\hat{b}_i$, $i = 1, \ldots, d$, $\hat{e}$ are smooth and with our assumptions on a, b, c, g, e and f we can follow the proof of Theorem 4.1 in É. Pardoux [13], which establishes the convergence of BSDEs. In fact considering the progressively measurable process $\{(Y_s^\varepsilon, Z_s^\varepsilon); 0 \leqslant s \leqslant t\}$ in $\mathbb{R} \times \mathbb{R}^{\mathrm{d}}$ solution of the BSDE :

$$Y_s^\varepsilon = g(X_t^\varepsilon) + \frac{1}{\varepsilon}\int_s^t e(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr + \int_s^t f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dW_r,$$

with

$$\mathbb{E}(\sup_{0\leqslant s\leqslant t} |Y_s^\varepsilon|^2 + \int_0^t ||Z_s^\varepsilon||^2 ds) < \infty,$$

by É. Pardoux [14] the solution of (4.1) is given by

$$u^\varepsilon(t,x) = Y_0^\varepsilon.$$

In order to prove the above Theorem 4.1, it suffices to prove that

$$Y_0^\varepsilon \longrightarrow Y_0,$$

where Y_0 is the value at $t = 0$ of the solution of the FBSDE

$$\begin{cases} X_s = x + \int_0^s C(Y_r)dr + A^{\frac{1}{2}}B_s, \ 0 \leqslant s \leqslant t \\ Y_s = g(X_t) + \int_s^t D(Y_r)dr - \int_s^t Z_r dB_r, \ 0 \leqslant s \leqslant t. \end{cases}$$

5. Homogenization of a semilinear parabolic equation 2

We consider the semilinear parabolic equation

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t}(t,x) = L_\varepsilon u^\varepsilon(t,x) + f(\frac{x}{\varepsilon}, u^\varepsilon(t,x), \nabla u^\varepsilon(t,x)\sigma(\frac{x}{\varepsilon})) \\ u^\varepsilon(0,x) = g(x), x \in \mathbb{R}^d, \end{cases} \tag{5.1}$$

where

$$L_\varepsilon = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(\frac{x}{\varepsilon})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d (\frac{1}{\varepsilon}b_i(\frac{x}{\varepsilon}) + c_i(\frac{x}{\varepsilon}))\frac{\partial}{\partial x_i}.$$

The functions a, b, c verify the assumptions of the previous section, and we assume that

$$g \in W^{2,p}(\mathbb{R}^{\mathrm{d}}),$$

for some $p > d+1$, p even, and

$$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R},$$

is continuous, periodic of period one in each direction with respect to its first argument, of class C^1 with respect to its second and third arguments, uniformly with respect to the first argument, with f'_y bounded from above, and $\nabla_z f$ bounded. We assume that

$$\begin{array}{rcl} |f(x,y,z)| & \leqslant & K'(1+|y|+|z|), \\ |f(t,y,z) - f(t,y',z')| & \leqslant & K(|y-y'|+|z-z'|), \end{array}$$

and moreover that for all $x \in \mathbb{R}^d$, $f(x,\cdot,\cdot) \in C^2(\mathbb{R}\times\mathbb{R}^d)$, all the derivatives being bounded, uniformly with respect to x. Let us consider the progressively measurable process $\{(Y_s^\varepsilon, Z_s^\varepsilon); 0 \leqslant s \leqslant t\}$ with values in $\mathbb{R} \times \mathbb{R}^d$, solution of the BSDE:

$$Y_s^\varepsilon = g(X_t^\varepsilon) + \int_s^t f(\frac{X_r^\varepsilon}{\varepsilon}, Y_r^\varepsilon, Z_r^\varepsilon)dr - \int_s^t Z_r^\varepsilon dB_r,$$

with $\mathbb{E}[\sup\limits_{0\leqslant s\leqslant t} |Y_s^\varepsilon|^2 + \int_0^t ||Z_r^\varepsilon||^2 dr] < \infty$.

The solution of (5.1) is given by: $u^\varepsilon(t,x) = Y_0^\varepsilon$. We are going to study the limit of u^ε when ε tends to zero.

We first state and prove the (for the notion of S–tightness, see [8], [10])

Proposition 5.1. — *There exists a constant $C > 0$ such that*

$$|Y_s^\varepsilon(\omega)| \leqslant C, \quad \forall \varepsilon > 0,\ 0 \leqslant s \leqslant t,\ \omega \in \Omega.$$

Moreover, the collection of continuous processes $\{Y_s^\varepsilon,\ 0 \leqslant s \leqslant t\}_{0<\varepsilon<\varepsilon_0}$ is S–tight.

Proof. — Since g is bounded, it follows from Itô's formula that for any $\alpha \in \mathbb{R}$,

$$e^{\alpha s}|Y_s^\varepsilon|^2 + \int_s^t e^{\alpha r}(\alpha|Y_r^\varepsilon|^2 + |Z_r|^2)dr \leqslant c + 2\int_s^t e^{\alpha r} Y_r^\varepsilon f(\overline{X}_r^\varepsilon, Y_r^\varepsilon, Z_r^\varepsilon)dr$$

$$+2\int_s^t e^{\alpha r} Y_r^\varepsilon Z_r^\varepsilon dB_r$$

Now from our assumption on f,

$$\begin{array}{rcl} yf(x,y,z) & \leqslant & K'(|y|+y^2+|y|\times|z|) \\ & \leqslant & K' + (K' + \frac{K'^2}{2})y^2 + |z|^2, \end{array}$$

hence, combining this with the previous inequality where we choose $\alpha = 2(K' + \frac{K'^2}{2})$, and taking the conditional expectation given $\mathcal{F}_s$, we deduce that

$$|Y_s^\varepsilon|^2 \leqslant \frac{2K'}{\alpha}(e^{\alpha(t-s)} - 1) + ce^{-\alpha s},$$

from which the first result follows. It now follows easily from the above that

$$\sup_{\varepsilon>0} \mathbb{E}\left(\sup_{0\leqslant s\leqslant t} |Y_s^\varepsilon|^2 + \int_0^t |Z_s^\varepsilon|^2 ds \right) < \infty,$$

from which the S–tightness follows, since

$$|f(x,y,z)| \leqslant K'(1 + |y| + |z|).$$

□

The limiting PDE can be formulated as:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} C_i \frac{\partial u}{\partial x_i} + \overline{f}(u(t,x), \nabla u(t,x)) \\ u(0,x) = g(x), x \in \mathbb{R}^d, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} A &= \int_{\mathbf{T^d}} (I + \nabla\hat{b})a(I + \nabla\hat{b})^*(x)\mu(dx) \\ C &= \int_{\mathbf{T^d}} (I + \nabla\hat{b})c(x)\mu(dx) \\ \overline{f}(y,z) &= \int_{\mathbf{T^d}} f(x, y, z(I + \nabla\hat{b})\sigma(x))\mu(dx). \end{aligned}$$

It follows from the above assumptions on f that $\overline{f} \in C^2(\mathbb{R} \times \mathbb{R}^d)$, with bounded derivatives. We shall assume w. l. o. g. that the orthonormal basis of $\mathbb{R}^d$ has been choosen in such a way that the matrix A is of the form

$$A = \begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix},$$

where A' is a $d' \times d'$ positive definite matrix, with $d' \leqslant d$. We set $\mathbb{R}^{\mathrm{d}} = E_{d'} \oplus E_{d-d'}$, where $E_{d'}$ is the subspace of $\mathbb{R}^{\mathrm{d}}$ of dimension d' generated by the vectors $e_i, i = 1, 2, \cdots, d'$ after a new arrangement of the basis vectors of $\mathbb{R}^{\mathrm{d}}$ so we can obtain the wished form of A.

