PROBABILISTIC INTERPRETATION OF A SYSTEM OF QUASILINEAR PARABOLIC PDES

A.B. SOW AND E. PARDOUX

LERSTAD, UFR S.A.T, Université Gaston Berger, B.P. 234 Saint-Louis, Sénégal.
email: absow@ugb.sn
LATP, CMI, 39, Rue F-Joliot Curie, 13453, Marseille cedex 13, France.
email: pardoux@cmi.univ-mrs.fr

Abstract. *Using a FBSDE associated to a transmutation process driven by a finite sequence of Poisson processes, we obtain a probabilistic interpretation for a non degenerate system of quasilinear parabolic PDEs. The novetly is that the linear second order differential operator is different on each line of the system.*

Keywords: System of parabolic quasilinear PDEs; forward-backward stochastic differential equations; Poisson process; Malliavin Calculus.

1. Introduction

In this paper we are interested in the problem of providing a stochastic representation of the solutions of the following system of quasilinear parabolic PDEs

$$(\mathcal{E}) \quad \begin{cases} \forall\, (t,x) \in [0,T] \times \mathbf{R}^d, \ \ \forall\, i \in \mathcal{K} \\ \dfrac{\partial u_i}{\partial t}(t,x) \ + L^i\, u_i(t,x) + \bar{f}_i(t,x,u(t,x), \nabla_x\, u_i(t,x)\sigma_i(t,x,u_i(t,x))) = 0 \\ u_i(T,x) = h_i(x) \end{cases}$$

where the coefficients $\bar{b}, \bar{f}, \sigma$ and h are given from above, the second order differential operator L is defined by

$$\begin{aligned} \forall\, (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R}, \quad L^i\, u_i(t,x) \ &= \ \frac{1}{2} \sum_{j,m=1}^{d} a_i(t,x,u_i(t,x))_{jm}\, \frac{\partial^2\, u_i(t,x)}{\partial x_j \partial x_m} \\ &+ \ \sum_{j=1}^{d} (\bar{b}_i)^j(t,x,u(t,x), \nabla_x\, u_i(t,x)\, \sigma_i(t,x,u_i(t,x)))\, \frac{\partial\, u_i(t,x)}{\partial x_j} \end{aligned}$$

and for $i \in \mathcal{K}$, $\forall\, (t,x,y) \in [0,T] \times \mathbf{R}^d \times \mathbf{R}$, the matrix $a_i(t,x,y) = (\sigma_i\, \sigma_i^*)(t,x,y)$ is supposed to be uniformly elliptic. Moreover we give sufficient conditions which ensures existence and uniqueness of a solution the system of forward-backward stochastic differential equations $(E^{0,\xi})$ (see (2.1)) whatever the time duration T may be.
It is well known that Backward Stochastic Differential Equations (BSDEs in short) are closely connected to Partial Differential Equations (PDEs in short). This link between PDEs and BSDEs has been used in both ways. In the first case, regularity results for solutions of PDEs have been used in the study of solutions of BSDEs. On th eother hand, there appeared recently some probabilistic methods for systems of PDEs. An essential tool in this new method is the Malliavin Calculus. For example Pardoux and Peng [7] prove that BSDEs provide a probabilistic formula for solutions of certain classes of quasilinear parabolic PDEs.

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BSDEs with Poisson Process were first discussed by Tang and Li [12]. Studying such equations, Barles, Buckdahn and Pardoux [1] generalized the result of Pardoux Peng [7] and obtained a probabilistic interpretation of a solution of a parabolic integral-partial differential equation.
Soon after appeared fully coupled forward-backward stochastic differential equations (FBSDEs in short). They were originally motivated by stochastic optimal control theory. In his PhD thesis Antonelli obtained the first result on the solvability of a FBSDE over a small time duration. Using the strong link between the FBSDEs and a quasilinear parabolic PDEs, Ma, Protter and Yong in their so called "four step scheme" prove successfully an existence and uniqueness result of a FBSDE. But they require non–degeneracy assumption on the diffusion of the forward equation, non–randomness and rather stringent smoothness of the coefficients.
Recently in the same spirit, Delarue [2] proved a similar result under rather weaker assumptions than those required in the "four step scheme", and provided in a probabilistic manner a solution of a non–degenerate system of quasilinear parabolic PDEs. The key device of his method based on an iterative scheme is an efficient control of the length of the interval on which the result on small time duration holds, which ensures existence and uniqueness of solutions on arbitrary time intervals. For this to be true, the non–degeneracy of the forward diffusion coefficient is crucial.
All those systems of PDEs studied so far have the same linear second order differential operator on each line. Pardoux, Pardeilles and Rao [9] removed this restriction in the case of BSDEs and provided a stochastic representation of a viscosity solution of a system of semilinear parabolic PDEs with a different second order differential operator from one line to another. This is done by coupling the forward diffusion with a transmutation process driven by a finite sequence of Poisson Processes. We aim in this work to get a similar result in the case of non–degenerate parabolic quasilinear systems through the study of a corresponding system of FBSDEs following the method developed by Delarue [2].
The paper is organized as follows. In section 2 we prove the existence and uniqueness of a solution of a system of FBSDE with respect to a Brownian motion and a finite sequence of Poisson Processes for a small time duration. We also present some properties of the solutions before extending this result to the case of random coefficients. Studying the case of smooth coefficients, we establish by a stochastic method our main result (existence and uniqueness under the non–degeneracy assumption) of the solution of the system of PDEs. Finally in Section 3 we extend the local solution of the FBSDE $(E^{0,\xi})$ obtained in the previous section to a global one.

2. FBSDE with Poisson Process in small time duration

$\forall\ Q \in \mathbf{N}^*$, $|\ .\ |$ stands for the euclidian norm in $\mathbf{R}^Q$.
Let $k \geq 2$ be an integer and $\mathcal{K} = \{1, 2, ..., k\}$, $L = \mathcal{K} - \{k\}$.

We consider a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ such that the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is generated by two mutually independent processes:

- a d-dimensional Brownian motion $(B_t)_{0\leq t\leq T}$.
- a Poisson random measure N on $\mathbf{R}_+ \times L$ where L is the set of marks equipped with the field $\mathcal{L}$ of all subsets of L such that $M([0,t]\times A) = N\big((0,t)\times A\big) - t\,\lambda\, cardA$ is a $\mathcal{F}_t-$ martingale for all $A \subset L$ and some fixed $\lambda > 0$.

Moreover we consider a filtration $(\mathcal{G}_t)_{0\leq t\leq T}$ satisfying the usual conditions such that $(B_t)_{0\leq t\leq T}$ still a $(\mathcal{G}_t)$-Brownian motion and the filtration $(\mathcal{G}^0_t)_{0\leq t\leq T}$ defined by $\forall\, t \in [0,T]$ $\mathcal{G}^0_t = \mathcal{G}_0 \vee \mathcal{F}_t$.

For $0 \leq t < s \leq T$, $i \in \mathcal{K}$ and $l \in L$, we define

$$N_s = N((0,s]\times L), \quad N_s(l) = N((0,s]\times\{l\}), \quad M_s(l) = N_s(l) - \lambda s$$

We define the Markov process $N^{t,i}_s$ by

$$N^{t,i}_s = i + \sum_{l=1}^{k-1} l N((t,s]\times\{l\}) \quad mod[k] \ \text{ and let } \ N^{0,i}_s = N^i_s$$

Let $\mathcal{P}$ denote the $\sigma-$–algebra of $\mathcal{G}_t$–predictable subsets of $\Omega \times [0,T]$.
For each $i \in \mathcal{K}$, we are given $\bar{b}_i \in \mathcal{C}([0,T]\times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \mathbf{R}^d)$, $\bar{f}_i \in \mathcal{C}([0,T]\times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \mathbf{R})$, $\sigma_i \in \mathcal{C}([0,T]\times \mathbf{R}^d \times \mathbf{R}, \mathbf{R}^d \times \mathbf{R}^d)$ and $h_i \in \mathcal{C}(\mathbf{R}^d, \mathbf{R})$.
We define the following continuous functions on their domains

$$\begin{array}{lcl} \bar{b} : [0,T]\times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K} & \longrightarrow & \mathbf{R}^d \\ \bar{f} : [0,T]\times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K} & \longrightarrow & \mathbf{R} \\ \sigma : [0,T]\times \mathbf{R}^d \times \mathbf{R} \times \mathcal{K} & \longrightarrow & \mathbf{R}^d \times \mathbf{R}^d \\ h : \mathbf{R}^d \times \mathcal{K} & \longrightarrow & \mathbf{R} \end{array}$$

by putting $\forall\ (t,x,y,u,z,i) \in [0,T]\times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}^d \times \mathcal{K}$, $\delta(x,u,z,i) = \delta_i(x,u,z)$ for $\delta = \bar{b}, \bar{f}$ and $\sigma(\text{resp}\, h)(t,x,y,i) = \sigma_i(t,x,y)(\text{resp}\, h_i(x))$.
Let $i \in \mathcal{K}$; we define the functions b_i, f_i and $\tilde{f}_i \in \mathcal{C}([0,T]\times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d)$ such that

$$\begin{aligned} &\forall\ (t,x) \in [0,T]\times \mathbf{R}^d,\ \ \forall\ u \in \mathbf{R}^k, \\ &\qquad \bar{f}_i(t,x,u_1,u_2,...,u_k,z) = \tilde{f}_i(t,x,u_i,h^i,z) = f_i(t,x,u_i,h^i,z) + \lambda \sum_{l\in L} h^i(l) \\ &\qquad \bar{b}_i(t,x,u_1,u_2,...,u_k,z) = b_i(t,x,u_i,h^i,z) \\ &\qquad \text{where} \qquad h^i_j = \begin{cases} u_{i+j} - u_i & 1 \le j \le k-i \\ u_{i+j-k} - u_i & k-i+1 \le j \le k-1 \end{cases} \end{aligned}$$

For $i \in \mathcal{K}$ and $\xi \in \mathbf{R}^d$ a $\mathcal{G}_0$-measurable random vector satisfying $\mathbf{E}(|\xi|^2) < \infty$, we intend to find a $\mathcal{G}_t$–progressively measurable quartet of processes $(\Theta^{0,\xi,i}_s)_{0\le s\le T} \triangleq (X^{0,\xi,i}_s, Y^{0,\xi,i}_s, H^{0,\xi,i}_s, Z^{0,\xi,i}_s)_{0\le s\le T}$ with values in $\mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$ solution of the problem

$$(2.1)\quad (E^{0,\xi}) \begin{cases} \forall\ s \in [0,T] \\ X^{0,\xi,i}_s = \xi + \displaystyle\int_0^s b(r,\Theta^{0,\xi,i}_r, N^i_r)dr + \int_0^s \sigma(r, X^{0,\xi,i}_r, Y^{0,\xi,i}_r, N^i_r)\, dB_r \\ Y^{0,\xi,i}_s = h_{N^i_T}(X^{0,\xi,i}_T) + \displaystyle\int_s^T f(r,\Theta^{0,\xi,i}_r, N^i_r)dr - \int_s^T Z^{0,\xi,i}_r\, dB_r - \int_s^T \sum_{l\in L} H^{0,\xi,i}_r(l)\, dM_r(l) \\ \mathbf{E}\displaystyle\int_0^T (|X^{0,\xi,i}_t|^2 + |Y^{0,\xi,i}_t|^2 + |Z^{0,\xi,i}_t|^2 + |H^{0,\xi,i}_t|^2)dt < \infty \end{cases}$$

where $\forall\ s \in [0,T]$, $H^{0,\xi,i}_s = (H^{0,\xi,i}_s(1),\ H^{0,\xi,i}_s(2), ...,\ H^{0,\xi,i}_s(k-1))$.
The superscript ${}^{0,\xi,i}$ indicates the dependence of the solution on the initial date $(0,\xi,i)$ and will often be omitted for notational simplicity.
This section is devoted to the study of $(E^{0,\xi})$ in small time duration. In all of what follows, we assume that $T \le 1$.

2.1. **The case of non smooth coefficients.**

2.1.1. *Existence and uniqueness of solutions.* Let us first prove existence and uniqueness of solutions. The idea is based on a fixed point argument on a suitable space in the spirit of the method developped by Delarue [2], Théorème 1.1.
We say that the functions b, f, h and σ satisfy assumption **(H1)**, if there exist two constants K and Λ such that

(H1.1) : $\forall t \in [0,T], \forall (x,y,u,z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K}$ and $\forall (x',y',u',z') \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$,

$$\begin{array}{rcl} |b_i(t,x,y,u,z) - b_i(t,x,y',u',z')| & \leq & K(|y-y'| + |z-z'| + |u-u'|) \\ |f_i(t,x,y,u,z) - f_i(t,x',y,u',z')| & \leq & K(|x-x'| + |z-z'| + |u-u'|) \\ |\sigma_i(t,x,y) - \sigma_i(t,x',y')|^2 & \leq & K^2(|x-x'|^2 + |y-y'|^2) \\ |h_i(x) - h_i(x')| & \leq & K|x-x'| \end{array}$$

(H1.2) $\forall t \in [0,T], \forall (x,y,u,z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K}$ et $\forall (x',y') \in \mathbf{R}^d \times \mathbf{R}$,

$$\begin{array}{rcl} < x-x', b_i(t,x,y,u,z) - b_i(t,x',y,u,z) > & \leq & K|x-x'|^2 \\ (y-y')(f_i(t,x,y,u,z) - f_i(t,x,y',u,z)) & \leq & K|y-y'|^2 \end{array}$$

(H1.3) $\forall t \in [0,T], \forall (x,y,u,z) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d, \forall i \in \mathcal{K}$,

$$|b_i(t,x,y,u,z)| + |f_i(t,x,y,u,z)| + |\sigma_i(t,x,y)| + |h_i(x)| \leq \Lambda$$

Theorem 2.1. *Assume that* **(H1)** *holds and let $i \in \mathcal{K}$. Then for every $\mathcal{G}_0-$measurable random vector ξ satisfying $\mathbf{E}|\xi|^2 < \infty$, a solution of $(E^{0,\xi})$ satisfies*
i) $(X_s)_{0\leq s\leq T}$ is continuous and $(Y_s)_{0\leq s\leq T}$ has a càdàg version.

ii) $\mathbf{E}(\sup_{0\leq t\leq T} |X_t|^2 + \sup_{0\leq t\leq T} |Y_t|^2) < \infty.$

Moreover there exists a constant $C^{(i1)}_{K,\lambda} > 0$ depending on K and λ such that for every $T \leq C^{(i1)}_{K,\lambda}$, the problem $(E^{0,\xi})$ admits a unique solution.

Proof : As explained, our aim is to construct a contraction in a suitable space. To this end for $T > 0$ and $Q \in \mathbf{N}$, we consider the following sets

- $H^2_T(\mathbf{R}^Q)$ the space of $\{\mathcal{G}^0_t\}$ progressively measurable processes
$$\Psi : [0,T] \times \Omega \longrightarrow \mathbf{R}^Q \ , ||\Psi||^2 = \mathbf{E} \int_0^T |\Psi_t|^2 \, dt < \infty$$
- $S^2_T(\mathbf{R}^Q)$ the space of continuous $\{\mathcal{G}^0_t\}$ adapted processes
$$\Psi : [0,T] \times \Omega \longrightarrow \mathbf{R}^Q \ , ||\Psi||^2_2 = \mathbf{E}(\sup_{0\leq t\leq T} |\Psi_t|^2) < \infty$$
- $\mathcal{S}^2_T(\mathbf{R}^Q)$ the space of $\{\mathcal{G}^0_t\}$ adapted càdlàg processes
$$\Psi : [0,T] \times \Omega \longrightarrow \mathbf{R}^Q \ , ||\Psi||^2_2 = \mathbf{E}(\sup_{0\leq t\leq T} |\Psi_t|^2) < \infty$$
- $[L^2(\mathcal{P} \otimes \mathcal{L})]^{k-1}$ the space of mappings $H : \Omega \times [0,T] \times L \longrightarrow \mathbf{R}$ which are $\mathcal{P} \otimes \mathcal{L}$-measurable such that
$$||H||^2_{[L^2(\mathcal{P}\otimes\mathcal{L})]^{k-1}} = \mathbf{E} \sum_{l\in L} \int_0^T (H_t(l))^2 \, dt < \infty$$
- $\forall \, p \geq 1$, $\mathcal{B}^{2p}_{[0,T]}$ the space $S^2_T(\mathbf{R}^d) \times \mathcal{S}^2_T(\mathbf{R}) \times [L^2(\mathcal{P} \otimes \mathcal{L})]^{k-1} \times H^2_T(\mathbf{R}^d)$ endowed with the norm

$$\begin{array}{rcl} \left\|(X_s,Y_s,H_s,Z_s)\right\|_{\mathcal{B}^{2p}_{[0,T]}} & = & \left(\mathbf{E} \sup_{0\leq s\leq T} |X_s|^{2p}\right)^{\frac{1}{2p}} + \left(\mathbf{E} \sup_{0\leq s\leq T} |Y_s|^{2p}\right)^{\frac{1}{2p}} + \left(\mathbf{E}\left[\left(\int_0^T \sum_{l\in L} (H_s(l))^2 \, ds\right)^{1/2}\right]^p\right)^{\frac{1}{p}} \\ & + & \left(\mathbf{E}\left[\left(\int_0^T |Z_s|^2 \, ds\right)^{1/2}\right]^p\right)^{\frac{1}{p}} \end{array}$$

Notice that $\mathcal{B}^{2p}_{[0,T]}$ endowed with this norm is a Banach space.
We define a map φ on $\mathcal{B}^2_{[0,T]}$ into itself as follows

$$\begin{array}{ccc} \mathcal{B}^2_{[0,T]} & \longrightarrow & \mathcal{B}^2_{[0,T]} \\ (X,Y,H,Z) & \longrightarrow & (\bar{X}^{0,\xi,i}, \bar{Y}^{0,\xi,i}, \bar{H}^{0,\xi,i}, \bar{Z}^{0,\xi,i}) \end{array}$$

where the quartet $(\bar{\Theta}^{0,\xi,i}_s)_{0\le s\le T} \triangleq (\bar{X}^{0,\xi,i}_s, \bar{Y}^{0,\xi,i}_s, \bar{H}^{0,\xi,i}_s, \bar{Z}^{0,\xi,i}_s)_{0\le s\le T}$ is defined by

$$\left\{ \begin{array}{l} \forall t \in [0,T] \\ \bar{X}^{0,\xi,i}_t = \xi + \displaystyle\int_0^t b(s, \bar{X}^{0,\xi,i}_s, Y_s, H_s, Z_s, N^i_s) ds + \int_0^t \sigma(s, \bar{X}^{0,\xi,i}_s, Y_s, N^i_s) dB_s \end{array} \right. \tag{2.2}$$

and

$$\left\{ \begin{array}{l} \forall t \in [0,T] \\ \bar{Y}^{0,\xi,i}_t = h_{N^i_T}(\bar{X}^{0,\xi,i}_T) + \displaystyle\int_t^T f(s, \bar{\Theta}^{0,\xi,i}_s, N^i_s) ds - \int_t^T \bar{Z}^{0,\xi,i}_s \, dB_s - \int_t^T \sum_{l\in L} \bar{H}^{0,\xi,i}_s(l) \, dM_s(l) \end{array} \right. \tag{2.3}$$

Note that the process $\bar{X}^{0,\xi,i}_s$ is a solution of a SDE with initial time 0 and initial values $(\xi,\ i)$ (i initial value of the process $N^{0,i}_s$), whereas the triplet $(\bar{Y}^{0,\xi,i}_s, \bar{H}^{0,\xi,i}_s, \bar{Z}^{0,\xi,i}_s)$ solves a BSDE with Poisson process.

In the sequel of the proof, we denote

$$\varphi(X,Y,H,Z) = (\bar{X}_s, \bar{Y}_s, \bar{H}_s, \bar{Z}_s) = \bar{\Theta}_s; \quad \varphi(U,V,G,W) = (\bar{U}_s, \bar{V}_s, \bar{G}_s, \bar{W}_s) = \bar{\Theta}'_s$$

Using Itô's formula, we get

$$\begin{array}{rcl} |\bar{X}_s - \bar{U}_s|^2 & = & 2\displaystyle\int_0^s < \bar{X}_r - \bar{U}_r, b(r, \bar{X}_r, Y_r, Z_r, H_r, N^i_r) - b(r, \bar{U}_r, V_r, W_r, G_r, N^i_r) > dr \\ & + & 2\displaystyle\int_0^s < \bar{X}_r - \bar{U}_r, (\sigma(r, \bar{X}_r, Y_r, N^i_r) - \sigma(r, \bar{U}_r, V_r, N^i_r)) dB_r > \\ & + & \displaystyle\int_0^s |\sigma(r, \bar{X}_r, Y_r, N^i_r) - \sigma(r, \bar{U}_r, V_r, N^i_r)|^2 dr \end{array}$$

Applying the Burkhölder-Davis-Gundy inequality and assumption (H1.1), we find $\delta > 0$ such that

$$\begin{array}{rcl} \mathbf{E} \displaystyle\sup_{0\le t\le T} |\bar{X}_t - \bar{U}_t|^2 & \le & 2K\mathbf{E}\displaystyle\int_0^T |\bar{X}_r - \bar{U}_r| \Big(|\bar{X}_r - \bar{U}_r| + |Y_r - V_r| + |Z_r - W_r| + |H_r - G_r| \Big) dr \\ & + & 2\,\delta\, K\mathbf{E}\Big[\displaystyle\int_0^T |\bar{X}_r - \bar{U}_r|^2 (|\bar{X}_r - \bar{U}_r|^2 + |Y_r - V_r|^2) dr \Big]^{\frac{1}{2}} \\ & + & K^2\mathbf{E}\displaystyle\int_0^T (|\bar{X}_r - \bar{U}_r|^2 + |Y_r - V_r|^2) dr \end{array}$$

so using standard estimates, we prove the existence of a constant γ_K depending on K such that

$$\begin{array}{rcl} \mathbf{E} \displaystyle\sup_{0\le t\le T} |\bar{X}_t - \bar{U}_t|^2 & \le & \gamma_K T^{1/2} \Big[\mathbf{E} \displaystyle\sup_{0\le t\le T} |\bar{X}_t - \bar{U}_t|^2 + \mathbf{E} \sup_{0\le t\le T} |Y_t - V_t|^2 + \mathbf{E} \int_0^T |Z_r - W_r|^2 \, dr \\ & + & \mathbf{E} \displaystyle\sum_{l\in L} \int_0^T ((H_r - G_r)(l))^2 \, dr \Big] \end{array}$$

which implies

$$(2.4)\qquad (1-\gamma_K T^{1/2})\mathbf{E}\sup_{0\le t\le T}|\bar{X}_t-\bar{U}_t|^2 \le \gamma_K T^{1/2}\Big[\mathbf{E}\sup_{0\le t\le T}|Y_t-V_t|^2+\mathbf{E}\int_0^T|Z_r-W_r|^2\,dr + \mathbf{E}\sum_{l\in L}\int_0^T((H_r-G_r)(l))^2\,dr\Big]$$

Moreover Itô's formula yields from (2.3), for all $t\in[0,T]$

$$(2.5)\qquad |\bar{Y}_t-\bar{V}_t|^2 + \int_t^T|\bar{Z}_s-\bar{W}_s|^2\,ds+\lambda\int_t^T\sum_{l\in L}((\bar{H}_s-\bar{G}_s)(l))^2ds+\sum_{t<s\le T}(\Delta(\bar{Y}_s-\bar{V}_s))^2$$
$$= |h_{N_T^i}(\bar{X}_T)-h_{N_T^i}(\bar{U}_T)|^2+2\int_t^T(\bar{Y}_s-\bar{V}_s)(f(s,\bar{\Theta}_s,N_s^i)-f(s,\bar{\Theta}'_s,N_s^i))ds$$
$$- 2\int_t^T(\bar{Y}_s-\bar{V}_s)(\bar{Z}_s-\bar{W}_s)\,dB_s-2\int_t^T\sum_{l\in L}(\bar{Y}_{s^-}-\bar{V}_{s^-})(\bar{H}_s-\bar{G}_s)(l)dM_s(l)$$

