Potential Analysis **9:** 27–64, 1998.

Malliavin Calculus for White Noise Driven Parabolic SPDEs

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(Received: 21 December 1995; accepted in final form: 2 September 1996)

Abstract. We consider the parabolic SPDE

$$\partial_t X(t,x) = \partial_x^2 X(t,x) + \psi(X(t,x)) + \varphi(X(t,x))\dot{W}(t,x), (t,x) \in R_+ \times [0,1],$$

with the Neuman boundary condition

$$\frac{\partial X}{\partial x}(t,0) = \frac{\partial X}{\partial x}(t,1) = 1$$

and some initial condition.

We use the Malliavin calculus in order to prove that, if the coefficients φ and ψ are smooth and $\varphi > 0$, then the law of any vector $(X(t,x_1),\dots,X(t,x_d)), 0 \leqslant x_1 \leqslant \cdots \leqslant x_d \leqslant 1$, has a smooth, strictly positive density with respect to Lebesgue measure.

Mathematics Subject Classifications (1991): 60H15, 35R60, 60H07.

Key words: Stochastic partial differential equations, space time white noise, Malliavin calculus.

1. Introduction

Consider the following stochastic partial differential equation

$$\frac{\partial X}{\partial t}(t,x) = \frac{\partial^2 X}{\partial x^2}(t,x) + \psi(X(t,x)) + \varphi(X(t,x))\dot{W}(t,x), \qquad \begin{matrix} 0 \leqslant x \leqslant 1 \\ 0 \leqslant t, \end{matrix} \tag{1.1}$$

with boundary conditions

$$X(0,x) = X_0(x); \qquad \frac{\partial X}{\partial x}(t,0) = \frac{\partial X}{\partial x}(t,1) = 0.$$

This equation can be rigorously formulated as an integral equation

$$X(t,x) = \int_0^1 G_t(x,y)X_0(y)\,\mathrm{d}y + \int_0^t\int_0^1 G_{t-s}(x,y)\psi(X(s,y))\,\mathrm{d}y\,\mathrm{d}s + \int_0^t\int_0^1 G_{t-s}(x,y)\varphi(X(s,y))W(\mathrm{d}y,\mathrm{d}s), \tag{1.2}$$

where G is the fundamental solution of the heat equation on $\mathbb{R}_+ \times (0,1)$ with Neumann boundary conditions, $\dot{W}$ is the white noise on $\mathbb{R}_+ \times [0,1]$ and φ and ψ are smooth and bounded functions.

The aim of this paper is to prove that for any $t > 0, d \in \mathbb{N}$ and any $0 \leqslant x_1 < \cdots < x_d \leqslant 1$, the law of $(X(t,x_1),\ldots,X(t,x_d)) \in \mathbb{R}^d$ has a smooth and strictly positive density with respect to Lebesgue measure on the set $\{\varphi \neq 0\}^d$. This may be seen as a regularity result for the marginal distributions of the $C([0,1])$-valued random variable $X(t,\cdot)$.

In order to prove this result we use the Malliavin calculus associated to the white noise $\dot{W}$. We refer the reader to Ikeda and Watanabe [5] for a general presentation of Malliavin's calculus and Nualart and Zakai [11] for the special case of a space-time white noise. With similar aims and in the frame of two-parameter stochastic processes also, the same machinery has already been used by Carmona and Nualart [3] and Nualart and Sanz [10]. Anyway the situation is rather different here. Unusually, there is a difficulty in establishing the differentiability of $X(t,y)$ as a Wiener functional and the L^p evaluations for the associated Sobolev semi-norms. This difficulty comes from the singularity of $s \to G_{t-s}(x,y)$ as $s \uparrow t$, and the equations satisfied by the Malliavin derivatives contain this singularity in their initial condition. On the other hand, the relative ease with which we obtain the evaluations of the covariance matrix is due to our strong local nondegeneracy assumption on the diffusion coefficient.

Our proof of the strict positivity is inspired by an analogous result for SDEs due to Ben Arous and Léandre [2], see also Millet and Sanz-Solé [7] for the case of hyperbolic SPDEs, Aida et al. [1] and Nualart [8] for the case of random variables defined on an abstract Wiener space. However, the abstract results do not apply directly here, essentially because our SPDE cannot be written in Stratonovich form: this is due to the infinite trace of the covariance operator of white noise. On the other hand, our situation is a sense simpler than those considered in the above references, since we are in a locally elliptic situation.

The existence of the density for the law of $X(t,x)$ has already been proved by Pardoux and Zhang Tusheng [12] under a weaker nondegeneracy assumption. However, we have not been able to prove our results under the same assumption.

The paper is organized as follows. In Section 1, we state our main result. In Section 2, we introduce the tools from the Malliavin Calculus, we give a local criterion for the existence of a smooth density and a condition for its strict positivity. In Section 3, we study the smoothness of the solution $X(t,x)$ of our SPDE. In

Section 4, we estimate the Malliavin covariance matrix. Finally, in Section 5, we prove the strict positivity of the density.

2. Statement of the Main Result

For the sake of making some of the notations below unambiguous, let us assume that our probability space $(\Omega, \mathcal{F}, P)$ is defined as follows

$$\Omega = \{\varphi \in C([0,T] \times [0,1]), \varphi(0,x) = \varphi(t,0) = 0\},$$

$\mathcal{F}$ is the Borel σ-field of Ω, and P is such that the canonical process

$$\{W_{t,x}(\omega) \triangleq \omega(t,x); (t,x) \in [0,T] \times [0,1]\}$$

is a Brownian sheet under P, i.e. it is a zero mean Gaussian continuous random field with correlation given by

$$E(W_{t,x}W_{s,y}) = (t \wedge s) \times (x \wedge y).$$

Let $\mathcal{F}_t = \sigma(W(A); A \in \mathcal{B}([0,t] \times [0,1])) \vee \mathcal{N}$ where $\mathcal{N}$ is the class of P-null sets in $\mathcal{F}$, and $\mathcal{P}$ denote the σ-algebra of $\mathcal{F}_t$-progressively measurable subsets of $\Omega \times [0,T]$.

Let $X = (X(t,x))$ be the solution of the parabolic SPDE

$$\begin{aligned}\frac{\partial X}{\partial t}(t,x) \;=\; & \frac{\partial^2 X}{\partial x^2}(t,x) + \psi(X(t,x)) \\ & + \varphi(X(t,x))\dot{W}(t,x), (t,x) \in [0,T] \times [0,1], \qquad (2.1)\end{aligned}$$

with initial condition $X(0,x) = X_0(x), 0 \leqslant x \leqslant 1$, where $X_0 \in C([0,1])$, and boundary conditions

$$\frac{\partial X}{\partial x}(t,0) = \frac{\partial X}{\partial x}(t,1) = 0; \quad t \in [0,T].$$

This means that X satisfies the following: for any $f \in C^2([0,1])$ such that $f'(0) = f'(1) = 0$

$$\begin{aligned}\int_0^1 X(t,x)f(x)\,\mathrm{d}x \;=\; & \int_0^1 X_0(x)f(x)\,\mathrm{d}x \\ & + \int_0^t \int_0^1 \left[X(s,x)\frac{\partial^2 f}{\partial x^2}(x) + \psi(X(s,x))f(x)\right] \mathrm{d}x\,\mathrm{d}s \\ & + \int_0^t \int_0^1 \varphi(X(s,x))f(x)W(\mathrm{d}x,\mathrm{d}s), \qquad (2.2)\end{aligned}$$

or equivalently (see Walsh [13])

$$X(t,x) = \int_0^1 G_t(x,y)X_0(y)\,\mathrm{d}y + \int_0^t\int_0^1 G_{t-s}(x,y)\psi(X(s,y))\,\mathrm{d}y\,\mathrm{d}s + \int_0^t\int_0^1 G_{t-s}(x,y)\varphi(X(s,y))W(\mathrm{d}y,\mathrm{d}s), \tag{2.3}$$

where

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left(-\frac{(y-x-2n)^2}{4t}\right) + \exp\left(-\frac{(y+x-2n)^2}{4t}\right)\right\} \tag{2.4}$$

is the fundamental solution of the heat equation on $\mathbb{R}_+ \times (0,1)$ with Neumann boundary conditions.

REMARK 2.1. One could consider Dirichlet boundary conditions (i.e. $X(t,0) = X(t,1) = 0$, for $t \in [0,T]$) instead of the Neumann boundary conditions (i.e. $(\partial/\partial x)X(t,0) = (\partial/\partial x)X(t,1) = 0$). In this case the Green function is

$$G_t(x,y) = \frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{\infty}\left\{\exp\left(-\frac{(y-x-2n)^2}{4t}\right) - \exp\left(-\frac{(y+x-2n)^2}{4t}\right)\right\}. \tag{2.5}$$

All the results in the paper hold in this case also, with minor modifications. Actually we do not use the explicit form of G but only the properties (A.1), (A.3), (A.4) from the Appendix and the Hölder continuity of $(t,x) \to X(t,x)$ (see Walsh [13]) which hold for G in (1.5) as well.

We assume that the coefficients fulfill the hypothesis

$$\varphi,\psi\colon \mathbb{R} \to \mathbb{R} \quad \text{are infinitely differentiable functions, which are bounded together with their derivatives of all order.} \tag{2.6}$$

Our result is the following

THEOREM 2.2. *For every* $0 \leqslant x_1 < x_2 < \cdots < x_d \leqslant 1, t \in (0,T]$, *the law of* $(X(t,x_1),\ldots,X(t,x_d))$ *admits a strictly positive smooth density* p *on* $\{\varphi \neq 0\}^d$, *that is there exists* $p \in C^\infty(\{\varphi \neq 0\}^d, \mathbb{R})$ *such that*

(i) *For every* $f \in C_b(\mathbb{R}^d, \mathbb{R})$ *with* supp $f \subseteq \{\varphi \neq 0\}^d$

$$E(f(X(t, x_1), \ldots, X(t, x_d)) = \int_{\mathbb{R}^d} f(y)p(y)\,\mathrm{d}y. \tag{2.7}$$

(ii) $p(y) > 0, \quad \forall y \in \{\varphi \neq 0\}^d$.

Note that in the case of Dirichlet boundary conditions Theorem 2.2 would be true only if $0 < x_1 < \cdots < x_d < 1$.

REMARK 2.3. The densities $p_m, m \in \mathbb{N}$, which we shall use as approximations of p (see the proof of Theorem 3.1) are not equally bounded, so we can say nothing about the boundedness of p. Nevertheless, under the strong ellipticity assumption $|\varphi| \geqslant c > 0$, it is clear from our proof that p and all its derivatives are bounded.

The proof of Theorem 2.2 goes through several steps contained in Sections 3, 4, 5 and 6. Note that (A.1) , . . . , (A.7) refer to inequalities which are stated and proved in the Appendix, at the end of the paper.

3. The Malliavin Calculus

Let us recall the objects involved in the Malliavin calculus associated to W. We denote by $\mathcal{S}$ the space of simple (or smooth) functionals, that is functionals of the form

$$F = f(W(h_1), \ldots, W(h_m)),$$

with $f \in C^\infty(\mathbb{R}^m)$ with at most polynomial growth at infinity, and $h_1, \ldots, h_m$ is an orthonormal sequence in $L^2(\Lambda_T; \mathrm{d}t\,\mathrm{d}x)$, where $\Lambda_t =: [0, t] \times [0, 1]$, and for $h \in L^2(\Lambda_T)$

$$W(h) =: \int_0^T \int_0^1 h(s, y) W(\mathrm{d}y, \mathrm{d}s).$$

For $F \in \mathcal{S}$ one defines the first order Malliavin derivative to be the $L^2(\Lambda_T)$-valued random variable

$$D^1_{(t,x)}F = \sum_{i=1}^m \partial_i f(W(h_1), \ldots, W(h_m))h_i(t, x),$$

where $\partial_i = \partial/\partial x^i$.