Note that from Jensen's inequality

$$\begin{aligned}|\overline{f}(y,z)| &\leqslant \int |f(x,y,z(I+\nabla\hat{b})\sigma(x))|\mu(dx)\\ &\leqslant K'(1+|y|+\int |z(I+\nabla\hat{b})\sigma(x)|\mu(dx)\\ &\leqslant K'(1+|y|+\sqrt{< Az,z >}),\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}|\overline{f}(y,z)-\overline{f}(y',z')| &\leqslant \int_{\mathbf{T^d}} |f(x,y,z(I+\nabla\hat{b})\sigma(x))\\ &\qquad -f(x,y',z'(I+\nabla\hat{b})\sigma(x))|\mu(dx)\\ &\leqslant K\left(|y-y'|+\int_{\mathbf{T^d}}|(z-z')(I+\nabla\hat{b})\sigma(x)|\mu(dx)\right)\\ &\leqslant K\left(|y-y'|+\sqrt{\langle A(z-z'),z-z'\rangle}\right)\end{aligned} \tag{5.4}$$

We define $H_A(\mathbb{R}^{\mathrm{d}}) = \{u \in L^2(\mathbb{R}^{\mathrm{d}}); \sqrt{A}\nabla u \in (L^2(\mathbb{R}^{\mathrm{d}}))^d\}$, and we define the following norm on $H_A(\mathbb{R}^{\mathrm{d}})$

$$||v||_{H_A(\mathbb{R}^{\mathrm{d}})} = \left(||v||^2_{L^2(\mathbb{R}^{\mathrm{d}})} + ||\sqrt{A}\nabla v||^2_{(L^2(\mathbb{R}^{\mathrm{d}}))^d}\right)^{\frac{1}{2}}.$$

We have by (5.3), $||\overline{f}(v,\nabla v)||_{L^2(\mathbb{R}^{\mathrm{d}})} \leqslant C(1+||v||_{H_A(\mathbb{R}^{\mathrm{d}})})$.
We can show the

THEOREM 5.2. — *Equation (5.2) has a unique solution u in $L^2((0,T);H^1(\mathbb{R}^d))$, such that for all $1 \leqslant k \leqslant d$,*

$$< A\nabla u_k, \nabla u_k > \in L^1((0,T)\times\mathbb{R}^d),$$

where

$$\frac{\partial u}{\partial x_k} = u_k \in L^2((0,T)\times\mathbb{R}^d).$$

Moreover

$$u \in C(\mathbb{R}_+; L^2(\mathbb{R}^{\mathrm{d}})).$$

Proof. — **Step 1:**
We first assume that the matrix A is elliptic, and we look for a solution $u \in L^2(0,T;H^1(\mathbb{R}^{\mathrm{d}})) \cap C([0,T],L^2(\mathbb{R}^{\mathrm{d}}))$. Let us prove the existence and uniqueness of the solution of the PDE. We set $F = L^2((0,T);H^1(\mathbb{R}^{\mathrm{d}}))$, and consider the map

$$\Phi : F \longrightarrow F,$$

defined as follows. For $v \in F, u = \Phi(v)$ is the unique solution in F of the linear parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = & \frac{1}{2}\sum_{i,j=1}^{d} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^{d} C_i \frac{\partial u}{\partial x_i}(t,x) \\ & & +\overline{f}(v(t,x),\nabla v(t,x)) \\ u(0,x) & = & g(x), x \in \mathbb{R}^d, \end{cases}$$

Let us show that Φ is a contraction. For $v, v' \in F$, $u = \Phi(v)$, $u' = \Phi(v')$, $(\overline{u},\overline{v}) = (u-u', v-v')$, we have, for any $\alpha > 0$, if we denote by ν the ellipticity constant of the matrix A,

$$\begin{aligned} & \frac{1}{2}e^{-\alpha t}||\overline{u}(t)||^2_{L^2(\mathbb{R}^d)} + \nu\int_0^t e^{-\alpha s}||\nabla\overline{u}(s)||^2_{(L^2(\mathbb{R}^d))^d}ds \\ \leqslant & -\frac{\alpha}{2}\int_0^t e^{-\alpha s}||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)}ds \\ & + \int_0^t e^{-\alpha s}\langle \overline{f}(v(s),\nabla v(s)) - \overline{f}(v'(s),\nabla v'(s)), \overline{u}(s)\rangle_{L^2(\mathbb{R}^d)}ds \end{aligned}$$

By the inequality (5.4), we get

$$\begin{aligned} & \nu\int_0^t e^{-\alpha s}||\nabla\overline{u}(s)||^2_{(L^2(\mathbb{R}^d))^d}ds + \frac{\alpha}{2}\int_0^t e^{-\alpha s}||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)}ds \\ & \leqslant C\int_0^t e^{-\alpha s}\left(||\overline{v}(s)||_{L^2(\mathbb{R}^d)} + \|\nabla\overline{v}(s)\|_{(L^2(\mathbb{R}^d))^d}\right)||\overline{u}(s)||_{L^2(\mathbb{R}^d))}ds \\ & \qquad \leqslant \frac{\nu}{2}\int_0^t e^{-\alpha s}(||\overline{v}(s)||^2_{L^2(\mathbb{R}^d)} + ||\nabla\overline{v}(s)||^2_{(L^2(\mathbb{R}^d))^d})ds \\ & \qquad\qquad + \frac{C^2}{2\nu}\int_0^t e^{-\alpha s}||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)}ds, \end{aligned}$$

hence we get

$$\begin{aligned} & \int_0^t e^{-\alpha s}(\nu||\nabla\overline{u}(s)||^2_{(L^2(\mathbb{R}^d))^d} + \frac{\nu\alpha - C^2}{2\nu}||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)})ds \\ & \qquad \leqslant \frac{\nu}{2}\int_0^t e^{-\alpha s}(||\overline{v}(s)||^2_{L^2(\mathbb{R}^d)} + ||\nabla\overline{v}(s)||^2_{(L^2(\mathbb{R}^d))^d})ds. \end{aligned}$$

If we choose $\alpha = 2\nu + \frac{C^2}{\nu}$ and divide the last inequality by ν, we obtain

$$\begin{aligned} & \int_0^t e^{-\alpha s}(||\nabla\overline{u}(s)||^2_{L^2(\mathbb{R}^d)} + ||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)})ds \\ & \qquad \leqslant \frac{1}{2}\int_0^t e^{-\alpha s}(||\overline{v}(s)||^2_{L^2(\mathbb{R}^d)} + ||\nabla\overline{v}(s)||^2_{(L^2(\mathbb{R}^d))^d})ds, \end{aligned}$$

from which we deduce that Φ is a strict contraction on F equiped with the norm:

$$||u||_\alpha = \left(\int_0^t e^{-\alpha s}(||\nabla\overline{u}(s)||^2_{(L^2(\mathbb{R}^d))^d} + ||\overline{u}(s)||^2_{L^2(\mathbb{R}^d)})ds\right)^{\frac{1}{2}}, \alpha = 2\nu + \frac{C^2}{\nu}.$$

Hence Φ has a unique fixed point.