Applying (H1.1) and (H1.2), we deduce

$$\mathbf{E}\int_0^T|\bar{Z}_s-\bar{W}_s|^2\,ds + \lambda\mathbf{E}\int_0^T\sum_{l\in L}((\bar{H}_s-\bar{G}_s)(l))^2ds \le K^2\,\mathbf{E}\sup_{0\le t\le T}|\bar{X}_t-\bar{U}_t|^2$$
$$+ 2K\Big[\mathbf{E}\int_0^T|\bar{Y}_s-\bar{V}_s||\bar{X}_s-\bar{U}_s|ds+\mathbf{E}\int_0^T|\bar{Y}_s-\bar{V}_s|^2\,ds$$
$$+ \mathbf{E}\int_0^T|\bar{Y}_s-\bar{V}_s||\bar{Z}_s-\bar{W}_s|\,ds+\mathbf{E}\int_0^T|\bar{Y}_s-\bar{V}_s||\bar{H}_s-\bar{G}_s|\,ds\Big]$$

Hence using standard estimates, we find a constant $\gamma_{(K,\lambda)}$ depending on K and λ such that

$$(2.6)\qquad \mathbf{E}\int_0^T|\bar{Z}_s-\bar{W}_s|^2\,ds + \lambda\mathbf{E}\int_0^T\sum_{l\in L}((\bar{H}_s-\bar{G}_s)(l))^2\,ds \le$$
$$\gamma_{(K,\lambda)}\Big[(1+T)\mathbf{E}\sup_{0\le t\le T}|\bar{X}_t-\bar{U}_t|^2 + T\mathbf{E}\sup_{0\le t\le T}|\bar{Y}_t-\bar{V}_t|^2\Big]$$

Futhermore using (2.5) we obtain

$$(2.7)\qquad \mathbf{E}\sup_{0\le t\le T}|\bar{Y}_t-\bar{V}_t|^2 \le \mathbf{E}(|h_{N_T^i}(\bar{X}_T)-h_{N_T^i}(\bar{U}_T)|^2)$$
$$+ 2\,\mathbf{E}\sup_{0\le t\le T}\left(\int_t^T(\bar{Y}_s-\bar{V}_s)(f(s,\bar{\Theta}_s,N_s^i)-f(s,\bar{\Theta}'_s,N_s^i))ds\right)$$
$$+ 2\mathbf{E}\sup_{0\le t\le T}|\int_t^T(\bar{Y}_s-\bar{V}_s)(\bar{Z}_s-\bar{W}_s)dB_s|$$
$$+ 2\,\mathbf{E}\sup_{0\le t\le T}|\int_t^T\sum_{l\in L}(\bar{Y}_{s^-}-\bar{V}_{s^-})(\bar{H}_s-\bar{G}_s)(l)dM_s(l)|$$

From the Burkhölder-Davis-Gundy inequality, there exists a constant $C>0$ which can change from line to line such that

$$2\mathbf{E}\sup_{0\le t\le T}|\int_t^T(\bar{Y}_s-\bar{V}_s)(\bar{Z}_s-\bar{W}_s)dB_s| \le C\mathbf{E}\left(\int_0^T|\bar{Y}_s-\bar{V}_s|^2\,|\bar{Z}_s-\bar{W}_s|^2\right)^{\frac{1}{2}}$$
$$\le \frac{1}{4}\mathbf{E}\sup_{0\le s\le T}|\bar{Y}_s-\bar{V}_s|^2+4C^2\mathbf{E}\int_0^T|\bar{Z}_s-\bar{W}_s|^2\,ds$$

Similarly, using again the Burkhölder-Davis-Gundy inequality for discontinuous processes and standard estimates where

$$\hat{M}_t(l) = (\bar{Y}_{s^-} - \bar{V}_{s^-})(\bar{H}_s - \bar{G}_s)(l)\, dM_s(l),$$

there exists another constant which we note C again changing from line to line such that

$$\begin{aligned}
(2.8) \qquad 2\mathbf{E} \sup_{0\le t\le T} |\int_t^T \sum_{l\in L} (\bar{Y}_{s^-} &- \bar{V}_{s^-})(\bar{H}_s - \bar{G}_s)(l)\, dM_s(l)| \le C\,\mathbf{E}\left[\left(\sum \Delta\hat{M}_t(l)^2\right)^{\frac{1}{2}}\right] \\
&= C\,\mathbf{E}\left[\int_0^T \sum_{l\in L} (\bar{Y}_{s^-} - \bar{V}_{s^-})^2((\bar{H}_s - \bar{G}_s)(l))^2\, dN_s(l)\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\mathbf{E} \sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2 + C^2\,\lambda\mathbf{E}\sum_{l\in L}\int_0^T ((\bar{H}_s - \bar{G}_s)(l))^2\, ds
\end{aligned}$$

Hence combining these two previous inequalities with (2.6), we deduce from (2.7)

$$\begin{aligned}
\mathbf{E} \sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2 &+ \mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2\, ds + \lambda\mathbf{E}\int_0^T \sum_{l\in L} ((\bar{H}_s - \bar{G}_s)(l))^2\, ds \le \\
& 2K\mathbf{E}\int_0^T |\bar{Y}_s - \bar{V}_s|\big(|\bar{X}_s - \bar{U}_s| + |\bar{Y}_s - \bar{V}_s| + |\bar{Z}_s - \bar{W}_s| + |\bar{H}_s - \bar{G}_s|\big) ds \\
&+ K^2\,\mathbf{E}\sup_{0\le s\le T} |\bar{X}_t - \bar{U}_t|^2 + \frac{1}{2}\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2 \\
&+ C\,\gamma_{(K,\lambda)}\left[(1+T)\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2\right]
\end{aligned}$$

which implies using standard estimates and modifying $\gamma_{(K,\lambda)}$

$$\begin{aligned}
(2.9) \qquad \mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2 &\le 2\gamma_{(K,\lambda)}\left[(1+T)\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2\right] \\
&+ C\,\gamma_{(K,\lambda)}\left[(1+T)\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2 + T\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2\right] \\
&+ \frac{1}{2}\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2
\end{aligned}$$

hence putting $\delta_{(K,\lambda)} = 2(2\,\gamma_{(K,\lambda)} + C\,\gamma_{(K,\lambda)})$, we get

$$(2.10) \qquad (1 - \delta_{(K,\lambda)}T)\,\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2 \le \delta_{(K,\lambda)}(1+T)\,\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2$$

and the following inequalities

$$\begin{aligned}
(2.11) \qquad \mathbf{E}\int_0^T |\bar{Z}_s - \bar{W}_s|^2\, ds + \mathbf{E}\sum_{l\in L}\int_0^T ((\bar{H}_s &- \bar{G}_s)(l))^2\, ds \le \gamma_{(K,\lambda)}\Big[(1+T)\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2 \\
&+ T\mathbf{E}\sup_{0\le s\le T} |\bar{Y}_s - \bar{V}_s|^2\Big]
\end{aligned}$$

$$\begin{aligned}
(2.12) \qquad (1 - \gamma_K\,T^{1/2})\,\mathbf{E}\sup_{0\le s\le T} |\bar{X}_s - \bar{U}_s|^2 &\le \gamma_K\,T^{1/2}\Big[\mathbf{E}\sup_{0\le s\le T} |Y_s - V_s|^2 + \mathbf{E}\int_0^T |Z_s - W_s|^2\, ds \\
&+ \mathbf{E}\sum_{l\in L}\int_0^T ((H_s - G_s)(l))^2\, ds\Big]
\end{aligned}$$

This proves that there exists a constant $C^{(i1)}_{(K,\lambda)} > 0$ depending on K and λ such that for $T \le C^{(i1)}_{(K,\lambda)}$, the map φ is a contraction from $\mathcal{B}^2_{[0,T]}$ into itself.

Consequently by the fixed point theorem, there exists a unique $\{\mathcal{G}_t^0\}_{0\le t\le T}$ -progressively measurable quartet of processes with values in $(\mathbf{R}^d, \mathbf{R}, \mathbf{R}^{k-1}, \mathbf{R}^d)$ solution to $(E^{0,\xi})$.
Note that this solution is obviously $\mathcal{G}_t-$progressively measurable. ■

Remark: This theorem shows that for every $T \le C^{(i1)}_{(K,\lambda)}$ and for every $x \in \mathbf{R}^d$ the problem $(E^{0,x})$ with x as initial value of the SDE admits a unique solution $\Theta_t^{0,x,i} \triangleq (X_t^{0,x,i}, Y_t^{0,x,i}, H_t^{0,x,i}, Z_t^{0,x,i})_{0\le t\le T}$ which is a $\{\mathcal{F}_t\}_{0\le t\le T}-$ progressively measurable quartet of processes. In particular the $\mathcal{F}_0$-progressively measurable random variable $Y_0 = Y_0^{0,x,i}$ is deterministic.

Now we establish some properties of the solution, which will be useful in the sequel.

2.1.2. *A priori estimates.*

Theorem 2.2. *Suppose that (b, f, h, σ) and $(\tilde{b}, \tilde{f}, \tilde{h}, \tilde{\sigma})$ are two quartets of functions satisfying* **(H1)** *with the same constants K and Λ. For $i \in \mathcal{K}$, let $\Theta_s^{0,\xi,i} = (X_s^{0,\xi,i}, Y_s^{0,\xi,i}, H_s^{0,\xi,i}, Z_s^{0,\xi,i})_{(0\le s\le T)}$ (resp $\tilde{\Theta}_s^{0,\tilde{\xi},i} = (\tilde{X}_s^{0,\tilde{\xi},i}, \tilde{Y}_s^{0,\tilde{\xi},i}, \tilde{H}_s^{0,\tilde{\xi},i}, \tilde{Z}_s^{0,\tilde{\xi},i})_{(0\le s\le T)}$) be the associated solution to (b, f, h, σ) (resp $(\tilde{b}, \tilde{f}, \tilde{h}, \tilde{\sigma})$) with initial condition $(0, \xi, i)$ (resp $(0, \tilde{\xi}, i)$).*
Then there exists two constants $C^{(i2)}_{(K,\lambda)}$ and $\beta^{(i1)}_{(K,\lambda)}$ depending on K and λ such that for $T \le C^{(i2)}_{(K,\lambda)}$ and for every $A \in \mathcal{G}_0$, the following estimate holds

$$
\begin{aligned}
(2.13)\qquad \mathbf{E}\,(1_A \sup_{0\le s\le T} |X_s &- \tilde{X}_s|^2) + \mathbf{E}\,(1_A \sup_{0\le s\le T} |Y_s - \tilde{Y}_s|^2) + \mathbf{E} \sum_{l\in L} \int_0^T 1_A((H_s - \tilde{H}_s)(l))^2\, ds \\
&+ \mathbf{E} \int_0^T 1_A\, |Z_s - \tilde{Z}_s|^2\, ds \le \beta^{(i1)}_{(K,\lambda)} \Big[\mathbf{E}(1_A|\xi - \tilde{\xi}|^2) + \mathbf{E}(1_A|(h - \tilde{h})(X_T)|^2) \\
&+ \mathbf{E} \int_0^T 1_A|(\sigma - \tilde{\sigma})(s, X_s, Y_s, N_s^i)|^2\, ds \\
&+ \mathbf{E} \Big(\int_0^T 1_A \big(|b - \tilde{b}| + |f - \tilde{f}|\big)(s, \Theta_s^{0,\xi,i}, N_s^i)\, ds \Big)^2 \Big]
\end{aligned}
$$

Proof : Let $A \in \mathcal{G}_0$. Using Itô's formula we have

$$
\begin{aligned}
\mathbf{E}\,(1_A \sup_{0\le s\le T} |X_s - \tilde{X}_s|^2) \quad &\le \quad \mathbf{E}\,(\,1_A|\xi - \tilde{\xi}|^2\,) + \mathbf{E} \int_0^T 1_A|\sigma_{N_s^i}(s, X_s, Y_s) - \tilde{\sigma}_{N_s^i}(s, \tilde{X}_s, \tilde{Y}_s)|^2\, ds \\
&+ \quad 2\mathbf{E} \sup_{0\le t\le T} \Big(\int_0^t 1_A < X_s - \tilde{X}_s, b(s, \Theta_s^{0,\xi,i}, N_s^i) - \tilde{b}(s, \tilde{\Theta}_s^{0,\tilde{\xi},i}, N_s^i) > ds \Big) \\
&+ \quad 2\mathbf{E} \sup_{0\le t\le T} \Big(\int_0^t 1_A < X_s - \tilde{X}_s, \big(\sigma_{N_s^i}(s, X_s, Y_s) - \tilde{\sigma}_{N_s^i}(s, \tilde{X}_s, \tilde{Y}_s)\big)\, dB_s > \Big)
\end{aligned}
$$

Thanks to Burkhölder-Davis-Gundy inequalities, assumption (H1) and standard estimates, there exists a constant γ_K depending on K such that

$$
\begin{aligned}
(2.14)\quad \mathbf{E}(1_A \sup_{0\le s\le T} |\tilde{X}_s &- X_s|^2) \le \gamma_K \Big[\mathbf{E}(1_A|\xi - \tilde{\xi}|^2) + \mathbf{E} \int_0^T 1_A\big(|\tilde{X}_s - X_s|^2 + |\tilde{Y}_s - Y_s|^2\big) ds \\
&+ \mathbf{E} \int_0^T 1_A|\tilde{X}_s - X_s|\, (|\tilde{Z}_s - Z_s| + |\tilde{H}_s - H_s|)\, ds \\
&+ \mathbf{E} \int_0^T 1_A|(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N_s^i)|^2\, ds + \mathbf{E} \Big(\int_0^T 1_A|(\tilde{b} - b)(s, \Theta_s^{0,\xi,i}, N_s^i)|\, ds \Big)^2 \Big]
\end{aligned}
$$

Futhermore applying Itô's formula to (2.3), we have $\forall\, t \in [0,T]$,

$$\begin{aligned}(2.15)\qquad |\tilde{Y}_t &- Y_t|^2 + \int_t^T |\tilde{Z}_s - Z_s|^2\,ds + \lambda \sum_{l\in L} \int_t^T (\tilde{H}_s - H_s)(l))^2\,ds + \sum_{t<s\le T} (\Delta(\tilde{Y}_s - Y_s))^2 \\ &= |\tilde{h}(\tilde{X}_T, N_T^i) - h(X_T, N_T^i)|^2 + 2\int_t^T (\tilde{Y}_s - Y_s)(\tilde{f}(s, \tilde{\Theta}_s^{0,\tilde{\xi},i}, N_s^i) - f(s, \Theta_s^{0,\xi,i}, N_s^i))ds \\ &- 2\int_t^T (\tilde{Y}_s - Y_s)((\tilde{Z}_s - Z_s)\,dB_s) - 2\int_t^T \sum_{l\in L} (\tilde{Y}_{s^-} - Y_{s^-})(\tilde{H}_s(l) - H_s(l))dM_s(l)\end{aligned}$$

Hence we deduce $\forall\, t \in [0,T]$,

$$\begin{aligned}(2.16)\quad \mathbf{E}\int_t^T 1_A|\tilde{Z}_s - Z_s|^2\,ds \;&+\; \lambda\mathbf{E}\sum_{l\in L}\int_t^T 1_A((\tilde{H}_s - H_s)(l))^2\,ds \;\le\; \mathbf{E}\big(1_A|\tilde{h}(\tilde{X}_T, N_T^i) - h(X_T, N_T^i)|^2\big) \\ &+\; 2\mathbf{E}\int_t^T 1_A(\tilde{Y}_s - Y_s)(\tilde{f}(s, \tilde{\Theta}_s^{0,\tilde{\xi},i}, N_s^i) - f(s, \Theta_s^{0,\xi,i}, N_s^i))\,ds\end{aligned}$$

Moreover from Burkhölder-Davis-Gundy inequalities, there exists $\gamma > 0$, changing from line to line such that

$$\begin{aligned}(2.17)\qquad 2\,\mathbf{E}\sup_{0\le t\le T}|\int_t^T 1_A(\tilde{Y}_s - Y_s)((\tilde{Z}_s &- Z_s)\,dB_s)| \le 2\gamma\,\mathbf{E}\left(\int_0^T 1_A|\tilde{Y}_s - Y_s|^2\,|\tilde{Z}_s - Z_s|^2\,ds\right)^{1/2} \\ &\le \frac{1}{4}\,\mathbf{E}(1_A \sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^2) + \gamma\mathbf{E}\int_0^T 1_A|\tilde{Z}_s - Z_s|^2\,ds\end{aligned}$$

and similarly to (2.9),

$$\begin{aligned}(2.18)\qquad 2\,\mathbf{E}\sup_{0\le t\le T}|\int_t^T \sum_{l\in L} 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})((\tilde{H}_s &- H_s)(l)dM_s(l))| \le \frac{1}{4}\,\mathbf{E}(1_A \sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^2) \\ &+ \lambda\gamma\,\mathbf{E}\sum_{l\in L}\int_0^T 1_A\,((\tilde{H}_s - H_s)(l))^2\,ds\end{aligned}$$

Hence combining (2.16), (2.17) and (2.18), we find from (2.15) a constant $\beta_{(K,\lambda)}$ depending on λ and K such that

$$\begin{aligned}(2.19)\quad \mathbf{E}(1_A \sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^2) \;&+\; \mathbf{E}\int_0^T 1_A|\tilde{Z}_s - Z_s|^2\,ds + \mathbf{E}\sum_{l\in L}\int_0^T 1_A((\tilde{H}_s - H_s)(l))^2\,ds \\ &\le\; \beta_{(K,\lambda)}\Big[\mathbf{E}\big(1_A|\tilde{h}_{N_T^i}(\tilde{X}_T) - h_{N_T^i}(X_T)|^2\big) \\ &+\; \mathbf{E}\Big(\sup_{0\le t\le T}\int_t^T 1_A\,(\tilde{Y}_s - Y_s)(\tilde{f}(s, \tilde{\Theta}_s^{0,\tilde{\xi},i}, N_s^i) - f(s, \Theta_s^{0,\xi,i}, N_s^i))ds\Big)\Big]\end{aligned}$$

Using assumptions (H1.1), (H1.2) and standard estimates, there exists $\tilde{\beta}_K > 0$ only depending on K satisfying

$$\begin{aligned}2\mathbf{E}\Big(\sup_{0\le t\le T}\int_t^T 1_A\,(\tilde{Y}_s &- Y_s)(\tilde{f}(s, \tilde{\Theta}_s^{0,\tilde{\xi},i}, N_s^i) - f(s, \Theta_s^{0,\xi,i}, N_s^i))ds\Big) \le \\ &+ \tilde{\beta}_K\Big[\mathbf{E}\int_0^T 1_A\,|\tilde{X}_s - X_s|^2\,ds + \mathbf{E}\int_0^T 1_A|\tilde{Y}_s - Y_s|(|\tilde{Z}_s - Z_s| + |\tilde{H}_s - H_s|)\,ds \\ &+ \mathbf{E}\int_0^T 1_A\,|\tilde{Y}_s - Y_s|^2\,ds + \mathbf{E}\Big(\int_0^T 1_A\,|(\tilde{f} - f)(s, \Theta_s^{0,\xi,i}, N_s^i)|\,ds\Big)^2\Big]\end{aligned}$$

So, plugging this inequality into (2.19) and adding with (2.14), we obtain modifying $\beta_{(K,\lambda)}$

$$\begin{aligned}
\mathbf{E}(1_A \sup_{0\le s\le T} |\tilde{X}_s & - X_s|^2) + \mathbf{E}(1_A \sup_{0\le s\le T} |\tilde{Y}_s - Y_s|^2) + \mathbf{E}\sum_{l\in L}\int_0^T 1_A\,((\tilde{H}_s - H_s)(l))^2\,ds \\
&+ \mathbf{E}\int_0^T 1_A|\tilde{Z}_s - Z_s|^2\,ds \\
&\le \beta_{(K,\lambda)}\Big[\mathbf{E}(1_A|\tilde{\xi}-\xi|^2) + \mathbf{E}(1_A|\tilde{h}_{N_T^i}(\tilde{X}_T) - h_{N_T^i}(X_T)|^2) \\
&+ \mathbf{E}\int_0^T 1_A(|\tilde{X}_s - X_s|^2)ds + \mathbf{E}\int_0^T 1_A|\tilde{Z}_s - Z_s|(|\tilde{X}_s - X_s|\,ds + |\tilde{Y}_s - Y_s|)\,ds \\
&+ \mathbf{E}\int_0^T 1_A(|\tilde{Y}_s - Y_s|^2)ds + \mathbf{E}\int_0^T 1_A|\tilde{H}_s - H_s|(|\tilde{X}_s - X_s| + |\tilde{Y}_s - Y_s|)\,ds \\
&+ \mathbf{E}\int_0^T 1_A|(\tilde{\sigma}-\sigma)(s, X_s, Y_s, N_s^i)|^2\,ds \\
&+ \mathbf{E}\Big(\int_0^T 1_A\,(|\tilde{b}-b| + |\tilde{f}-f|)(s, \Theta_s^{0,\xi,i}, N_s^i)\,ds\Big)^2\Big]
\end{aligned}$$

Hence using once again standard estimates, we prove the existence of two constants $\beta^{(i1)}_{(K,\lambda)}$ and $C^{(i2)}_{(K,\lambda)}$ depending on K and λ such that for $T \le C^{(i2)}_{(K,\lambda)}$,

$$\begin{aligned}
\mathbf{E}(\sup_{0\le s\le T} 1_A\,|\tilde{X}_s & - X_s|^2) + \mathbf{E}(\sup_{0\le s\le T} 1_A\,|\tilde{Y}_s - Y_s|^2) + \mathbf{E}\sum_{l\in L}\int_0^T 1_A\,((\tilde{H}_s - H_s)(l))^2\,ds \qquad (2.20)\\
&+ \mathbf{E}\int_0^T 1_A|\tilde{Z}_s - Z_s|^2\,ds \qquad (2.21)\\
&\le \beta^{(i1)}_{(K,\lambda)}\Big[\mathbf{E}(1_A|\tilde{\xi}-\xi|^2) + \mathbf{E}(1_A|(\tilde{h}_{N_T^i} - h_{N_T^i})(X_T)|^2) \\
&+ \mathbf{E}\Big(\int_0^T 1_A\,(|\tilde{b}-b| + |\tilde{f}-f|)(s, \Theta_s^{0,\xi,i}, N_s^i)\,ds\Big)^2 \\
&+ \mathbf{E}\int_0^T 1_A|(\tilde{\sigma}-\sigma)(s, X_s, Y_s, N_s^i)|^2\,ds\Big]
\end{aligned}$$

■

Corollary 2.1. *Assume that* **(H1)** *holds and* $T \le C^{(i2)}_{(K,\lambda)}$. *Then* $\forall\,(t,i) \in [0,T]\times\mathcal{K}$ *and for every* $x \in \mathbf{R}^d$, *the unique solution* $\Theta_s^{t,x,i} = (X_s^{t,x,i}, Y_s^{t,x,i}, H_s^{t,x,i}, Z_s^{t,x,i})_{t\le s\le T}$ *of the problem* $(E^{t,x})$ *extended on whole* $[0,T]$ *by putting* $\forall\, s\in[0,t]$, $X_s^{t,x,i} = x$; $Y_s^{t,x,i} = Y_t^{t,x,i}$; $H_s^{t,x,i} = Z_s^{t,x,i} = 0$
satisfies
i) $\exists\, C^{(i.1)}_{(K,\Lambda,\lambda)}$ *such that* $\forall\,(t,x)\in[0,T]\times\mathbf{R}^d$,

$$\mathbf{E}\sup_{0\le s\le T}|X_s^{t,x,i}|^2 + \mathbf{E}\sup_{0\le s\le T}|Y_s^{t,x,i}|^2 + \mathbf{E}\sum_{l\in L}\int_0^T (H_s^{t,x,i}(l))^2\,ds + \mathbf{E}\int_0^T |Z_s^{t,x,i}|^2\,ds \le C^{(i.1)}_{(K,\Lambda,\lambda)}(1+|x|^2) \qquad (2.22)$$

ii) $\forall\ \big((t,x),\ (t',x')\big) \in ([0,T]\times\mathbf{R}^d)^2\ ,\ \exists\, C^{(i.2)}_{(K,\Lambda,\lambda)}$ *such that*

$$\mathbf{E}\sup_{0\le s\le T}|X^{t,x,i}_s - X^{t',x',i}_s|^2 + \mathbf{E}\sup_{0\le s\le T}|Y^{t,x,i}_s - Y^{t',x',i}_s|^2 + \mathbf{E}\int_0^T |Z^{t,x,i}_s - Z^{t',x',i}_s|^2\, ds \tag{2.23}$$

$$+\ \mathbf{E}\sum_{l\in L}\int_0^T \big(H^{t,x,i}_s(l) - H^{t',x',i}_s(l)\big)^2\, ds \tag{2.24}$$

$$\le\ \beta^{(i1)}_{(K,\lambda)}\,|x-x'|^2\ +\ C^{(i.2)}_{(K,\Lambda,\lambda)}\,|t-t'|$$