The derivative of order k of F is the $L^2(\Lambda_T^k)$-valued random variable given by

$$D^k_\alpha F = \sum_{i_1,\ldots,i_k=1}^m \partial_{i_1} \ldots \partial_{i_k} f(W(h_1), \ldots, W(h_m))h_{i_1}(\alpha_1) \ldots h_{i_k}(\alpha_k),$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_k); \qquad \alpha_i = (r_i, z_i) \in \Lambda_T; \quad 1 \leqslant i \leqslant k.$$

For $p \geqslant 1$ and $k \in \mathbb{N}$ the space $\mathcal{D}_{p,k}$ is the completion of S with respect to the seminorm

$$\|F\|_{p,k} = (E|F|^p)^{1/p} + \sum_{i=1}^{k} (E|D^i F|_2^p)^{1/p},$$

with

$$|D^i F|_2^2 = \int_{\Lambda_T^i} |D_\alpha^i F|^2 \,\mathrm{d}\alpha \quad (\mathrm{d}\alpha \text{ denotes Lebesgue measure}).$$

We also define

$$\mathcal{D}_\infty = \bigcap_{p \geqslant 1} \bigcap_{k \in \mathbb{N}} \mathcal{D}_{p,k}.$$

On the other hand one defines on S the Ornstein–Uhlenbeck operator

$$\begin{aligned} LF \ = \ & \sum_{i=1}^{m} [\partial_i \, \partial_i f(W(h_1), \ldots, W(h_m)) \\ & - \partial_i f(W(h_1), \ldots, W(h_m)) W(h_i)]. \end{aligned}$$

This operator is closable and $\mathcal{D}_\infty \subseteq \mathrm{Dom}(L) \subseteq L^2(\Omega, \mathcal{F}, P)$ (see Ikeda and Watanabe [5]). The covariance matrix associated to F is the matrix $\sigma = (\sigma^{ij})_{1 \leqslant i,j \leqslant d}$ defined by

$$\sigma^{ij} = \langle D^1 F^i, D^1 F^j \rangle; \quad 1 \leqslant i, j \leqslant d.$$

We shall use the following 'localized' variant of Malliavin's absolute continuity theorem,

THEOREM 3.1. *Let* $\Gamma_m \subseteq \mathbb{R}^d, m \in \mathbb{N}$, *be a sequence of open sets such that* $\bar{\Gamma}_m \subset \Gamma_{m+1}$ *and let* $F\colon \Omega \to \mathbb{R}^d$ *be a measurable functional. If* $F \in (\mathcal{D}_\infty)^d$ *and*

$$E((\det \sigma)^{-q}, F \in \Gamma_m) < \infty; \quad \forall q \geqslant 1, \quad m \in N, \tag{3.1}$$

then $P \circ F^{-1}$ *has a smooth density on* $\Gamma = \bigcup_m \Gamma_m$, *i.e., there exists* $p \in C^\infty(\Gamma)$ *such that*

$$E(f(F)) = \int_{\mathbb{R}^d} f(x) p(x) \,\mathrm{d}x, \tag{3.2}$$

for any $f: \mathbb{R}^d \to \mathbb{R}$ *which is bounded, measurable and such that* supp $f \subseteq \Gamma$.

Proof. Let $\{f_m, m \in \mathbb{N}\}$ be a sequence in $C^\infty(\mathbb{R}^d)$ such that $0 \leqslant f_m \leqslant 1$, $f_m|_{\Gamma_m} = 1$ and $f_m|_{\Gamma^c_{m+1}} = 0$.

Define $u^j = DF^j, G^{i,j} = f_m(F)\langle DF^i, DF^j\rangle$ and $A = \{F \in \Gamma_m\}$. With these notations, the result follows from Theorem 2.1.4 in Nualart [9]. □

We now give a criterion for the strict positivity of the density at a given point $y_0 \in \mathbb{R}^d$.

We first state a technical lemma, which can be found e.g. in Aida et al. [1] and Nualart [8].

LEMMA 3.2. *For any* $\beta > 0, \delta > 0$, *there exist constants* $c, \rho > 0$ *such that any mapping*

$$g: \mathbb{R}^d \to \mathbb{R}^d,$$

satisfying

$$(j) \quad |\det g'(0)| \geqslant \frac{1}{\beta},$$

$$(jj) \quad \sup_{|x|\leqslant\delta} (|g(x)| + |g'(x)| + |g''(x)|) \leqslant \beta$$

is a diffeomorphism from a neighborhood of 0 *contained in the ball* $B(0, c)$ *onto the ball* $B(g(0), \rho)$.

To each $z = (z^1, \ldots, z^d) \in \mathbb{R}^d$, and $h = (h^1, \ldots, h^d) \in H^d = L^2((0, T) \times (0, 1))^d$, we associate a transformation T_z on Ω, defined by

$$[T_z(\omega)](t, x) = \omega(t, x) + \sum_{i=1}^{d} z_i \int_0^t \int_0^x h^i(s, y)\, \mathrm{d}y\, \mathrm{d}s.$$

In the next statement, we shall consider a sequence $\{h_n\}_{n=1,2,\ldots} \subset H^d$, and we define

$$[T^n_z(\omega)](t, x) = \omega(t, x) + \sum_{i=1}^{d} z_i \int_0^t \int_0^x h^i_n(s, y)\, \mathrm{d}y\, \mathrm{d}s. \tag{3.3}$$

THEOREM 3.3. *Let* F *be a* d*-dimensional random vector, such that for some* $p \in C(\mathbb{R}^d)$ *and* Γ *an open subset of* $\mathbb{R}^d$

$$1_\Gamma(y) P \circ F^{-1}(\mathrm{d}y) = 1_\Gamma(y) p(y)\, \mathrm{d}y.$$

We assume that there exists a sequence $\{h_n\}_{n=1,2,\ldots,} \subset H^d$ *such that the associated sequence of random fields defined by*

$$\phi_n(z) := F \circ T^n_z$$

satisfies the two following conditions, for some $y_0 \in \Gamma, c, \delta, k > 0$:

(i) $\limsup_{n\to\infty} P(\{|F-y_0| \leqslant r\} \cap \{\det \partial_z \phi_n(0) \geqslant c\}) > 0, \quad \forall r > 0.$
(ii) $\lim_{n\to\infty} P\left(\sup_{|z|\leqslant\delta}[\|\partial_z \phi_n(z)\| + \|\partial_z^2 \phi_n(z)\|] \leqslant k\right) = 1.$

Then $p(y_0) > 0$.
Proof. We define

$$\Lambda_n = \{|F-y_0| \leqslant r\} \bigcap \{\det \partial_z \phi_n(0) \geqslant c\} \bigcap \left\{ \sup_{|z|\leqslant\delta} (\|\partial_z \phi_n(z)\| + \|\partial_z^2 \phi_n(z)\|) \leqslant k \right\}.$$

It follows from (i), (ii) that for each $r > 0$, there exists $n \in \mathbb{N}$ such that

(iii) $P(\Lambda_n) > 0$.

From now on, r and n will be fixed, such that (iii) holds. Note that on Λ_n,

$$\sup_{|z|\leqslant\delta} |\phi_n(z)| \leqslant k' = |y_0| + r + \delta k.$$

It then follows from Lemma 3.2 that there exists $\alpha > 0$ such that for all $\omega \in \Lambda_n$, the mapping

$$z \to \phi_n(z, \omega)$$

is a diffeomorphism between an open neighborhood $V_n(\omega)$ of 0 in $\mathbb{R}^d$, contained in some ball $B(0, R)$, and the ball $B(F, \alpha)$. We can and do assume that $r < \alpha$, since α depends only on c, δ, k, and r can be chosen arbitrarily small, and that R is chosen small enough such that $\omega \in \Lambda_n$ and $z \in V_n(\omega)$ imply that

$$\det \partial_z \phi_n(z, \omega) \geqslant \frac{c}{2}.$$

From Girsanov's Theorem, for each $n \in \mathbb{N}, z \in \mathbb{R}^d$

$$(P \circ F^{-1})(\mathrm{d}z) = e_n(z)(P \circ \phi_n^{-1})(\mathrm{d}z),$$

where

$$e_n(z) = \exp\left[\langle z, W(h_n)\rangle - \frac{1}{2}\sum_1^d z_i^2 \|h_n^i\|^2_{L^2((0,T)\times(0,1))}\right].$$

Let $\psi(z) = (2\pi)^{-(d/2)} \exp(-|z|^2/2)$, and $f \in C_b^\infty(\mathbb{R}^d; \mathbb{R}_+)$.

We have

$$\begin{aligned}E[f(F)] &= \int_{\mathbb{R}^d} \psi(z)E[e_n(z)f(\phi_n(z))]\,\mathrm{d}z\\ &\geqslant E\left(\int_{\mathbb{R}^d} \psi(z)e_n(z)f(\phi_n(z))\,\mathrm{d}z;\Lambda_n\right)\\ &\geqslant E\left(\int_{V_n} \psi(z)e_n(z)f(\phi_n(z))\,\mathrm{d}z;\Lambda_n\right)\\ &= E\left(\int_{B(F,\alpha)} f(y)\left(\frac{\psi e_n}{|\det\partial_z\phi_n|}\right)(\phi_n^{-1}(y))\,\mathrm{d}y;\Lambda_n\right)\\ &\geqslant \int_{\mathbb{R}^d} f(y)\theta_n(y)\,\mathrm{d}y,\end{aligned}$$

where

$$\theta_n(y) = E\left(\varphi(|F-y|)\psi\circ\left(\frac{e_n\psi}{|\det\partial_z\phi_n|}\right)(\phi_n^{-1}(y));\Lambda_n\right),$$

$\varphi\colon\mathbb{R}_+\to[0,1]$ is continuous, $1_{[0,r]}\leqslant\varphi\leqslant 1_{[0,\alpha]}$, and $\psi(r)=\inf(r,1)$. $\theta_n(y_0)>0$ follows easily from (iii) and the fact that $\{|F-y_0|\leqslant r\}\subset\Lambda_n$.

But

$$y\to\varphi(|F-y|)\psi\circ\left(\frac{\psi}{|\det\partial_z\phi_n|}\right)(\phi_n^{-1}(y))$$

is a.s. continuous, and bounded by 1.

Hence form Lebesgue's dominated convergence theorem, θ_n is continuous.

Finally, if supp $f\subset\Gamma$,

$$E(F) = \int_{\mathbb{R}^d} f(y)p(y)\,\mathrm{d}y \geqslant \int_{\mathbb{R}^d} f(y)\theta_n(y)\,\mathrm{d}y.$$

The theorem is proved. □

4. Differentiability of the Solution

The aim of this section is to prove that $X(t,x)\in\mathcal{D}_\infty$ for each $(t,x)\in[0,T]\times[0,1]$. To this end we shall construct a sequence of simple functionals in the following way. Let $\{\mathrm{e}_k\}_{k\in\mathbb{N}}$, be an orthonormal basis of $L^2(0,T)$ and

$$W^k(t) = \int_0^t\int_0^1 \mathrm{e}_k(y)W(\mathrm{d}y,\mathrm{d}s),\quad k\in\mathbb{N}.$$

Then, for every $\mathcal{P} \otimes \mathcal{B}([0,1])$ measurable, square integrable process $f\colon \Omega \times [0,T] \times [0,1] \to \mathbb{R}$, one has

$$\int_0^t \int_0^1 f(\omega,s,y) W(\mathrm{d}y,\mathrm{d}s) = \sum_{k=0}^{\infty} \int_0^t \left(\int_0^1 f(\omega,s,y)\, \mathrm{e}_k(y)\,\mathrm{d}y \right) \mathrm{d}W^k(s). \tag{4.1}$$

This is easily seen for $f(s,y,\omega) = g(\omega)1_{(t,t']}(s)1_A(y)$ where $0 \leqslant t < t' \leqslant T, A \in \mathcal{B}([0,1])$ and g is a $\mathcal{F}_t$ measurable random variable. Then one takes linear combinations and L^2 limits.

The Equation (2.3) may be written in the form

$$\begin{aligned} X(t,x) &= \int_0^1 G_t(x,y) X_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G_{t-s}(x,y)\psi(X(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \sum_{k=0}^{\infty} \int_0^t \left(\int_0^1 G_{t-s}(x,y)\varphi(X(s,y))\,\mathrm{e}_k(y)\,\mathrm{d}y \right) \mathrm{d}W^k(s). \end{aligned} \tag{4.2}$$

The approximations are constructed in the following way: for $n \in N$ and $t \in [i/n,(i+1)/n)$ we denote

$$t_n^+ = \frac{i+1}{n} \quad \text{and} \quad t_n^- = \frac{i}{n}.$$

We also set

$$\begin{aligned} \Delta_{i,k}^n &= W^k\left(\frac{i+1}{n}\right) - W^k\left(\frac{i}{n}\right) \\ &= \int_0^T \int_0^1 \mathrm{e}_k(y) 1_{[i/n,i+1/n)}(s) W(\mathrm{d}y,\mathrm{d}s) \end{aligned}$$

and define

$$X_n(0,x) = X_0(x), \quad x \in [0,1]$$

$$\begin{aligned} X_n\left(\frac{l}{n},x\right) &= \int_0^1 G_{l/n}(x,y) X_0(y)\,\mathrm{d}y \\ &\quad + \sum_{i=0}^{l-1} 1/n \int_0^1 G_{(l+1/n)-(i/n)}(x,y)\psi\left(X_n\left(\frac{i}{n},y\right)\right)\,\mathrm{d}y \\ &\quad + \sum_{k=0}^{n}\sum_{i=0}^{l-1} \int_0^1 G_{(l+1/n)-(i/n)}(x,y)\varphi\left(X_n\left(\frac{i}{n},y\right)\right)\,\mathrm{e}_k(y)\,\mathrm{d}y \cdot \Delta_{i,k}^n \end{aligned} \tag{4.3}$$

and

$$X_n(t,x) = X_n(t_n^-,x); \quad 0 \leqslant t \leqslant T, \quad x \in [0,1].$$

It is then clear that $X_n(t,x)$ solves the equation

$$\begin{aligned} X_n(t,x) \ = \ & \int_0^1 G_{t_n^-}(x,y) X_0(y)\,\mathrm{d}y \\ & + \int_0^{t_n^-}\int_0^1 G_{t_n^+ - s_n^-}(x,y)\psi(X_n(s_n^-,y))\,\mathrm{d}y\,\mathrm{d}s \\ & + \sum_{k=0}^{n}\int_0^{t_n^-}\int_0^1 G_{t_n^+ - s_n^-}(x,y)\varphi(X_n(s_n^-,y))\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s); \\ & 0 \leqslant t \leqslant T, \quad x \in [0,1]. \end{aligned} \tag{4.4}$$

It is also clear that $X_n(t,x) \in S$. Starting with (4.3) we may calculate the derivatives of $X_n(t,x)$ and further – by taking limits – those of $X(t,x)$. In order to state the equations satisfied by the derivatives we have to introduce some more notation.