Step 2 :
We now drop the assumption that A be elliptic, and set $A^n = A + \frac{1}{n}I_d$. Let u^n denote the unique solution of the equation (5.2), with A replaced by A^n. Multiplying the equation by u^n, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|u^n(t,x)|^2dx + \frac{1}{2}\int_{\mathbb{R}^d}\langle A^n\nabla u^n(t,x), \nabla u^n(t,x)\rangle dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^d}\sum_{i=1}^d C_i\frac{\partial}{\partial x_i}(u^n(t,x)^2)dx + \int_{\mathbb{R}^d}\overline{f}(u^n(t,x),\nabla u^n(t,x))u^n(t,x)dx.$$

We know that $\int_{\mathbb{R}^d}\sum_{i=1}^d C_i\frac{\partial}{\partial x_i}(u^n(t,x)^2)dx = 0, t$ a.e. and $\forall\delta > 0$

$$\int_{\mathbb{R}^d}\overline{f}(u^n(t,x),\nabla u^n(t,x))u^n(t,x)dx \leqslant (K' + \frac{K'^2}{2\delta})(1 + \int_{\mathbb{R}^d}||u^n(t,x)||^2dx)$$
$$+\frac{\delta}{2}\int_{\mathbb{R}^d}\langle A\nabla u^n(t,x), \nabla u^n(t,x)\rangle dx.$$

Choosing $\delta = \frac{1}{2}$, then by Gronwall's lemma we deduce that

$$\int_{\mathbb{R}^d}|u^n(t,x)|^2dx \leqslant Ce^{Ct}$$

and

$$\int_0^T\int_{\mathbb{R}^d}\langle A\nabla u^n(t,x), \nabla u^n(t,x)\rangle dxdt \leqslant C(T).$$

Let us set $\Lambda(x) = (I + \nabla\hat{b})\sigma(x)$. Now we differentiate the equation for u^n with respect to x_k. Then $u^n_k = \frac{\partial u^n}{\partial x_k}$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}u^n_k(t,x) = \frac{1}{2}\sum_{i,j=1}^d A^n_{ij}\frac{\partial^2 u^n_k}{\partial x_i\partial x_j}(t,x) + \sum_{i=1}^d C_i\frac{\partial u^n_k}{\partial x_i}(t,x) \\ \qquad +\overline{f}_y(u^n(t,x),\nabla u^n(t,x))u^n_k(t,x) \\ + \int_{\mathbf{T^d}}\mu(dx')^t\nabla u^n_k(t,x)\Lambda(x')\nabla_z f(x', u^n(t,x), \nabla u^n(t,x)\Lambda(x')), \\ \quad u^n_k(0,x) = \frac{\partial g}{\partial x_k}(x). \end{cases} \tag{5.5}$$

Multiplying this equation by u_k^n we have

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u_k^n(t,x)|^2dx+\frac{1}{2}\int_{\mathbb{R}^d}\langle A^n\nabla u_k^n(t,x),\nabla u_k^n(t,x)\rangle dx$$
$$=\frac{1}{2}\int_{\mathbb{R}^d}\sum_{i=1}^d C_i\frac{\partial[(u_k^n(t,x))^2]}{\partial x_i}dx$$
$$+\int_{\mathbb{R}^d}\overline{f}_y(u^n(t,x),\nabla u^n(t,x))[u_k^n(t,x)]^2dx$$
$$+\int_{\mathbb{R}^d}\int_{\mathbf{T^d}}\mu(dx')^t\nabla u_k^n(t,x)\Lambda(x')\nabla_z f(x',u^n(t,x),\nabla u^n(t,x)\Lambda(x'))u_k^n(t,x)dx. \tag{5.6}$$

We know that $\displaystyle\int_{\mathbb{R}^d}\sum_{i=1}^d C_i\frac{\partial[u_k^n(t,x)^2]}{\partial x_i}dx=0$, t a.e. and for any $\delta>0$, since $\nabla_z f$ is bounded,

$$\int_{\mathbb{R}^d}\int_{\mathbf{T^d}}\mu(dx')^t\nabla u_k^n(t,x)\Lambda(x')\nabla_z f(x',u^n(t,x),\nabla u^n(t,x)\Lambda(x'))u_k^n(t,x)dx$$
$$\leqslant C\delta\int_{\mathbb{R}^d}\langle A\nabla u_k^n(t,x),\nabla u_k^n(t,x)\rangle dx+\frac{C}{\delta}\int_{\mathbb{R}^d}|u_k^n(t,x)|^2dx.$$

By an appropriate choice of δ, we deduce that for $1\leqslant k\leqslant d$, $t>0$,

$$\int_{\mathbb{R}^d}|u_k^n(t,x)|^2dx\leqslant Ce^{Ct}.$$

We have proved that u^n is bounded in $L^\infty([0,T],H^1(\mathbb{R}^d))$, and also that each u_k^n is bounded in $L^2(0,T;H_A)$. Let us now show that u^n is a Cauchy sequence in $L^2(0,T;H_A)$. We have

$$\frac{\partial(u^n-u^m)}{\partial t}(t,x)=\frac{1}{2}\sum_{i,j=1}^d A_{ij}\frac{\partial^2(u^n-u^m)}{\partial x_i\partial x_j}(t,x)+\frac{1}{2n}\sum_{i,j=1}^d\frac{\partial^2u^n}{\partial x_i\partial x_j}(t,x)$$
$$-\frac{1}{2m}\sum_{i,j=1}^d\frac{\partial^2u^m}{\partial x_i\partial x_j}(t,x)+\sum_{i=1}^d C_i\frac{\partial(u^n-u^m)}{\partial x_i}(t,x)$$
$$+(\overline{f}(u^n(t,x),\nabla u^n(t,x))-\overline{f}(u^m(t,x),\nabla u^m(t,x))),$$