Proof : i) Let $x\in\mathbf{R}^d$ and assume $T\le C^{(i2)}_{(K,\lambda)}$. The process $(\Theta^{t,x,i}_s)_{0\le s\le T}$ is a solution of the problem

$$\begin{cases} \forall\ s\in[0,T] \\ X^{t,x,i}_s = x + \int_0^s 1_{[t,T]}(r) b(r,\Theta^{t,x,i}_r, N^{t,i}_r)\, dr + \int_0^s 1_{[t,T]}(r)\,\sigma(r,X_r,Y_r,N^{t,i}_r)\, dB_r \\ Y^{t,x,i}_s = h_{N^{t,i}_T}(X^{t,x,i}_T) + \int_s^T 1_{[t,T]}(r)\, f(r,\Theta^{t,x,i}_r, N^{t,i}_r)\, dr - \int_s^T Z_r dB_r - \int_s^T \sum_{l\in L} H_r(l)\, dM_r(l) \\ \mathbf{E}\int_0^T (|X_r|^2+|Y_r|^2+|Z_r|^2+|H_r|^2)\, dr < +\infty \end{cases}$$

Applying theorem 2.2 to the quartet of functions $(1_{[t,T]}f_{N^{t,i}_s}, 1_{[t,T]}g_{N^{t,i}_s}, 1_{[t,T]}\sigma_{N^{t,i}_s}, h_{N^{t,i}_T})$ and $(0,0,0,0)$, we have

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le T}|X^{t,x,i}_s|^2 + \mathbf{E}\sup_{0\le s\le T}|Y^{t,x,i}_s|^2 \ &+\ \mathbf{E}\int_0^T |Z^{t,x,i}_s|^2\, ds + \mathbf{E}\sum_{l\in L}\int_0^T (H^{t,x,i}_s(l))^2\, ds \\ &\le\ \beta^{(i1)}_{(K,\lambda)}\Big[|x|^2 + \mathbf{E}(|h_{N^i_T}(0)|) + \mathbf{E}\int_0^T |\sigma_{N^i_s}(s,0,0)|^2\, ds \\ &+\ \mathbf{E}\Big(\int_0^T (|b_{N^i_s}| + |f_{N^i_s}|)(s,0,0,0,0)\Big)^2\Big]\end{aligned}$$

Applying assumption (H1.3), there exists a constant $C^{(i.1)}_{(K,\Lambda,\lambda)}$ depending on K,Λ and λ such that

$$\mathbf{E}\sup_{0\le s\le T}|X^{t,x,i}_s|^2 + \mathbf{E}\sup_{0\le s\le T}|Y^{t,x,i}_s|^2 \ +\ \mathbf{E}\int_0^T |Z^{t,x,i}_s|^2\, ds + \mathbf{E}\sum_{l\in L}\int_0^T (H^{t,x,i}_s(l))^2\, ds \le C^{(i.1)}_{(K,\Lambda,\lambda)}(1+|x|^2)$$

ii) Let $(t,x)\in[0,T]\times\mathbf{R}^d$ and $(t',x')\in[0,T]\times\mathbf{R}^d$, $t'<t$.
Noting that the two quartets of functions $(1_{[t,T]}b_{N^{t,i}_s}, 1_{[t,T]}f_{N^{t,i}_s}, 1_{[t,T]}\sigma_{N^{t,i}_s}, h_{N^{t,i}_T})$ and $(1_{[t',T]}b_{N^{t',i}_s}, 1_{[t',T]}f_{N^{t',i}_s}, 1_{[t',T]}\sigma_{N^{t',i}_s}, h_{N^{t',i}_T})$, satisfy **(H1)**, using of theorem 2.2 we obtain

$$\mathbf{E}\ \sup_{0\le s\le T}|X^{t,x,i}_s - X^{t',x',i}_s|^2 + \mathbf{E}\sup_{0\le s\le T}|Y^{t,x,i}_s - Y^{t',x',i}_s|^2 + \mathbf{E}\int_0^T |Z^{t,x,i}_s - Z^{t',x',i}_s|^2\, ds \tag{2.25}$$

$$+\ \mathbf{E}\sum_{l\in L}\int_0^T \Big(H^{t,x,i}_s(l) - H^{t',x',i}_s(l)\Big)^2\, ds \tag{2.26}$$

$$\begin{aligned}&\le\ \beta^{(1)}_{(K,\lambda)}\Big[|x-x'|^2 + \mathbf{E}|(h_{N^{t,i}_T} - h_{N^{t',i}_T})(X_T)|^2) \\ &+\ \mathbf{E}\int_0^T |(1_{[t',T]}\sigma_{N^{t',i}_r} - 1_{[t,T]}\sigma_{N^{t,i}_r})(r, X^{t,x,i}_r, Y^{t,x,i}_r)|^2\, dr \\ &+\ \mathbf{E}\Big(\int_0^T (|1_{[t',T]}b_{N^{t',i}_r} - 1_{[t,T]}b_{N^{t,i}_r}| + |1_{[t',T]}\, f_{N^{t',i}_r} - 1_{[t,T]}\, f_{N^{t,i}_r}|)(r,\Theta^{t,x,i}_r)dr\Big)^2\Big]\end{aligned}$$

and from assumption (H1.3) there exists a constant $C_{\Lambda,\lambda}$ depending only on Λ and λ such that

$$\begin{aligned}\mathbf{E}|h_{N_T^{t,i}}-h_{N_T^{t',i}})(X_T)|^2 &= \mathbf{E}(|h_{N_T^{t,i}}-h_{N_T^{t',i}})(X_T)|^2 1_{\{N_T^{t',i}\neq N_T^{t,i}\}})\\ &\leq \mathbf{E}(|h_{N_T^{t,i}}-h_{N_T^{t',i}})(X_T)|^2 1_{\{\nu(]t',t[)\neq 0\}})\\ \mathbf{E}(|h_{N_T^{t,i}}-h_{N_T^{t',i}})(X_T)|^2) &\leq 2\Lambda^2\,\mathbf{P}(\nu(]t',t[)\neq 0)\leq C_{\Lambda,\lambda}|t'-t|\end{aligned}$$

where ν is the law of the first jump between t and t'.
So, using the same argument in the two last integrals and assumption (H1.3), we easily find from (2.25) a constant $C^{(i.2)}_{(K,\Lambda,\lambda)}$ such that

$$\begin{aligned}\mathbf{E}\sup_{0\leq s\leq T}|X_s^{t,x,i} &- X_s^{t',x',i}|^2+\mathbf{E}\sup_{0\leq s\leq T}|Y_s^{t,x,i}-Y_s^{t',x',i}|^2+\mathbf{E}\int_0^T|Z_s^{t,x,i}-Z_s^{t',x',i}|^2\,ds\\ &+ \mathbf{E}\sum_{l\in L}\int_0^T(H_s^{t,x,i}(l)-H_s^{t',x',i}(l))^2\,ds\\ &\leq \beta^{(i1)}_{(K,\lambda)}\,|x-x'|^2\;+\;C^{(i.2)}_{(K,\Lambda,\lambda)}\,|t-t'|\quad\blacksquare\end{aligned}$$

We deduce the following result which is an obvious consequence of the previous corollary.

Corollary 2.2. *Assume that* **(H1)** *holds. Then there exists some constants* $C^{(1.1)}_{(K,\Lambda,\lambda)}$, $C^{(1.2)}_{(K,\Lambda,\lambda)}$, $\beta^{(1)}_{(K,\lambda)}$ *and* $C^{(2)}_{(K,\lambda)}$ *such that* $\forall\, T\leq C^{(2)}_{(K,\lambda)}$, *the map*

$$\begin{aligned}\theta:[0,T]\times\mathbf{R}^d\times\mathcal{K} &\longrightarrow \mathbf{R}\\ (t,x,i) &\longrightarrow Y_t^{t,x,i}\end{aligned}$$

which defines also a mapping

$$\begin{aligned}\theta:[0,T]\times\mathbf{R}^d &\longrightarrow \mathbf{R}^k\\ (t,x) &\longrightarrow (\theta(t,x,1),\ \theta(t,x,2),\ ...,\theta(t,x,k))\end{aligned}$$

satisfies
i) $\forall(t,x)\in[0,T]\times\mathbf{R}^d,\ \forall(t',x')\in[0,T]\times\mathbf{R}^d,\ \ \forall\, i\in\mathcal{K},$

$$|\theta(t,x,i)|^2 \leq C^{(1.1)}_{(K,\Lambda,\lambda)}\,(1+|x|^2) \tag{2.27}$$

$$|\theta(t,x,i)-\theta(t',x',i)|^2 \leq \beta^{(1)}_{(K,\lambda)}\,|x-x'|^2\;+\;C^{(1.2)}_{(K,\Lambda,\lambda)}\,|t-t'| \tag{2.28}$$

ii) $\forall\, i\in\mathcal{K},\ \ \forall\,(t,x)\in[0,T]\times\mathbf{R}^d$ *and for every* $\mathcal{G}_t$ *measurable random vector* ξ *with finite second moment, there exists a P-null set* $G_Y^{t;\xi}$ *such that* $\forall\ s\in[t,T]$,

$$\forall\omega\notin G_Y^{t,\xi}\quad \theta(s,X_s^{t,\xi,i}(\omega),N_s^{t,i}) = Y_s^{t,\xi,i}(\omega) \tag{2.29}$$

$$\forall\, l\in L\quad H_s^{t,x,i}(l) = \theta(s,X_s^{t,x,i},N_{s^-}^{t,i}+l)-\theta(s,X_s^{t,x,i},N_{s^-}^{t,i}) \tag{2.30}$$

$$f_{N_s^{t,i}}(s,X_s^{t,x,i},Y_s^{t,x,i},H_s^{t,x,i},Z_s^{t,x,i}) = \bar{f}_{N_s^{t,i}}(s,X_s^{t,x,i},\theta(s,X_s^{t,x,i}),Z_s^{t,x,i}) \tag{2.31}$$

$$b_{N_s^{t,i}}(s,X_s^{t,x,i},Y_s^{t,x,i},H_s^{t,x,i},Z_s^{t,x,i}) = \bar{b}_{N_s^{t,i}}(s,X_s^{t,x,i},\theta(s,X_s^{t,x,i}),Z_s^{t,x,i}) \tag{2.32}$$

Proof : Putting

$$C^{(1.1)}_{(K,\Lambda,\lambda)} = \max_{i=1,\dots,k} C^{(i.1)}_{(K,\Lambda,\lambda)} \quad ; \quad \beta^{(1)}_{(K,\lambda)} = \max_{i=1,\dots,k} \beta^{(i1)}_{(K,\lambda)}$$

$$C^{(1.2)}_{(K,\Lambda,\lambda)} = \max_{i=1,\dots,k} C^{(i.2)}_{(K,\Lambda,\lambda)} \quad ; \quad C^{(2)}_{(K,\lambda)} = \min_{i=1,\dots,k} C^{(i2)}_{(K,\lambda)}$$

then we get easily (2.27) and (2.28) as immediate consequences of the previous corollary.
Let us recall the proof of (2.29) which is an adaptation of one given in [2]. We nevertheless include a complete proof for the convenience of the reader.
Let $(t,x) \in [0,T] \times \mathbf{R}^d$, $i \in \mathcal{K}$ and ξ a $\mathcal{G}_t$ measurable random vector with finite second moment. Applying theorem 2.2 (using the same coefficients and different initial values), we have

$$\forall\ s \in [t,T],\ \forall\ \varepsilon > 0 \quad \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |Y^{t,x,i}_s - Y^{t,\xi,i}_s|^2) \le \beta^{(i1)}_{(K,\lambda)} \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\xi - x|^2)$$

Hence

$$\forall\ \varepsilon > 0 \quad \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\theta(t,x,i) - Y^{t,\xi,i}_t|^2) \le \beta^{(1)}_{(K,\lambda)} \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\xi - x|^2)$$

Futhermore we have

$$\mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\theta(t,\xi,i) - Y^{t,\xi,i}_t|^2) \le 2\Big[\mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\theta(t,\xi,i) - \theta(t,x,i)|^2 + \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\theta(t,x,i) - Y^{t,\xi,i}_t|^2)\Big]$$

Using the Lipschitz property (2.28), we obtain

$$\mathbf{E}\,(1_{\{|\xi - x| < \varepsilon\}} |\theta(t,\xi,i) - \theta(t,x,i)|^2) \le \beta^{(1)}_{(K,\lambda)} \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\xi - x|^2)$$

Hence

$$\forall\ i = 1,...,k \quad \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\theta(t,\xi,i) - Y^{t,\xi,i}_t|^2) \le 4\ \beta^{(1)}_{(K,\lambda)} \mathbf{E}(1_{\{|\xi - x| < \varepsilon\}}\ |\xi - x|^2).$$

Choosing $\varepsilon = \frac{1}{m}$, $x = \frac{p}{m}$, $p \in \mathbf{Z}^d$, we have

$$\forall\ m \in \mathbf{N} \sum_{p \in \mathbf{Z}^d} \mathbf{E}(1_{\{|\xi - \frac{p}{m}| < \varepsilon\}}\ |\theta(t,\xi,i) - Y^{t,\xi,i}_t|^2) \le \frac{4}{m^2}\, \beta^{(1)}_{(K,\lambda)} \sum_{p \in \mathbf{Z}^d} \mathbf{E}(1_{\{|\xi - \frac{p}{m}|_\infty < \frac{1}{m}\}})$$

therefore

$$\mathbf{E}(\ |\theta(t,\xi,i) - Y^{t,\xi,i}_t|^2) \le \frac{2^{d+2}}{m^2}\, \beta^{(1)}_{(K,\lambda)} \quad \forall\ i = 1,...,k; \quad \forall\ m \in \mathbf{N}$$

We deduce

$$p.s \quad \theta(t,\xi,i) = Y^{t,\xi,i}_t \qquad \forall\ i = 1,...,k. \tag{2.33}$$

Elsewhere $\forall\ s \in [t,T]$, $(\Theta^{t,\xi,i}_u)_{s \le u \le T} \triangleq (X^{t,\xi,i}_u, Y^{t,\xi,i}_u, H^{t,\xi,i}_u, Z^{t,\xi,i}_u)_{s \le u \le T}$ is solution of the problem

$$\begin{cases} \forall\ s \in [t,T] \\ X^{t,\xi,i}_u = X^{t,\xi,i}_s + \displaystyle\int_s^u b(r, \Theta^{t,\xi,i}_r, N^{s,N^{t,i}_s}_r)\, dr + \int_s^u \sigma(r, X_r, Y_r, N^{s,N^{t,i}_s}_r)\, dB_r \\ Y^{t,\xi,i}_u = h_{N^{s,N^{t,i}_s}_T}(X^{t,\xi,i}_T) + \displaystyle\int_u^T f(r, \Theta^{t,\xi,i}_r, N^{s,N^{t,i}_s}_r)\, dr - \int_u^T Z_r dB_r - \int_u^T \sum_{l \in L} H_r(l) dM_r(l) \\ \mathbf{E} \displaystyle\int_s^T (|X_r|^2 + |Y_r|^2 + |Z_r|^2 + |H_r|^2)\, dr < +\infty \end{cases}$$

hence using (2.33) and uniqueness of solutions of our system, we obtain

$$\forall\ s \in [t,T], \quad \theta(s, X^{t,\xi,i}_s, N^{t,i}_s) = Y^{s,X^{t,\xi,i}_s,N^{t,i}_s}_s = Y^{t,\xi,i}_s \quad p.s$$

Using [9], lemma 2.2 we get (2.30), (2.31) and (2.32). ■

In the next section we extend our results to the case of random coefficients in order to get some regularity properties of the map θ given in the previous corollary.
However the proofs being the same as in the previous section, we only give the statments of the results.

2.1.3. *Extension to the random coefficients case.* We assume $\forall\, i \in \mathcal{K}$ the functions b_i, f_i, σ_i et h_i are respectively $\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^d)\otimes\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R}^{k-1})\otimes\mathcal{B}(\mathbf{R}^d)/\mathcal{B}(\mathbf{R}^d)$, $\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^d)\otimes\mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R}^{k-1})\otimes\mathcal{B}(\mathbf{R}^d))/\mathcal{B}(\mathbf{R})$, $\mathcal{P}\otimes\mathcal{B}(\mathbf{R}^d)\otimes\mathcal{B}(\mathbf{R})/\mathcal{B}(\mathbf{R}^{d\times d})$ and $\mathcal{G}_T^0\otimes\mathbf{R}^d/\mathcal{B}(\mathbf{R}^d)$ measurable.

Theorem 2.3 (Solution in small time duration)**.** *Let $i\in\mathcal{K}$ and assume that the coefficients satisfy assumption* **(H1)**. *Then for every $\mathcal{G}_0-$ measurable random vector ξ with finite second moment a solution of $(E^{0,\xi})$ satisfies*
i) $(X_s)_{0\le s\le T}$ is continuous and $(Y_s)_{0\le s\le T}$ has a càdlàg version.

ii) $\mathbf{E}\,(\sup\limits_{0\le t\le T}|X_t|^2+\sup\limits_{0\le t\le T}|Y_t|^2)<\infty.$

Moreover there exists a constant $\tilde{C}^{(i1)}_{(K,\lambda)}$ depending on K and λ such that for $T\le\tilde{C}^{(i1)}_{(K,\lambda)}$, the problem $(E^{0,\xi})$ admits a unique solution.

Theorem 2.4 (A priori Estimates)**.** *Let $i\in\mathcal{K}$, (b,f,h,σ) and $(\tilde{b},\tilde{f},\tilde{h},\tilde{\sigma})$ two quartets of functions satisfying assumption* **(H1)** *with the same constants K and Λ.*
Let $(\Theta^{0,\xi,i}_s)_{(0\le s\le T)}$ (resp $(\tilde{\Theta}^{0,\tilde{\xi},i}_s)_{(0\le s\le T)}$ the associated solution to (b,f,h,σ) (resp $(\tilde{b},\tilde{f},\tilde{h},\tilde{\sigma})$) with initial condition $(0,\xi,i)$ (resp $(0,\tilde{\xi},i)$) .
There exists two constants $\tilde{C}^{(i2)}_{(K,\lambda)}\le\tilde{C}^{(i1)}_{(K,\lambda)}$ and $\tilde{\beta}^{(i1)}_{(K,\lambda)}$ depending on K and λ such that for every $T\le\tilde{C}^{(i2)}_{(K,\lambda)}$ and for every $A\in\mathcal{G}_0$, the following holds

$$\begin{aligned}\mathbf{E}\,(1_A\sup_{0\le s\le T}|X_s-\tilde{X}_s|^2) &+ \mathbf{E}\,(1_A\sup_{0\le s\le T}|Y_s-\tilde{Y}_s|^2)+\mathbf{E}\sum_{l\in L}\int_0^T 1_A|(H_s-\tilde{H}_s)(l)|^2\,ds\\ &+ \mathbf{E}\int_0^T 1_A\,|Z_s-\tilde{Z}_s|^2\,ds\le\tilde{\beta}^{(i1)}_{(K,\lambda)}\Big[\mathbf{E}(1_A|\xi-\tilde{\xi}|^2)\\ &+ \mathbf{E}(1_A|(h-\tilde{h})(X_T)|^2)+\mathbf{E}\int_0^T 1_A|(\sigma-\tilde{\sigma})(s,X_s,Y_s,N^i_s)|^2\,ds\\ &+ \mathbf{E}\Big(\int_0^T 1_A\big(|b-\tilde{b}|+|f-\tilde{f}|\big)(s,\Theta^{0,\xi,i}_s,N^i_s)\,ds\Big)^2\Big]\end{aligned}$$

Corollary 2.3. $\forall\,T\le\tilde{C}^{(i2)}_{(K,\lambda)}$, $\forall\,(t,i)\in[0,T]\times\mathcal{K}$ *under the assumption* **(H1)**, *for every $x\in\mathbf{R}^d$ we define the process $(\Theta^{t,x,i}_s)_{t\le s\le T}$ as the unique solution of the problem*

$$\begin{cases}\forall\,s\in[t,T]\\ X^{t,x,i}_s=x+\displaystyle\int_t^s b(r,\Theta^{t,x,i}_r,N^{t,i}_r)\,dr+\int_t^s\sigma(r,X_r,Y_r,N^{t,i}_r)\,dB_r\\ Y^{t,x,i}_s=h_{N^{t,i}_T}(X^{t,x,i}_T)+\displaystyle\int_s^T f(r,\Theta^{t,x,i}_r,N^{t,i}_r)\,dr-\int_s^T Z^{t,x,i}_r dB_r-\int_s^T\sum_{l\in L}H^{t,x,i}_r(l)dM_r(l)\\ \mathbf{E}\displaystyle\int_t^T(|X_r|^2+|Y_r|^2+|H_r|^2+|Z_r|^2)\,dr<+\infty\end{cases}$$

extended on $[0,T]$ by putting

$$\begin{cases}\forall\,s\in[0,t],\quad X^{t,x,i}_s=x\quad;Y^{t,x,i}_s=\mathbf{E}(Y^{t,x,i}_t/\mathcal{G}_s)\\ Y^{t,x,i}_t=Y^{t,x,i}_0+\int_0^t Z^{t,x,i}_s dB_u+\int_0^t\sum_{l\in L}H^{t,x,i}_r(l)dM_r(l)\end{cases}$$

then
i) $\exists\, \tilde{C}^{(i.1)}_{(K,\Lambda,\lambda)}$ *such that* $\forall\,(t,x)\in[0,T]\times\mathbf{R}^d$,

$$\mathbf{E}\sup_{0\le s\le T}|X_s^{t,x,i}|^2+\mathbf{E}\sup_{0\le s\le T}|Y_s^{t,x,i}|^2+\mathbf{E}\sum_{l\in L}\int_0^T(H_s^{t,x,i}(l))^2\,ds+\mathbf{E}\int_0^T|Z_s^{t,x,i}|^2\,ds\ \le\ \tilde{C}^{(i.1)}_{(K,\Lambda,\lambda)}(1+|x|^2)$$

ii) $\forall\,\big((t,x),\ (t',x')\big)\in([0,T]\times\mathbf{R}^d)^2$, $\exists\,\tilde{C}^{(i.2)}_{(K,\Lambda,\lambda)}$ *such that*

$$\mathbf{E}\sup_{0\le s\le T}|X_s^{t,x,i}\ -\ X_s^{t',x',i}|^2+\mathbf{E}\sup_{0\le s\le T}|Y_s^{t,x,i}-Y_s^{t',x',i}|^2+\mathbf{E}\sum_{l\in L}\int_0^T\big(H_s^{t,x,i}(l)-H_s^{t',x',i}(l)\big)^2\,ds$$
$$+\ \ \mathbf{E}\int_0^T|Z_s^{t,x,i}-Z_s^{t',x',i}|^2\,ds\le\tilde{\beta}^{(i1)}_{(K,\lambda)}\ |x-x'|^2\ +\ \tilde{C}^{(i.2)}_{(K,\Lambda,\lambda)}\ |t-t'|$$

Theorem 2.5. *$\forall\, i\in\mathcal{K}$, $\forall\, p\ge 1$ there exists two constants $\beta^{(i)}_{(p,K,\lambda,\Lambda)}$ and $\beta^{(i)}_{(p,K,\lambda)}$ such that $\forall\, T\le\tilde{C}^{(i2)}_{(K,\lambda)}$, and for every $\mathcal{G}_0-$measurable random vector ξ such that $\mathbf{E}|\xi|^{2p}<\infty$,*

i) the process $(\Theta_s^{0,\xi,i})_{0\le s\le T}$ solution of $(E^{0,\xi})$ satisfies

$$\mathbf{E}\sup_{0\le s\le T}|X_s^{0,\xi,i}|^{2p}+\mathbf{E}\sup_{0\le s\le T}|Y_s^{0,\xi,i}|^{2p}\ \ +\ \ \mathbf{E}\left(\int_0^T\sum_{l\in L}(H_s^{0,\xi,i}(l))^2ds\right)^p$$
$$+\ \ \mathbf{E}\left(\int_0^T|Z_s^{0,\xi,i}|^2ds\right)^p\ \le\ \beta^{(i)}_{(p,K,\lambda,\Lambda)}\ (1+\mathbf{E}|\xi|^{2p}) \tag{2.34}$$

ii) For every quartet $(\tilde{b},\tilde{f},\tilde{\sigma},\tilde{h})$ (resp (b,f,σ,h)) satisfying **(H1)** *with the constants K,Λ, for every $A\in\mathcal{G}_0$ and for every random vector $\tilde{\xi}$ (resp ξ) with finite $2p^{th}$ moment we have*

$$\mathbf{E}(\ \ 1_A\sup_{0\le s\le T}|\tilde{X}_s-X_s|^{2p})+\mathbf{E}(1_A\sup_{0\le s\le T}|\tilde{Y}_s-Y_s|^{2p})+\mathbf{E}\left(1_A\int_0^T|\tilde{Z}_s-Z_s|^2ds\right)^p$$
$$+\ \ \mathbf{E}\left(1_A\int_0^T\sum_{l\in L}((\tilde{H}_s-H_s)(l))^2ds\right)^p\le\beta^{(i)}_{(p,K,\lambda)}\mathbf{E}\Big[1_A\Big(|\tilde{\xi}-\xi|^{2p}+|(\tilde{h}_{N_T^i}-h_{N_T^i})(X_T)|^{2p}$$
$$+\ \ \left(\int_0^T|\tilde{\sigma}-\sigma|^2(s,X_s,Y_s,N_s^i)ds\right)^p+\left(\int_0^T|\tilde{b}-b|+|\tilde{f}-f|(s,\Theta_s,N_s^i)ds\right)^{2p}\Big)\Big] \tag{2.35}$$

where $(\tilde{X}_s,\tilde{Y}_s,\tilde{H}_s,\tilde{Z}_s)$ (resp (X_s,Y_s,H_s,Z_s)) is the solution of $(E^{0,\tilde{\xi}})$ (resp $(E^{0,\xi})$) associated to $(\tilde{b},\tilde{f},\tilde{\sigma},\tilde{h})$ (resp (b,f,σ,h)) and with initial values $(0,\tilde{\xi},i)$ (resp $(0,\xi,i)$).