For $\alpha = (\alpha_1,\ldots,\alpha_M) \in \Lambda_T^M$ we denote

$$|\alpha| = M \quad \text{(the length of } \alpha).$$

Let $\alpha_i = (r_i, z_i) \in [0,T]\times[0,1], 1 \leqslant i \leqslant M$, and let i_0 be such that $r_{i_0} > r_i$ for every $i \neq i_0$ (such an i_0 exists for every $\alpha \in \Lambda_T^M$ outside a Lebesgue null set). We denote

$$\begin{aligned} \bar\alpha &= (\bar r, \bar z) = (r_{i_0}, z_{i_0}) = \alpha_{i_0}, \\ \hat\alpha &= (\alpha_1,\ldots,\alpha_{i_0-1},\alpha_{i_0+1},\ldots,\alpha_M). \end{aligned} \tag{4.5}$$

Then we define

$$\Gamma_\alpha^{(n)}(\varphi)(t,x) = \sum_{m=1}^{M}\sum \varphi^{(m)}(X_n(t_n^-,x))\prod_{i=1}^{m} D_{p_i}^{|p_i|}X_n(t_n^-,x), \tag{4.6}$$

where the second sum $\sum$ is taken over all the partitions $p_1,\ldots,p_m$ of length m of α, and

$$\begin{aligned} \Delta_\alpha^{(n)}(\varphi)(t,x) \ &= \ \Gamma_\alpha^{(n)}(\varphi)(t,x) - \varphi'(X_n(t_n^-,x))D_\alpha^M X_n(t_n^-,x) \\ &= \ \sum_{m=2}^{M}\sum \varphi^{(m)}(X_n(t_n^-,x))\prod_{i=1}^{m} D_{p_i}^{|p_i|}X_n(t_n^-,x). \end{aligned} \tag{4.7}$$

Let us start with (4.3) and calculate explicitly the first order derivatives. One has

$$
\begin{aligned}
&D^1_{(r,z)}X_n\left(\frac{l}{n},x\right)\\
&\quad=\sum_{i=0}^{l-1}\sum_{k=0}^{n}\mathrm{e}_k(z)1_{[i/n,i+1/n)}(r)\int_0^1 G_{(l+1/n)-(i/n)}(x,y)\\
&\qquad\times\varphi\bigg(X_n\left(\frac{i}{n},y\right)\bigg)\ \mathrm{e}_k(y)\,\mathrm{d}y\\
&\qquad+\sum_{i=0}^{l-1}1_{[i/n,i+1/n)}(r)\sum_{k=0}^{n}\sum_{j=i+1}^{l-1}\int_0^1 G_{(l+1/n)-(j/n)}(x,y)\\
&\qquad\times\varphi'\bigg(X_n\left(\frac{j}{n},y\right)\bigg)\ D^1_{(r,z)}X_n\left(\frac{j}{n},y\right)\ \mathrm{e}_k(y)\,\mathrm{d}y\Delta^n_{j,k}\\
&\qquad+\sum_{i=0}^{l-1}1_{[i/n,i+1/n)}(r)\sum_{j=i+1}^{l-1}1/n\int_0^1 G_{(l+1/n)-(j/n)}(x,y)\\
&\qquad\times\varphi'\bigg(X_n\left(\frac{j}{n},y\right)\bigg)\ D^1_{(r,z)}X_n\left(\frac{j}{n},y\right)\ \mathrm{d}y,
\end{aligned}
$$

with the convention that $\sum_{j=l}^{l-1}=0$.

The above formula shows that

$$
D^1_{(r,z)}X_n\left(\frac{l}{n},z\right)=0\quad\text{if}\quad r_n^+>l/n.
$$

On the other hand, since

$$
X_n(t,x)=X_n\left(\frac{l}{n},x\right)\quad\text{for}\quad\frac{l}{n}\leqslant t<\frac{l+1}{n},
$$

one may rewrite the above equation in the form

$$
\begin{aligned}
&D^1_{(r,z)}X_n(t,x)\\
&\quad=\sum_{k=0}^{n}\mathrm{e}_k(z)\int_0^1 G_{t_n^+-r_n^-}(x,y)\varphi(X_n(r_n^-,y))\,\mathrm{e}_k(y)\,\mathrm{d}y\\
&\qquad+\sum_{k=0}^{n}\int_{r_n^+}^{t_n^-}\int_0^1 G_{t_n^+-s_n^-}(x,y)\varphi'(X_n(s_n^-,y))\\
&\qquad\times D^1_{(r,z)}X_n(s_n^-,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s)
\end{aligned}
$$

$$+\int_{r_n^+}^{t_n^-}\int_0^1 G_{t_n^+-s_n^-}(x,y)\varphi'(X_n(s_n^-,y))$$
$$\times D^1_{(r,z)}X_n(s_n^-,y)\,\mathrm{d}y\,\mathrm{d}s \quad \text{for} \quad r_n^+\leqslant t_n^-. \tag{4.8}$$

One checks by induction that the derivatives of order M of $X_n(t,x)$ solve the equations

$$D^M_\alpha X_n(t,x)=0 \quad \text{if} \quad t_n^-<r_n^+$$

and

$$D^M_\alpha X_n(t,x)$$
$$=\sum_{k=0}^{n}\mathrm{e}_k(\bar z)\int_0^1 G_{t_n^+-\bar r_n^-}(x,y)\Gamma^{(n)}_{\hat\alpha}(\varphi)(\bar r_n^-,y)\,\mathrm{e}_k(y)\,\mathrm{d}y$$
$$+\sum_{k=0}^{n}\int_{\bar r_n^+}^{t_n^-}\int_0^1 G_{t_n^+-s_n^-}(x,y)\Gamma^{(n)}_{\alpha}(\varphi)(s_n^-,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s)$$
$$+\int_{\bar r_n^+}^{t_n^-}\int_0^1 G_{t_n^+-s_n^-}(x,y)\Gamma^{(n)}_{\alpha}(\psi)(s_n^-,y)\,\mathrm{d}y\,\mathrm{d}s, \quad \text{for } t_n^-\geqslant r_n^+, \tag{4.9}$$

where $\bar\alpha=(\bar r,\bar z)$ and $\hat\alpha$ are defined in (4.5).

We now write the equations satisfied by the derivatives of $X(t,x)$. Actually $D^1_{(r,z)}X(t,x)$ (resp. $D^M_\alpha X(t,x)$) is defined to be the solution of Equation (3.10) (resp. (3.11)) and their significance as 'Malliavin derivatives of $X(t,x)$' will be established after proving the convergence in Proposition 4.3 below.

$$D^1_{(r,z)}X(t,x)$$
$$=\sum_{k=0}^{\infty}\mathrm{e}_k(z)\int_0^1 G_{t-r}(x,y)\varphi(X(r,y))\,\mathrm{e}_k(y)\,\mathrm{d}y$$
$$+\sum_{k=0}^{\infty}\int_r^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y))D^1_{(r,z)}X(s,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s)$$
$$+\int_r^t\int_0^1 G_{t-s}(x,y)\psi'(X(s,y))D^1_{(r,z)}X(s,y)\,\mathrm{d}y\,\mathrm{d}s, \quad \text{for} \quad r\leqslant t$$
$$=0 \quad \text{for} \quad r\geqslant t \tag{4.10}$$

and, for α such that $|\alpha|=M$

$$\begin{aligned} & D_\alpha^M X(t,x) \\ & \quad = \sum_{k=0}^{\infty} \mathrm{e}_k(\bar z) \int_0^1 G_{t-\bar r}(x,y)\Gamma_{\hat\alpha}(\varphi)(\bar r,y)\,\mathrm{e}_k(y)\,\mathrm{d}y \\ & \qquad + \sum_{k=0}^{\infty} \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\Gamma_\alpha(\varphi)(s,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s) \\ & \qquad + \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\Gamma_\alpha(\psi)(s,y)\,\mathrm{d}y\,\mathrm{d}s, \quad \text{for} \quad \bar r < t; \\ & \quad = 0 \quad \text{for} \quad \bar r \geqslant t, \end{aligned} \tag{4.11}$$

with

$$\Gamma_\alpha(\varphi)(s,y) = \sum_{m=1}^{M} \sum \varphi^{(m)}(X(s,y)) \prod_{i=1}^{m} D_{p_i}^{|p_i|} X(s,y), \tag{4.12}$$

where the above second sum is extended over all the partitions $p_1,\dots,p_m$ of α.

REMARK 4.1. In order to see that (4.11) is a true equation which defines recursively $D_\alpha^M X(t,x)$ one writes

$$\Gamma_\alpha(\varphi)(s,y) = \Delta_\alpha(\varphi)(s,y) + \varphi'(X(s,y)) D_\alpha^M X(s,y), \tag{4.13}$$

with

$$\Delta_\alpha(\varphi)(s,y) = \sum_{m=2}^{M} \sum \varphi^{(m)}(X(s,y)) \prod_{i=1}^{m} D_{p_i}^{|p_i|} X(s,y). \tag{4.14}$$

REMARK 4.2. Coming back to $W(\mathrm{d}y,\mathrm{d}s)$ (by means of (3.1)) one may write Equations (4.10) and (4.11) in the form

$$\begin{aligned} & D_{(r,z)}^1 X(t,x) \\ & \quad = G_{t-r}(x,z)\varphi(X(r,z)) \\ & \qquad + \int_r^t \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)) D_{(r,z)}^1 X(s,y) W(\mathrm{d}y,\mathrm{d}s) \\ & \qquad + \int_r^t \int_0^1 G_{t-s}(x,y)\psi'(X(s,y)) D_{(r,z)}^1 X(s,y)\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&D_\alpha^M X(t,x) \\
&\quad = G_{t-\bar{r}}(x,\bar{z})\Gamma_{\hat{\alpha}}(\varphi)(\bar{\alpha}) \\
&\qquad + \int_{\bar{r}}^t \int_0^1 G_{t-s}(x,y)\Delta_\alpha(\varphi)(s,y)W(\mathrm{d}y,\mathrm{d}s) \\
&\qquad + \int_{\bar{r}}^t \int_0^1 G_{t-s}(x,y)\Delta_\alpha(\psi)(s,y)\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad + \int_{\bar{r}}^t \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y))D_\alpha^M X(s,y)W(\mathrm{d}y,\mathrm{d}s) \\
&\qquad + \int_{\bar{r}}^t \int_0^1 G_{t-s}(x,y)\psi'(X(s,y))D_\alpha^M X(s,y)\,\mathrm{d}y\,\mathrm{d}s. \qquad (4.16)
\end{aligned}$$

PROPOSITION 4.3. *For every* $p > 1, M \in \mathbb{N}$, *and* $\alpha \in \Lambda_T^M$,

(i) $\lim_{n\to\infty} \sup_{(t,x)\in\Lambda_T} E|X(t,x) - X_n(t,x)|^{2p} = 0$.

(ii) $\lim_{n\to\infty} \sup_{(t,x)\in\Lambda_T} E\left|\int_{\Lambda_T^M} |D_\alpha^M X(t,x) - D_\alpha^M X_n(t,x)|^2\,\mathrm{d}\alpha\right|^p = 0$.

As a consequence $X(t,x) \in \mathcal{D}_\infty$ *and* $D_\alpha^M X(t,x), \alpha \in \Lambda_T^M$, *represents the Malliavin derivative of order* M *of* $X(t,x)$.

Proof. Let us first check that

$$\sup_{(t,x)\in\Lambda_T} E\left|\int_{\Lambda_t^M} |D_\alpha^M X(t,x)|^2\,\mathrm{d}\alpha\right|^p < \infty; \quad \forall p \geqslant 1, \quad M \in \mathbb{N}. \qquad (4.17)$$

We proceed by induction on M. Assume (3.17) holds for every $M' < M$ and $p \geqslant 1$.

We shall use Burkholder's inequality for Hilbert space valued martingales (see Metivier [6], E. 2. p. 212) in the following form.