then by multiplying this equation by $u^n - u^m$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||u^n - u^m||^2_{L^2}(t) + \frac{1}{2}\int_{\mathbb{R}^d}\langle A\nabla(u^n - u^m), \nabla(u^n - u^m)\rangle(t,x)dx \\
&+\frac{1}{2}\int_{\mathbb{R}^d}\langle\frac{1}{n}\nabla u^n - \frac{1}{m}\nabla u^m, \nabla(u^n - u^m)\rangle(t,x)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\sum_{i=1}^{d} C_i \frac{\partial[(u^n - u^m)^2]}{\partial x_i}(t,x)dx \\
&+ \int_{\mathbb{R}^d}\langle\overline{f}(u^n(t,x), \nabla u^n(t,x)) - \overline{f}(u^m(t,x), \nabla u^m(t,x)), u^n - u^m\rangle(t,x)dx,
\end{aligned}$$

and integrating with respect to t we have

$$\begin{aligned}
&\frac{1}{2}||u^n - u^m||^2_{L^2}(t) + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\langle A(\nabla u^n - \nabla u^m), \nabla(u^n - u^m)\rangle(s,x)dxds \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\langle\frac{1}{n}\nabla u^n - \frac{1}{m}\nabla u^m, \nabla(u^n - u^m)\rangle(s,x)dxds \\
&= \int_0^t\int_{\mathbb{R}^d}\langle\overline{f}(u^n(s,x), \nabla u^n(s,x)) - \overline{f}(u^m(s,x), \nabla u^m(s,x)), u^n - u^m\rangle(s,x)dxds.
\end{aligned}$$

Since ∇u^n and ∇u^m are bounded in $L^2((0,T)\times\mathbb{R}^d)^d$,

$$\int_0^T\int_{\mathbb{R}^d}\langle\frac{1}{n}\nabla u^n - \frac{1}{m}\nabla u^m, \nabla(u^n - u^m)\rangle(t,x)dxdt$$

tends to zero when n and m tend to infinity.

For $\varepsilon > 0$, there exists N_ε such that for $n, m \geqslant N_\varepsilon$, all $\delta > 0$,

$$\begin{aligned}
&\frac{1}{2}||u^n - u^m||^2_{L^2}(t) + \frac{1-\delta}{2}\int_0^t\int_{\mathbb{R}^d}\langle A(\nabla u^n - \nabla u^m), \nabla(u^n - u^m)\rangle(s,x)dxds \\
&\qquad\leqslant \varepsilon + (K + \frac{K^2}{2\delta})\int_0^t ||u^n - u^m||^2_{L^2}(s)ds.
\end{aligned}$$

Hence choosing $\delta = \frac{1}{2}$, and exploiting Gronwall's lemma we have

$$\begin{aligned}
&\frac{1}{2}||u^n - u^m||^2_{L^2}(t) + \frac{1}{4}\int_0^t\int_{\mathbb{R}^d}\langle A(\nabla u^n - \nabla u^m), \nabla(u^n - u^m)\rangle(s,x)dxds \\
&\qquad\leqslant \varepsilon e^{Ct}, \forall n, m \geqslant N_\varepsilon, 0 < t \leqslant T.
\end{aligned}$$

Hence u^n is a Cauchy sequence in $L^2(0,T;H_A)$, and there exists $u \in L^2(0,T;H_A)$ such that

$$u^n \longrightarrow u \ \text{ in } \ L^2(0,T;H_A).$$

Moreover since

$$\int_0^T \int_{\mathbb{R}^d} |\overline{f}(u^n(t,x), \nabla u^n(t,x)) - \overline{f}(u(t,x), \nabla u(t,x))|^2 dtdx$$
$$\leqslant C(\int_0^T \int_{\mathbb{R}^d} ||u^n - u||^2(t,x) + \langle A(\nabla u^n - \nabla u), \nabla(u^n - u)\rangle(t,x))dtdx,$$

then

$$\overline{f}(u^n(t,x), \nabla u^n(t,x)) \longrightarrow \overline{f}(u(t,x), \nabla u(t,x)), \ in \ L^2((0,T) \times \mathbb{R}^d).$$

Moreover the sequence $\{u^n\}$ is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$, hence $u \in L^2(0,T;H^1(\mathbb{R}^d))$.

We finally show the uniqueness of the solution in the space $L^2(0,T;H^1(\mathbb{R}^d))$. Let u, u' be two solutions of the PDE (5.2), then $u - u'$ solves

$$\frac{\partial(u-u')}{\partial t}(t,x) = \frac{1}{2}\sum_{i,j=1}^d A_{ij}\frac{\partial^2(u-u')}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^d C_i \frac{\partial(u-u')}{\partial x_i}(t,x)$$
$$+(\overline{f}(u(t,x), \nabla u(t,x)) - \overline{f}(u'(t,x), \nabla u'(t,x))).$$

Multiplying this equation $u - u'$, we obtain

$$\frac{1}{2}||u-u'||^2_{L^2}(t) + \frac{1-\delta}{2}\int_0^t \int_{\mathbb{R}^d} \langle A(\nabla u - \nabla u'), \nabla(u-u)\rangle(s,x)dxds$$
$$\leqslant (K' + \frac{K'^2}{2\delta})\int_0^t ||u-u'||^2_{L^2}(s)ds,$$

since we know that $\int_{\mathbb{R}^d} \sum_{i=1}^d C_i \frac{\partial(u-u')^2}{\partial x_i}(t,x)dx = 0, \ t$ a.e, because $u(t), \ u'(t) \in H^1(\mathbb{R}^d)$, t a.e.

If we choose $\delta = \frac{1}{2}$, and by Gronwall lemma we have

$$||u-u'||^2_{L^2}(t) = 0,$$

which proves the uniqueness. □

We now prove some additional regularity

Proposition 5.3. — *Assume that $g \in W^{2,p}(\mathbb{R}^d)$, for some $p > d$, $p \geqslant 4$ and p even. Then for all $T > 0$, $\nabla^{d'} u \in L^\infty(0,T; C_b(\mathbb{R}^d, \mathbb{R}^{d'}))$.*

Proof. — Had we multiplied equation (5.5) by $(u_k^n(t,x))^{p-1}$, p even, we would deduce by similar arguments as for the case $p = 2$ that for each p

even, there exists a constant C_p such that for all $1 \leqslant k \leqslant d$, all n,

$$\int_{\mathrm{IR^d}} |u_k^n(t,x)|^p dx \leqslant C_p \| \frac{\partial g}{\partial x_k} \|^p_{L^p(\mathrm{IR^d})} e^{C_p t}. \tag{5.7}$$

We differentiate the equation (5.5) with respect to $x_\ell, 1 \leqslant \ell \leqslant d'$. Given the form of A, only the gradient in the direction of $E_{d'}$ has some effect on the nonlinear term of the PDE, we will denote $\nabla^{d'}$ this gradient.