Proof : Let us recall that we assume and $T\le 1$. Using the same technique developed in [2], we suppose in a first step that the processes $(\Theta_s^{0,\xi,i})_{0\le s\le T}$ and $(\tilde{\Theta}_s^{0,\tilde{\xi},i})_{0\le s\le T}$ satisfy $p-$ integrability conditions defined below

$$(L_p)\quad \mathbf{E}\sup_{0\le s\le T}|X_s^{0,\xi,i}|^{2p}+\mathbf{E}\sup_{0\le s\le T}|Y_s^{0,\xi,i}|^{2p}+\mathbf{E}\left(\int_0^T\sum_{l\in L}(H_s^{0,\xi,i}(l))^2ds\right)^p+\mathbf{E}\left(\int_0^T|Z_s^{0,\xi,i}|^2ds\right)^p<\infty$$

$$(\tilde{L}_p)\quad \mathbf{E}\sup_{0\le s\le T}|\tilde{X}_s^{0,\tilde{\xi},i}|^{2p}+\mathbf{E}\sup_{0\le s\le T}|\tilde{Y}_s^{0,\tilde{\xi},i}|^{2p}+\mathbf{E}\left(\int_0^T\sum_{l\in L}(\tilde{H}_s^{0,\tilde{\xi},i}(l))^2ds\right)^p+\mathbf{E}\left(\int_0^T|\tilde{Z}_s^{0,\tilde{\xi},i}|^2ds\right)^p<\infty$$

Let us prove the existence of a constant $\tilde{C}_{(p,K,\lambda)}$ such that for every $T\le\tilde{C}_{(p,K,\lambda)}$ inequality (2.35) holds. For convenience in the proof, we omit the superscript ${}^{0,\xi,i}$ and ${}^{0,\tilde{\xi},i}$

Let $A \in \mathcal{G}_0$. Thanks to Itô's formula with the function $\varphi_p(x) = x^{2p}$, $p \geq 1$, we have $\forall\, t \in [0,T]$,

$$\begin{aligned}
\mathbf{E}(1_A \sup_{0\leq s\leq T} |\tilde{X}_s - X_s|^{2p}) &\leq \mathbf{E}(1_A|\tilde{\xi}-\xi|^{2p}) \\
&+ 2p\,\mathbf{E}\sup_{0\leq t\leq T}\left(\int_0^t 1_A\,|\tilde{X}_s - X_s|^{2p-1}(\tilde{b}(s,\tilde{\Theta}_s,N_s^i) - b(s,\Theta_s,N_s^i))ds\right) \\
&+ 2p\,\mathbf{E}\sup_{0\leq t\leq T}\left(\int_0^t 1_A\,|\tilde{X}_s - X_s|^{2p-1}(\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s,N_s^i) - \sigma(s,X_s,Y_s,N_s^i))dB_s\right) \\
&+ p\,(2p-1)\,\mathbf{E}\int_0^T 1_A\,|\tilde{X}_s - X_s|^{2p-2}|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s,N_s^i) - \sigma(s,X_s,Y_s,N_s^i)|^2 ds
\end{aligned}$$

Using Burkholder-Davis-Gundy inequalities and assumption (H1.1)-(H1.2), we obtain (with $\gamma > 0$)

$$\begin{aligned}
\mathbf{E}(1_A \sup_{0\leq s\leq T} |\tilde{X}_s - X_s|^{2p}) &\leq \mathbf{E}(1_A|\tilde{\xi}-\xi|^{2p}) + 2p\,K\mathbf{E}\bigg[\int_0^T 1_A|\tilde{X}_s - X_s|^{2p-1}|(\tilde{b}-b)(s,\Theta_s,N_s^i)|ds \\
&+ \int_0^T 1_A|\tilde{X}_s - X_s|^{2p-1}(|\tilde{X}_s - X_s| + |\tilde{Y}_s - Y_s| + |\tilde{H}_s - H_s| + |\tilde{Z}_s - Z_s|)ds\bigg] \\
&+ 2p\gamma\mathbf{E}\bigg(\int_0^T 1_A\,|\tilde{X}_s - X_s|^{4p-2}\,|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s,N_s^i) - \sigma(s,X_s,Y_s,N_s^i)|^2 ds\bigg)^{\frac{1}{2}} \\
&+ p\,(2p-1)\,\mathbf{E}\int_0^T 1_A\,|\tilde{X}_s - X_s|^{2p-2}|\tilde{\sigma}(s,\tilde{X}_s,\tilde{Y}_s,N_s^i) - \sigma(s,X_s,Y_s,N_s^i)|^2 ds
\end{aligned}$$

Hence there exists a constant $c_{(p,K)}$ depending on p and K such that

$$\begin{aligned}
\mathbf{E}(1_A \sup_{0\leq s\leq T} |\tilde{X}_s - X_s|^{2p}) &\leq c_{(p,K)}\bigg[\mathbf{E}(1_A|\tilde{\xi}-\xi|^{2p}) + \mathbf{E}\int_0^T 1_A|\tilde{X}_s - X_s|^{2p-1}|(\tilde{b}-b)(s,\Theta_s,N_s^i)|ds \\
&+ \mathbf{E}\int_0^T 1_A|\tilde{X}_s - X_s|^{2p-1}(|\tilde{X}_s - X_s| + |\tilde{Y}_s - Y_s| + |\tilde{H}_s - H_s| + |\tilde{Z}_s - Z_s|)ds \\
&+ \mathbf{E}\bigg(\int_0^T 1_A|\tilde{X}_s - X_s|^{4p-2}|(\tilde{\sigma}-\sigma)(s,X_s,Y_s,N_s^i)|^2 ds\bigg)^{\frac{1}{2}} \\
&+ \mathbf{E}\bigg(\int_0^T 1_A|\tilde{X}_s - X_s|^{2p-2}|(\tilde{\sigma}-\sigma)(s,X_s,Y_s,N_s^i)|^2 ds\bigg]
\end{aligned}$$

which implies using Young inequalities and modifying $c_{(p,K)}$

$$\begin{aligned}
\mathbf{E}\,(\;1_A \sup_{0\leq s\leq T} |\tilde{X}_s - X_s|^{2p}) &\leq c_{(p,K)}\bigg[\mathbf{E}(1_A|\tilde{\xi}-\xi|^{2p}) + \mathbf{E}\int_0^T 1_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})ds \\
&+ T^p\mathbf{E}\left(\int_0^T 1_A|\tilde{Z}_s - Z_s|^2 ds\right)^p + T^p\mathbf{E}\left(\sum_{l\in L}\int_0^T 1_A(\tilde{H}_s(l) - H_s(l))^2 ds\right)^p \\
&+ \mathbf{E}\left(\int_0^T 1_A|(\tilde{b}-b)(s,\Theta_s,N_s^i)|ds\right)^{2p} + \mathbf{E}\left(\int_0^T 1_A|(\tilde{\sigma}-\sigma)(s,X_s,Y_s,N_s^i)|^2 ds\right)^p\;\bigg]
\end{aligned} \tag{2.36}$$

Furthermore using the same function, from Itô's formula for discontinuous processes we have

$$
\begin{aligned}
(2.37)\qquad |\tilde{Y}_t - Y_t|^{2p} &= |\tilde{Y}_T - Y_T|^{2p} + 2p\int_t^T (\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}(\tilde{f}(s,\tilde{\Theta}_s,N_s^i) - f(s,\Theta_s,N_s^i))ds \\
&- p(2p-1)\int_t^T (\tilde{Y}_{s^-} - Y_{s^-})^{2p-2}\,|\tilde{Z}_s - Z_s|^2\,ds \\
&- 2p\int_t^T (\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\,(\tilde{Z}_s - Z_s)\,dB_s \\
&- 2p\int_t^T \sum_{l\in L}(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\,(\tilde{H}_s(l) - H_s(l))\,dM_s(l) \\
&- \sum_{t<s\le T}\Big[|\tilde{Y}_s - Y_s|^{2p} - |\tilde{Y}_{s^-} - Y_{s^-}|^{2p} - 2p(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\Delta(\tilde{Y}_s - Y_s)\Big]
\end{aligned}
$$

Putting $V_s = |\tilde{Y}_s - Y_s|^2 - |\tilde{Y}_{s^-} - Y_{s^-}|^2 = (2(\tilde{Y}_{s^-} - Y_{s^-}) + (\tilde{H}_s - H_s))(\tilde{H}_s - H_s)$, we deduce

$$
\begin{aligned}
|\tilde{Y}_s - Y_s|^{2p} &- |\tilde{Y}_{s^-} - Y_{s^-}|^{2p} - 2p\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\Delta(\tilde{Y}_s - Y_s) \\
&= \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s\Big)^p - \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^p - 2\,p\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}(\tilde{H}_s(l) - H_s(l)) \\
&= \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s\Big)^p - \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^p - p\Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^{p-1} V_s \\
&+ p\,(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}(\tilde{H}_s - H_s)^2
\end{aligned}
$$

Hence substituting the last sum in (2.37) by the quantity

$$
\begin{aligned}
\sum_{t<s\le T}\Big[\Big(|\tilde{Y}_{s^-} &- Y_{s^-}|^2 + V_s\Big)^p - \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^p - p\Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^{p-1} V_s\Big] \\
&+ p\sum_{t<s\le T}(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}(\Delta(\tilde{Y}_s - Y_s))^2 + \lambda\,p\int_t^T (\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}|\tilde{H}_s - H_s|^2 ds
\end{aligned}
$$

we obtain $\forall\, t \in [0,T]$ and $A \in \mathcal{G}_0$,

$$
\begin{aligned}
1_A|\tilde{Y}_t &- Y_t|^{2p} + p(2p-1)\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-2}\,|\tilde{Z}_s - Z_s|^2\,ds + p\sum_{t<s\le T} 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}(\Delta(\tilde{Y}_s - Y_s))^2 \\
&+ \lambda\,p\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}|\tilde{H}_s - H_s|^2 ds \\
&= |\tilde{Y}_T - Y_T|^{2p} \\
&+ 2p\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}(\tilde{f}(s,\tilde{\Theta}_s,N_s^i) - f(s,\Theta_s,N_s^i))ds - 2p\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\,(\tilde{Z}_s - Z_s)\,dB_s \\
&- 2p\int_t^T \sum_{l\in L} 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}\,(\tilde{H}_s(l) - H_s(l))dM_s(l) \\
&- \sum_{t<s\le T} 1_A\,\Big[\Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2 + V_s\Big)^p - \Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^p - p\Big(|\tilde{Y}_{s^-} - Y_{s^-}|^2\Big)^{p-1} V_s\Big]
\end{aligned}
$$

Using the convexity property of φ_p, we prove that the last sum of the right hand side is positive. Therefore there exists $c_p > 0$ such that $\forall\, t \in [0,T]$,

$$\begin{aligned}
(2.38)\qquad \mathbf{E}\,(1_A\,|\tilde{Y}_t &- Y_t|^{2p}) + \mathbf{E}\int_t^T 1_A\,|\tilde{Y}_s - Y_s|^{2(p-1)}(|\tilde{Z}_s - Z_s|^2 + \lambda|\tilde{H}_s - H_s|^2)\,ds \\
&\leq c_p\Big[\mathbf{E}\int_t^T 1_A\,(\tilde{Y}_s - Y_s)^{2p-1}(\tilde{f}(s,\tilde{\Theta}_s,N_s^i) - f(s,\Theta_s,N_s^i))ds + \mathbf{E}\,1_A\,|\tilde{Y}_T - Y_T|^{2p}\Big]
\end{aligned}$$

Moreover since $\varphi_p^{''} > 0$ by the same argument, the last sum in (2.37) is positive.
Using the Burkhölder-Davis-Gundy inequality, we find a constant $\tilde{c}_p$ only depending on p which can change line to line such that

$$\begin{aligned}
\mathbf{E}\sup_{0\leq t\leq T}|\int_t^T \sum_{l\in L} 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}(\tilde{H}_s(l) &- H_s(l)))dM_s(l)| \leq \frac{1}{4}\mathbf{E}(1_A \sup_{0\leq s\leq T}|\tilde{Y}_s - Y_s|^{2p}) \\
+\ \lambda\,\tilde{c}_p\mathbf{E}\int_t^T \sum_{l\in L} 1_A(\tilde{Y}_{s^-} &- Y_{s^-})^{2(p-1)}(\tilde{H}_s(l) - H_s(l))^2 ds \\
\mathbf{E}\sup_{0\leq t\leq T}|\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2p-1}(\tilde{Z}_s - Z_s)dB_s| &\leq \frac{1}{4}\mathbf{E}(1_A \sup_{0\leq s\leq T}|\tilde{Y}_s - Y_s|^{2p}) \\
&+ \tilde{c}_p\mathbf{E}\int_t^T 1_A\,(\tilde{Y}_{s^-} - Y_{s^-})^{2(p-1)}|\tilde{Z}_s - Z_s|^2 ds
\end{aligned}$$

Hence using these two inequalities and modifying $\tilde{c}_p$ if necessary we deduce from (2.37)

$$\begin{aligned}
(2.39)\qquad \mathbf{E}(1_A \sup_{0\leq s\leq T}|\tilde{Y}_s - Y_s|^{2p}) &\leq \tilde{c}_p\Big[\mathbf{E}(1_A\,|\tilde{Y}_T - Y_T|^{2p}) \\
&+ \mathbf{E}\int_0^T 1_A\,|\tilde{Y}_s - Y_s|^{2(p-1)}(|\tilde{Z}_s - Z_s|^2 + \lambda\,|\tilde{H}_s - H_s|^2)\,ds \\
&+ \int_0^T 1_A\,|\tilde{Y}_s - Y_s|^{2p-1}(\tilde{f}(s,\tilde{\Theta}_s,N_s^i) - f(s,\Theta_s,N_s^i))\,ds\Big]
\end{aligned}$$

Combining (2.38), (2.39) and using assumptions (H1.1)-(H1.2), we find a constant $c_{(p,K)}$ depending on p and K satisfying

$$\begin{aligned}
\mathbf{E}\,(1_A \sup_{0\leq s\leq T}|\tilde{Y}_s - Y_s|^{2p}) &+ \mathbf{E}\int_0^T 1_A\,|\tilde{Y}_s - Y_s|^{2(p-1)}\,(|\tilde{Z}_s - Z_s|^2 + \lambda\,|\tilde{H}_s - H_s|^2)\,ds \\
&\leq c_{(p,K)}\Big[\mathbf{E}(1_A\,|\tilde{Y}_T - Y_T|^{2p}) + \mathbf{E}\int_0^T 1_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})\,ds \\
&+ \mathbf{E}\int_0^T 1_A\,|\tilde{Y}_s - Y_s|^{2p-1}\,(|\tilde{Z}_s - Z_s|\, + |\tilde{H}_s - H_s|\,)\,ds \\
&+ \mathbf{E}\left(\int_0^T 1_A\,|(\tilde{f} - f)(s,\Theta_s,N_s^i)|\,ds\right)^{2p}\Big]
\end{aligned}$$

so, using standard estimates, we find $c_{(p,K,\lambda)}$ depending on p, λ and K such that

$$\begin{aligned} (2.40)\quad & \mathbf{E}\quad (1_A \sup_{0\le s\le T} |\tilde{Y}_s - Y_s|^{2p}) \;+\; \frac{1}{2}\mathbf{E}\int_0^T 1_A\,|\tilde{Y}_s - Y_s|^{2(p-1)}(|\tilde{Z}_s - Z_s|^2 + \;\lambda\,|\tilde{H}_s - H_s|^2)\;ds \\ \le \quad & c_{(p,K,\lambda)}\Big[\mathbf{E}(1_A|\tilde{Y}_T - Y_T|^{2p}) + \mathbf{E}\int_0^T 1_A\,(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})\;ds \\ + \quad & \mathbf{E}\left(\int_0^T 1_A\,|\tilde{f} - f|(s,\Theta_s,N_s^i)\;ds\right)^{2p}\;\Big] \end{aligned}$$

Futhermore $\forall\; s \in [0,T]$,

$$\begin{aligned} \int_s^T (\tilde{Z}_r - Z_r)dB_r \;+\; \int_s^T \sum_{l\in L}(\tilde{H}_r(l) - H_r(l))dM_r(l) \quad &= \quad (\tilde{Y}_T - Y_T) - (\tilde{Y}_s - Y_s) \\ &+ \quad \int_s^T (\tilde{f}(r,\tilde{\Theta}_r,N_r^i) - f(r,\Theta_r,N_r^i))dr \end{aligned}$$

The Burkhölder-Davis-Gundy inequality gives us a constant $\alpha_{p,\lambda}$ depending on p and λ such that

$$\begin{aligned} \mathbf{E}\left(\int_0^T (|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2)dr\right)^p \quad &\le \quad \alpha_{p,\lambda}\Big[\,2\;\mathbf{E}\sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^{2p} \\ &+ \quad \mathbf{E}\left(\int_0^T |\tilde{f}(r,\tilde{\Theta}_r,N_r^i) - f(r,\Theta_r,N_r^i)|dr\right)^{2p}\;\Big] \end{aligned}$$

Thanks to assumptions (H1.1)-(H1.2), there exists a constant $\alpha_{(p,K,\lambda)}$ depending on p, λ and K such that

$$\begin{aligned} (2.41)\qquad \mathbf{E}\bigg(1_A\int_0^T (|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2)dr\bigg)^p \quad &\le \quad \alpha_{(p,K,\lambda)}\Big[\mathbf{E}\int_0^T 1_A(|\tilde{X}_s - X_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})ds \\ &+ \quad \mathbf{E}\left(\int_0^T 1_A|(\tilde{f} - f)(s,\Theta_s,N_s^i)|ds\right)^{2p} \\ &+ \quad T^p\mathbf{E}\left(\int_0^T 1_A(|\tilde{Z}_s - Z_s|^2 + |\tilde{H}_s - H_s|^2)dr\right)^p\;\Big] \end{aligned}$$

Hence using (2.36), (2.40) and (2.41), we prove the existence of two constants $\tilde{C}_{(p,K,\lambda)}$ and $\beta^{(i)}_{(p,K,\lambda)}$ depending on p, K and λ such that for $T \le \tilde{C}_{(p,K,\lambda)}$, we have

$$\begin{aligned} (2.42)\qquad \mathbf{E}(1_A\sup_{0\le s\le T}|\tilde{X}_s \quad &- \quad X_s|^{2p}) + \mathbf{E}(1_A\sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^{2p}) + \mathbf{E}\bigg(\int_0^T\sum_{l\in L}1_A(\tilde{H}_s(l) - H_s(l))^2ds\bigg)^p \\ (2.43)\qquad &+ \quad \mathbf{E}\bigg(\int_0^T 1_A\,|\tilde{Z}_s - Z_s|^2\;ds\bigg)^p \\ &\le \quad \beta^{(i)}_{(p,K,\lambda)}\mathbf{E}\Big[1_A\bigg(|\tilde{\xi} - \xi|^{2p} \\ &+ \quad |(\tilde{h}_{N_T^i} - h_{N_T^i})\,(\tilde{X}_T)|^{2p} + \left(\int_0^T (|\tilde{\sigma} - \sigma|^2(s,X_s,Y_s,N_s^i)\;ds\right)^p \\ &+ \quad \left(\int_0^T (|\tilde{b} - b| + |\tilde{f} - f|)(s,X_s,Y_s,H_s,Z_s,N_s^i)\;ds\right)^{2p}\bigg)\Big] \end{aligned}$$

For every process $(\Theta_s)_{0\leq s\leq T}$ and $(\tilde{\Theta}_s)_{0\leq s\leq T}$ satisfying (L_p) and $(\tilde{L}_p)$, thanks to theorem 2.3, the iterated scheme

$$\begin{cases} X_s^{n+1} &= \xi + \int_0^s b(r, X_r^{n+1}, Y_r^n, H_r^n, Z_r^n, N_r^i)dr + \int_0^s \sigma(r, X_r^{n+1}, Y_r^n, N_r^i)dB_r \\ Y_s^{n+1} &= h_{N_T^i}(X_T^{n+1}) + \int_s^T f(r, \Theta_r^{n+1}, N_r^i)dr - \int_s^T Z_r^{n+1}\, dB_r - \int_s^T \sum_{l\in L} H_r^{n+1}(l) dM_r(l) \end{cases}$$

satisfies $||\Theta_r^n - \Theta_r||_{\mathcal{B}^2} \to 0$ as $n \to \infty$.