If $H_{s,y}$ is a previsible $L^2(\Lambda_t^M)$-valued process, then

$$\begin{aligned}
&E\left|\int_{\Lambda_t^M} \left(\int_0^t \int_0^1 H_{s,y}(\alpha)W(\mathrm{d}y,\mathrm{d}s)\right)^2 \mathrm{d}\alpha\right|^p \\
&\quad \leqslant KE\left|\int_0^t \int_0^1 \left(\int_{\Lambda_t^M} H_{s,y}^2(\alpha)\,\mathrm{d}\alpha\right) \mathrm{d}y\,\mathrm{d}s\right|^p. \qquad (4.18)
\end{aligned}$$

The above inequality and the Equation (4.16) yield

$$\begin{aligned}
&E\left|\int_{\Lambda_t^M}|D_\alpha^M X(t,x)|^2\,\mathrm{d}\alpha\right|^p \\
&\quad\leqslant K\left\{E\left|\int_{\Lambda_t^M}(G_{t-\bar r}(x,\bar z)\Gamma_{\hat\alpha}(\varphi)(\bar\alpha))^2\,\mathrm{d}\alpha\right|^p\right. \\
&\qquad +E\left|\int_0^t\int_0^1 G_{t-s}^2(x,y)\int_{\Lambda_t^M}1_{[\bar r,\infty]}(s)\right. \\
&\qquad \left.\times\,[(\Delta_\alpha(\varphi)(s,y))^2+(\Delta_\alpha(\psi)(s,y))^2]\,\mathrm{d}\alpha\,\mathrm{d}y\,\mathrm{d}s\right|^p \\
&\qquad +E\left|\int_0^t\int_0^1 G_{t-s}^2(x,y)[(\varphi')^2(X(s,y))+(\psi')^2(X(s,y))]\right. \\
&\qquad \left.\left.\times\int_{\Lambda_t^M}(D_\alpha^M X(s,y))^2\,\mathrm{d}\alpha\,\mathrm{d}y\,\mathrm{d}s\right|^p\right\} \\
&\quad =: K(\mathrm{I}_1(t,x)+\mathrm{I}_2(t,x)+\mathrm{I}_3(t,x)).
\end{aligned}$$

Let us evaluate $\mathrm{I}_1(t,x)$. Since all the derivatives of φ are bounded, (3.12) yields

$$\mathrm{I}_1(t,x)\leqslant K\sum E\left|\int_{\Lambda_t^M}\left(G_{t-\bar r}(x,\bar z)\prod_{i=1}^m D_{p_i}^{\lambda_i}X(\bar r,\bar z)\right)^2\,\mathrm{d}\alpha\right|^p,$$

where the sum extends over all the partitions $p_1,\dots,p_m$ of $\hat\alpha$, and $\lambda_i=|p_i|$, where m varies from 1 to $M-1$.

Note that $(\bar r,\bar z),p_1,\dots,p_m$ is a partition of α. Then, the integral over Λ_t^M splits into integrals over Λ_t and $\Lambda_t^{\lambda_i}$, $1\leqslant i\leqslant m$, and so one dominates the above term by

$$K\sum E\left(\left|\int_{\Lambda_t}G_{t-r}^2(x,z)\left(\prod_{i=1}^m\int_{\Lambda_t^{\lambda_i}}|D_{p_i}^{\lambda_i}X(r,z)|^2\,\mathrm{d}p_i\right)\,\mathrm{d}z\,\mathrm{d}r\right|^p\right).$$

By (A.4) and Hölder's inequality, assuming $p>3$, so $q=p/(p-1)<\frac{3}{2}$,

$$\begin{aligned}
&\mathrm{I}_1(t,x)\\
&\quad\leqslant K\sum\int_{\Lambda_t}\prod_{i=1}^m\left(E\left|\int_{\Lambda_{t_i}^{\lambda_i}}|D_{p_i}^{\lambda_i}X(r,z)|^2\,\mathrm{d}p_i\right|^{2^{m-1}p}\right)^{1/2^{m-1}}\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned}$$

Since $\lambda_i<M, 1\leqslant i\leqslant m$, one applies the induction hypothesis to get

$$\sup_{(t,x)\in\Lambda_T}\mathrm{I}_1(t,x)<\infty.$$

By using (4.14) and the boundedness of the derivatives of φ and ψ one gets

$$\begin{aligned}
&\mathrm{I}_2(t,x)\\
&\quad\leqslant K\sum E\left|\int_0^t\int_0^1 G_{t-s}^2(x,y)\int_{\Lambda_t^M} 1_{[\bar r,\infty)}(s)\prod_{i=1}^m |D_{p_i}^{\lambda_i}X(s,y)|^2\,\mathrm{d}\alpha\,\mathrm{d}y\,\mathrm{d}s\right|^p\\
&\quad= K\sum E\left|\int_0^t\int_0^1 G_{t-s}^2(x,y)\int_{\Lambda_s^M}\prod_{i=1}^m |D_{p_i}^{\lambda_i}X(s,y)|^2\,\mathrm{d}\alpha\,\mathrm{d}y\,\mathrm{d}s\right|^p,
\end{aligned}$$

where the sum extends over all the partitions of α such that $\lambda_i = |p_i| < M, 1 \leqslant i \leqslant m$. Assume again $p > 3$. By Hölder's inequality

$$\begin{aligned}
&\mathrm{I}_2(t,x)\\
&\quad\leqslant K\sum\left(\int_0^t\int_0^1 G_{t-s}^{2q}(x,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{p/q}\\
&\qquad\times\int_0^t\int_0^1 E\left|\int_{\Lambda_s^M}\left|\prod_{i=1}^m D_{p_i}^{\lambda_i}X(s,y)\right|^2\mathrm{d}\alpha\right|^p\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

One uses (A.4) to dominate the first term in the above product. Next, the same reasoning as above (Schwarz's inequality and the induction hypothesis) permits to dominate the second term. So we have proved that

$$\sup_{(t,x)\in\Lambda_T}\mathrm{I}_2(t,x) < \infty.$$

By using Hölder's inequality as above one gets

$$\begin{aligned}
&\mathrm{I}_3(t,x)\\
&\quad\leqslant K\left(\int_0^t\int_0^1 G_{t-s}^{2q}(x,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{p/q}\\
&\qquad\times\int_0^t\int_0^1 E\left|\int_{\Lambda_s^M}|D_\alpha^M X(s,y)|^2\,\mathrm{d}\alpha\right|^p\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

So we have proved that

$$F(t,x) \leqslant K + K'\int_0^t\int_0^1 F(s,y)\,\mathrm{d}y\,\mathrm{d}s, \tag{4.19}$$

where K and K' are constants independent of $(x,t)\in\Lambda_T$ and

$$F(t,x) = E\left|\int_{\Lambda_t^M}|D_\alpha^M X(t,x)|^2\,\mathrm{d}\alpha\right|^p.$$

The inequality (4.19) tells nothing about the boundedness of F ($F \equiv \infty$ is a solution of (4.19)), so we have to use a truncation argument. Take $\varepsilon > 0$ and $\phi_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ a smooth function such that $\phi_\varepsilon(x) = x$ for $|x| \leqslant 1/\varepsilon$, $|\phi_\varepsilon(x)| \leqslant 1+1/\varepsilon$ and $|\phi'_\varepsilon(x)| \leqslant 1$ for every $x \in \mathbb{R}$. Let $Y^\varepsilon_\alpha(t,x)$ be the solution of Equation (4.16) in which the last two terms are replaced by

$$\int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\phi_\varepsilon(Y^\varepsilon_\alpha(s,y))\varphi'(X(s,y))W(\mathrm{d}y,\mathrm{d}s) \\ + \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\phi_\varepsilon(Y^\varepsilon_\alpha(s,y))\psi'(X(s,y))\,\mathrm{d}y\,\mathrm{d}s.$$

Since $|\phi_\varepsilon(x)| \leqslant |x|$, the same reasoning as above gives

$$F_\varepsilon(t,x) \leqslant K + K'\int_0^t \int_0^1 E\left|\int_{\Lambda_t^M} \phi_\varepsilon^2(Y^\varepsilon_\alpha(s,y))\,\mathrm{d}\alpha\right|^p \mathrm{d}y\,\mathrm{d}s, \tag{4.20}$$

with K and K' constants independent of $(t,x) \in \Lambda_T$ and $\varepsilon > 0$ and

$$F_\varepsilon(t,x) = E\left|\int_{\Lambda_t^M} |Y^\varepsilon_\alpha(t,x)|^2\,\mathrm{d}\alpha\right|^p.$$

This guarantees that

$$\sup_{t,x} |F_\varepsilon(t,x)| < \infty. \tag{4.21}$$

Since $|\phi_\varepsilon(x)| \leqslant |x|$, (4.20) yields

$$F_\varepsilon(t,x) \leqslant K + K'\int_0^t \int_0^1 F_\varepsilon(s,y)\,\mathrm{d}y\,\mathrm{d}s$$

and further

$$f_\varepsilon(t) \leqslant K + K'\int_0^t f_\varepsilon(s)\,\mathrm{d}s,$$

where

$$f_\varepsilon(t) = \sup_x |F_\varepsilon(t,x)|.$$

By (4.21)

$$\sup_{t\leqslant T} |f_\varepsilon(t)| < \infty,$$

so one may apply Gronwall's Lemma to get

$$f_\varepsilon(t) \leqslant Ke^{K'T} \quad \text{for} \quad t \leqslant T,$$

that is

$$\sup_{\varepsilon>0} \sup_{(t,x)\in\Lambda_T^m} |F_\varepsilon(t,x)| \leqslant Ke^{K'T}.$$

It is standard to prove that

$$\lim_{\varepsilon\to 0} Y_\alpha^\varepsilon(t,x) = D_\alpha X(t,x)$$

a.s., for every fixed $\alpha \in \Lambda_T^M$ and $(t,x) \in \Lambda_T$. So, by Fatou's Lemma

$$\begin{aligned} E\left|\int_{\Lambda_t^M} |D_\alpha^M X(t,x)|^2 \,\mathrm{d}\alpha\right|^p &\leqslant \overline{\lim_{\varepsilon\to 0}}\, E\left|\int_{\Lambda_t^M} |Y_\alpha^\varepsilon(t,x)|^2 \,\mathrm{d}\alpha\right|^p, \\ &\leqslant Ke^{K'T}. \end{aligned}$$

(4.17) is proved.

In the same way one gets

$$\begin{aligned} &\sup_n \sup_{(t,x)\in\Lambda_T} E\left|\int_{\Lambda_t^M} |D_\alpha^M X_n(t,x)|^2 \,\mathrm{d}\alpha\right|^p < \infty; \\ &\quad \forall M \in \mathbb{N} \quad \text{and} \quad p \geqslant 1. \end{aligned} \tag{4.22}$$

We are now ready to prove the point (ii) in Proposition 4.3 (the point (i) is analogous, but simpler, so we leave it out).

We shall prove the following by induction on M

$$(A_{M,p}) \lim_n \sup_{t,x} E\left|\int_{\Lambda_T^M} [D_\alpha^M X_n(t,x) - D_\alpha^M X(t,x)]^2 \,\mathrm{d}\alpha\right|^p = 0.$$

Let us first note that, if one proves $(A_{M,p})$ for $p=1$, then the same follows for every $p \geqslant 1$. This is because of (4.17) and (4.22).

Let us now assume that $(A_{M',p})$ holds for every $M' < M$ and every $p \geqslant 1$, and let us prove $(A_{M,1})$.

Going back to Equations (4.9) and (4.11) one writes

$$|D_\alpha^M X_n(t,x) - D_\alpha^M X(t,x)| \leqslant \sum_{i=1}^{8} |\mathrm{I}_\alpha^{n,i}(t,x)|, \tag{4.23}$$

with

$$\mathrm{I}^{n,1}_{\alpha}(t,x) = \sum_{k=1}^{n} \mathrm{e}_k(\bar{z}) \int_0^1 (G_{t_n^+ - \bar{r}_n^-}(x,y)\Gamma^{(n)}_{\hat{\alpha}}(\varphi)(\bar{r}_n^-, y) - G_{t-\bar{r}}(x,y)\Gamma_{\hat{\alpha}}(\varphi)(\bar{r},y))\, \mathrm{e}_k(y)\, \mathrm{d}y,$$

$$\mathrm{I}^{n,2}_{\alpha}(t,x) = \sum_{k=n+1}^{\infty} \mathrm{e}_k(\bar{z}) \int_0^1 G_{t-\bar{r}}(x,y)\Gamma_{\hat{\alpha}}(\varphi)(\bar{r},y)\, \mathrm{e}_k(y)\, \mathrm{d}y,$$

$$\mathrm{I}^{n,3}_{\alpha}(t,x) = \sum_{k=1}^{n} \int_{\bar{r}}^{t} \int_0^1 [G_{t_n^+ - s_n^-}(x,y) 1_{[\bar{r}_n^+, t_n^-)}(s) \Delta^{(n)}_{\alpha}(\varphi)(\bar{s}_n, y) - G_{t-s}(x,y)\Delta_{\alpha}(\varphi)(s,y)]\, \mathrm{e}_k(y)\, \mathrm{d}y\, \mathrm{d}W^k(s),$$

$$\mathrm{I}^{n,4}_{\alpha}(t,x) = \sum_{k=n+1}^{\infty} \int_{\bar{r}}^{t} \int_0^1 G_{t-s}(x,y)\Delta_{\alpha}(\varphi)(s,y)\, \mathrm{e}_k(y)\, \mathrm{d}y\, \mathrm{d}W^k(s),$$

$$\mathrm{I}^{n,5}_{\alpha}(t,x) = \sum_{k=n+1}^{\infty} \int_{\bar{r}}^{t} \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)) \times D^M_{\alpha} X(s,y)\, \mathrm{e}_k(y)\, \mathrm{d}y\, \mathrm{d}W^k(s),$$

$$\mathrm{I}^{n,6}_{\alpha}(t,x) = \int_{\bar{r}}^{t} \int_0^1 [G_{t_n^+ - s_n^-}(x,y) 1_{[\bar{r}_n^+, t_n^-)}(s) \Delta^{(n)}_{\alpha}(\psi)(s_n^-, y) - G_{t-s}(x,y)\Delta_{\alpha}(\psi)(s,y)]\, \mathrm{d}y\, \mathrm{d}s,$$

$$\mathrm{I}^{n,7}_{\alpha}(t,x) = \sum_{k=1}^{n} \int_{\bar{r}}^{t} \int_0^1 [G_{t_n^+ - s_n^-}(x,y) 1_{[\bar{r}_n^+, t_n^-)}(s) \varphi'(X_n(s_n^-, y)) D^M_{\alpha} X_n(s_n^-, y) - G_{t-s}(x,y)\varphi'(X(s,y)) D^M_{\alpha} X(s,y)]\, \mathrm{e}_k(y)\, \mathrm{d}y\, \mathrm{d}W^k(s),$$

$$\mathrm{I}^{n,8}_{\alpha}(t,x) = \int_{\bar{r}}^{t} \int_0^1 [G_{t_n^+ - s_n^-}(x,y) 1_{[\bar{r}_n^+, t_n^-)}(s) \psi'(X_n(s_n^-, y)) D^M_{\alpha} X_n(s_n^-, y) - G_{t-s}(x,y)\psi'(X(s,y)) D^M_{\alpha} X(s,y)]\, \mathrm{d}y\, \mathrm{d}s.$$