$$\begin{cases} \dfrac{\partial u^n_{k\ell}}{\partial t}(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^d A^n_{ij} \dfrac{\partial^2 u^n_{k\ell}}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^d C_i \dfrac{\partial u^n_{k\ell}}{\partial x_i}(t,x) \\ +\overline{f}_y(u^n(t,x), \nabla u^n(t,x)) u^n_{k\ell}(t,x) \\ +\overline{f}_{yy}(u^n(t,x), \nabla u^n(t,x)) u^n_k(t,x) u^n_\ell(t,x) \\ +{}^t\nabla^{d'} u^n_\ell(t,x) \nabla_z \overline{f}_y(u^n(t,x), \nabla u^n(t,x)) u^n_k(t,x) \\ +\dfrac{\partial}{\partial x_\ell}(\displaystyle\int_{\mathbf{T^d}} \mu(dx')^t \nabla u^n_k(t,x) \Lambda(x') \nabla_z f(x', u^n(t,x), \nabla u^n(t,x)\Lambda(x'))), \\ u^n_{k\ell}(0,x) = \dfrac{\partial^2 g}{\partial x_k \partial x_\ell}(x). \end{cases} \tag{5.8}$$

Multiplying this equation by $u^n_{k\ell}$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathrm{IR^d}} |u^n_{k\ell}(t,x)|^2 dx + \frac{1}{2}\int_{\mathrm{IR^d}} \langle A \nabla u^n_{k\ell}(t,x), \nabla u^n_{k\ell}(t,x)\rangle dx \\ &= \int_{\mathrm{IR^d}} \overline{f}_y(u^n(t,x), \nabla u^n(t,x))(u^n_{k\ell})^2(t,x) dx \\ &+ \int_{\mathrm{IR^d}} \overline{f}_{yy}(u^n(t,x), \nabla u^n(t,x)) u^n_k(t,x) u^n_\ell(t,x) u^n_{k\ell}(t,x) dx \\ &+ \int^t_{\mathrm{IR^d}} \nabla^{d'} u^n_\ell(t,x) \nabla_z \overline{f}_y(u^n(t,x), \nabla u^n(t,x)) u^n_k(t,x) u^n_{k\ell}(t,x)) dx \\ &- \int_{\mathrm{IR^d}} (\int_{\mathbf{T^d}} \mu(dx')^t \nabla u^n_k(t,x) \Lambda(x') \nabla_z f(x', u^n(t,x), \nabla u^n(t,x)\Lambda(x'))) \frac{\partial u^n_{k\ell}}{\partial x_\ell} dx. \end{aligned}$$

We have the following estimates :

$$|\int_{\mathrm{IR^d}} \overline{f}_y(u^n(t,x), \nabla u^n(t,x))(u^n_{k\ell})^2(t,x) dx| \leqslant C \int_{\mathrm{IR^d}} |u^n_{k\ell}|^2(t,x) dx,$$

$$\begin{aligned} &\int_{\mathrm{IR^d}} \overline{f}_{yy}(u^n(t,x), \nabla u^n(t,x)) u^n_k(t,x) u^n_\ell(t,x) u^n_{k\ell}(t,x) dx \\ &\leqslant C \int_{\mathrm{IR^d}} (|u^n_k|^4(t,x) + |u^n_\ell|^4(t,x) + |u^n_{k\ell}|^2(t,x)) dx \end{aligned}$$

$$|\int_{\mathbb{R}^d}^{t} \nabla^{d'} u_\ell^n(t,x) \nabla_z \overline{f}_y(u^n(t,x), \nabla u^n(t,x)) u_k^n(t,x) u_{k\ell}^n(t,x))dx|$$
$$\leqslant C \int_{\mathbb{R}^d} (\delta' |\nabla^{d'} u_\ell^n|^2(t,x)dx + \frac{1}{\delta'} \int_{\mathbb{R}^d} |u_{k\ell}^n|^2(t,x))dx,$$

$$\int_{\mathbb{R}^d} (\int_{\mathbf{T^d}} \mu(dx') \nabla^t u_k^n(t,x) \Lambda(x') \nabla_z f(x', u^n(t,x), \nabla u^n(t,x)\Lambda(x'))) \frac{\partial u_{k\ell}^n}{\partial x_\ell} dx$$
$$\leqslant C(\frac{1}{\delta} \int_{\mathbb{R}^d} \langle A \nabla u_k^n, \nabla u_k^n \rangle (t,x)dx + \delta \int_{\mathbb{R}^d} |\frac{\partial u_{k\ell}^n}{\partial x_\ell}|^2(t,x)dx).$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d} |u_{k\ell}^n(t,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} \langle A \nabla u_{k\ell}^n(t,x), \nabla u_{k\ell}^n(t,x) \rangle dx \\
&\leqslant C \int_{\mathbb{R}^d} (|u_k^n|^4(t,x) + |u_\ell^n|^4(t,x) + |u_{k\ell}^n|^2(t,x))dx \\
&+ C\delta' \int_{\mathbb{R}^d} |\nabla^{d'} u_\ell^n|^2(t,x)dx \\
&+ C(\frac{1}{\delta} \int_{\mathbb{R}^d} \langle A \nabla u_k^n, \nabla u_k^n \rangle (t,x)dx + \delta \int_{\mathbb{R}^d} |\frac{\partial u_{k\ell}^n}{\partial x_\ell}|^2(t,x)dx)
\end{aligned}$$

But in the subspace $E_{d'}$ we have $\langle A' \nabla^{d'} u_k^n, \nabla^{d'} u_k^n \rangle \geqslant \alpha |\nabla^{d'} u_k^n|^2$, where α is the ellipticity constant of A'. Hence, since $1 \leqslant \ell \leqslant d'$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d} |u_{k\ell}^n(t,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} \langle A \nabla u_{k\ell}^n(t,x), \nabla u_{k\ell}^n(t,x) \rangle dx \\
&\leqslant C \int_{\mathbb{R}^d} (|u_k^n|^4(t,x) + |u_\ell^n|^4(t,x) + |u_{k\ell}^n|^2(t,x))dx \\
&+ C\frac{\delta'}{\alpha} \int_{\mathbb{R}^d} \langle A \nabla u_\ell^n, \nabla u_\ell^n \rangle (t,x)dx + \frac{C}{\delta} \int_{\mathbb{R}^d} \langle A \nabla u_k^n, \nabla u_k^n \rangle (t,x)dx \\
&+ C\frac{\delta}{\alpha} \int_{\mathbb{R}^d} \langle A \nabla u_{k\ell}^n(t,x), \nabla u_{k\ell}^n(t,x) \rangle dx
\end{aligned}$$