Hence choosing a quartet $(X_s^0, Y_s^0, H_s^0, Z_s^0)$ satisfying (L_p) conditions and using Pardoux-Buckdahn [8] we prove that the process $(\Theta_s^n)_{0\leq s\leq T}$ verifies (L_p).
Let us prove the existence of a constant $\gamma_{(p,K,\lambda)}$ such that for $T \leq \gamma_{(p,K,\lambda)}$ the sequence $(\Theta_s^n)_{s\in[0,T],n\in\mathbf{N}}$ satisfies

$$||\Theta_r^n - \Theta_r^m||_{\mathcal{B}^{2p}_{[0,T]}} \to 0 \text{ whenever } m, n \to \infty$$

We assume $T \leq \tilde{C}_{(p,K,\lambda)}$ and let $n \in \mathbf{N}$ be fixed. Applying (2.42) to the previous iterated scheme, we obtain

$$\begin{aligned} \mathbf{E}(\sup_{0\leq s\leq T} |X_s^{n+2} &- X_s^{n+1}|^{2p}) + \mathbf{E}(\sup_{0\leq s\leq T} |Y_s^{n+2} - Y_s^{n+1}|^{2p}) + \mathbf{E}\Big(\int_0^T \sum_{l\in L}(H_s^{n+2}(l) - H_s^{n+1}(l))^2 ds\Big)^p \\ &+ \mathbf{E}\Big(\int_0^T |Z_s^{n+2} - Z_s^{n+1}|^2\, ds\Big)^p \leq \\ &\beta^{(i)}_{(p,K,\lambda)} \mathbf{E}\Big[\Big(\int_0^T |\sigma(s, X_s^{n+1}, Y_s^{n+1}, N_s^i) - \sigma(s, X_s^{n+1}, Y_s^n, N_s^i)|^2\, ds\Big)^p \\ &+ \Big(\int_0^T (|b(s, X_s^{n+1}, Y_s^{n+1}, H_s^{n+1}, Z_s^{n+1}, N_s^i) - b(s, X_s^{n+1}, Y_s^n, H_s^n, Z_s^n, N_s^i)|\, ds\Big)^{2p}\Big] \end{aligned}$$

and with Schwartz inequality, we deduce

$$\begin{aligned} \mathbf{E}(\sup_{0\leq s\leq T} |X_s^{n+2} &- X_s^{n+1}|^{2p}) + \mathbf{E}(\sup_{0\leq s\leq T} |Y_s^{n+2} - Y_s^{n+1}|^{2p}) + \mathbf{E}\Big(\int_0^T \sum_{l\in L}(H_s^{n+2}(l) - H_s^{n+1}(l))^2 ds\Big)^p \\ &+ \mathbf{E}\Big(\int_0^T |Z_s^{n+2} - Z_s^{n+1}|^2\, ds\Big)^p \\ &\leq K^{2p}(T^p + T^{2p})\beta^{(i)}_{(p,K,\lambda)} \mathbf{E}\Big[\sup_{0\leq s\leq T} |Y_s^{n+1} - Y_s^n|^{2p} \\ &+ \Big(\int_0^T |Z_s^{n+1} - Z_s^n|^2\, ds\Big)^p + \Big(\int_0^T \sum_{l\in L}(H_s^{n+1}(l) - H_s^n(l))^2 ds\Big)^p\Big] \end{aligned}$$

Therefore with a backward induction, there exists a constant $\gamma_{(p,K,\lambda)}$ such that the following series

$$\begin{aligned} \Big(\mathbf{E} \sup_{0\leq s\leq T} |X_s^{n+1} - X_s^n|^{2p}\Big)^{\frac{1}{2p}} &+ \Big(\mathbf{E} \sup_{0\leq s\leq T} |Y_s^{n+1} - Y_s^n|^{2p}\Big)^{\frac{1}{2p}} + \Big(\mathbf{E}\Big(\int_0^T |Z_s^{n+1} - Z_s^n|^2\, ds\Big)^p\Big)^{\frac{1}{2p}} \\ &+ \Big(\mathbf{E}\Big(\int_0^T \sum_{l\in L}(H_s^{n+1}(l) - H_s^n(l))^2 ds\Big)^p\Big)^{\frac{1}{2p}} \end{aligned}$$

converges if $T \leq \gamma_{(p,K,\lambda)}$, which is enough to conclude.
Hence applying (2.35) to the quartet (X_s, Y_s, H_s, Z_s) and $(0,0,0,0)$, we obtain

$$\begin{aligned}\mathbf{E}\sup_{0\leq s\leq T}|X_s^{0,\xi,i}|^{2p}+\mathbf{E}\sup_{0\leq s\leq T}|Y_s^{0,\xi,i}|^{2p} &+ \mathbf{E}\Big(\int_0^T\sum_{l\in L}(H_s^{0,\xi,i}(l))^2ds\Big)^p+\mathbf{E}\Big(\int_0^T|Z_s^{0,\xi,i}|^2\,ds\Big)^p\\ &\leq \beta^{(i)}_{(p,K,\lambda)}\mathbf{E}\Big[|\xi|^{2p}+|h(0)|^{2p}+\Big(\int_0^T|\sigma(s,0,0,N_s^i)|^2\,ds\Big)^p\\ &+ \Big(\int_0^T|b(s,0,0,0,0,N_s^i)|+|f(s,0,0,0,0,N_s^i)|\,ds\Big)^{2p}\Big]\end{aligned}$$

and with the help of assumption (H1.3), we find a constant $\beta^{(i)}_{(p,K,\lambda,\Lambda)}$ such that for every $T\leq\gamma_{(p,K,\lambda)}$,

$$\begin{aligned}\mathbf{E}\sup_{0\leq s\leq T}|X_s^{0,\xi,i}|^{2p}+\mathbf{E}\sup_{0\leq s\leq T}|Y_s^{0,\xi,i}|^{2p}+\mathbf{E}\Big(\sum_{l\in L}(H_s^{0,\xi,i}(l))^2ds\Big)^p &+ \mathbf{E}\Big(\int_0^T|Z_s^{0,\xi,i}|^2\,ds\Big)^p\\ &\leq \beta^{(i)}_{(p,K,\lambda,\Lambda)}\mathbf{E}(1+|\xi|^{2p})\end{aligned}$$

Using the same method as in [2] theorem 4.5, we extend this result to $T\leq\tilde{C}^{(i2)}_{(k,\lambda)}$. ■

Taking a deterministic initial value, we get the following

Corollary 2.4. *$\forall\ i\in\mathcal{K},\ p\geq 1$, there exists a constant $\tilde{c}^{K,\Lambda,i}_{p,\lambda}$ depending on $K,\Lambda,\lambda,$ and p such that for every $T\leq\tilde{C}^{(i2)}_{(K,\lambda)}$, we have*
i) For every $(t,x)\in[0,T]\times\mathbf{R}^d$

$$\begin{aligned}\mathbf{E}\sup_{0\leq s\leq T}|X_s^{t,x,i}|^{2p}+\mathbf{E}\sup_{0\leq s\leq T}|Y_s^{t,x,i}|^{2p} &+ \mathbf{E}\left(\int_0^T\sum_{l\in L}(H_s^{t,x,i}(l))^2ds\right)^p\\ &+ \mathbf{E}\left(\int_0^T|Z_s^{t,x,i}|^2ds\right)^p\ \leq\ \beta^{(i)}_{(p,K,\lambda,\Lambda)}\,(1+|x|^{2p})\end{aligned}\tag{2.44}$$

ii) For every $((t,x),(t',x'))\in([0,T]\times\mathbf{R}^d)^2$

$$\begin{aligned}\mathbf{E}\sup_{0\leq s\leq T}|X_s^{t,x,i} &- X_s^{t',x',i}|^{2p}+\mathbf{E}\sup_{0\leq s\leq T}|Y_s^{t,x,i}-Y_s^{t',x',i}|^{2p}\\ &+ \mathbf{E}\left(\int_0^T\sum_{l\in L}((H_s^{t,x,i}-H_s^{t',x',i})(l))^2ds\right)^p\\ &+ \mathbf{E}\left(\int_0^T|Z_s^{t,x,i}-Z_s^{t',x',i}|^2ds\right)^p\ \leq\ \beta^{(i)}_{(p,K,\lambda)}\,|x-x'|^{2p}+\tilde{c}^{K,\Lambda,i}_{p,\lambda}\,|t-t'|^p\end{aligned}\tag{2.45}$$ ■

In the sequel to this section, for all $i\in\mathcal{K}$, we consider the following system for the use of Malliavin Calculus.

$$(E^*)\begin{cases}\forall\ s\in[0,T]\\ X_s=\xi_s+\displaystyle\int_0^s b(r,\Theta_r^{0,\xi_t,i},N_r^i)dr+\int_0^s\sigma(r,X_r,Y_r,N_r^i)dB_r\\ Y_s=h_{N_T^i}(X_T^{t,x,i})+\displaystyle\int_s^T f(r,\Theta_r^{0,\xi_t,i},N_r^i)dr-\int_s^T Z_rdB_r-\int_s^T\sum_{l\in L}H_r(l)dM_r(l)\\ \mathbf{E}\displaystyle\int_0^T(|X_t|^2+|Y_t|^2+|H_t|^2+|Z_t|^2)\,dt<\infty\end{cases}$$

where $(\xi_s)_{0\leq s\leq T}$ is continous and $\mathcal{G}_s^0-$ adapted process.
Then we have the following

Theorem 2.6. *Assume that* **(H1)** *holds. Then* $\forall\, i \in \mathcal{K}$, *there exists a constant* $C^{*,i}_{(K,\lambda)}$ *depending on* K *and* λ *such that* $\forall\, T \leq C^{*,i}_{(K,\lambda)}$, *and for every* $\mathcal{G}_s^0-$ *adapted process* $(\xi_s)_{0\leq s\leq T}$ *such that* $\mathbf{E}\sup_{0\leq s\leq T}|\xi_s|^2 < \infty$, (E^*) *admits a unique solution* $(\Theta_s)_{0\leq s\leq T} = (X_s, Y_s, H_s, Z_s)_{0\leq s\leq T}$ *in* $\mathcal{B}^2$ *satisfying*

i) $(X_s)_{0\leq s\leq T}$ *is continuous and* $(Y_s)_{0\leq s\leq T}$ *has a càdlàg version.*

ii) $\mathbf{E}(\sup_{0\leq s\leq T}|X_s|^2 + \sup_{0\leq s\leq T}|Y_s|^2) < \infty.$

Moreover for every $p \geq 1$ *there exists two constants* $C^{*,i}_{(p,K,\lambda)} \leq C^{*,i}_{(K,\lambda)}$ *and* $\beta^{*,i}_{(p,K,\lambda)}$ *depending on* p, λ *and* K *such that for every* $T \leq C^{*,i}_{(p,K,\lambda)}$ *and for every* $\mathcal{G}_s^0$ *adapted process* $(\xi_s)_{0\leq s\leq T}$ *such that* $\mathbf{E}\sup_{0\leq s\leq T}|\xi_s|^{2p} < \infty$ *the unique solution of* (E^*) *satisfies*

$$\begin{aligned}
\mathbf{E}\sup_{0\leq s\leq T}|X_s|^{2p} + \mathbf{E}\sup_{0\leq s\leq T}|Y_s|^{2p} + \mathbf{E}\left(\int_0^T \sum_{l\in L}(H_s(l))^2 ds\right)^p &+ \mathbf{E}\left(\int_0^T |Z_s|^2 ds\right)^p \\
&\leq \beta^{*,i}_{(p,K,\lambda)}\ (1 + \mathbf{E}\sup_{0\leq s\leq T}|\xi_s|^{2p})
\end{aligned} \tag{2.46}$$

ii) For every quartet $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ *satisfying* **(H1)** *with the same constants* K, Λ *and for every* $\mathcal{G}_s^0$ *adapted process* $(\tilde{\xi}_s)_{0\leq s\leq T}$ *such that* $\mathbf{E}\sup_{0\leq s\leq T}|\tilde{\xi}_s|^{2p} < \infty$ *we have*

$$\begin{aligned}
\mathbf{E}\sup_{0\leq s\leq T}|\tilde{X}_s &- X_s|^{2p} + \mathbf{E}\sup_{0\leq s\leq T}|\tilde{Y}_s - Y_s|^{2p} + \mathbf{E}\left(\int_0^T \sum_{l\in L}((\tilde{H}_s - H_s)(l))^2 ds\right)^p \\
&+ \mathbf{E}\left(\int_0^T |\tilde{Z}_s - Z_s|^2 ds\right)^p \leq \beta^{*,i}_{(p,K,\lambda)}\mathbf{E}\Big[\sup_{0\leq s\leq T}|\tilde{\xi}_s - \xi_s|^{2p} + |(\tilde{h}_{N^i_T} - h_{N^i_T})(X_T)|^{2p} \\
&+ \left(\int_0^T |\tilde{\sigma} - \sigma|^2(s, X_s, Y_s, N^i_s)ds\right)^p + \left(\int_0^T |\tilde{b} - b| + |\tilde{f} - f|(s, \Theta_s, N^i_s)ds\right)^{2p}\Big]
\end{aligned} \tag{2.47}$$

where $(\tilde{X}_s, \tilde{Y}_s, \tilde{H}_s, \tilde{Z}_s)$ *is the solution of* (E^*) *associated to* $(\tilde{b}, \tilde{f}, \tilde{\sigma}, \tilde{h})$ *with initial values* $(0, \tilde{\xi}_s, i)$.

Proof : The proof is the same as in theorem 2.5 with some alterations to the forward equation.
We suppose that the processes (Θ_s) and $(\tilde{\Theta}_s)$ are solutions of (E^*) and satisfy the (L^p) conditions.
The process defined by $\forall\, s \in [0,T],\quad X^*_s = X_s - \xi_s$ solves the SDE

$$X^*_s = \int_0^s b^*(r, X^*_r, Y_r, H_r, Z_r, N^i_r)dr + \int_0^s \sigma(r, X_r, Y_r, N^i_r)dB_r$$

where the function b^* defined by

$$\forall\, (t,x,y,u,z) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d,\ \forall\, i \in \mathcal{K}\quad b^*_i(t,x,y,u,z) = b_i(t, x+\xi_s, y, u, z)$$

satisfies assumption **(H1)**. Then using the same technique as in the proof of (2.36), we find a constant $\alpha_{(p,K)}$ depending on p and K such that

$$\begin{aligned}\mathbf{E} &\quad (\sup_{0\le s\le T}|\tilde{X}^*_s - X^*_s|^{2p}) \le \alpha_{(p,K)}\Big[\mathbf{E}\int_0^T 1_A(|\tilde{X}^*_s - X^*_s|^{2p} + |\tilde{Y}_s - Y_s|^{2p})ds \\ &+ \quad T^p\mathbf{E}\left(\int_0^T 1_A|\tilde{Z}_s - Z_s|^2 ds\right)^p + T^p\mathbf{E}\left(\sum_{l\in L}\int_0^T 1_A(\tilde{H}_s(l) - H_s(l))^2 ds\right)^p \\ &+ \quad \mathbf{E}\left(\int_0^T 1_A|(\tilde{b}^* - b^*)(s, X^*_s, Y_s, H_s, Z_s, N^i_s)|ds\right)^{2p} + \mathbf{E}\left(\int_0^T 1_A|(\tilde{\sigma} - \sigma)(s, X_s, Y_s, N^i_s)|^2 ds\right)^p\Big]\end{aligned}$$

Using standard computations and modifying $\alpha_{(p,K)}$, we deduce

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le T}|\tilde{X}_s - X_s|^{2p} &\le \alpha_{p,K}\Big[\mathbf{E}\sup_{0\le s\le T}|\tilde{\xi}_s - \xi_s|^{2p} + T\mathbf{E}\Big(\sup_{0\le s\le T}|\tilde{X}_s - X_s|^{2p} + \sup_{0\le s\le T}|\tilde{Y}_s - Y_s|^{2p}\Big) \\ &+ T^p\mathbf{E}\left(\int_0^T\sum_{l\in L}((\tilde{H}_s - H_s)(l))^2 ds\right)^p + T^p\mathbf{E}\left(\int_0^T|\tilde{Z}_s - Z_s|^2 ds\right)^p \\ &+ \mathbf{E}\Big(\int_0^T|\tilde{b} - b|(s, \Theta_s, N^i_s)ds\Big)^{2p} + \mathbf{E}\Big(\int_0^T|(\tilde{\sigma} - \sigma)(s, \Theta_s, N^i_s)|^2 ds\Big)^p\Big]\end{aligned}$$

Similarly to the proof of theorem 2.5, we find for any $i \in \mathcal{K}$ a constant $C^{*,i}_{(p,K,\lambda)}$ such that for $T \le C^{*,i}_{(p,K,\lambda)}$, inequality (2.47) holds.
Therefore for an $i \in \mathcal{K}$ fixed, applying this inequality with $p = 1$ to the following iterated procedure

$$\begin{cases}\forall\, s \in [0,T], \\ X^{n+1}_s = \xi_s + \displaystyle\int_0^s b(r, X^{n+1}_r, Y^n_r, H^n_r, Z^n_r, N^i_r)dr + \int_0^s \sigma(r, X^{n+1}_r, Y^n_r, N^i_r)dB_r \\ \\ Y^{n+1}_s = h_{N^i_T}(X^{n+1}_T) + \displaystyle\int_s^T f(r, \Theta^{n+1}_r, N^i_r)dr - \int_s^T Z^{n+1}_r\, dB_r - \int_s^T \sum_{l\in L} H^{n+1}_r(l)dM_r(l)\end{cases}$$

we prove existence and uniqueness of solution to (E^*) in a small time duration. Similarly to theorem 2.5, we show the two last inequalities.

2.2. **The case of smooth coefficients.** In what follows we reinforce our assumptions on the deterministic coefficients of the problem $(E^{0,x})$ in order to establish some differentiability properties of the map θ defined in corollary 2.2.

2.2.1. *Regularity of* Θ. We introduce $D : L^2(\Omega) \to L^2(\Omega \times [0,T], \mathbf{R}^d)$ the Malliavin derivative operator with respect to the Brownian motion and the following space

$$\mathbf{D}^{1,2} = \{\xi \in L^2(\Omega)\ /\ \mathbf{E}(|\xi|^2) + \mathbf{E}\int_0^T |D_r\xi|^2\, dr < \infty\}$$

For a differentiable function in $\mathbf{R}^Q$ g, g'_x stand for its partial derivative with respect to x.
We say that the coefficients b, f, σ and h satisfy assumption **(H2)** if

(H2.1) (H1.3) holds.
(H2.2) All the functions are twice continously differentiable with respect to x, y, u and z.
(H2.2) All the functions and their derivatives up to order two are K Lipschitz with respect to x, y, u and z.

We claim the following

Theorem 2.7. *Assume that* **(H2)** *holds. Then for all* $i \in \mathcal{K}$ *there exists a constant* $C^{*,i}_{(K,\lambda)}$ *such that* $\forall\, T \leq C^{*,i}_{(K,\lambda)}$, $\forall\,(t,x) \in [0,T]\times \mathbf{R}^d$ *the process* $(\Theta^{t,x,i}_s)_{t\leq s\leq T}$ *solution of* $(E^{t,x})$ *satisfies*

i) $\forall\, s \in [t,T], \qquad X^{t,x,i}_s \in (\mathbf{D}^{1,2})^d,\;\; Y^{t,x,i}_s \in \mathbf{D}^{1,2},\;\; H^{t,x,i}_s \in (\mathbf{D}^{1,2})^{k-1},\;\; Z^{t,x,i}_s \in (\mathbf{D}^{1,2})^d$

ii) $\forall\, r \in [0,T]\backslash(t,s], \qquad D_rX^{t,x,i}_s = D_rY^{t,x,i}_s = D_rH^{t,x,i}_s = D_rZ^{t,x,i}_s = 0$

iii) $\forall\, j \in \{1,...,d\}$, *the process* $(D^j_r\Theta^{t,x,i}_s)_{r\leq s\leq T} \triangleq (D^j_rX^{t,x,i}_s, D^j_rY^{t,x,i}_s, D^j_rH^{t,x,i}_s, D^j_rZ^{t,x,i}_s)_{r\leq s\leq T}$ *is the unique solution of the problem*

$$\begin{cases} D^j_rX_s = \sigma^{(j)}(r,X_r,Y_r,N^{t,i}_r) + \displaystyle\int_r^s \mathcal{B}^{t,x,i}(u,D^j_rX_u,D^j_rY_u,,D^j_rH_u,D^j_rZ_u,N^{t,i}_u)du \\ \qquad\qquad + \displaystyle\int_r^s \Sigma^{t,x,i}(u,D^j_rX_u,D^j_rY_u,N^{t,i}_u)dB_u \\ D^j_rY_s = \mathcal{H}^{t,x,i}(D^j_rX_T,N^{t,i}_T) - \displaystyle\int_s^T F^{t,x,i}(u,D^j_rX_u,D^j_rY_u,D^j_rH_u,D^j_rZ_u,N^{t,i}_u)du - \int_s^T D^j_rZ_u dB_u \\ \qquad\qquad - \displaystyle\int_s^T \sum_{l\in L} D^j_rH_u(l)dM_u(l) \\ \mathbf{E}\displaystyle\int_r^T (|D^j_rX_u|^2 + |D^j_rY_u|^2 + |D^j_rH_u|^2 + |D^j_rZ_u|^2)du < \infty \end{cases}$$

where the functions $\mathcal{B}^{t,x,i}, \Sigma^{t,x,i}, F^{t,x,i}$ *and* $\mathcal{H}^{t,x,i}$ *are defined by*
$\forall\,(r,\tilde{x},v,w,q) \in [0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^{k-1}\times\mathbf{R}^d$

$$\begin{aligned} \mathcal{B}^{t,x,i}(r,\tilde{x},v,w,q,N^{t,i}_r) &= b'_x(r,\Theta^{t,x,i}_r,N^{t,i}_r)\tilde{x} + b'_y(r,\Theta^{t,x,i}_r,N^{t,i}_r)v + b'_u(r,\Theta^{t,x,i}_r,N^{t,i}_r)w + b'_z(r,\Theta^{t,x,i}_r,N^{t,i}_r)q \\ F^{t,x,i}(r,\tilde{x},v,w,q,N^{t,i}_r) &= f'_x(r,\Theta^{t,x,i}_r,N^{t,i}_r)\tilde{x} + f'_y(r,\Theta^{t,x,i}_r,N^{t,i}_r)v + f'_u(r,\Theta^{t,x,i}_r,N^{t,i}_r)w + f'_z(r,\Theta^{t,x,i}_r,N^{t,i}_r)q \\ \Sigma^{t,x,i}(r,\tilde{x},v,N^{t,i}_r) &= \sigma'_x(r,X^{t,x,i}_r,Y^{t,x,i}_r,N^{t,i}_r)\tilde{x} + \sigma'_y(r,X^{t,x,i}_r,Y^{t,x,i}_r,N^{t,i}_r)v \\ \mathcal{H}^{t,x,i}(\tilde{x},N^{t,i}_T) &= h'(X^{t,x,i}_T,N^{t,i}_T)\tilde{x} \end{aligned}$$

and $\sigma^{(j)}$ *is the* j^{th} *column of the matrix* σ.

iv) $\forall\, j = 1,...,d$, *the process* $\{D^j_sY^{t,x,i}_s\ ,\ s\in[t,T]\}$ *is a version of* $\{(Z^{t,x,i}_s)^j\ ,\ t\leq s\leq T\}$.