The first step is to prove that, for $1 \leqslant i \leqslant 6$

$$\lim_n \sup_{t,x} E \int_{\Lambda_t^M} |\mathrm{I}^{n,i}_{\alpha}(t,x)|^2\, \mathrm{d}\alpha = 0. \tag{4.24}$$

We begin with $i = 3$. One has

$$
\begin{aligned}
&E|\mathrm{I}_\alpha^{n,3}(t,x)|^2 \\
&= E\int_{\bar r}^{t}\sum_{k=1}^{n}\left|\int_0^1 (G_{t_n^+-s_n^-}(x,y)1_{[\bar r_n^+,t_n^-)}(s)\Delta_\alpha^{(n)}(\varphi)(s_n^-,y)\right. \\
&\qquad\qquad \left. - G_{t-s}(x,y)\Delta_\alpha(\varphi)(s,y))\,\mathrm{e}_k(y)\,\mathrm{d}y\right|^2 \mathrm{d}s \\
&\leqslant E\int_{\bar r}^{t}\int_0^1 |\,G_{t_n^+-s_n^-}(x,y)1_{[\bar r_n^+,t_n^-)}(s)\Delta_\alpha^{(n)}(\psi)(s_n^-,y) \\
&\qquad\qquad -G_{t-s}(x,y)\Delta_\alpha(\psi)(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant 2E\int_{\bar r_n^+}^{t_n^-}\int_0^1 G^2_{t_n^+-s_n^-}(x,y)|\Delta_\alpha^{(n)}(\varphi)(s_n^-,y)-\Delta_\alpha(\varphi)(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s \\
&\quad +2E\int_{\bar r}^{t}\int_0^1 |\,G_{t_n^+-s_n^-}(x,y)1_{[\bar r_n^+,t_n^-)}(s) \\
&\qquad\qquad -G_{t-s}(x,y)|^2(\Delta_\alpha(\varphi)(s,y))^2\,\mathrm{d}y\,\mathrm{d}s \\
&=: U_\alpha^{(n)}(t,x)+V_\alpha^{(n)}(t,x).
\end{aligned}
$$

Let $J_1,\dots,J_m$ be a partition of $\{1,\dots,M\}, \lambda_i = |J_i|$ and $J_i(\alpha) = \{\alpha_{j_1},\dots,\alpha_{j_{\lambda_i}}\}$ for $J_i = \{j_1,\dots,j_{\lambda_i}\}$. Since the derivatives of φ are bounded, one gets

$$
\begin{aligned}
&\int_{\Lambda_t^M} U_\alpha^{(n)}(t,x)\,\mathrm{d}\alpha \\
&\quad\leqslant \int_0^t\int_0^1 \mathrm{d}s\,\mathrm{d}y G^2_{t_n^+-s_n^-}(x,y) \\
&\qquad\times E\int_{\Lambda_s^M} |\Delta_\alpha^{(n)}(\varphi)(s_n^-,y)-\Delta_\alpha(\varphi)(s,y)|^2\,\mathrm{d}\alpha \\
&\quad\leqslant K\sum\int_0^t\int_0^1 \mathrm{d}s\,\mathrm{d}y G^2_{t_n^+-s_n^-}(x,y) \\
&\qquad\times E\int_{\Lambda_s^M}\left(\prod_{i=1}^m D^{\lambda_i}_{J_i(\alpha)}X_n(s,y)-\prod_{i=1}^m D^{\lambda_i}_{J_i(\alpha)}X(s,y)\right)^2\mathrm{d}\alpha,
\end{aligned}
$$

with the sum over all the partitions $J_1,\dots,J_m$ of $\{1,\dots,M\}$ (see (4.7) and (4.14)). We write the above terms in the form

$$
\begin{aligned}
\sum\int_0^t\int_0^1 G^2_{t_n^+-s_n^-}(x,y)\int_{\Lambda_s^M} E\Bigg(&\sum_{l=1}^m\prod_{i=1}^{l-1}(D^{\lambda_i}_{J_i(\alpha)}X_n(s,y))^2 \\
&\cdot (D^{\lambda_l}_{J_l(\alpha)}X_n(s,y)-D^{\lambda_l}_{J_l(\alpha)}X(s,y))^2 \\
&\cdot \prod_{i=l+1}^m (D^{\lambda_i}_{J_i(\alpha)}X(s,y))^2\Bigg)\,\mathrm{d}\alpha\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= \sum \sum_{l=1}^{m} \int_0^t \int_0^1 G_{t_n^+ - s_n^-}(x,y) E \left(\prod_{i=1}^{l-1} \int_{\Lambda_s^{\lambda_l}} (D_{J_i(\alpha)}^{\lambda_i} X_n(s,y))^2 \, \mathrm{d}J_i(\alpha) \right. \\
&\quad \times \int_{\Lambda_s^{\lambda_l}} (D_{J_l(\alpha)}^{\lambda_l} X_n(s,y) - D_{J_l(\alpha)}^{\lambda_l} X(s,y))^2 \, \mathrm{d}J_l(\alpha) \\
&\quad \left. \times \prod_{i=l+1}^{m} \int_{\Lambda_s^{\lambda_i}} (D_{J_i(\alpha)}^{\lambda_i} X(s,y))^2 \, \mathrm{d}J_i(\alpha) \right) \mathrm{d}y \, \mathrm{d}s,
\end{aligned}
$$

with the convention that $\prod_{i=1}^{0} = \prod_{i=m+1}^{m} = 1$. Now, by using Schwarz's inequality we dominate the above sum by

$$
\sum \sum_{l=1}^{m} \int_0^t \int_0^1 G_{t_n^+ - s_n^-}^2(x,y) \prod_{i=1}^{m} U_{n,i}^{(\ell)}(s,y) \, \mathrm{d}y \, \mathrm{d}s,
$$

where

$$
U_{n,i}^{(\ell)}(s,y) = \left(E \left| \int_{\Lambda_s^{\lambda_i}} (D_\alpha^{\lambda_i} X_n(s,y))^2 \, \mathrm{d}\alpha \right|^{2^{m-1}} \right)^{1/2^{m-1}} ; \quad 1 \leqslant i \leqslant l-1,
$$

$$
U_{n,l}^{(\ell)}(s,y) = \left(E \left| \int_{\Lambda_s^{\lambda_l}} (D_\alpha^{\lambda_l} X_n(s,y) - D_\alpha^{\lambda_l} X(s,y))^2 \, \mathrm{d}\alpha \right|^{2^{m-1}} \right)^{1/2^{m-1}} ,
$$

$$
U_{n,i}^{(\ell)}(s,y) = \left(E \left| \int_{\Lambda_s^{\lambda_i}} (D_\alpha^{\lambda_i} X(s,y))^2 \, \mathrm{d}\alpha \right|^{2^{m-1}} \right)^{1/2^{m-1}} ; \quad l+1 \leqslant i \leqslant m.
$$

By (4.17) and (4.22)

$$
\sup_n \sup_{s,y} U_{n,i}^{(\ell)}(s,y) < \infty \quad \text{for } i \neq l
$$

and, by the induction hypothesis

$$
\lim_n \sup_{s,y} U_{n,l}^{(\ell)}(s,y) = 0.
$$

Since

$$
\sup_n \sup_{t,x} \int_0^t \int_0^1 G_{t_n^+ - s_n^-}^2(x,y) \, \mathrm{d}y \, \mathrm{d}s < \infty,
$$

we have finally proved that

$$
\lim_n \sup_{t,x} \int_{\Lambda_s^M} U_\alpha^{(n)}(t,x) \, \mathrm{d}\alpha = 0.
$$

Let us now evaluate $V_\alpha^{(n)}(t,x)$. Note first that

$$\sup_{s,y} E\int_{\Lambda_s^M}\left(\prod_{i=1}^m D_{J_i(\alpha)}^{\lambda_i}X(s,y)\right)^2 \mathrm{d}\alpha<\infty.$$

Consequently

$$\begin{aligned} &E\int_{\Lambda_s^M} V_\alpha^{(n)}(t,x)\,\mathrm{d}\alpha\\ &\quad\leqslant K\sum\int_0^t\int_0^1 \mathrm{d}s\,\mathrm{d}y(G_{t_n^+-s_n^-}(x,y)1_{[\bar r_n^+,t_n^-)}(s)-G_{t-s}(x,y))^2\\ &\qquad\times E\int_{\Lambda_s^M}\left(\prod_{i=1}^m D_{J_i(\alpha)}^{\lambda_i}X(s,y)\right)^2\mathrm{d}\alpha\\ &\quad\leqslant K\int_0^t\int_0^1(G_{t_n^+-s_n^-}(x,y)1_{[\bar r_n^+,t_n^-)}(s)-G_{t-s}(x,y))^2\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

So, by (A.6) one gets

$$\lim_n\sup_{t,x}E\int_{\Lambda_t^M}V_\alpha^{(n)}(t,x)\,\mathrm{d}\alpha=0,$$

which finishes the proof of (4.24) for $i=3$.

Let us now evalute $\mathrm{I}_\alpha^{n,4}(t,x)$. One writes

$$\begin{aligned} h_n(t,x) \;&=:\; E\int_{\Lambda_t^M}(\mathrm{I}_\alpha^{n,4}(t,x))^2\,\mathrm{d}\alpha\\ &=\; E\int_{\Lambda_t^M}\int_{\bar r}^t\int_0^1(\mathrm{Pr}_n^\perp(G_{t-s}(x,\cdot)\Delta_\alpha(\varphi)(s,\cdot))(y))^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\alpha\\ &=\; E\int_{\Lambda_t^M}\int_0^T\int_0^1(\mathrm{Pr}_n^\perp(G_{t-s}(x,\cdot)1_{[0,t)}(s)\\ &\qquad\times 1_{\Lambda_s^m}(\alpha)\Delta_\alpha(\varphi)(s,\cdot))(y))^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\alpha,\end{aligned}$$

where $\mathrm{Pr}_n^\perp$ denotes the projection on the sub-space of $L^2([0,1])$ spanned by $e_i, i\geqslant n+1$. We have to prove that $h_n, n\in N$ converges to zero, uniformly with respect to $(t,x)\in\Lambda_T$. To this end we shall first check that they are equally continuous and then prove the pointwise convergence.

By using the inequality $|\|f\|^2 - \|g\|^2| \leqslant (\|f\| + \|g\|)\|f-g\|$ (in $L^2(\Lambda_T^M \times \Lambda_T)$ here), one gets

$$\begin{aligned}
&|h_n(t,x) - h_n(t',x')| \\
&\quad \leqslant K(h_n^{1/2}(t,x) + h_n^{1/2}(t',x')) \\
&\qquad \times \Bigg(E\int_{\Lambda_T^M}\int_0^T\int_0^1 (\mathrm{Pr}_n^{\perp}((G_{t-s}(x,\cdot)1_{[0,t)}(s) \\
&\qquad\qquad -G_{t'-s}(x',\cdot)1_{[0,t')}(s))1_{\Lambda_s^M}(\alpha) \\
&\qquad\qquad \times\Delta_\alpha(\varphi)(s,\cdot))(y))^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\alpha \Bigg)^{1/2} \\
&\quad \leqslant K(h_n^{1/2}(t,x) + h_n^{1/2}(t',x')) \\
&\qquad \times \Bigg(E\int_{\Lambda_T^M}\int_0^T\int_0^1 ((G_{t-s}(x,y)1_{[0,t)}(s) \\
&\qquad\qquad -G_{t'-s}(x',y))1_{\Lambda_s^M}(\alpha)\Delta_\alpha(\varphi)(s,y))^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\alpha \Bigg)^{1/2} \\
&\quad \leqslant K(h_0^{1/2}(t,x) + h_0^{1/2}(t',x'))\cdot \sup_{(s,y)\in\Lambda_T} \left(E\int_{\Lambda_s^M} (\Delta_\alpha(\varphi)(s,y))^2\,\mathrm{d}\alpha \right)^{1/2} \\
&\qquad \times \left(\int_0^T\int_0^1 (G_{t-s}(x,y)1_{[0,t)}(s) - G_{t'-s}(x',y)1_{[0,t')}(s))^2\,\mathrm{d}y\,\mathrm{d}s \right)^{1/2},
\end{aligned}$$

which converges to zero as $x' \to x$ and $t' \to t$.