Choosing $\delta = \frac{\alpha}{4C}$ we have by the Gronwall's lemma

$$\int_{\mathbb{R}^d} |u_{k\ell}^n(t,x)|^2 dx \leqslant ce^{ct}.$$

If we multiply now the equation (5.8) by $(u^n_{k\ell})^{p-1}$, p even, we get

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^d}|u^n_{k\ell}(t,x)|^p dx+\frac{p-1}{2}\int_{\mathbb{R}^d}\langle A\nabla u^n_{k\ell}(t,x),\nabla u^n_{k\ell}(t,x)\rangle(u^n_{k\ell})^{p-2}(t,x)dx$$
$$=\int_{\mathbb{R}^d}\overline{f}_y(u^n(t,x),\nabla u^n(t,x))(u^n_{k\ell})^p(t,x)dx$$
$$+\int_{\mathbb{R}^d}\overline{f}_{yy}(u^n(t,x),\nabla u^n(t,x))u^n_k(t,x)u^n_\ell(t,x)(u^n_{k\ell})^{p-1}(t,x)dx$$
$$+\int_{\mathbb{R}^d}\nabla^{d'}u^n_\ell(t,x)\nabla_z\overline{f}_y(u^n(t,x),\nabla u^n(t,x))u^n_k(t,x)(u^n_{k\ell})^{p-1}(t,x))dx$$
$$-(p-1)\int_{\mathbb{R}^d}(\int_{\mathbf{T^d}}\mu(dx')\nabla^t u^n_k(t,x)\Lambda(x')\nabla_z f(x',u^n(t,x),\nabla u^n(t,x)\Lambda(x')))$$
$$\frac{\partial u^n_{k\ell}}{\partial x_\ell}(t,x)(u^n_{k\ell}(t,x))^{p-2}dx.$$

From arguments similar to those above, we can deduce that for all p even, there exists $C_p>0$ such that for all $1\leqslant k\leqslant d$, $1\leqslant \ell\leqslant d'$, all n,

$$\int_{\mathbb{R}^d}|u^n_{k\ell}(t,x)|^p dx\leqslant C_p\|\frac{\partial^2 g}{\partial x_k\partial x_\ell}\|^p_{L^p(\mathbb{R}^d)}e^{C_p t}. \tag{5.9}$$

Now from (5.7) and (5.9), we deduce by taking the limit as $n\to\infty$ that for all $1\leqslant k\leqslant d$, $1\leqslant \ell\leqslant d'$, $t>0$,

$$\int_{\mathbb{R}^d}\left(\left|\frac{\partial u}{\partial x_k}(t,x)\right|^p+\left|\frac{\partial^2 u}{\partial x_k\partial x_\ell}(t,x)\right|^p\right)dx\leqslant C_p\|g\|^p_{W^{2,p}(\mathbb{R}^d)}e^{C_p t}. \tag{5.10}$$

The result now follows from the Sobolev embedding theorem (see e. g. Theorem IX.12, Corollary IX.13 in H. Brézis [4]). □

We can now deduce from Proposition 5.3 the

PROPOSITION 5.4. — *If $g\in W^{2,p}(\mathbb{R}^d)$ for some $p>d+1$, p even, then for all $T>0$, $u\in C_b([0,T]\times\mathbb{R}^d)$.*

Proof. — We deduce from similar (but simpler) arguments as those in the proof of Proposition 5.3 that for each $t>0$, p as above,

$$\int_0^t\int_{\mathbb{R}^d}|u(s,x)|^p dxds<\infty.$$

Moreover, it follows from (5.10) and the equation for u that

$$\int_0^t\int_{\mathbb{R}^d}\left|\frac{\partial u}{\partial t}(s,x)\right|^p dxds<\infty.$$

Using in addition (5.10), the result follows again from Theorem IX.12 in H. Brézis [4]. $\square$

We now define a new sequence $\{u^n(t,x),\ n \in \mathbb{N}\}$ (no relation with the sequence constructed in the proof of Theorem 5.2) of smooth approximations of $u(t,x)$ as follows

$$u^n(t,x) = \int\int u(s,y)\rho_n(t-s)\varphi_n(x-y)dsdy$$

where

$$\begin{array}{rcl} \rho_n(t) & = & n\rho(nt), \\ \varphi_n(x) & = & n^d\varphi(nx), \end{array}$$

ρ and φ are functions respectively of $C_c^\infty(\mathbb{R},\mathbb{R}^+)$ and $C_c^\infty(\mathbb{R}^{\mathrm{d}},\mathbb{R}+)$ with compact support, and

$$\int_{\mathbb{R}} \rho(t)dt = 1; \int_{\mathbb{R}^{\mathrm{d}}} \varphi(x)dx = 1.$$

We assume moreover that the support of ρ is included in $\mathbb{R}_-$.

The functions u^n are smooth and solve the equation

$$\left\{ \begin{array}{rcl} \dfrac{\partial u^n}{\partial t}(t,x) & = & \dfrac{1}{2}\displaystyle\sum_{i,j=1}^d A_{ij}(t)\dfrac{\partial^2 u^n}{\partial x_i \partial x_j}(t,x) + \sum_{i=1}^d C_i(t)\dfrac{\partial u^n}{\partial x_i}(t,x) \\ & & + \displaystyle\int\int \overline{f}(u(s,y),\nabla u(s,y))\rho_n(t-s)\varphi_n(x-y)dsdy, \\ & & t>0,\ x\in\mathbb{R}^d,\ n\in\mathbb{N}. \end{array} \right.$$

Let us set $\overline{X}^\varepsilon_t = \frac{1}{\varepsilon}X^\varepsilon_t$; $\hat{X}^\varepsilon_t = X^\varepsilon_t + \varepsilon\hat{b}(\overline{X}^\varepsilon_t)$. We note that for $s \geqslant 0$,

$$\hat{X}^\varepsilon_s = x + \int_0^s [(I+\nabla\hat{b})c](\overline{X}^\varepsilon_r)dr + \int_0^s [(I+\nabla\hat{b})\sigma](\overline{X}^\varepsilon_r)dB_r.$$

We define moreover

$$\begin{array}{l} \tilde{Y}^{\varepsilon,n}_s = Y^\varepsilon_s - u^n(t-s,\hat{X}^\varepsilon_s) \\ \tilde{Z}^{\varepsilon,n}_s = Z^\varepsilon_s - \nabla u^n(t-s,\hat{X}^\varepsilon_s)[(I+\nabla\hat{b})\sigma](\overline{X}^\varepsilon_s). \end{array}$$