Proof : Let $i \in \mathcal{K}$ be fixed. Thanks to theorem 2.5, there exists a constant $\tilde{C}^{i,3}_{K,\lambda}$ such that for $T \leq \tilde{C}^{i,3}_{K,\lambda}$ the sequence (we omit the superscript "^{t,x,i}" for a sake of simplicity) defined by

$$\begin{cases} \forall\, s\in[t,T] \\ X^{n+1}_s = x + \displaystyle\int_t^s b(r,X^{n+1}_r,Y^n_r,H^n_r,Z^n_r,N^{t,i}_r)dr + \int_t^s \sigma(r,X^{n+1}_r,Y^n_r,N^{t,i}_r)dB_r \\ Y^{n+1}_s = h_{N^i_T}(X^{n+1}_T) + \displaystyle\int_s^T f(r,\Theta^{n+1}_r,N^{t,i}_r)dr - \int_s^T Z^{n+1}_r\,dB_r - \int_s^T \sum_{l\in L} H^{n+1}_r dM_r(l) \end{cases}$$

satisfies

$$\sup_{n\in\mathbf{N}}\left[\mathbf{E}\sup_{t\leq s\leq T}|X^n_s|^4 + \mathbf{E}\sup_{t\leq s\leq T}|X^n_s|^4 + \mathbf{E}\Big(\int_t^T |Z^n_s|^2\,ds\Big)^2 + \mathbf{E}\Big(\int_t^T \sum_{l\in L}(H^n_s(l))^2\,ds\Big)^2\right] < \infty$$

and

$$\mathbf{E}\sup_{t\le s\le T}|X_s^m-X_s^n|^4 + \mathbf{E}\sup_{t\le s\le T}|Y_s^m-Y_s^n|^4+\mathbf{E}\Big(\int_t^T|Z_s^m-Z_s^n|^2\,ds\Big)^2 + \mathbf{E}\Big(\int_t^T\sum_{l\in L}(H_s^m(l)-H_s^n(l)^2\,ds\Big)^2 \to 0 \quad \text{whenever } n,m\to\infty. \tag{2.48}$$

For $n\in\mathbf{N}$, we consider the following property $(\mathcal{P}_n)$

$$(\mathcal{P}_n):\begin{cases}\forall\, s\in[t,T],\ X_s^n\in(\mathbf{D}^{1,2})^d,\ \ Y_s^n\in\mathbf{D}^{1,2},\ \ H_s^n\in L^2([t,T],(\mathbf{D}^{1,2})^{k-1}),\ \ Z_s^n\in L^2([t,T],(\mathbf{D}^{1,2})^d)\\ \text{There exists a version of } (D_r\Theta_s^n)=(D_rX_s^n,D_rY_s^n,D_rH_s^n,D_rZ_s^n)_{0\le r\le T,\ t\le s\le T} \text{ such that}\\ \sup_{0\le r\le T}\Big[\mathbf{E}\sup_{t\le s\le T}(|D_rX_s^n|^2+|D_rY_s^n|^2)+\mathbf{E}\Big(\int_t^T\sum_{l\in L}|D_rH_s^n(l)|^2\,ds\Big)^2\\ \qquad +\ \mathbf{E}\Big(\int_t^T|D_rZ_s^n|^2\,ds\Big)^2\Big]<\infty\end{cases}$$

Using Pardoux-Pradeilles-Rao [9], proposition 3.1, we prove $(\mathcal{P}_n)\Longrightarrow(\mathcal{P}_{n+1})$. Therefore choosing a quartet (X^0,Y^0,H^0,Z^0) satisfying $(\mathcal{P}_0)$ then $\forall\, n\in\mathbf{N}$, $(\mathcal{P}_n)$ holds. Moreover,

$$\forall\, r\in[0,T]\ (t,s],\quad D_rX_s^n=0\,;D_rY_s^n=0\,;D_rH_s^n=0\,;D_rZ_s^n=0.$$

and $\forall\, r\in[t,T]$, $(D_r\Theta_s^{t,x,n+1})_{r\le s\le T}$ is solution of the system, $\forall\, j=1,...,d$

$$\begin{cases}D_r^jX_s^{n+1} &= \sigma^{(j)}(r,X_r^{n+1},Y_r^n,N_r^{t,i})+\int_r^s\mathcal{B}^{n,i}(u,D_r^jX_u^{n+1},D_r^jY_u^n,D_r^jH_u^n,D_r^jZ_u^n,N_u^{t,i})du\\ &+\ \int_r^s\Sigma^{n,i}(u,D_r^jX_u^{n+1},D_r^jY_u^n,N_u^{t,i})dB_u\\ D_r^jY_s^{n+1} &= \mathcal{H}^{n,i}(D_r^jX_T^{n+1},N_T^{t,i})-\int_s^TF^{n,i}(u,D_r^j\Theta_u^{n+1},N_u^{t,i})du-\int_s^TD_r^jZ_u^{n+1}dB_u\\ &-\ \int_s^T\sum_{l\in L}D_r^jH_u^{n+1}(l)dM_u(l)\end{cases}$$

where for $n\in\mathbf{N}$, $\mathcal{B}^{n,i},F^{n,i}$, $\Sigma^{n,i}$ and $\mathcal{H}^{n,i}$ are defined by $\forall\,(r,\tilde{x},v,w,q)\in[0,T]\times\mathbf{R}^d,\times\mathbf{R}\times\mathbf{R}^{k-1}\times\mathbf{R}^d$

$$\begin{aligned}\mathcal{B}^{n,i}(r,\tilde{x},v,w,q,N_r^{t,i}) &= b'_x(r,X_r^{n+1},Y_r^n,H_r^n,Z_r^n,N_r^{t,i})\tilde{x}+b'_y(r,X_r^{n+1},Y_r^n,H_r^n,Z_r^n,N_r^{t,i})v\\ &+\ b'_u(r,X_r^{n+1},Y_r^n,H_r^n,Z_r^n,N_r^{t,i})w+b'_z(r,X_r^{n+1},Y_r^n,H_r^n,Z_r^n,N_r^{t,i})q\\ F^{n,i}(r,\tilde{x},v,w,q,N_r^{t,i}) &= f'_x(r,\Theta_r^{n+1},N_r^{t,i})\tilde{x}+f'_y(r,\Theta_r^{n+1},N_r^{t,i})v+f'_u(r,\Theta_r^{n+1},N_r^{t,i})w+f'_z(r,\Theta_r^{n+1},N_r^{t,i})q\\ \Sigma^{n,i}(r,\tilde{x},v,N_r^{t,i}) &= \sigma'_x(r,X_r^{n+1},Y_r^n,N_r^{t,i})\tilde{x}+\sigma'_y(r,X_r^{n+1},Y_r^n,N_r^{t,i})v\\ \mathcal{H}^{n,i}(\tilde{x},N_T^{t,i}) &= h'(X_T^{n+1},N_T^{t,i})\tilde{x}\end{aligned}$$

Noting that the functions $\mathcal{B}^{n,i},F^{n,i}$, $\Sigma^{n,i}$ and $\mathcal{H}^{n,i}$ are $K-$ Lipschitz with respect to $(r,\tilde{x},v,w,q)$, thanks to theorem 2.6, there exists two constants which we note again $\tilde{C}^{i,3}_{(K,\lambda)}$ and $\tilde{\beta}^i_{(K,\lambda)}$ such that for

$T \leq \tilde{C}^{i,3}_{(K,\lambda)}, \quad \forall\, r \in [t,T],$

$$\begin{aligned}
\mathbf{E}\sup_{r\leq s\leq T}|D^j_r X^{n+1}_s|^4 &+ \mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^{n+1}_s|^4 + \mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^{n+1}_s(l))^2 ds\Big)^2\\
&+ \mathbf{E}\Big(\int_r^T |D^j_r Z^{n+1}_s|^2 ds\Big)^2 \leq \tilde{\beta}^i_{(K,\lambda)}\Big[\mathbf{E}\sup_{r\leq s\leq T}|\sigma^{(j)}(r, X^{n+1}_r, Y^n_r, N^{t,i}_r)|^4\\
&+ \mathbf{E}\Big(\int_r^T |\Sigma^{n,i}(u,0,D^j_r Y^n_u, N^{t,i}_u) - \Sigma^{n,i}(u,0,0,N^{t,i}_u)|^2 du\Big)^2\\
&+ \mathbf{E}\Big(\int_r^T |\mathcal{B}^{n,i}(u,0,D^j_r Y^n_u, D^j_r H^n_u, D^j_r Z^n_u, N^{t,i}_u) - \mathcal{B}^{n,i}(u,0,0,0,0,N^{t,i}_u)|du\Big)^4\Big]
\end{aligned}$$

Using the Lipschitz property of the coefficients and (H1.3), we deduce $\forall\, r \in [t,T]$

$$\begin{aligned}
\mathbf{E}\sup_{r\leq s\leq T}|D^j_r X^{n+1}_s|^4 &+ \mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^{n+1}_s|^4 + \mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^{n+1}_s(l))^2 ds\Big)^2 + \mathbf{E}\Big(\int_r^T |D^j_r Z^{n+1}_s|^2 ds\Big)^2\\
&\leq \tilde{\beta}^i_{(K,\lambda)}\Big[\Lambda^4 + K^4T^2\mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^n_s|^4\\
&+ K^4T^2\Big(\mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^n_s|^4 + \mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^n_s(l))^2 ds\Big)^2 + \mathbf{E}\Big(\int_r^T |D^j_r Z^n_s|^2 ds\Big)^2\Big)\Big]
\end{aligned}$$

Hence applying an induction and modifying $\tilde{C}^{i,3}_{(K,\lambda)}$ we show that for $T \leq \tilde{C}^{i,3}_{(K,\lambda)}$, the following series

$$\begin{aligned}
\sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r X^n_s|^4 + \sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^n_s|^4 &+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^n_s(l))^2 ds\Big)^2\\
&+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T |D^j_r Z^n_s|^2 ds\Big)^2
\end{aligned}$$

converges. Which implies for $T \leq \tilde{C}^{i,3}_{(K,\lambda)}$,

$$\begin{aligned}
\sup_{n\in\mathbf{N}}\Big[\sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r X^n_s|^4 + \sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^n_s|^4 &+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^n_s(l))^2 ds\Big)^2\\
&+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T |D^j_r Z^n_s|^2 ds\Big)^2\Big] < \infty
\end{aligned}$$

Similarly since the derivatives of functions are K-Lipschitz and (2.48) holds, we deduce using again theorem 2.6, for all $T \leq \tilde{C}^{i,3}_{(K,\lambda)}$ (modifying $\tilde{C}^{i,3}_{(K,\lambda)}$ if necessary)

$$\begin{aligned}
\sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r X^n_s - D^j_r X^m_s|^4 &+ \sup_{0\leq r\leq T}\mathbf{E}\sup_{r\leq s\leq T}|D^j_r Y^n_s - D^j_r Y^m_s|^4\\
&+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T \sum_{l\in L}(D^j_r H^n_s(l) - D^j_r H^m_s(l))^2 ds\Big)^2\\
&+ \sup_{0\leq r\leq T}\mathbf{E}\Big(\int_r^T |D^j_r Z^n_s - D^j_r Z^m_s|^2 ds\Big)^2 \;\to 0 \text{ whenever } m,n\to\infty
\end{aligned}$$

Hence we deduce i), ii) and iii).
Futhermore iv) is proved using Proposition 3.1 of Pardoux-Pradeilles-Rao [9]. ■

Elsewhere since the functions $\mathcal{B}^{t,x,i}, F^{t,x,i}, \Sigma^{t,x,i}$ and $\mathcal{H}^{t,x,i}$ are $K-$Lipschitz with respect to $\tilde{x}, v, w, q$, the following result which can be proved analogously to theorem 2.3.

Theorem 2.8. *Assume that* **(H2)** *holds and let* $i \in \mathcal{K}$. *Then* $\forall\ T \leq \tilde{C}^{(i2)}_{(K,\lambda)}$, $\forall\ (t,x) \in [0,T]\times \mathbf{R}^d$, $\forall\ j \in \{1,...,d\}$, *the system*

$$\begin{cases} \forall\ s \in [t,T] \\ \partial^j X^{t,x,i}_s = e_j + \displaystyle\int_t^s \mathcal{B}^{t,x,i}(u,\partial^j X_u,\partial^j Y_u,\partial^j H_u,\partial^j Z_u, N^{t,i}_u)du + \int_t^s \Sigma^{t,x,i}(u,\partial^j X_u,\partial^j Y_u, N^{t,i}_u)dB_u \\ \partial^j Y^{t,x,i}_s = \mathcal{H}^{t,x,i}(\partial^j X_T, N^{t,i}_T) - \displaystyle\int_s^T F^{t,x,i}(u,\partial^j X_u,\partial^j Y_u,\partial^j H_u,\partial^j Z_u, N^{t,i}_u)du - \int_s^T \partial^j Z_u dB_u \\ \qquad\qquad - \displaystyle\int_s^T \sum_{l\in L} \partial^j H_u(l)dM_u(l) \\ \mathbf{E}\displaystyle\int_t^T (|\partial^j X_u|^2 + |\partial^j Y_u|^2 + |\partial^j H_u|^2 + |\partial^j Z_u|^2)du < \infty \end{cases}$$

admits a unique solution $(\partial\Theta^{t,x,i}_s)_{t\leq s\leq T} \triangleq (\partial X^{t,x,i}_s, \partial Y^{t,x,i}_s, \partial H^{t,x,i}_s, \partial Z^{t,x,i}_s)_{t\leq s\leq T}$ *where* e_j *is the* j^{th} *vector of the canonical basis of* $\mathbf{R}^d$.

From uniqueness of solutions of this system we deduce the following

Corollary 2.5. *Assume that* **(H2)** *holds. Let* $i \in \mathcal{K}$ *and* $T \leq \tilde{C}^{(i2)}_{(K,\lambda)}$. *Then* $\forall\ (t,x) \in [0,T]\times\mathbf{R}^d$, $\forall\ t \leq r \leq s \leq T$, *we have*

$$\begin{aligned} D_r X^{t,x,i}_s &= (\partial X^{t,x,i}_s)(\partial X^{t,x,i}_r)^{-1}\sigma(r, X^{t,x,i}_r, Y^{t,x,i}_r, N^{t,i}_r) \\ D_r Y^{t,x,i}_s &= (\partial Y^{t,x,i}_s)(\partial X^{t,x,i}_r)^{-1}\sigma(r, X^{t,x,i}_r, Y^{t,x,i}_r, N^{t,i}_r) \\ D_r H^{t,x,i}_s &= (\partial H^{t,x,i}_s)(\partial X^{t,x,i}_r)^{-1}\sigma(r, X^{t,x,i}_r, Y^{t,x,i}_r, N^{t,i}_r) \\ D_r Z^{t,x,i}_s &= (\partial Z^{t,x,i}_s)(\partial X^{t,x,i}_r)^{-1}\sigma(r, X^{t,x,i}_r, Y^{t,x,i}_r, N^{t,i}_r). \end{aligned}$$

If $\tilde{h}$ is a function of $x \in \mathbf{R}^d$, for $\rho \in \mathbf{R} - \{0\}$, let

$$\Delta^j_\rho \tilde{h}(x) = \frac{1}{\rho}(\tilde{h}(x+\rho e_j) - \tilde{h}(x)), \qquad 1 \leq j \leq d$$

Proposition 2.1. *Assume* **(H2)** *holds and* $i \in \mathcal{K}$ *be fixed. For* $T \leq \tilde{C}^{(i2)}_{(K,\lambda)}$, $(t,x) \in [0,T]\times\mathbf{R}^d$ *we consider the processes* $(\Theta^{t,x,i}_s)_{(t\leq s\leq T)}$ *and* $(\partial\Theta^{t,x,i}_s)_{(t\leq s\leq T)}$ *extended on* $[0,T]$ *as in corollary 2.3.*
Then $\forall\ p \geq 1$, $\forall\ j = 1,...,d$,
i) There exists a constant $\gamma^{(i1)}_{(p,K,\lambda)}$ *depending on* p, K *and* λ *such that* $\forall\ (t,x) \in [0,T]\times\mathbf{R}^d$

$$\begin{aligned} \mathbf{E}\sup_{0\leq s\leq T}(|\Delta^j_\rho X^{t,x,i}_s|^{2p} &+ |\partial^j X^{t,x,i}_s|^{2p}) + \mathbf{E}\sup_{0\leq s\leq T}(|\Delta^j_\rho Y^{t,x,i}_s|^{2p} + |\partial^j Y^{t,x,i}_s|^{2p}) \\ &+ \mathbf{E}\left(\sum_{l\in L}\int_0^T (|\Delta^j_\rho H^{t,x,i}_s|^2 + |\partial^j H^{t,x,i}_s|^2)(l)ds\right)^p \\ &+ \mathbf{E}\left(\int_0^T (|\Delta^j_\rho Z^{t,x,i}_s|^2 + |\partial^j Z^{t,x,i}_s|^2)ds\right)^p \leq \gamma^{(i1)}_{(p,K,\lambda)} \end{aligned} \tag{2.49}$$

ii) There exists a constant $\gamma^{(i2)}_{(p,K,\lambda)}$ *depending on* p, K *and* λ *such that* $\forall\ (t,x,\rho)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}^*$

$$
\begin{aligned}
\text{(2.50)}\qquad \mathbf{E}\sup_{0\le s\le T}|\Delta^j_\rho X^{t,x,i}_s &- \partial^j X^{t,x,i}_s|^{2p}+\mathbf{E}\sup_{0\le s\le T}|\Delta^j_\rho Y^{t,x,i}_s-\partial^j Y^{t,x,i}_s|^{2p}\\
&+ \mathbf{E}\left(\sum_{l\in L}\int_0^T\left((\Delta^j_\rho H^{t,x,i}_s-\partial^j H^{t,x,i}_s)(l)\right)^2ds\right)^p\\
&+ \mathbf{E}\left(\int_0^T|\Delta^j_\rho Z^{t,x,i}_s-\partial^j Z^{t,x,i}_s|^2ds\right)^p \ \le\ \gamma^{(i2)}_{(p,K,\lambda)}\ |\rho|^{2p}
\end{aligned}
$$

iii) There exists two constants $\gamma^{(i3)}_{(p,K,\lambda)}$ *depending on* p, K,λ *and* $\gamma^{(i4)}_{(p,K,\lambda,\Lambda)}$ *depending on* p, K,λ *and* Λ *such that* $\forall\ ((t,x),(t',x'))\in([0,T]\times\mathbf{R}^d)^2$

$$
\begin{aligned}
\text{(2.51)}\qquad \mathbf{E}\sup_{0\le s\le T}|\partial^j X^{t,x,i}_s &- \partial^j X^{t',x',i}_s|^{2p}+\mathbf{E}\sup_{0\le s\le T}|\partial^j Y^{t,x,i}_s-\partial^j Y^{t',x',i}_s|^{2p}\\
&+ \mathbf{E}\left(\sum_{l\in L}\int_0^T((\partial^j H^{t,x,i}_s-\partial^j H^{t',x',i}_s)(l))^2ds\right)^p\\
&+ \mathbf{E}\left(\int_0^T|\partial^j Z^{t,x,i}_s-\partial^j Z^{t',x',i}_s|^2ds\right)^p \le\gamma^{(i3)}_{(p,K,\lambda)}\ |x-x'|^{2p}+\gamma^{(i4)}_{(p,K,\lambda)}\ |t-t'|^p
\end{aligned}
$$

Proof : Let $i\in\mathcal{K}$ and $T\le\tilde{C}^{(i2)}_{(K,\lambda)}$.
We consider the following functions defined for all $(r,\tilde{x},v,w,q)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^{k-1}\times\mathbf{R}^d$ by

$$
\begin{aligned}
\mathcal{B}^{t,x,\rho}(r,\tilde{x},v,w,q,N^{t,i}_r) &= \int_0^1\left(b'_x(\zeta^{t,x,\rho}_{r,\alpha})\tilde{x}+b'_y(\zeta^{t,x,\rho}_{r,\alpha})v+b'_u(\zeta^{t,x,\rho}_{r,\alpha})w+b'_z(\zeta^{t,x,\rho}_{r,\alpha})q\right)d\alpha\\
F^{t,x,\rho}(r,\tilde{x},v,w,q,N^{t,i}_r) &= \int_0^1\left(f'_x(\zeta^{t,x,\rho}_{r,\alpha})\tilde{x}+f'_y(\zeta^{t,x,\rho}_{r,\alpha})v+f'_u(\zeta^{t,x,\rho}_{r,\alpha})w+f'_z(\zeta^{t,x,\rho}_{r,\alpha})q\right)d\alpha\\
\Sigma^{t,x,\rho}(r,\tilde{x},v,N^{t,i}_r) &= \int_0^1\left(\sigma'_x(\tilde{\zeta}^{t,x,\rho}_{r,\alpha})\tilde{x}+\sigma'_y(\tilde{\zeta}^{t,x,\rho}_{r,\alpha})v\right)d\alpha\\
\mathcal{H}^{t,x,\rho}(\tilde{x},N^{t,i}_T) &= \int_0^1\left(h'(X^{t,x,i}_T+\alpha\rho\Delta^j_\rho X^{t,x,i}_T,N^{t,i}_T)\tilde{x}\right)d\alpha\\
\text{where}\qquad \zeta^{t,x,\rho}_{r,\alpha} &= (r,X_r+\alpha\rho\Delta^j_\rho X^{t,x,i}_r,Y_r+\alpha\rho\Delta^j_\rho Y^{t,x,i}_r,H_r+\alpha\rho\Delta^j_\rho H^{t,x,i}_r,Z_r+\alpha\rho\Delta^j_\rho Z^{t,x,i}_r,N^{t,i}_r)\\
\tilde{\zeta}^{t,x,\rho}_{r,\alpha} &= (r,X_r+\alpha\rho\Delta^j_\rho X^{t,x,i}_r,Y_r+\alpha\rho\Delta^j_\rho Y^{t,x,i}_r,N^{t,i}_r)
\end{aligned}
$$

Note that for all $(t,x,\rho,i)\in[0,T]\times\mathbf{R}^d\times\mathbf{R}^*\times\mathcal{K}$, $\ \forall\ j=1,...,d$, the quartet of processes

$(\Delta_\rho^j \Theta_r^{t,x,i})_{(0\leq s\leq T)} \triangleq (\Delta_\rho^j X_s^{t,x,i}, \Delta_\rho^j Y_s^{t,x,i}, \Delta_\rho^j H_s^{t,x,i}, \Delta_\rho^j Z_s^{t,x,i})_{(0\leq s\leq T)}$ is solution of the system

$$\begin{cases} \Delta_\rho^j X_s^{t,x,i} = e_j + \int_0^s 1_{[t,T]}(r)\mathcal{B}^{t,x,\rho}(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, \Delta_\rho^j H_r, \Delta_\rho^j Z_r, N_r^{t,i})dr \\ \qquad\qquad + \int_0^s 1_{[t,T]}(r)\Sigma^{t,x,\rho}(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, N_r^{t,i})dB_r \\ \Delta_\rho^j Y_s^{t,x,i} = \mathcal{H}^{t,x,\rho}(\Delta_\rho^j X_T, N_T^{t,i}) + \int_s^T 1_{[t,T]}(r)F^{t,x,\rho}(r, \Delta_\rho^j X_r, \Delta_\rho^j Y_r, \Delta_\rho^j H_r, \Delta_\rho^j Z_r, N_r^{t,i})dr - \int_s^T \Delta_\rho^j Z_r dB_r \\ \qquad\qquad - \int_s^T \sum_{l\in L} \Delta_\rho^j H_r(l) dM_r(l) \\ \mathbf{E}\int_0^T (|\Delta_\rho^j X_r|^2 + |\Delta_\rho^j Y_r|^2 + |\Delta_\rho^j H_r|^2 + |\Delta_\rho^j Z_r|^2)dr < \infty \end{cases}$$

and the two quartets of functions $(1_{[t,T]}(r)\mathcal{B}^{t,x,\rho},\ 1_{[t,T]}(r)F^{t,x,\rho},\ 1_{[t,T]}(r)\Sigma^{t,x,\rho},\ \mathcal{H}^{t,x,\rho})$ and $(\mathcal{B}^{t,x,i},\ F^{t,x,i},\ \Sigma^{t,x,i},\ \mathcal{H}^{t,x,i})$ satisfy assumption **(H2)**, then applying (2.34) to the associated solutions, we easily deduce (2.49).
Futhermore using once again theorem 2.5, we obtain $\forall\ p \geq 1$,

$$\begin{aligned} \mathbf{E}\sup_{0\leq s\leq T} |\Delta_\rho^j X_s^{t,x,i} &- \partial^j X_s^{t,x,i}|^{2p} + \mathbf{E}\sup_{0\leq s\leq T} |\Delta_\rho^j Y_s^{t,x,i} - \partial^j Y_s^{t,x,i}|^{2p} \\ &+ \mathbf{E}\left(\sum_{l\in L}\int_0^T ((\Delta_\rho^j H_s^{t,x,i} - \partial^j H_s^{t,x,i})(l)^2 ds\right)^p \\ &+ \mathbf{E}\left(\int_0^T |\Delta_\rho^j Z_s^{t,x,i} - \partial^j Z_s^{t,x,i}|^2 ds\right)^p \leq \beta_{(p,K,\lambda)}^{(i)}\Big[\mathbf{E}|(\mathcal{H}^{t,x,\rho} - \mathcal{H}^{t,x,i})(\partial^j X_T, N_T^{t,i})|^{2p} \\ &+ \mathbf{E}\Big(\int_t^T |\Sigma^{t,x,\rho} - \Sigma^{t,x,i}|^2(s, \partial^j X_s, \partial^j Y_s, N_s^{t,i})ds\Big)^p \\ &+ \mathbf{E}\Big(\int_t^T (|\mathcal{B}^{t,x,\rho} - \mathcal{B}^{t,x,i}| + |F^{t,x,\rho} - F^{t,x,i}|)(s, \partial^j X_s, \partial^j Y_s, \partial^j H_s, \partial^j Z_s, N_s^{t,i})ds\Big)^{2p}\Big] \end{aligned}$$

Since $\forall\ i \in \mathcal{K},\ h'(\cdot, i)$ is K-Lipschitz, there exists a constant c_p depending on p such that

$$\begin{aligned} |(\mathcal{H}^{t,x,\rho} - \mathcal{H}^{t,x,i})(\partial^j X_T, N_T^{t,i})|^{2p} &= \left|\partial^j X_T\Big(\int_0^1 (h'(X_T + \alpha\rho\Delta_\rho^j X_T, N_T^{t,i}) - h'(X_T, N_T^{t,i}))d\alpha\Big)\right|^{2p} \\ &\leq c_p K^{2p}|\rho|^{2p}(|\Delta_\rho^j X_T|\ |\partial^j X_T|)^{2p} \end{aligned}$$