Consequently $h_n, n \in \mathbb{N}$, are equi-continuous. Let us now prove the pointwise convergence. Clearly $\mathrm{Pr}_n^{\perp}(G_{t-s}(x,\cdot)\cdot\Delta_\alpha(\varphi)(s,\cdot))(\cdot) \to 0$ as $n \to \infty$ in $L^2(0,1;\mathrm{d}y)$. Since $h_0(t,x) < \infty$, we may use the dominated convergence theorem to get $\lim_{n\to\infty} h_n(t,x) = 0$. So we have proved (4.24) for $i = 4$. For $i = 2,5$ (resp. $i = 1,6$) the proof is analogous to that for $i = 4$ (resp. $i = 3$).

Let us now write

$$\mathrm{I}_\alpha^{n,7}(t,x) = r_\alpha^{(n)}(t,x) + H_\alpha^{(n)}(t,x),$$

with

$$\begin{aligned}
r_\alpha^{(n)}(t,x) \;=\; &\sum_{k=1}^n \int_{\bar r}^t\int_0^1 (G_{t_n^+ - s_n^-}(x,y)1_{[\bar r_n^+, t_n^-)}(s)\varphi'(X_n(s,y)) \\
&-G_{t-s}(x,y)\varphi'(X(s,y)))D_\alpha^M X_n(s,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s)
\end{aligned}$$

and

$$H_\alpha^{(n)}(t,x) = \sum_{k=1}^n \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)) \times (D_\alpha^M X_n(s,y) - D_\alpha^M X(s,y))\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s).$$

Similar arguments as for $V_\alpha^{(n)}(t,x)$ (based on (A.6) and the induction hypothesis $X_n \to X$) show that $r_\alpha^{(n)}(t,x)$ satisfies (4.24). A similar decomposition works for $\mathrm{I}_\alpha^{n,8}(t,x)$. Let us now denote

$$F_\alpha^{(n)}(t,x) = D_\alpha^M X_n(t,x) - D_\alpha^M X(t,x).$$

We have proved that

$$F_\alpha^{(n)}(t,x) = J_\alpha^{(n)}(t,x) + \sum_{k=1}^n \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)) \times F_\alpha^{(n)}(s,y)\,\mathrm{e}_k(y)\,\mathrm{d}y\,\mathrm{d}W^k(s) + \int_{\bar r}^t \int_0^1 G_{t-s}(x,y)\psi'(X(s,y))F_\alpha^{(n)}(s,y)\,\mathrm{d}y\,\mathrm{d}s,$$

where $J_\alpha^{(n)}(t,x)$ satisfies (4.24).

By taking expectations and by using the boundedness of φ' and ψ' one gets

$$\begin{aligned} &E\int_{\Lambda_t^M} |F_\alpha^{(n)}(t,x)|^2\,\mathrm{d}\alpha \\ &\leqslant K_n + K'E\int_{\Lambda_t^M}\int_{\bar r}^t \int_0^1 G_{t-s}^2(x,y)|F_\alpha^{(n)}(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\alpha \\ &= K_n + K'\int_0^t\int_0^1 \mathrm{d}s\,\mathrm{d}y G_{t-s}(x,y)E\int_{\Lambda_s^M}|F_\alpha^{(n)}(s,y)|^2\,\mathrm{d}\alpha, \end{aligned}$$

where $K_n \to 0$ as $n \to \infty$.

Let us denote

$$f_n(t) = \sup_x E\int_{\Lambda_t^M}|F_\alpha^{(n)}(t,x)|^2\,\mathrm{d}\alpha.$$

Then, the above inequality gives

$$\begin{aligned} f_n(t) &\leqslant K_n + K'\int_0^t \mathrm{d}s f_n(s)\int_0^1 G_{t-s}^2(x,y)\,\mathrm{d}y \\ &\leqslant K_n + K''\int_0^t f_n(s)\frac{1}{\sqrt{t-s}}\,\mathrm{d}s. \end{aligned}$$

One iterates this inequality and uses Fubini's Theorem to get

$$f_n(t) \leqslant K_n + K''' \int_0^t f_n(u)\,\mathrm{d}u.$$

Now, by Gronwall's Lemma

$$f_n(t) \leqslant K_n\,\mathrm{e}^{K'''t} \leqslant K_n\,\mathrm{e}^{K'''T} \to 0 \quad \text{as} \quad n \to \infty. \qquad \square$$

5. Evaluation of the Covariance Matrix

The aim of this section is to prove Part (i) of Theorem 2.2. We shall apply Theorem 3.1 to the functional $F = (X(t, x_1), \ldots, X(t, x_d))$. Actually it suffices to prove that

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} P((\det \sigma)^{1/d} < a\sqrt{\varepsilon}, F \in \Gamma_c) = 0; \quad \forall c > 0, \quad k \in \mathbb{N}, \tag{5.1}$$

where $a = \frac{4}{3}K_0 c$ (K_0 is the constant in (A.3)), σ is the Malliavin covariance matrix attached to F and $\Gamma_c = \{\varphi^2 \geqslant c\}^d$.

It follows from Theorem 3.1 and the results in the preceding section that Part (i) of Theorem 2.2 is a consequence of (5.1), which we now prove.

Let $S_{r,z}(t, x)$ be the solution of the equation

$$\begin{aligned} S_{r,z}(t,x) \;=\; & G_{t-r}(x,z) + \int_r^t \int_0^1 G_{t-r}(x,y)\varphi'(X(s,y)) \\ & \times S_{r,z}(s,y) W(\mathrm{d}y, \mathrm{d}s) \\ & + \int_r^t \int_0^1 G_{t-s}(x,y)\psi'(X(s,y)) S_{r,z}(s,y)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{5.2}$$

Then, a standard uniqueness argument shows that

$$D_{(r,z)}X(t,x) = S_{r,z}(t,x)\varphi(X(r,z)). \tag{5.3}$$

It follows that the covariance matrix σ is given by

$$\sigma^{ij} = \int_0^t \int_0^1 \varphi^2(X(r,z)) S_{r,z}(t,x_i) S_{r,z}(t,x_j)\,\mathrm{d}z\,\mathrm{d}r.$$

In order to evaluate det σ we have to get a lower bound for the quadratic form associated to σ. Let $\frac{1}{4}\min_{ij}|x_i - x_j|^2 > \varepsilon > 0$ and $\xi \in \mathbb{R}^d$. One has

$$\begin{aligned} \langle \sigma\xi, \xi\rangle \;=\; & \int_0^t \int_0^1 \varphi^2(X(r,z)) \left(\sum_{i=1}^d S_{r,z}(t,x_i)\xi_i\right)^2 \mathrm{d}z\,\mathrm{d}r \\ \geqslant\; & \sum_{j=1}^d \int_{t-\varepsilon}^t \int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \varphi^2(X(r,z)) \left(\sum_{i=1}^d S_{r,z}(t,x_i)\xi_i\right)^2 \mathrm{d}z\,\mathrm{d}r \\ \geqslant\; & \tfrac{2}{3}\mathrm{I}_\varepsilon^{(2)}(\xi) - 2\mathrm{I}_\varepsilon^{(1)}(\xi) \end{aligned}$$

with

$$\begin{aligned}
\mathrm{I}_\varepsilon^{(1)}(\xi) &= \sum_{j=1}^{d}\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \varphi^2(X(r,z)) \\
&\quad\times\left(\sum_{i\neq j} S_{r,z}(t,x_i)\xi_i S_{r,z}(t,x_j)\xi_j\right)\,\mathrm{d}z\,\mathrm{d}r \\
\mathrm{I}_\varepsilon^{(2)}(\xi) &= \sum_{j=1}^{d}\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \varphi^2(X(r,z))S_{r,z}^2(t,x_j)\xi_j^2\,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

Let us also denote

$$\begin{aligned}
Q_{r,z}(t,x) &= \int_r^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y))D_{(r,z)}X(s,y)W(\mathrm{d}y,\mathrm{d}s) \\
&\quad+\int_r^t\int_0^1 G_{t-s}(x,y)\psi'(X(s,y))D_{(r,z)}X(s,y)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Then

$$\mathrm{I}_\varepsilon^{(2)}(\xi)\geqslant \tfrac{2}{3}\mathrm{I}_\varepsilon^{(4)}(\xi)-2\mathrm{I}_\varepsilon^{(3)}(\xi),$$

with

$$\begin{aligned}
\mathrm{I}_\varepsilon^{(3)}(\xi) &= \sum_{j=1}^{d}\int_{t-\varepsilon}^{t}\int_0^1 \varphi^2(X(r,z))Q_{r,z}^2(t,x_j)\xi_j^2\,\mathrm{d}z\,\mathrm{d}r, \\
\mathrm{I}_\varepsilon^{(4)}(\xi) &= \sum_{j=1}^{d}\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} \varphi^2(X(r,z))G_{t-r}^2(x_j,z)\xi_j^2\,\mathrm{d}z\,\mathrm{d}r
\end{aligned}$$

and further

$$\mathrm{I}_\varepsilon^{(4)}(\xi)\geqslant \mathrm{I}_\varepsilon^{(6)}(\xi)-\mathrm{I}_\varepsilon^{(5)}(\xi),$$

with

$$\begin{aligned}
\mathrm{I}_\varepsilon^{(5)}(\xi) &= \sum_{j=1}^{d}\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} |\varphi^2(X(r,z)) \\
&\quad-\varphi^2(X(t,x_j))|G_{t-r}^2(x_j,z)\xi_j^2\,\mathrm{d}z\,\mathrm{d}r \\
\mathrm{I}_\varepsilon^{(6)}(\xi) &= \sum_{j=1}^{d}\varphi^2(X(t,x_j))\xi_j^2\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} G_{t-r}^2(x_j,z)\,\mathrm{d}z\,\mathrm{d}r.
\end{aligned}$$

By (A.3),

$$\mathrm{I}_\varepsilon^{(6)}(\xi) \geqslant K_0\sqrt{\varepsilon}\sum_{j=1}^{d}\varphi^2(X(t,x_j))\xi_j^2.$$

Assume that $(X(t,x_j)) \in \{\varphi^2 \geqslant c\}, 1 \leqslant j \leqslant d$. Then

$$(\det\sigma)^{1/d} \geqslant \inf_{|\xi|=1}\langle\sigma\xi,\xi\rangle \geqslant \tfrac{2}{3}K_0C\sqrt{\varepsilon}c - \sum_{j=1}^{3} 2\sup_{|\xi|=1}|\mathrm{I}_\varepsilon^{(2j-1)}(\xi)|.$$

So (4.1) will be proved as soon as we check that

$$E\left(\sup_{|\xi|=1}|\mathrm{I}_\varepsilon^{(2i-1)}(\xi)|^p\right) \leqslant K\varepsilon^{p(1/2+1/8)}; \quad \forall p > 5, \quad i = 1,2,3, \tag{5.4}$$

Let us evaluate $\mathrm{I}_\varepsilon^{(5)}(\xi)$. By using Hölder's inequality $(q = p/(p-1))$ and the fact that $|\xi_j| \leqslant 1$ one gets

$$\begin{aligned} E|\mathrm{I}_\varepsilon^{(5)}(\xi)|^p &\leqslant K\sum_{j=1}^{d}\left(\int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} G_{t-r}^{2q}(x_j,z)\,\mathrm{d}z\,\mathrm{d}r\right)^{p/q} \\ &\quad \times \int_{t-\varepsilon}^{t}\int_{x_j-\sqrt{\varepsilon}}^{x_j+\sqrt{\varepsilon}} E|\varphi^2(X(r,z)) - \varphi^2(X(t,x_j))|^p\,\mathrm{d}z\,\mathrm{d}r \\ &\leqslant K\varepsilon^{((3/2)-q)p/q} \times (\varepsilon\sqrt{\varepsilon}) \times (\sqrt{\varepsilon})^{p/4} = K\varepsilon^{p(1/2+1/8)} \end{aligned}$$

the last inequality being a consequence of (A.5) and of the Hölder property of $(t,x) \to X(t,x)(\frac{1}{2}-\delta$ in x and $\frac{1}{4}-\delta$ in $t, \delta > 0$: see Walsh [13]).

We use now Burkholder's inequality (i.e. (4.18)) and the boundedness of φ' in order to get

$$\begin{aligned} &E\left|\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}z\,\mathrm{d}r\left(\int_r^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y))S_{r,z}(s,y)W(\mathrm{d}y,\mathrm{d}s)\right)^2\right|^p \\ &\leqslant KE\left|\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}z\,\mathrm{d}r\int_r^t\int_0^1 G_{t-s}^2(x,y)S_{r,z}^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \\ &= KE\left|\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}y\,\mathrm{d}sG_{t-s}^2(x,y)\int_{t-\varepsilon}^{s}\int_0^1 S_{r,z}^2(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \\ &\leqslant K\left(\int_{t-\varepsilon}^{t}\int_0^1 G_{t-s}^{2q}(x,y)\,\mathrm{d}y\,\mathrm{d}s\right)^{p/q}\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}y\,\mathrm{d}s \end{aligned}$$

$$\times E\left|\int_{t-\varepsilon}^{s}\int_0^1 S^2_{r,z}(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p$$

$$\leqslant K\varepsilon^{((3/2)-q)p/q}\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}y\,\mathrm{d}sE\left|\int_{t-\varepsilon}^{s}\int_0^1 S^2_{r,z}(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p, \qquad (5.5)$$

the last inequality being a consequence of (A.5).