It follows from Propositions 5.1 and 5.4 that there exists $C>0$ such that $\tilde{Y}^{\varepsilon,n}_s \leqslant C$ a. s., for all $\varepsilon>0$, $n\in\mathbb{N}$, $0\leqslant s\leqslant t$. Using the Itô formula we have

$$\begin{array}{rcl} u^n(t-s,\hat{X}^\varepsilon_s) & = & u^n(0,\hat{X}^\varepsilon_t) - \displaystyle\int_s^t (-\frac{\partial u^n}{\partial r}(t-r,\hat{X}^\varepsilon_r) + \hat{L}_{\varepsilon,n}(r))dr \\ & & - \displaystyle\int_s^t \nabla u^n(t-r,\hat{X}^\varepsilon_r)[(I+\nabla\hat{b})\sigma](\overline{X}^\varepsilon_r)dB_r, \end{array}$$

where

$$\hat{L}_{\varepsilon,n}(r) = \frac{1}{2}\sum_{i,j=1}^{d}[((I+\nabla\hat{b})a(I+\nabla\hat{b})^*)(\overline{X}_r^{\varepsilon})]_{ij}\frac{\partial^2 u^n}{\partial x_i \partial x_j}(t-r,\hat{X}_r^{\varepsilon}) + \sum_{i=1}^{d}[((I+\nabla\hat{b})c)(\overline{X}_r^{\varepsilon})]_i\frac{\partial u^n}{\partial x_i}(t-r,\hat{X}_r^{\varepsilon}).$$

Then

$$\begin{aligned}
\tilde{Y}_s^{\varepsilon,n} &= Y_s^{\varepsilon} - u^n(t-s,\hat{X}_s^{\varepsilon}) \\
&= g(X_t^{\varepsilon}) - u^n(0,\hat{X}_t^{\varepsilon}) \\
&+ \int_s^t [f(\overline{X}_r^{\varepsilon}, \tilde{Y}_r^{\varepsilon,n} + u^n(t-r,\hat{X}_r^{\varepsilon}), \tilde{Z}_r^{\varepsilon,n} + \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon})) \\
&- \int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy \\
&+ Lu^n(t-r,\hat{X}_r^{\varepsilon}) - \hat{L}_{\varepsilon,n}(r)]dr - \int_s^t \tilde{Z}_r^{\varepsilon,n}dB_r,
\end{aligned}$$

where $L = \frac{1}{2}\sum_{i,j=1}^d A_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^d C_i\frac{\partial}{\partial x_i}$. We have

$$\begin{aligned}
&|\tilde{Y}_s^{\varepsilon,n}|^2 + \int_s^t ||\tilde{Z}_r^{\varepsilon,n}||^2 dr = |\xi^{\varepsilon,n}|^2 \\
&+2\int_s^t [\langle \tilde{Y}_r^{\varepsilon,n}, f(\overline{X}_r^{\varepsilon}, \tilde{Y}_r^{\varepsilon,n} + u^n(t-r,\hat{X}_r^{\varepsilon}), \tilde{Z}_r^{\varepsilon,n} + \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon})) \\
&-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_v^{\varepsilon}-y)dvdy \\
&+L^n u^n(t-r,\hat{X}_r^{\varepsilon}) - \hat{L}_{\varepsilon,n}(r)\rangle]dr - 2\textstyle\int_s^t \langle \tilde{Y}_r^{\varepsilon,n}, \tilde{Z}_r^{\varepsilon,n}dB_r\rangle,
\end{aligned}$$

where $\xi^{\varepsilon,n} = g(X_t^{\varepsilon}) - u^n(0,\hat{X}_t^{\varepsilon})$.

By the assumptions on f we have

$$\begin{aligned}
&\langle \tilde{Y}_r^{\varepsilon,n}, f(\overline{X}_r^{\varepsilon}, \tilde{Y}_r^{\varepsilon,n} + u^n(t-r,\hat{X}_r^{\varepsilon}), \tilde{Z}_r^{\varepsilon,n} + \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon}) \\
&-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\rangle \\
&\leqslant K|\tilde{Y}_r^{\varepsilon,n}|^2 + K|\tilde{Y}_r^{\varepsilon,n}| \times ||\tilde{Z}_r^{\varepsilon,n}|| \\
&+\langle \tilde{Y}_r^{\varepsilon,n}, [f(\overline{X}_r^{\varepsilon}, u^n(t-r,\hat{X}_r^{\varepsilon}), \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon})) \\
&-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\rangle.
\end{aligned}$$

Then (see the proof of Proposition 5.1) there exists α (which depends only on the constant K) such that

$$\begin{aligned}
&|\tilde{Y}_0^{\varepsilon,n}|^2 \leqslant e^{\alpha t}\mathbb{E}[|\xi^{\varepsilon,n}|^2] \\
&+2\mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, f(\overline{X}_r^{\varepsilon}, u^n(t-r,\hat{X}_r^{\varepsilon}), \nabla u^n(r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon})) \\
&-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\rangle dr \\
&+2\mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, L^n u^n(t-r,\hat{X}_r^{\varepsilon}) - \hat{L}_{\varepsilon,n}(r)\rangle dr.
\end{aligned} \tag{5.11}$$

Recall that

$$\begin{aligned}
\tilde{Y}_0^{\varepsilon,n} &= Y_0^{\varepsilon} - u^n(t,x) \\
&= u^{\varepsilon}(t,x) - u^n(t,x),
\end{aligned}$$

and we have that $u^n(t,x) \to u(t,x)$, since u is continuous from Proposition 5.4. Then the desired result will follow from the

THEOREM 5.5. — *For all $\delta > 0$, there exists $n(\delta)$ such that for all $n \geqslant n(\delta)$,*

$$\limsup_{\varepsilon\to 0} |\tilde{Y}_0^{\varepsilon,n}| \leqslant \delta.$$

Proof. — All we have to do is to show that if $V_1^{\varepsilon,n}$, $V_2^{\varepsilon,n}$ and $V_3^{\varepsilon,n}$ denote the three terms in the right hand side of (5.11), then for $i = 1,2,3$ and for all $\delta > 0$, there exists $n(\delta)$ such that for all $n \geqslant n(\delta)$,

$$\limsup_{\varepsilon\to 0} |\tilde{V}_i^{\varepsilon,n}| \leqslant \delta. \tag{5.12}$$

Step 1 : Proof of (5.12) for $i = 1$. We note that for any $\beta > 0$,

$$\begin{aligned}
\xi^{\varepsilon,n} &= g(X_t^{\varepsilon}) - u^n(0,\hat{X}_t^{\varepsilon}), \\
\mathbb{P}(|\xi^{\varepsilon,n}| > \beta) &\leqslant \mathbb{P}(|g(X_t^{\varepsilon}) - g(\hat{X}_t^{\varepsilon}| > \beta/2) \\
&\quad +\mathbb{P}(|g(\hat{X}_t^{\varepsilon}) - u^n(0,\hat{X}_t^{\varepsilon})| > \beta/2, |\hat{X}_t^{\varepsilon}| \leqslant M) \\
&\quad +\mathbb{P}(|\hat{X}_t^{\varepsilon}| > M) \\
&= \mathbb{P}(|\hat{X}_t^{\varepsilon}| > M),
\end{aligned}$$

provided $\varepsilon \leqslant (2K\|\hat{b}\|_\infty)^{-1}\beta$ if K is the sup of $|\nabla g|$, and $n \geqslant n(\beta, M)$, since $u^n(0,\cdot)$ converges locally uniformly to g, as $n \to \infty$. Now $\rho_M := \sup_{0<\varepsilon\leqslant 1}\mathbb{P}(|\hat{X}_t^{\varepsilon}| > M)$ tends to 0 as $M \to \infty$. Since moreover $|\xi^{\varepsilon,n}| \leqslant K'$ a. s., for all $\varepsilon > 0$, $n \in \mathbb{N}$, some K',

$$\mathbb{E}\left[|\xi^{\varepsilon,n}|^2\right] \leqslant \beta^2(1-\rho_M) + K'^2\rho_M,$$

provided $\varepsilon \leqslant (2K\|\hat{b}\|_\infty)^{-1}\beta$ and $n \geqslant n(\beta, M)$. Step 1 follows.