Using the same argument in the two last integrals, we prove the existence of a constant $\gamma^{(i)}_{(p,K,\lambda)}$ depending on p, λ and K satisfying

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le T}|\Delta^j_\rho X^{t,x,i}_s &- \partial^j X^{t,x,i}_s|^{2p}+\mathbf{E}\sup_{0\le s\le T}|\Delta^j_\rho Y^{t,x,i}_s-\partial^j Y^{t,x,i}_s|^{2p}\\ &+ \mathbf{E}\left(\sum_{l\in L}\int_0^T((\Delta^j_\rho H^{t,x,i}_s-\partial^j H^{t,x,i}_s)(l)^2ds\right)^p\\ &+ \mathbf{E}\left(\int_0^T|\Delta^j_\rho Z^{t,x,i}_s-\partial^j Z^{t,x,i}_s|^2ds\right)^p\le\gamma^{(i)}_{(p,K,\lambda)}\;|\rho|^{2p}\Big[\mathbf{E}(|\Delta^j_\rho X_T||\partial^j X_T|)^{2p}\\ &+ \mathbf{E}\Big(\int_0^T(|\Delta^j_\rho X_s|^2+|\Delta^j_\rho Y_s|^2)(|\partial^j X_s|^2+|\partial^j Y_s|^2)ds\Big)^{2p}\\ &+ \mathbf{E}\Big(\int_0^T(|\Delta^j_\rho X_s|+|\Delta^j_\rho Y_s|+|\Delta^j_\rho H_s|+|\Delta^j_\rho Z_s|)\\ &\times\;(|\partial^j X_s|+|\partial^j Y_s|+|\partial^j H_s|+|\partial^j Z_s|)ds\Big)^{2p}\Big]\end{aligned}$$

Whence thanks to standard estimates and inequality (2.34), we deduce (2.50).
Using once again theorem 2.5, we have $\forall\;((t,x),(t',x'))\in([0,T]\times\mathbf{R}^d)^2\;\;\forall\;j=1,...,d$

$$\begin{aligned}\mathbf{E}\sup_{0\le s\le T}|\partial^j X^{t,x,i}_s &- \partial^j X^{t',x',i}_s|^{2p}+\mathbf{E}\sup_{0\le s\le T}|\partial^j Y^{t,x,i}_s-\partial^j Y^{t',x',i}_s|^{2p}\\ &+ \mathbf{E}\left(\sum_{l\in L}\int_0^T((\partial^j H^{t,x,i}_s-\partial^j H^{t',x',i}_s)(l)^2ds\right)^p\\ &+ \mathbf{E}\left(\int_0^T|\partial^j Z^{t,x,i}_s-\partial^j Z^{t',x',i}_s|^2ds\right)^p\le\beta^{(i)}_{(p,K,\lambda)}\Big[|t-t'|^p\\ &+ \mathbf{E}(|X^{t,x}_T-X^{t',x'}_T||\partial^j X_T|)^{2p}\\ &+ \mathbf{E}\Big(\int_{t'}^T(|X^{t,x}_s-X^{t',x'}_s|^2+|Y^{t,x}_s-Y^{t',x'}_s|^2)(|\partial^j X^{t,x}_s|^2+|\partial^j Y^{t,x}_s|^2)ds\Big)^p\\ &+ \mathbf{E}\Big(\int_{t'}^T(|X^{t,x}_s-X^{t',x'}_s|+|Y^{t,x}_s-Y^{t',x'}_s|+|H^{t,x}_s-H^{t',x'}_s|+|Z^{t,x}_s-Z^{t',x'}_s|)\\ &\times(|\partial^j X^{t,x}_s|+|\partial^j Y^{t,x}_s|+|\partial^j H^{t,x}_s|+|\partial^j Z^{t,x}_s|)ds\Big)^{2p}\Big]\end{aligned}$$

Thanks to standard estimates and corollary 2.3, we find two constants $\gamma^{(i3)}_{(p,K,\lambda)}$ and $\tilde\gamma^{(i4)}_{(p,K,\lambda,\Lambda)}$ satisfying (2.51). ■

Corollary 2.6. *Assume that* **(H2)** *holds and let $i\in\mathcal{K}$ be fixed. Then for every $T\le\tilde C^{(i2)}_{(K,\lambda)}$, the map θ is twice differentiable with respect to x and for $j=1,...,d$, the functions θ, $(\frac{\partial\theta}{\partial x_j})_j$ and $(\frac{\partial\theta}{\partial x_m\partial x_j})_{m,j}$ are continuous on $[0,T]\times\mathbf{R}^d$.*

Proof : Let $i\in\mathcal{K},\;T\le\tilde C^{(i2)}_{(K,\lambda)}$ and $(t,x)\in[0,T]\times\mathbf{R}^d$. Inequality (2.28) implies θ is Lipschitz continuous with respect to x and we easily deduce from (2.50)

$$\forall\;j=1,...,d\;,\qquad\lim_{\rho\longrightarrow0}\Delta^j_\rho Y^{t,x,i}_t=\partial^j Y^{t,x,i}_t$$

hence $\theta(t,\cdot,i)$ is differentiable with respect to x and its partial derivatives are defined by

$$\forall\, j=1,...,d\,, \qquad \frac{\partial\theta}{\partial x_j}(t,x,i)=\partial^j Y_t^{t,x,i}$$

Moreover using (2.51), we deduce $\frac{\partial\theta}{\partial x_j}(t,\cdot,i)=\partial^j Y_t^{t,\cdot,i}$ is continuous and $\gamma_{p,K,\lambda}^{(i3)}$ Lipschitz.
With assumptions **(H2)**, applying the same method as [3], corollary 5.4 to the derivatives of the functions, we can prove that for $T\le \tilde{C}_{(K,\lambda)}^{(i2)}$ and $\forall\, j=1,...,d$, $\frac{\partial\theta(t,\cdot,i)}{\partial x_j}$ is continuously differentiable with respect to x. ■

Corollary 2.7. *Let $i\in\mathcal{K}$ and $T\le\tilde{C}_{(K,\lambda)}^{(i2)}$. Then $\forall\,(t,x)\in[0,T]\times\mathbf{R}^d$, $\forall\, t\le s\le T$,*

$$Z_s^{t,x,i}=\nabla_x\,\theta_{N_s^{t,i}}(s,X_s^{t,x,i})\;\tilde{\sigma}_{N_s^{t,i}}(s,X_s^{t,x,i})$$

with $\tilde{\sigma}_{N_s^{t,i}}(s,X_s^{t,x,i})=\sigma_{N_s^{t,i}}(s,X_s^{t,x,i},\;\theta_{N_s^{t,i}}(s,X_s^{t,x,i}))$.

Proof : It is an obvious consequence or Corollary 2.5 and Theorem 2.7 ■
We are now in position to prove our main result.

2.2.2. *Solution of a parabolic system of PDEs.* In this section we are interested to the system of PDEs $(\mathcal{E})$ and we need to specify the Lipschitz constant of σ and h.
We say that the coefficients $b,\ \sigma,\ f$ and h satisfy assumption **(H3)** if **(H2)** holds with $\tilde{k}$ as Lipschitz constant of σ and h and moreover the following non degeneracy condition is satisfy

(H3.1) $\exists\delta>0$ such that $\forall\, i\in\mathcal{K},\ \ \forall\, v\in\mathbf{R}^d,\ \ \forall\,(t,x,y)\in[0,T]\times\mathbf{R}^d\times\mathbf{R},\quad <v,a_i(t,x,y)\,v>\ \ge\delta\,|v|^2$.

Let us recall that

$$\theta\,(t,x)=(\,\theta(t,x,1),\theta(t,x,2),...,\theta(t,x,k)\,),\qquad (t,x)\in[0,T]\times\mathbf{R}^d \tag{2.52}$$

We claim the following

Proposition 2.2. *Assume* **(H3)** *is in force. Then there exists a constant $\tilde{C}_{(K,\lambda)}^{(2)}$ such that for all $T\le\tilde{C}_{(K,\lambda)}^{(2)}$, the map $\theta\in\mathcal{C}^{1,2}([0,T]\times\mathbf{R}^d,\mathbf{R}^k)$ and is solution to $(\mathcal{E})$.*
Moreover there exists some constants C (depending on Λ and T) and Γ (depending on $\Lambda,\tilde{k},\delta,d$ and T) such that

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d}|\theta(t,x)|\ \le\ C \tag{2.53}$$

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d}|\nabla_x\theta(t,x)|\ \le\ \Gamma. \tag{2.54}$$

Proof : Let us define $\tilde{C}_{(K,\lambda)}^{(2)}=\min_{i\in\mathcal{K}}\,\tilde{C}_{(K,\lambda)}^{(i2)}$ and we assume that $T\le\tilde{C}_{(K,\lambda)}^{(2)}$. For some $t\in[0,T]$ we have $\forall\, s\in[t,T]$ and $i\in\mathcal{K}$,

$$\theta(s,x,i)-\theta(t,x,i)=\theta(s,x,i)-\ \theta(s,X_s^{t,x,i},N_s^{t,i})+\ \theta(s,X_s^{t,x,i},N_s^{t,i})-\ \theta(t,x,i)$$

Applying Itô's formula to the function $\theta(s,\cdot,\cdot)$, and putting $\theta(s,x,i)=\theta_i(s,x)$ we obtain $\forall\ s\in[t,T]$,

$$\begin{aligned}
\theta_{N_s^{t,i}}(s,X_s^{t,x,i}) &= \theta_i(s,x)+\int_t^s \nabla_x\theta_{N_r^{t,i}}(s,X_r)\,b(r,\Theta_r^{t,x,i},N_r^{t,i})dr\\
&+ \int_t^s \nabla_x\theta_{N_r^{t,i}}(s,X_r)\,\sigma(r,X_r,Y_r,N_r^{t,i})\,dB_r\\
&+ \frac{1}{2}\int_t^s Tr(\partial^2_{xx}\,\theta_{N_r^{t,i}}(s,X_r))(\sigma\,\sigma^*)(r,X_r,Y_r,N_r^{t,i})dr\\
&+ \int_t^s\sum_{l\in L}\Big[(\theta_{N_{r^-}^{t,i}+l}-\theta_{N_{r^-}^{t,i}})(s,X_r^{t,x,i})\Big]dN_r(l)
\end{aligned}$$

Futhermore we have $\forall\ s\in[t,T]$,

$$\begin{aligned}
\theta_{N_s^{t,i}}(s,X_s^{t,x,i})-\ \theta_i(t,x) &= Y_s^{t,x,i}-Y_t^{t,x,i}\\
&= -\int_t^s f(r,\Theta_r^{t,x,i},N_r^{t,i})dr+\int_t^s Z_r^{t,x,i}\,dB_r+\int_t^s\sum_{l\in L}H_r^{t,x,i}(l)\,dM_r(l)
\end{aligned}$$

Hence we deduce $\forall\ s\in[t,T]$

$$\begin{aligned}
\theta_i(s,x)-\theta_i(t,x) &= -\int_t^s\Big[\frac{1}{2}Tr(\partial^2_{xx}\,\theta_{N_r^{t,i}}(s,X_r))(\sigma\,\sigma^*)(r,X_r,Y_r,N_r^{t,i})\\
&+ \nabla_x\theta_{N_r^{t,i}}(s,X_r)\,b(r,\Theta_r^{t,x,i},N_r^{t,i})\\
&+ f(r,\Theta_r^{t,x,i},N_r^{t,i})+\lambda\sum_{l\in L}\Big((\theta_{N_{r^-}^{t,i}+l}-\theta_{N_{r^-}^{t,i}})(s,X_r^{t,x,i})\Big)\Big]dr\\
&+ \int_t^s\Big(Z_r^{t,x,i}-\ \nabla_x\theta_{N_r^{t,i}}(s,X_r)\,\sigma(r,X_r,Y_r,N_r^{t,i})\Big)dB_r\\
&+ \int_t^s\sum_{l\in L}\Big[H_r^{t,x,i}(l)-(\theta_{N_{r^-}^{t,i}+l}-\theta_{N_{r^-}^{t,i}})(s,X_r^{t,x,i})\Big]dM_r(l)
\end{aligned}$$

So, considering a subdivision $t=t_0<t_1<...<t_n=T$ of $[t,T]$, we deduce

$$\begin{aligned}
\theta_i(T,x)\ &-\ \theta_i(t,x)\ =\ \sum_{j=0}^{n-1}\theta_i(t_{j+1},x)-\theta_i(t_j,x)\\
&= -\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\Big[\frac{1}{2}Tr(\partial^2_{xx}\,\theta_{N_r^{t_j,i}}(t_{j+1},X_r^{t_j,x,i}))(\sigma\,\sigma^*)(r,X_r^{t_j,x,i},Y_r^{t_j,x,i},N_r^{t_j,i})\\
&+ \nabla_x\theta_{N_r^{t,i}}(t_{j+1},X_r^{t_j,x,i})\,b(r,\Theta_r^{t_j,x,i},N_r^{t_j,i})\\
&+ f(r,\Theta_r^{t_j,x,i},N_r^{t_j,i})+\lambda\sum_{l\in L}(\theta_{N_{r^-}^{t_j,i}+l}-\theta_{N_{r^-}^{t_j,i}})(t_{j+1},X_r^{t_j,x,i})\Big]dr\\
&+ \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\Big(\nabla_x\theta_{N_r^{t_j,i}}(r,X_r)-\ \nabla_x\theta_{N_r^{t_j,i}}(t_{j+1},X_r)\Big)\sigma(r,X_r^{t_j,x,i},Y_r^{t_j,x,i},N_r^{t_j,i})dB_r\\
&+ \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\sum_{l\in L}\Big[(\theta_{N_{r^-}^{t_j,i}+l}-\theta_{N_{r^-}^{t_j,i}})(r,X_r^{t_j,x,i})-(\theta_{N_{r^-}^{t_j,i}+l}-\theta_{N_{r^-}^{t_j,i}})(t_{j+1},X_r^{t_j,x,i})\Big]dM_r(l)
\end{aligned}$$

Since for $i\in\mathcal{K}$, $(X_s^{t,x,i})_{0\le t\le T,0\le s\le T,x\in R^d}$ is continuous and $(Y_s^{t,x,i})_{0\le t\le T,0\le s\le T,x\in R^d}$ is a càdlàg process, hence choosing a subdivision $\{t_0^n,t_1^n,...,t_n^n\}_{n\in\mathbf{N}}$ of $[t,T]$ such that $\lim_{n\to\infty}\ \sup_{0\le j\le n-1}(t_{j+1}^n-t_j^n)\ =0$, we obtain from corollary 2.2 and corollary 2.7

$$\theta(t,x,i) = h_i(x) + \int_t^T \Big(L^i\theta_i(r,x) + \bar{f}_i(r,x,\theta(r,x),\nabla_x\theta_i(r,x)\sigma_i(r,x,\theta_i(r,x))\Big) dr$$

This is enough to conclude that θ is solution of $(\mathcal{E})$. Consequently considering each line of the system of PDEs $(\mathcal{E})$ we note that $\forall\, i \in \mathcal{K},\ \theta(\cdot,\cdot,i)$ solves the parabolic quasilinear equations

$$(\tilde{\mathcal{E}}) \quad \begin{cases} \forall\,(t,x) \in [0,T]\times \mathbf{R}^d, \\ \\ \frac{\partial \tilde{u}}{\partial t}(t,x)\ + L_i^{(1)}\,\tilde{u}(t,x)\ + \bar{f}_i^{(1)}(t,x,\tilde{u}(t,x),\nabla_x\,\tilde{u}(t,x)\sigma_i(t,x,\tilde{u}(t,x))) = 0 \\ \\ \tilde{u}(T,x) = h_i(x) \end{cases}$$

where $\bar{b}_i^{(1)}$ and $\bar{f}_i^{(1)}$ are define by $\forall\ (t,x,y,z) \in [0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d$

$$\bar{f}_i^{(1)}(t,x,y,z) = \bar{f}_i(t,x,\theta_1(t,x),\theta_2(t,x),...,\theta_{i-1}(t,x),y,\theta_{i+1}(t,x),...,\theta_k(t,x),z)$$
$$\bar{b}_i^{(1)}(t,x,y,z) = b_i(t,x,\theta_1(t,x),\theta_2(t,x),...,\theta_{i-1}(t,x),y,\theta_{i+1}(t,x),...,\theta_k(t,x),z)$$

and $L_i^{(1)}$ is defined from L^i substituting $\bar{b}_i$ by $\bar{b}_i^{(1)}$. Using theorem 7.1, chapter VII of Ladyzenskaja et al. [4] théorème 7.1, under assumptions **(H3)** the quasilinear parabolic PDEs $(\tilde{\mathcal{E}})$ have a unique bounded solution $\tilde{u} \in \mathcal{C}^{1,2}([0,T]\times\mathbf{R}^d,\mathbf{R})$ with bounded derivatives. Therefore $\forall\, i\in\mathcal{K}$,

$$\forall\, j,m \in \{1,...,d\}^2,\ \ \frac{\partial\theta_i}{\partial x_j}\ \text{ and }\ \frac{\partial^2\theta_i}{\partial x_j\,\partial x_m}\ \text{ are bounded on }\ \mathbf{R}^d.$$

Let us prove (2.53) and (2.54). For $i\in\mathcal{K}$, we define for $(t,x)\in[0,T]\times\mathbf{R}^d$

$$\tilde{B}_i(t,x) \;=\; \tilde{B}(t,x,i) = \bar{b}_i(t,x,\theta(t,x),\nabla_x\theta_i(t,x)\sigma_i(t,x,\theta_i(t,x)))$$

$$\Sigma_i(t,x) \;=\; \Sigma(t,x,i) = \sigma_i(t,x,\theta_i(t,x))$$

Let $i\in\mathcal{K}$ and ξ be a $\mathcal{G}_t$–measurable random vector with finite second moment. We consider $X_s^{t,\xi,i}$ the diffusion process defined by $\forall\, s\in[t,T]$,

$$X_s^{t,\xi,i} = \xi + \int_t^s \tilde{B}(r,X_r,N_r^{t,i})\,dr + \int_t^s \Sigma(r,X_r,N_r^{t,i})\,dB_r$$

and the following processes $\forall\, s\in[t,T]$,

$$Y_s^{t,\xi,i} = \theta_{N_s^{t,i}}(s,X_s^{t,\xi,i})\ ;\quad Z_s^{t,\xi,i} \;=\; \nabla_x\,\theta_{N_s^{t,i}}(s,X_s^{t,\xi,i})\,\sigma_{N_s^{t,i}}(s,X_s^{t,\xi,i},Y_s^{t,\xi,i})$$

$$\forall\ l\in L\ ,\ \ H_s^{t,\xi,i}(l) \;=\; (\theta_{N_{s^-}^{t,i}+l} - \theta_{N_{s^-}^{t,i}})(s,X_s^{t,\xi,i})$$

Since $\theta_{N_s^{t,i}} \in \mathcal{C}^{1,2}([0,T]\times\mathbf{R}^d,\mathbf{R})$, Itô's formula yields for all $s\in[t,T]$,

$$\begin{aligned}
\theta_{N_s^{t,i}}(s,X_s^{t,\xi,i}) \;=\;& \theta_{N_T^{t,i}}(T,X_T^{t,\xi,i}) - \int_s^T \frac{\partial}{\partial t}\theta_{N_r^{t,i}}(r,X_r^{t,\xi,i})\,dr - \int_s^T \nabla_x\,\theta_{N_r^{t,i}}(r,X_r)\,\tilde{B}(r,X_r,N_r^{t,i})dr \\
-& \int_s^T \nabla_x\,\theta_{N_r^{t,i}}(r,X_r)\,\Sigma(r,X_r,N_r^{t,i})\,dB_r \\
-& \frac{1}{2}\int_s^T Tr(\partial_{xx}^2\,\theta_{N_r^{t,i}}(r,X_r))(\Sigma\,\Sigma^*)(r,X_r,N_r^{t,i})dr \\
-& \int_s^T \sum_{l\in L}\Big[(\theta_{N_{r^-}^{t,i}+l} - \theta_{N_{r^-}^{t,i}})(r,X_r^{t,\xi,i})\Big]dN_r(l)
\end{aligned}$$

Which implies $\forall\ s \in [t,T]$

$$\begin{aligned}\theta_{N_s^{t,i}}(s, X_s^{t,\xi,i}) &= \theta_{N_T^{t,i}}(T, X_T^{t,\xi,i}) - \int_s^T \left[\frac{\partial}{\partial t}\theta_{N_r^{t,i}}(r, X_r^{t,\xi,i}) + L^{N_r^{t,i}}\, \theta_{N_r^{t,i}}(r, X_r^{t,\xi,i})\right] dr \\ &- \int_s^T Z_r^{t,\xi,i}\, dB_r - \int_s^T \sum_{l\in L}\left[(\theta_{N_{r-}^{t,i}+l} - \theta_{N_{r-}^{t,i}})(r, X_r^{t,\xi,i})\right] dN_r(l)\end{aligned}$$

Hence using $(\mathcal{E})$ we deduce , $\forall\ s \in [t,T]$

$$\begin{aligned}Y_s^{t,\xi,i} &= h_{N_T^{t,i}}(X_T^{t,\xi,i}) + \int_s^T \left(\bar{f}_{N_r^{t,i}}(r, X_r, \theta(r, X_r), Z_r) + \lambda \sum_{l\in L} H_r^{t,\xi,i}(l)\right) dr - \int_s^T Z_r^{t,\xi,i}\, dB_r \\ &- \int_s^T \sum_{l\in L} H_r^{t,\xi,i}(l)\, dM_r(l)\end{aligned}$$

Therefore the $\mathcal{G}_s$-progressively measurable quartet of processes $(X_s^{t,\xi,i}, Y_s^{t,\xi,i}, H_s^{t,\xi,i}, Z_s^{t,\xi,i})_{(t\le s\le T)}$ with values in $\mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{k-1} \times \mathbf{R}^d$ solves the FBSDE with transmutation associated to f, b, σ, h.