Further, by using the Equation (5.2) one gets

$$E\left|\int_{t-\varepsilon}^{s}\int_0^1 S^2_{r,z}(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \leqslant K\left|\int_{t-\varepsilon}^{s}\int_0^1 G^2_{s-r}(y,z)\,\mathrm{d}z\,\mathrm{d}r\right|^p$$
$$+KE\left|\int_{t-\varepsilon}^{s}\int_0^1 \mathrm{d}z\,\mathrm{d}r\int_r^s\int_0^1 G_{s-u}(y,v)\varphi'(X(u,v))\right.$$
$$\left.\times\ S_{r,z}(u,v)W(\mathrm{d}u\,\mathrm{d}v)\right|^p$$
$$+KE\left|\int_{t-\varepsilon}^{s}\int_0^1 \mathrm{d}z\,\mathrm{d}r\int_r^s\int_0^1 G_{s-u}(y,v)\psi'(X(u,v))S_{r,z}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right|^p.$$

By using (A.1) it is easy to see that the first term in the right-hand side of the above inequality is dominated by $K\varepsilon^{p/2}$. Then, by (4.5), the second term is dominated by

$$K\varepsilon^{((3/2)-q)p/q}\int_{t-\varepsilon}^{s}\int_0^1 \mathrm{d}z\,\mathrm{d}rE\left|\int_{t-\varepsilon}^{u}\int_0^1 S^2_{r,z}(u,v)\,\mathrm{d}u\,\mathrm{d}v\right|^p$$
$$\leqslant K\varepsilon^{((3/2)-q)(p/q)+1} = K\varepsilon^{(p-1)/2},$$

the last inequality being a consequence of (4.17)

The third term (containing ψ') is dominated in the same way. We conclude that

$$E\left|\int_{t-\varepsilon}^{s}\int_0^1 S^2_{r,z}(s,y)\,\mathrm{d}y\,\mathrm{d}s\right|^p \leqslant K\varepsilon^{(p-1)/2}. \qquad (5.6)$$

We plug this in (5.5) in order to get

$$E\left|\int_{t-\varepsilon}^{t}\int_0^1 \mathrm{d}z\,\mathrm{d}r\left(\int_r^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y))S_{r,z}(s,y)W(\mathrm{d}y,\mathrm{d}s)\right)^2\right|^p$$
$$\leqslant K\varepsilon^{p-2}. \qquad (5.7)$$

An analogous (but simpler) argument shows that (5.7) holds also if we replace φ' by ψ' and $W(\mathrm{d}y,\mathrm{d}s)$ by $\mathrm{d}y\,\mathrm{d}s$. We conclude that

$$E|\mathrm{I}^{(3)}_{\varepsilon}(\xi)|^p \leqslant K\varepsilon^{p-2} \leqslant K\varepsilon^{p(1/2+1/8)}.$$

Finally, by using Schwarz's inequality and (4.6) we prove that

$$E|\mathrm{I}_{\varepsilon}^{(1)}(\xi)|^p \leqslant K\varepsilon^{p-1}$$

and the proof is complete.

6. Strict Positivity of the Density

We now prove Part (ii) of Theorem 2.2. Let $t > 0$, and $0 \leqslant x_1 < x_2 < \cdots < x_d \leqslant 1$. We shall use the criterion from Theorem 3.3, with

$$F = (X(t, x_1), \ldots, X(t, x_d)).$$

First note that it suffices to prove the result for $y_0 \in \{\varphi \neq 0\}^d \cap \operatorname{supp}(P \circ F^{-1})$, since that result implies that

$$\{\varphi \neq 0\}^d \subset \operatorname{supp}(P \circ F^{-1}).$$

Indeed, if that inclusion would not hold, applying the result at $y_0 \in \{\varphi \neq 0\}^d \cap \Sigma$, where Σ denotes the boundary of the set $\operatorname{supp}(P \circ F^{-1})$, would lead to $0 < p(y_0) = 0$, since p is continuous

For each $n \geqslant 1, 1 \leqslant i \leqslant d$, let

$$h_n^i(r, z) = c_n^i 1_{[t-2^{-n}, t]}(r) 1_{[(x_i - 2^{-n}) \vee 0, (x_i + 2^{-n}) \wedge 1]}(z),$$

where

$$(c_n^i)^{-1} = \int_{t-2^{-n}}^{t} \int_{(x_i - 2^{-n}) \vee 0}^{(x_i + 2^{-n}) \wedge 1} G_{t-s}(x_i, y)\, \mathrm{d}y\, \mathrm{d}s.$$

We now define T_z^n by (3.3), and

$$\phi_n^i(z) = X(t, x_i) \circ T_z^n, \quad 1 \leqslant i \leqslant d.$$

It remains to show that the sequence of random vectors $\{\phi_n(z)\}_{n \in \mathbb{N}}$ indexed by z satisfies the conditions of Theorem 2.3, namely (i) and (ii). We first proceed to the

Proof of (i). Define

$$u_z^{n,i}(s, x) = \frac{\partial}{\partial z^i} X(s, x) \circ T_z^n.$$

We have

$$\begin{aligned} u_z^{n,i}(t,x) \;=\; & \theta_z^{n,i}(t,x) + \int_0^t \int_0^1 1_{[t-2^{-n},t]}(s) G_{t-s}(x,y) \\ & \times \psi'(X(s,y) \circ T_z^n) u_z^{n,i}(s,y)\, \mathrm{d}y\, \mathrm{d}s \\ & + \int_0^t \int_0^1 1_{[t-2^{-n},t]}(s) G_{t-s}(x,y) \varphi'(X(s,y) \circ T_z^n) u_z^{n,i}(s,y) \\ & \times [W(\mathrm{d}y, \mathrm{d}s) + \langle z, h_n(s,y)\rangle\, \mathrm{d}y\, \mathrm{d}s], \end{aligned} \tag{6.1}$$

where

$$\theta_z^{n,i}(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) \varphi(X(s,y) \circ T_z^n) h_n^i(s,y)\, \mathrm{d}y\, \mathrm{d}s.$$

We note that Equation (6.1) is obtained by integrating $D_{r,z}X(t,x) \circ T_z^n$ against $h_n^i(r,z)$, and using Fubini's Theorem to commute the integrals. The coefficient $1_{[t-2^{-n},t]}(s)$ is due to the facts that $h_n^i(r,z) = 1_{[t-2^{-n},t]}(r) h_n^i(r,z)$, and $D_{r,z}X(s,y) \circ T_z^n = 0$ for $s < r$.

From (6.1), Burkholder's and Hölder's inequalities and the boundedness of φ' and ψ', we deduce the estimate

$$\begin{aligned} & E|u_z^{n,i}(t,x)|^p \\ & \quad \leqslant KE|\theta_z^{n,i}(t,x)|^p \\ & \qquad + KE\left[\left(\int_{t-2^{-n}}^t \int_0^1 G_{t-s}^2(x,y) |u_z^{n,i}(s,y)|^2\, \mathrm{d}y\, \mathrm{d}s\right)^{p/2}\right] \\ & \qquad + KE\left[\left(\int_{t-2^{-n}}^t \int_0^1 G_{t-s}(x,y)(1 + |\langle z, h_n(s,y)\rangle|) \right.\right. \\ & \qquad\qquad \left.\left. \times\, |u_z^{n,i}(s,y)|\, \mathrm{d}y\, \mathrm{d}s\right)^p\right] \\ & \quad \leqslant KE|\theta_z^{n,i}(t,x)|^p \\ & \qquad + K\mu_n^{(p/2)-1} \int_{t-2^{-n}}^t \int_0^1 G_{t-s}^2(x,y) E|u_z^{n,i}(s,y)|^p\, \mathrm{d}y\, \mathrm{d}s \\ & \qquad + K\nu_n^{p-1} \int_{t-2^{-n}}^t \int_0^1 G_{t-s}(x,y)(1 + |\langle z, h_n(s,y)\rangle|) \\ & \qquad \times E|u_z^{n,i}(s,y)|^p\, \mathrm{d}y\, \mathrm{d}s, \end{aligned} \tag{6.2}$$

with

$$\begin{aligned}\mu_n &= \int_{t-2^{-n}}^{t}\int_0^1 G_{t-s}^2(x,y)\,\mathrm{d}y\,\mathrm{d}s\\ &\leqslant c\sqrt{2^{-n}},\\ \nu_n &= \int_{t-2^{-n}}^{t}\int_0^1 G_{t-s}(x,y)(1+|\langle z,h_n(s,y)\rangle|)\,\mathrm{d}y\,\mathrm{d}s.\end{aligned}$$

By using (A.1) one gets

$$\begin{aligned}&\int_{t-2^{-n}}^{t}\int_0^1 G_{t-s}(x,y)|h_n(s,y)|\,\mathrm{d}y\,\mathrm{d}s\\ &\quad\leqslant K\sum_{i=1}^{d} c_n^i \int_{t-2^{-n}}^{t}\int_{x_i-2^{-n}}^{x_i+2^{-n}} \frac{1}{\sqrt{2\pi(t-s)}}\,\mathrm{e}^{-(|x-y|^2/2(t-s))}\,\mathrm{d}y\,\mathrm{d}s\\ &\quad\leqslant K\sum_{i=1}^{d} c_n^i \int_{t-2^{-n}}^{t}\int_{x_i-2^{-n}}^{x_i+2^{-n}} \frac{1}{\sqrt{2\pi(t-s)}}\,\mathrm{e}^{-(|x_i-y|^2/2(t-s))}\,\mathrm{d}y\,\mathrm{d}s = Kd.\end{aligned}$$

Hence, for $|z|\leqslant\delta$,

$$\nu_n \leqslant K(2^{-n}+\delta).$$

It follows that

$$\begin{aligned}E|u_z^{n,i}(t,x)|^p &\leqslant KE|\theta_z^{n,i}(t,x)|^p\\ &\qquad + K(\mu_n^{p/2}+\nu_n^p)\sup_{(s,y)\in[0,t]\times[0,1]} E|u_z^{n,i}(s,y)|^p.\end{aligned}$$

For n large enough and δ small enough

$$K(\mu_n^{p/2}+\nu_n^p)\leqslant \tfrac{1}{2}$$

and then

$$\sup_{(t,x)\in[0,T]\times[0,1]} E|u_z^{n,i}(t,x)|^p \leqslant 2K \sup_{(t,x)\in[0,T]\times[0,1]} E|\theta_z^{n,i}(t,x)|^p. \tag{6.3}$$

Note that

$$\theta_z^{n,i}(t,x)\leqslant K\|\varphi\|_\infty \tag{6.4}$$

and for $i\neq j$,

$$\theta_z^{n,i}(t,x_j)\leqslant \|\varphi\|_\infty c_n^i \int_{t-2^{-n}}^{t}\int_{(x_i-2^{-n})\vee 0}^{(x_i+2^{-n})\wedge 1} G_{t-s}(x_j,y)\,\mathrm{d}y\,\mathrm{d}s. \tag{6.5}$$

By using (A.1) it is easy to check that $c_n^i \leqslant K2^{3n/2}$ and for $j \neq i$

$$\int_{t-2^{-n}}^{t} \int_{(x_i-2^{-n})\vee 0}^{(x_i+2^{-n})\wedge 1} G_{t-s}(x_j, y)\, \mathrm{d}y\, \mathrm{d}s \leqslant K \mathrm{e}^{-c2^n(x_i-x_j)^2},$$

for some $c < 1$.

It follows that, for $i \neq j, \theta_n^i(t, x_j) \to 0$ as $n \to \infty$.

It then follows from (6.3), (6.4) and (6.5) that

$$\sup_{(t,x)\in[0,T]\times[0,1]} E|u_z^{n,i}(t,x)|^p \leqslant K, \quad \forall n \in \mathbb{N}, \quad |z| \leqslant \delta.$$

A similar argument as used above shows that

$$\begin{aligned} E|u_z^{n,i}(t,x) - \theta_z^{n,i}(t,x)|^p &\leqslant K(\mu_n^{p/2} + \nu_n^p) \sup_{(s,y)\in[0,t]\times[0,1]} E|u_z^{n,i}(s,y)|^p \\ &\leqslant K'(\mu_n^{p/2} + \nu_n^p). \end{aligned}$$

Note that when $z = 0$, the term $\langle z, h_n \rangle$ in ν_n vanishes, and so we get

$$E|u_0^{n,i}(t,x) - \theta_0^{n,i}(t,x)|^p \leqslant K2^{-(np/4)}. \tag{6.6}$$

Since

$$\begin{aligned} \theta_0^{n,i}(t,x_i) &= \varphi(X(t,x_i)) + \int_0^t \int_0^1 G_{t-s}(x_i,y) \\ &\quad \times[\varphi(X(s,y)) - \varphi(X(t,x_i))] h_n^i(s,y)\, \mathrm{d}y\, \mathrm{d}s, \end{aligned}$$

we have that

$$E\left|\det[(\theta_0^{n,i}(t,x_j))_{i,j}] - \prod_1^d \varphi(X(t,x_i))\right|^p \to 0, \quad \text{as} \quad n \to \infty. \tag{6.7}$$

Since $y_0 \in \{\varphi \neq 0\}^d \cap \operatorname{supp}(P \circ F^{-1})$, there exists $r_0 > 0$ such that $\forall 0 < r \leqslant r_0$,

$$B(y_0, r) \subset \{\varphi \neq 0\}^d, \quad \text{and} \quad P(F \in B(y_0, r)) > 0.$$