Step 2 : Proof of (5.12) for $i = 2$. We have

$$\mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, f(\overline{X}_r^{\varepsilon}, u^n(t-r,\hat{X}_r^{\varepsilon}), \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon}))$$
$$-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\rangle dr$$
$$= \mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, f(\overline{X}_r^{\varepsilon}, u^n(t-r,\hat{X}_r^{\varepsilon}), \nabla u^n(t-r,\hat{X}_r^{\varepsilon})[(I+\nabla\hat{b})\sigma](\overline{X}_r^{\varepsilon}))$$
$$-\overline{f}(u^n(t-r,X_r),\nabla u^n(t-r,X_r))\rangle dr$$
$$+\mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, \overline{f}(u^n(t-r,X_r),\nabla u^n(t-r,X_r))$$
$$-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\rangle dr.$$

Since the sequence $\{\tilde{Y}^{\varepsilon,n}\}_n$ is S–tight, it follows from the same argument as that in Lemma 4.2 of Pardoux [13] and bounded convergence that the first term in the right hand side tends to zero, as $\varepsilon \to 0$, for each fixed n. The second term is bounded by a constant times

$$\mathbb{E}\int_0^t \Big|\overline{f}(u^n(t-r,X_r),\nabla u^n(t-r,X_r))$$
$$-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(\hat{X}_r^{\varepsilon}-y)dvdy\Big| dr,$$

which tends to

$$\mathbb{E}\int_0^t \Big|\overline{f}(u^n(t-r,X_r),\nabla u^n(t-r,X_r))$$
$$-\int\int \overline{f}(u(v,y),\nabla u(v,y))\rho_n(t-r-v)\varphi_n(X_r-y)dvdy\Big| dr,$$

as $\varepsilon \to 0$, and the latter tends to zero as $n \to \infty$, from Propositions 5.3 and 5.4.

Step 3 : Proof of (5.12) for $i = 3$ Since the sequence $\{\tilde{Y}^{\varepsilon,n}\}_n$ is S–tight, it follows again from the same argument as that in Lemma 4.2 of Pardoux [13] and bounded convergence that

$$\mathbb{E}\int_0^t e^{\alpha r}\langle \tilde{Y}_r^{\varepsilon,n}, L^n u^n(t-r,\hat{X}_r^{\varepsilon}) - \hat{L}_{\varepsilon,n}(r)\rangle dr \ \to\ 0$$

as ε tends to zero, for each fixed n.

Remark 5.6. — One would like to combine the difficulties of the two last sections, i. e. to homogenize a PDE with a nonlinear term of the form

$$\frac{1}{\varepsilon}e\left(\frac{x}{\varepsilon}, u^{\varepsilon}(t,x)\right) + f\left(\frac{x}{\varepsilon}, u^{\varepsilon}(t,x), \nabla u^{\varepsilon}(t,x)\sigma\left(\frac{x}{\varepsilon}\right)\right),$$

like in Delarue [5]. However, this would produce a term of the form

$$C(u(t,x)) \cdot \nabla u(t,x)$$

in the limiting equation, which must be controlled by a term of the form $\sqrt{A}\nabla u(t,x)$. We hope to treat this problem in subsequent paper, together with another type of degeneracy of the matrix $a(x)$.

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Bibliography

[1] De Arcangelis (R.), Serra Cassano (F.). — On the homogenization of degenerate elliptic equations in divergence form, J. Math. Pures Appl. 71, p. 119-138, (1992).

[2] Bellieud (M.), Bouchitté (G.). — Homogénéisation de problèmes elliptiques dégénérés, CRAS Sér. I Math. 327, p. 787-792, (1998).

[3] Biroli (M.), Mosco (U.), Tchou (N.). — Homogenization for degenerate operators with periodical coefficients with respect to the Heisenberg group, CRAS Sér. I Math. 322, p. 439-44, (1996).

[4] Brézis (H.). — Analyse fonctionnelle, Théorie et applications, Dunod, Paris, (1999).

[5] Delarue (F.). — Auxiliary SDEs for homogenization of quasilinear PDEs with periodic coefficients, Ann. Probab. 32, p. 2305-2361, (2004).

[6] Engström (J.), Persson (L.-E.), Piatnitski (A.), Wall (P.). — Homogenization of random degenerated nonlinear monotone operators, Preprint.

[7] Friedman (A.). — Foundations of Modern Analysis, Dover Publications, Inc. (1970).

[8] Jakubowski (A.). — A non–Skorohod topology on the Skorohod space, Elec. J. of Probability, 2, (1997).

[9] Jurdjevic (V.). — Geometric Control Theory, Cambridge Univ. Press, (1977).

[10] Meyer (P. A.), Zheng (W. A.). — Tightness criteria for laws of semi–martingales, Ann. Instit. Henri Poincaré 20, p. 353-372, (1984).

[11] Michel (D.), Pardoux (É.). — An Introduction to Malliavin calculus and some of its applications, in Recent advances in Stochastic Calculus, J.S. Baras, V. Mirelli eds., p. 65-104, Springer (1990).

[12] Pardoux (É.), Veretennikov (A. Yu.). — On Poisson equation and diffusion approximation 1, Ann. Probab. 29, p. 1061-1085, (2001).

[13] Pardoux (É.). — Homogenization of Linear and Semilinear Second Order Parabolic PDEs with Periodic Coefficients: A Probabilistic Approch, Journal of Functional Analysis 167, p. 498-520, (1999).

[14] Pardoux (É.). — BSDEs, weak convergence and homogenization of semilinear PDEs, in “ Nonlinear Analysis, Differential Equations and Control” (F. H. Clarke and R. J. Stern, EDs.), p. 503-549, Kluwer Acad. Pub., (1999).

[15] PARONETTO (F.). — Homogenization of degenerate elliptic–parabolic equations, Asymptot. Anal. 37, 21–56, 2004.

[16] STROOK (D.), and VARADHAN (S.R.S.). — On the support of diffusion processes with applications to the maximum principle. Proceedings of sith Berkeley Symposium on Mathematical Statistics and Probability, Vol. III, p. 333-360.

[17] TREVES (F.). — Introduction to Pseudodifferential and Fourier Integral Operators, Vol. 1, The Univ. series in Mathematics, Plenum Press, NY, (1980).