Let

$$c := \frac{1}{2} \inf_{y\in B(y_0,r_0)} \prod_1^d \varphi(y_i).$$

Then

$$P\left(\{|F - y_0| \leqslant r\} \cap \left\{\prod_1^d \varphi(X(t,x_i)) \geqslant 2c\right\}\right) > 0; \quad 0 < r \leqslant r_0,$$

hence from (6.6) and (6.7)

$$\limsup_{n\to\infty} P(\{|F-y_0|\leqslant r\}\cap\{\det\partial_z\phi_n(0)\geqslant c\})>0;\quad 0<r\leqslant r_0. \qquad \square$$

Proof of (ii) *for* $\partial_z\phi_n(z)$. Let $v_z^{n,i}$ denote the solution of the affine equation

$$\begin{aligned} v_z^{n,i}(t,x) &= \theta_z^{n,i}(t,x)+\int_0^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)\circ T_z^n) \\ &\quad \times v_z^{n,i}(s,y)\langle z,h_n(s,y)\rangle\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{6.8}$$

Define

$$\begin{aligned} \mathrm{I}(f,\omega) &= \int_0^t\int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)\circ T_z^n) \\ &\quad\times f(s,y)\langle z,h_n(s,y)\rangle\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

and note that

$$\begin{aligned} \|\mathrm{I}(f,\omega)\|_\infty &\leqslant \|\varphi'\|_\infty|z|\|f\|_\infty \\ &\leqslant \delta\|\varphi'\|_\infty\times\|f\|_\infty;\quad \omega\in\Omega,\quad |z|\leqslant\delta. \end{aligned}$$

Hence for δ small enough, $f\to\mathrm{I}(f,\omega)$ is a contraction for all $\omega\in\Omega$. Consequently equation (6.8) has a unique solution, and furthermore from (6.4), (6.5)

$$\begin{aligned} |v_z^{n,i}(t,x)| &\leqslant K|\theta_z^{n,i}(t,x)| \\ &\leqslant K';\quad n\in\mathbb{N},\quad |z|\leqslant\delta. \end{aligned} \tag{6.9}$$

Writing the equation for the random field $u_z^{n,i}-v_z^{n,i}$, and using similar inequalities as in (6.2), we deduce that

$$\begin{aligned} &E|u_z^{n,i}(t,x)-v_z^{n,i}(t,x)|^p \\ &\quad\leqslant K(\mu_n^{p/2}+2^{-np})\sup_{(s,y)\in[0,t]\times[0,1]}E|u_z^{n,i}(s,y)|^p \\ &\quad+K\nu_n^p\sup_{(s,y)\in[0,t]\times[0,1]}E|u_z^{n,i}(s,y)-v_z^{n,i}(s,y)|^p. \end{aligned}$$

Now since for δ small enough $K\nu_n^p\leqslant\frac{1}{2}$, and

$$\sup_{|z|<\delta}\sup_{(s,y)\in[0,t]\times[0,1]}E|u_z^{n,i}(s,y)|^p$$

is bounded, we have that

$$\sup_{|z|\leqslant\delta}\ \sup_{(s,x)\in[0,t]\times[0,1]} E|u_z^{n,i}(s,x)-v_z^{n,i}(s,x)|^p \leqslant K\mu_n^{p/2}$$
$$\leqslant K2^{-(np/4)}. \tag{6.10}$$

Moreover

$$\begin{aligned}
&E|u_z^{n,i}(t,x)-u_{z'}^{n,i}(t,x)|^p\\
&\quad\leqslant E|\theta_z^{n,i}(t,x)-\theta_{z'}^{n,i}(t,x)|^p\\
&\qquad+K(\mu_n^{p/2}+\nu_n^p)\sup_{(s,y)\in[0,t]\times[0,1]} E|u_z^{n,i}(s,x)-u_{z'}^{n,i}(s,x)|^p\\
&\qquad+K(\mu_n^{p/2}+\nu_n^p)\sup_{(s,y)\in[0,t]\times[0,1]} E(|u_z^{n,i}(s,y)(X(s,y)\circ T_z^n\\
&\qquad\qquad - X(s,y)\circ T_{z'}^n)|^p).
\end{aligned}$$

From Schwarz's inequality, the fact that $K(\mu_n^{p/2}+\nu_n^p)\leqslant\frac{1}{2}$, and the above bound for $u_z^{n,i}$, we deduce that

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times[0,1]} E|u_z^{n,i}(t,x)-u_{z'}^{n,i}(t,x)|^p\\
&\quad\leqslant \sup_{(t,x)\in[0,T]\times[0,1]} E|\theta_z^{n,i}(t,x)-\theta_{z'}^{n,i}(t,x)|^p\\
&\qquad+K\sup_{(t,x)\in[0,T]\times[0,1]} (E|X(t,x)\circ T_z^n - X(t,x)\circ T_{z'}^n|^{2p})^{1/2}
\end{aligned}$$

and the same inequality holds for $v_z^{n,i}(t,x)-v_{z'}^{n,i},(t,x)$.

It is not hard to show that

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|X(t,x)\circ T_z^n - X(t,x)\circ T_{z'}^n|^{2p})\leqslant K|z-z'|^{2p}$$

and the same inequality holds for $\theta_z^{n,i}(t,x)-\theta_{z'}^{n,i}(t,x)$.

These inequalities show that

$$E|u_z^{n,i}(t,x)-u_{z'}^{n,i}(t,x)|^p + E|v_z^{n,i}(t,x)-v_{z'}^{n,i}(t,x)|^p\leqslant k|z-z'|^p.$$

This, together with (6.10), shows that

$$E\left(\sup_{|z|\leqslant\delta}|u_z^{n,i}(t,x)-v_z^{n,i}(t,x)|^p\right)\to 0,$$

as $n\to\infty$. In view of (6.9), (ii) is proved for $\partial_z\phi_n(z)$. □

Proof of (ii) *for* $\partial_z^2\phi_n(z)$. This proof is analogous to the previous one, but the computations are more involved. Let us just write the equation for the quantity of interest and the main steps. Define

$$u_z^{n,i,k}(t,x) = \frac{\partial}{\partial z^k} u_z^{n,i}(t,x); \qquad v_z^{n,i,k}(t,x) = \frac{\partial}{\partial z^k} v_z^{n,i}(t,x);$$
$$\theta_z^{n,i,k}(t,x) = \frac{\partial}{\partial z^k} \theta_z^{n,i}(t,x).$$

We have

$$\begin{aligned}
&u_z^{n,i,k}(t,x)\\
&\quad = \theta_z^{n,i,k}(t,x)\\
&\qquad + \int_{t-2^{-n}}^t \int_0^1 G_{t-s}(x,y)\varphi'(X(s,y)\circ T_z^n) u_z^{n,i}(s,y) h_n^k(s,y)\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad + \int_{t-2^{-n}}^t \int_0^1 G_{t-s}(x,y) u_z^{n,i}(s,y) u_z^{n,k}(s,y)[\psi''(X(s,y)\circ T_z^n)\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad + \varphi''(X(s,y)\circ T_z^n)(W(\mathrm{d}y,\mathrm{d}s) + \langle z, h_n(s,y)\rangle\,\mathrm{d}y\,\mathrm{d}s)]\\
&\qquad + \int_{t-2^{-n}}^t \int_0^1 G_{t-s}(x,y) u_z^{n,i,k}(s,y)[\psi'(X(s,y)\circ T_z^n)\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad + \varphi'(X(s,y)\circ T_z^n)(W(\mathrm{d}y,\mathrm{d}s) + \langle z, h_n(s,y)\rangle\,\mathrm{d}y\,\mathrm{d}s)]
\end{aligned}$$

and a similar (simpler) equation for $v_z^{n,i,k}$.

One first shows that

$$\sup_{|z|\leqslant\delta}\ \sup_{(t,x)\in[0,T]\times[0,1]} E|u_z^{n,i,k}(t,x) - v_z^{n,i,k}(t,x)|^p \to 0,$$

as $n\to\infty$, bound $v_z^{n,i,k}(t,x)$, and estimate the differences $u_z^{n,i,k}(t,x) - u_{z'}^{n,i,k}(t,x)$ and $v_z^{n,i,k}(t,x) - v_{z'}^{n,i,k}(t,x)$. The result follows as in the previous step. □

Appendix

We present here elementary facts related to the kernel $G_{t-s}(x,y)$.

$$G_{t-s}(x,y) \leqslant K e_{t,s}(x,y) \leqslant K' G_{t-s}(x,y), \tag{A.1}$$

where K, K' are some constants and $e_{t,s}(x,y)$ is the heat kernel

$$e_{t,s}(x,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{|x-y|^2}{2(t-s)}\right).$$

The first inequality follows by direct calculation and the second one by taking into account the term corresponding to $n = 0$ only in (2.4).

Let $f:\Omega \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ be a measurable function. Then, for every ε, $l > 0, x \in [0,1]$

$$\int_{t-\varepsilon}^{t} \int_{(x-l,x+l)^c} |G_{t-s}(x,y) f(s,y)|^2 \,\mathrm{d}y \,\mathrm{d}s \leqslant K\|f\|_2^2 \,\mathrm{e}^{-(l^2/2\varepsilon)}. \tag{A.2}$$

Since

$$\begin{aligned}\int_{t-\varepsilon}^{t} \frac{\mathrm{d}s}{\sqrt{2\pi(t-s)}} &\int_{x-\sqrt{\varepsilon}}^{x+\sqrt{\varepsilon}} e_{t,s}(x,y)\,\mathrm{d}y \\ &\geqslant \int_{t-\varepsilon}^{t} \frac{1}{\sqrt{2\pi}\sqrt{t-s}} \left(1 - \frac{t-s}{\varepsilon}\right) \mathrm{d}s = \frac{4\sqrt{\varepsilon}}{3\sqrt{2\pi}},\end{aligned}$$

one has

$$\int_{t-\varepsilon}^{t} \int_{x-\sqrt{\varepsilon}}^{x+\sqrt{\varepsilon}} G_{t-s}^2(x,y)\,\mathrm{d}y\,\mathrm{d}s \geqslant K_0\sqrt{\varepsilon}, \quad \forall x \in [0,1], \quad \varepsilon > 0. \tag{A.3}$$

A simple calculation based on (A.1) shows that for $q < \frac{3}{2}$

$$\sup_{(t,x)\in\Lambda_T} \int_0^t \int_0^1 G_{t-s}^{2q}(x,y)\,\mathrm{d}y\,\mathrm{d}s < \infty \tag{A.4}$$

and, for $0 \leqslant t - \varepsilon \leqslant t$

$$\int_{t-\varepsilon}^{t} \int_0^1 G_{t-s}^{2q}(x,y)\,\mathrm{d}y\,\mathrm{d}s \leqslant K\varepsilon^{(3/2)-q}. \tag{A.5}$$

Finally we give a discretization for $G_{t-s}(x,y)$. Let $t_n^+ = (l+1)/n$ and $t_n^- = l/n$, for $t \in [l/n, (l+1)/n)$. Then

$$\lim_{n\to\infty} \sup_{t,x} \int_0^{t_n^-} (G_{t_n^+ - s_n^-}(x,y) - G_{t-s}(x,y))^2 \,\mathrm{d}y\,\mathrm{d}s = 0. \tag{A.6}$$

Proof of (A.6). Let $\eta > 0$. One writes

$$\int_0^{t_n^-} \int_0^1 (G_{t_n^+ - s_n^-}(x,y) - G_{t-s}(x,y))^2 \,\mathrm{d}y\,\mathrm{d}s \leqslant \varepsilon_n(\eta) + \varepsilon_n'(\eta),$$

with

$$\varepsilon_n(\eta) = 4\int_{t_n^- - \eta}^{t_n^-} \int_0^1 (G_{t_n^+ - s_n^-}^2(x,y) + G_{t-s}^2(x,y))^2 \,\mathrm{d}y\,\mathrm{d}s$$

and

$$\varepsilon_n'(\eta) = \int_0^{t_n^- - \eta} \int_0^1 (G_{t_n^+ - s_n^-}(x,y) - G_{t-s}(x,y))^2 \,\mathrm{d}y\,\mathrm{d}s.$$

Then $\varepsilon_n(\eta) \leqslant K\eta^{1/2}$. On the other hand

$$|G_{t_n^+ - s_n^-}(x,y) - G_{t-s}(x,y)| \leqslant \sup \left| \frac{\partial}{\partial u} G_u(x,y) \right| \cdot \frac{2}{n},$$

with the sup over $u \in [t-s, t_n^+ - s_n^-]$.

An easy calculation shows that

$$\left| \frac{\partial}{\partial u} G_u(x,y) \right| \leqslant K(u^{-(3/2)} + u^{-(5/2)}) \leqslant K\eta^{-(5/2)} \quad \text{for} \quad u \geqslant \eta.$$

It follows that $\varepsilon_n'(\eta) \leqslant K\eta^{-5} \cdot n^{-2}$. So

$$\sup_{t,x} \int_0^{t_n^+} \int_0^1 (G_{t_n^+ - s_n^-}(x,y) - G_{t-s}(x,y))^2 \, \mathrm{d}y \, \mathrm{d}s \leqslant K(\eta^{1/2} + \eta^{-5} n^{-2}).$$

By taking $\overline{\lim}_{n\to\infty}$ first and letting $\eta \searrow 0$ then, (A.6) is proved. □

Acknowledgement

The authors wish to thank D. Nualart for a very fruitful discussion, which helped them to simplify an earlier version of the paper.

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