For some $a \in \mathbf{R}$, applying Itô's formula for discontinuous processes to the semimartingale $(e^{as}|Y_s|^2)_{t\le s\le T}$, we obtain $\forall\ s \in [t,T]$

$$\begin{aligned}e^{as}|Y_s|^2 + \int_s^T e^{ar}|Z_r|^2\, dr &+ \lambda \int_s^T e^{ar}|H_r|^2\, dr + \sum_{s<r\le T} e^{ar}(\Delta Y_r)^2 \le e^{aT}|Y_T|^2 \\ &- a\int_s^T e^{ar}|Y_r|^2\, dr + 2\int_s^T e^{ar} Y_r f(r, \Theta_r, N_r^{t,i}) dr \\ &- 2\int_s^T e^{ar} Y_r\, Z_r dB_r - 2\int_s^T \sum_{l\in L} e^{ar} Y_r\, H_r(l) dM_r(l)\end{aligned}$$

Using assumption (H1.3), we deduce $\forall\ s \in [t,T]$

$$\begin{aligned}e^{as}|Y_s|^2 + \int_s^T e^{ar}|Z_r|^2\, dr &+ \lambda \int_s^T e^{ar}|H_r|^2\, dr \le e^{aT}|Y_T|^2 + \int_s^T e^{ar}[\Lambda + (\Lambda - a)|Y_r|^2\,]\, dr \\ &- 2\int_s^T e^{ar}\, Y_r\, Z_r dB_r - 2\int_s^T \sum_{l\in L} e^{ar}\, Y_r\, H_r(l) dM_r(l)\end{aligned}$$

Choosing $a = \Lambda$ and taking the conditional expectation given $\mathcal{G}_t$, for such $i \in \mathcal{K}$ we find a constant C_i, only depending on Λ and T, such that

$$\forall\ (t,x) \in [0,T] \times \mathbf{R}^d \quad |\theta_i(t,x)| \ \le\ C_i$$

Using the same argument as in Delarue [3], lemma 2.1, we prove the existence of a constant Γ_i only depending on $C_i, \tilde{k}, \delta, \Lambda$ and d such that

$$\forall\ (t,x) \in [0,T] \times \mathbf{R}^d \quad |\nabla_x\, \theta_i(t,x)| \ \le\ \Gamma_i$$

We have established the proposition with $C = \max_{i=1,\dots,k} C_i$ and $\Gamma = \max_{i=1,\dots,k} \Gamma_i$ ■

In what follows we build a regular solution of the system of PDEs $(\mathcal{E})$ whatever the time duration T may be.
Let $\gamma = \tilde{C}^{(2)}_{(K,\lambda)}$, $m = E(\frac{T}{\gamma}) + 1$ and the partition $(t_j)_{0\le j\le m}$ of $[0,T]$ defined by

$$t_0 = 0,\ \ \forall\ j \ge 1,\quad t_j = T - (m-j)\gamma$$

Corollary 2.8. *Assume that* **(H3)** *holds. Then* $\forall\, T > 0$ *the system of PDEs* $(\mathcal{E})$ *admits a unique solution* $\tilde{u}$ *and there exists two constants* $\tilde{C}$ *(depending on* Λ *and* T*) and* $\tilde{\Gamma}$ *(depending on* $\Lambda, \tilde{k}, \delta, d$ *and* T*) such that*

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\tilde{u}(t,x)| \leq \tilde{C} \tag{2.55}$$

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\nabla_x\, \tilde{u}(t,x)| \leq \tilde{\Gamma} \tag{2.56}$$

Proof : Using proposition 2.2, we build $\tilde{u} : [T-\gamma, T] \times \mathbf{R}^d \to \mathbf{R}^k$ solution of the system

$$(\mathcal{E}) \quad \begin{cases} \forall\, (t,x) \in [T-\gamma, T] \times \mathbf{R}^d, \ \ \forall\, i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t,x) \ + L^i\, u_i(t,x) \ + \bar{f}_i(t,x,u(t,x), \nabla_x\, u_i(t,x)\sigma_i(t,x,u_i(t,x))) = 0 \\ u_i(T,x) = h_i(x) \end{cases}$$

and there exists two constants $\tilde{C}$ (depending on Λ and T) and $\tilde{\Gamma}$ (depending on $\Lambda, \tilde{k}, \delta, d$ and T) such that

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\tilde{u}(t,x)| \leq \tilde{C} \qquad \text{and} \qquad \sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\nabla_x\, \tilde{u}(t,x)| \leq \tilde{\Gamma}$$

Moreover the first and second derivatives of $\tilde{u}$ with respect to x are Lipschitz and the function $\tilde{u}(T-\gamma, \cdot)$ is $\tilde{\Gamma}$ Lipschitz.
Considering the system

$$(\mathcal{E}) \quad \begin{cases} \forall\, (t,x) \in [T-2\gamma, T-\gamma] \times \mathbf{R}^d, \ \ \forall\, i \in \mathcal{K} \\ \frac{\partial u_i}{\partial t}(t,x) \ + L^i\, u_i(t,x) \ + \bar{f}_i(t,x,u(t,x), \nabla_x\, u_i(t,x)\sigma_i(t,x,u_i(t,x))) = 0 \\ u_i(T,x) = \ \tilde{u}(T-\gamma, x) \end{cases}$$

we can build another solution satisfying the same properties as $\tilde{u}$ with two constants which we note again $\tilde{C}$ and $\tilde{\Gamma}$.
Therefore we can extend $\tilde{u}$ on $[T-2\gamma, T]$, so by induction we construct a bounded solution of $(\mathcal{E})$ whose gradient is uniformly bounded on $[0,T] \times \mathbf{R}^d$.
For the proof of uniqueness we consider v another solution of $(\mathcal{E})$ satisfying (2.55) and (2.56) .
We associate to v a family of solutions $(X_s^{v,t,x,i})_{(t,x)\in[0,T]\times\mathbf{R}^d}$ of the SDE

$$X_s^{v,t,x,i} = x + \int_t^s \bar{b}(r, X_r, v(r,X_r), (\nabla_x\, v\, \sigma)(r, X_r, Y_r, N_r^{t,i}), N_r^{t,i})dr + \int_t^s \sigma(r, X_r, v(r, X_r, N_r^{t,i}), N_r^{t,i})\, dB_r$$

Considering the processes

$$\forall\, s \in [t,T], \quad Y_s^{v,t,x,i} = v_{N_s^{t,i}}(s, X_s^{v,t,x,i})\ ; \quad Z_s^{v,t,x,i} = \nabla_x\, v_{N_s^{t,i}}(s, X_s^{v,t,x,i})\, \sigma_{N_s^{t,i}}(s, X_s^{v,t,x,i}, Y_s^{v,t,x,i})$$

$$\forall\, l \in L\ , \ \ H_s^{v,t,x,i}(l) = (v_{N_{s-}^{t,i}+l} - v_{N_{s-}^{t,i}})(s, X_s^{v,t,x,i})$$

we deduce thanks to Itô's formula,

$$\begin{aligned} \forall\, s \in [t,T], \quad Y_s^{v,t,x,i} &= h_{N^{t,i}}(X_T^{v,t,x,i}) + \int_s^T f(r, X_r^{v,t,x,i}, Y_r^{v,t,x,i}, H_r^{v,t,x,i}, Z_r^{v,t,x,i}, N_r^{t,i})dr \\ &- \int_s^T Z_r^{v,t,x,i} dB_r - \int_s^T \sum_{l\in L} H_r^{v,t,x,i}(l) dM_r(l) \end{aligned}$$

Therefore the quartet $(X_r^{v,t,x,i}, Y_r^{v,t,x,i}, H_r^{v,t,x,i}, Z_r^{v,t,x,i})$ solves the FBSDE with transmutation process $(E^{t,x})$ with initial values t, x, i. Hence thanks to Theorem 2.1 we have

$$\forall\ (t,x) \in [T-\gamma, T] \quad \tilde{u}(t,x) = v(t,x)$$

Then using again an induction we prove $\tilde{u} = v$ sur $[0,T] \times \mathbf{R}^d$. ■

Comment: The previous corollary gives us a regular solution of our non degenerate system of quasilinear parabolic PDEs whose gradient is uniformly bounded by a quantity which is independent of the Lipschitz constant of the drifts of the FBSDE $(E^{0,\xi})$.
Thanks to this result, we have a control on the Lipschitz constant of θ_i, $i \in \mathcal{K}$. Indeed θ_i plays the role of h_i in the latter application of theorem 2.1 on a appriopriate subdivision of $[0,T]$ in order to extended the solution in small time duration to a global one. This is what we do in the next section.

3. Global solution of the system of FBSDE

In order to extend the local solution of the FBSDE, we remove regularity required in the previous section and assume that the coefficients satisfy the following assumptions **(H4)**:

(H4.1) b and f satisfy **(H1)**.

(H4.2) σ and h satisfy (H1.3) and there exists a constant $\tilde{k} > 0$ such that
$\forall\ i \in \mathcal{K},\ \ \forall\ t \in [0,T],\ \forall\ ((x,y),(x',y')) \in (\mathbf{R}^d \times \mathbf{R})^2$

$$\begin{aligned} |\sigma_i(t,x,y) - \sigma_i(t,x',y')|^2 &\leq \tilde{k}^2(|x-x'|^2 + |y-y'|^2) \\ |h_i(x) - h_i(x')| &\leq \tilde{k}|x-x'| \end{aligned}$$

(H4.3) The function σ satisfies **(H3.1)**.

We have the following

Proposition 3.1. *Assume that* **(H4)** *hold and* $T \leq \tilde{C}^{(2)}_{(K,\lambda)}$. *Then there exists a sequence of bounded* C^∞ *functions* $(\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})_{n\in\mathbf{N}}$ *with bounded derivatives of every order such that*

$$\forall\ (t,x,u,z) \in [0,T] \times \mathbf{R}^d \times \mathbf{R}^k \times \mathbf{R}^d, \qquad (\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})(t,x,u,z) \xrightarrow{n \longrightarrow +\infty} (\bar{b}, \bar{f}, \sigma, h)(t,x,u,z)$$

Moreover if $(X_s^{t,n,x,i}, Y_s^{t,n,x,i}, H_s^{t,n,x,i}, Z_s^{t,n,x,i})_{0\leq s\leq T}$ *stands for the solution of* $(E^{0,x})$ *associated to* $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$*, the map* θ_n *defined by*

$$\begin{aligned} \theta_n : [0,T] \times \mathbf{R}^d \times \mathcal{K} &\longrightarrow \mathbf{R} \\ (t,x,i) &\longrightarrow Y_t^{t,n,x,i} \end{aligned}$$

also defines a function $\theta_n : [0,T] \times \mathbf{R}^d \longrightarrow \mathbf{R}^k$ *solution of the system of PDEs*

$$\left\{ \begin{array}{l} \forall\ (t,x) \in [0,T] \times \mathbf{R}^d,\ \ \forall\ i \in \mathcal{K} \\ \\ \dfrac{\partial u_i}{\partial t}(t,x)\ + L_n^i\, u_i(t,x)\ + \bar{f}_i^{(n)}(t,x,u(t,x), \nabla_x\, u_i(t,x) \sigma_i^{(n)}(t,x,u_i(t,x))) = 0 \\ \\ u_i(T,x) = h_i^{(n)}(x) \end{array} \right. \tag{3.1}$$

satisfying $\forall\ n \in \mathbf{N}$,

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\theta_n(t,x)| \ \leq\ \tilde{C} \tag{3.2}$$

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\nabla_x\, \theta_n(t,x)| \ \leq\ \tilde{\Gamma} \tag{3.3}$$

Moreover

$$\theta_n \longrightarrow \theta \qquad as\ n \longrightarrow +\infty \tag{3.4}$$

where θ is the map defined in Corollary 2.2.

Proof : There exists a regularization sequence of bounded C^∞ functions $(\bar{b}^{(n)}, \bar{f}^{(n)}, \sigma^{(n)}, h^{(n)})_{n\in\mathbf{N}}$ with bounded derivatives such that for every $n \in \mathbf{N}$, the quartet $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ satisfies **(H4)** with the constants $K + 4\Lambda$, $\tilde{k}$, 2Λ and $\frac{\delta}{2}$ (we refer to [3], proposition 2.2).
Moreover since $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ satisfy also **(H3)**, by virtue of Proposition 2.2, there exist two constants still noted $\tilde{C}$ and $\tilde{\Gamma}$ and not depending on n, such that for all $n \in \mathbf{N}$, the map θ_n is a solution of (3.1) which satisfies (3.2) and (3.3).
Futhermore applying Theorem 2.2 to the two quartets of functions $(b^{(n)}, f^{(n)}, \sigma^{(n)}, h^{(n)})$ and (b, f, σ, h), we deduce from the Lebesgue dominated convergence theorem that $\theta_n \longrightarrow \theta$ as $n \longrightarrow +\infty$. ■

Let $\tilde{K} = \max\ (\tilde{k}, \Lambda, \tilde{\Gamma})$, $\gamma = \tilde{C}^{(2)}_{(\tilde{K},\lambda)}$. For any arbitrary $T > 0$, let $m = E(\frac{T}{\gamma}) + 1$ and consider the subdivision $(t_j)_{0\le j\le m}$ of $[0,T]$ defined by

$$t_0 = 0,\ \ \forall\, j \ge 1, \quad t_j = T - (m-j)\gamma$$

Proposition 3.2. *Assume that* **(H4)** *holds. Then* $\forall\, T > 0$, *there exists a map* $\theta : [0,T] \times \mathbf{R}^d \times \mathcal{K} \to \mathbf{R}$ *satisfying*

i) $\forall\, (t,x,i) \in [0,T] \times \mathbf{R}^d \times \mathcal{K}, \quad |\theta(t,x,i)| \ \le\ \tilde{C}$

ii) $\forall\, (t,i) \in [0,T] \times \mathcal{K}, \quad \forall\, (x,x') \in (\mathbf{R}^d)^2 \quad |\theta(t,x,i) - \theta(t,x',i)| \ \le\ \tilde{\Gamma}\, |x - x'|$

iii) $\forall\, x \in \mathbf{R}^d$, $\forall\, i \in \mathcal{K}$, $\theta(T,x,i) \ =\ h_i(x)$

iv) $\forall\ j \in 0,...,m-1$, $\forall\ t \in [t_j, t_{j+1}]$, $\forall\ i \in \mathcal{K}$, *for every $\mathcal{G}_t-$measurable random vector ξ with finite second moment, the problem*

$$\begin{cases} \forall\, s \in [t, t_{j+1}], \\ X_s = \xi + \displaystyle\int_t^s b(r, \Theta_r^{(j)}, N_r^{t,i})dr + \int_t^s \sigma(r, X_r, Y_r, N_r^{t,i})dB_r \\ Y_s = \theta(t_{j+1}, X_{t_{j+1}}, N^{t,i}_{t_{j+1}}) + \displaystyle\int_s^{t_{j+1}} f(r, \Theta_r^{(j)}, N_r^{t,i})dr - \int_s^{t_{j+1}} Z_r dB_r - \int_s^{t_{j+1}} \sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

has a unique solution $\Theta_s^{(j)} = (X_s^{(j)}, Y_s^{(j)}, H_s^{(j)}, Z_s^{(j)})_{t\le s\le t_{j+1}}$ *in* $\mathcal{B}^2_{[t,t_{j+1}]}$ *such that* $\forall\, s \in [t, t_{j+1}]$,

$$Y_s^{(j)} \ = \ \theta(s, X_s^{(j)}, N_s^{t,i}) \tag{3.5}$$

$$\forall\, l \in L, \quad H_s^{(j)}(l) \ = \ \theta(s, X_s^{(j)}, N^{t,i}_{s^-} + l) \ -\ \theta(s, X_s^{(j)}, N^{t,i}_{s^-}) \tag{3.6}$$

$$Z_s^{(j)} \ = \ \nabla_x\, \theta(s, X_s^{(j)}, N_s^{t,i})\, \sigma(s, X_s^{(j)}, Y_s^{(j)}, N_s^{t,i}) \tag{3.7}$$

Proof : We build θ by a backward induction.
$\forall\, t \in [t_{m-1}, T]$, thanks to Theorem 2.1, $\forall\, i \in \mathcal{K}$, the problem

$$(3.8)\quad \begin{cases} \forall\, s \in [t,T], \\ X_s = \xi + \displaystyle\int_t^s b(r,\Theta_r^{(m-1)},N_r^{t,i})dr + \int_t^s \sigma(r,X_r,Y_r,N_r^{t,i})dB_r \\ Y_s = h_{N_T^{t,i}}(X_T) + \displaystyle\int_s^T f(r,\Theta_r^{(m-1)},N_r^{t,i})dr - \int_s^T Z_r dB_r - \int_s^T \sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

has a unique solution $(\Theta_s^{(m-2)})_{t_{m-1}\le s\le T} \triangleq (X_s^{(m-1)},Y_s^{(m-1)},H_s^{(m-1)},Z_s^{(m-1)})_{t_{m-1}\le s\le T}$ in $\mathcal{B}^2_{[t_{m-1},T]}$ and applying Corollary 2.2, the map θ defined by

$$\begin{array}{rcl} \theta : [t_{m-1},T]\times \mathbf{R}^d\times\mathcal{K} & \longrightarrow & \mathbf{R} \\ (t,x,i) & \longrightarrow & Y_t^{t,x,i} \end{array}$$

satisfies the following equalities for all $t \in [t_{m-1},T]$,

$$\begin{array}{rcl} \theta(T,x,i) & = & h_i(x) \\ \forall\, s\in[t,T],\quad Y_s^{(m-1)} & = & \theta(s,X_s^{(m-1)},N_s^{t,i}) \\ \forall\, l \in L,\quad H_s^{(m-1)}(l) & = & \theta(s,X_s^{(m-1)},\, N_{s^-}^{t,i}+l) - \theta(s,X_s^{(m-1)},\, N_{s^-}^{t,i}) \end{array}$$

Futhermore using the previous Proposition, the map

$$\begin{array}{rcl} \theta_n : [t_{m-1},T]\times \mathbf{R}^d\times\mathcal{K} & \longrightarrow & \mathbf{R} \\ (t,x,i) & \longrightarrow & Y_t^{t,n,x,i} \end{array}$$

defines a function $\theta_n \in \mathcal{C}^{1,2}([t_{m-1},T]\times\mathbf{R}^d,\mathbf{R}^k)$, a solution of (3.1) which satisfies

$$\begin{array}{rcl} \sup\limits_{(t,x)\in[t_{m-1},T]\times\mathbf{R}^d} |\theta_n(t,x)| & \le & \tilde{C}, \\ \sup\limits_{(t,x)\in[t_{m-1},T]\times\mathbf{R}^d} |\nabla_x\,\theta_n(t,x)| & \le & \tilde{\Gamma}. \end{array}$$

Let $n \longrightarrow +\infty$, we deduce easily i) and ii). Therefore the proposition holds on $[t_{m-1},T]$.
$\forall\, t\in[t_{m-2},t_{m-1}], \forall\, i\in\mathcal{K}$, using once again theorem 2.1, for every $\mathcal{G}_t-$measurable random vector ξ with finite second moment, the problem

$$\begin{cases} \forall\, s \in [t,t_{m-1}], \\ X_s = \xi + \displaystyle\int_t^s b(r,\Theta_r^{(m-2)},N_r^{t,i})dr + \int_t^s \sigma(r,X_r,Y_r,N_r^{t,i})dB_r \\ Y_s = \theta(t_{m-1},X_{t_{m-1}}^{(m-2)},N_{t_{m-1}}^{t,i}) + \displaystyle\int_s^{t_{m-1}} f(r,\Theta_r^{(m-2)},N_r^{t,i})dr - \int_s^{t_{m-1}} Z_r dB_r - \int_s^{t_{m-1}} \sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{(m-2)}) \triangleq (X_s^{(m-2)},Y_s^{(m-2)},H_s^{(m-2)},Z_s^{(m-2)})_{t_{m-2}\le s\le t_{m-1}}$ in $\mathcal{B}^2_{[t_{m-2},t_{m-1}]}$. Hence using the same method, there exists a map which we note again θ

$$\begin{array}{rcl} \theta : [t_{m-2},t_{m-1}]\times \mathbf{R}^d\times\mathcal{K} & \longrightarrow & \mathbf{R} \\ (t,x,i) & \longrightarrow & Y_t^{t,x,i} \end{array}$$

satisfying for all $t\in[t_{m-2},t_{m-1}]$,

$$\begin{array}{rcl} |\theta(t,x,i)-\theta(t,x',i)| & \le & \tilde{\Gamma}|x-x'| \\ |\theta(t,x,i)| & \le & \tilde{C} \\ \forall\, s\in[t,T],\quad Y_s^{(m-1)} & = & \theta(s,X_s^{(m-1)},N_s^{t,i}) \\ \forall\, l\in L,\quad H_s^{(m-1)}(l) & = & \theta(s,X_s^{(m-1)},N_{s^-}^{t,i}+l) - \theta(s,X_s^{(m-1)},N_{s^-}^{t,i}) \end{array}$$

So, we can define θ on $[t_{m-2},T]\times\mathbf{R}^d\times\mathcal{K}\longrightarrow\mathbf{R}$.
Therefore using an induction we build $\theta : [0,T]\times\mathbf{R}^d\times\mathcal{K}$. ■

Corollary 3.1. *Under the assumptions* **(H4)** *and the statments of proposition 3.2, for every* $(t,i) \in [0,T]\times\mathcal{K}$ *and* ξ *a* $\mathcal{G}_t-$ *measurable random vector with finite second moment, every solution* $(\Theta_s^{t,\xi,i})_{(t\leq s\leq T)}$ *of the problem* $(E^{t,\xi})$ *satisfies*
$\forall\ n \leq j \leq m-1,$

$$\begin{aligned}
\mathbf{E}\sup_{\tilde{t}_j\leq s\leq \tilde{t}_{j+1}}|X_s^{t,\xi,i}-X_s^{\tilde{t}_j,X_{\tilde{t}_j},N^{t,i}_{\tilde{t}_j}}|^2 &= \mathbf{E}\sup_{\tilde{t}_j\leq s\leq \tilde{t}_{j+1}}|Y_s^{t,\xi,i}-Y_s^{\tilde{t}_j,X_{\tilde{t}_j},N^{t,i}_{\tilde{t}_j}}|^2 = \mathbf{E}\int_{\tilde{t}_j}^{\tilde{t}_{j+1}}|Z_s^{t,\xi,i}-Z_s^{\tilde{t}_j,X_{\tilde{t}_j},N^{t,i}_{\tilde{t}_j}}|^2\,ds \\
&= \mathbf{E}\int_{\tilde{t}_j}^{\tilde{t}_{j+1}}\sum_{l\in L}(H_s^{t,\xi,i}(l)-H_s^{\tilde{t}_j,X_{\tilde{t}_j},N^{t,i}_{\tilde{t}_j}}(l))^2\,ds = 0
\end{aligned}$$

where n *is the unique integer such that* $t\in[t_n,t_{n+1}[$ *and* $(\tilde{t}_j)_{n\leq j\leq m}$ *is the sequence defined by*

$$\tilde{t}_j = t \ \ si\ \ j=n\ \ ;\ \ \tilde{t}_j = t_j \ \ si\ \ j>n$$

Proof : If $n=m-1$, using Proposition 3.2 the result is an easy consequence of the uniqueness of solution on $[t_{m-1},T]$.
If $n\leq m-2$, and $(X_s,Y_s,H_s,Z_s)_{t\leq s\leq T}$ is the unique solution of the problem, then $(X_s,Y_s,H_s,Z_s)_{t_{m-1}\leq s\leq T}$ is solution of the system

$$\begin{cases}
\forall\ s\in[t_{m-1},T] \\
X_s^{t,\xi,i} = X_{t_{m-1}} + \displaystyle\int_{t_{m-1}}^s b(r,\Theta_r^{t,\xi,i},N_r^{t_{m-1},i_1})\,dr + \int_{t_{m-1}}^s \sigma(r,X_r,Y_r,N_r^{t_{m-1},i_1})\,dB_r \\
Y_s^{t,\xi,i} = h_{N_T^{t_{m-1},i_1}}(X_T) + \displaystyle\int_s^T f(r,\Theta_r^{t,\xi,i},N_r^{t_{m-1},i_1})\,dr - \int_s^T Z_r^{t,\xi,i}dB_r - \int_s^T\sum_{l\in L}H_r^{t,\xi,i}(l)dM_r(l) \\
\mathbf{E}\displaystyle\int_{t_{m-1}}^T(|X_r^{t,\xi,i}|^2+|Y_r^{t,\xi,i}|^2+|H_r^{t,\xi,i}|^2+|Z_r^{t,\xi,i}|^2)\,dr < +\infty
\end{cases}$$

where $i_1 = N^{t,i}_{t_{m-1}}$.
From uniqueness of solution and Proposition 3.2, we deduce

$$\begin{aligned}
\mathbf{E}\sup_{t_{m-1}\leq s\leq T}|X_s^{t,\xi,i}-X_s^{t_{m-1},X_{t_{m-1}},i_1}|^2 &= \mathbf{E}\sup_{t_{m-1}\leq s\leq T}|Y_s-Y_s^{t_{m-1},X_{t_{m-1}},i_1}|^2 \\
&= \mathbf{E}\sum_{l\in L}\int_{t_{m-1}}^T(H_s(l)-H_s^{t_{m-1},X_{t_{m-1}},i_1}(l))^2\,ds \\
&= \mathbf{E}\int_{t_{m-1}}^T|Z_s-Z_s^{t_{m-1},X_{t_{m-1}},i_1}|^2\,ds = 0
\end{aligned}$$

and $\forall\ s\in[t_{m-1},T],\ \ Y_s = \theta_{N_s^{t_{m-1},i_1}}(s,X_s)$, in particular

$$Y_{t_{m-1}} = \theta_{i_1}(t_{m-1},X_{t_{m-1}}) = \theta_{N^{t,i}_{t_{m-1}}}(t_{m-1},X_{t_{m-1}})$$

hence the process $(X_s, Y_s, H_s, Z_s)_{t\leq s\leq t_{m-1}}$ solves the system

$$\begin{cases} \forall\, s\in[t,t_{m-1}] \\ X_s^{t,\xi,i} = \xi + \displaystyle\int_t^s b(r,\Theta_r^{t,\xi,i},N_r^{t,i})\,dr + \int_t^s \sigma(r,X_r,Y_r,N_r^{t,i})\,dB_r \\ Y_s^{t,\xi,i} = \theta_{N^{t,i}_{t_{m-1}}}(t_{m-1},X_{t_{m-1}}) + \displaystyle\int_s^{t_{m-1}} f(r,\Theta_r^{t,\xi,i},N_r^{t,i})\,dr - \int_s^{t_{m-1}} Z_r^{t,\xi,i}dB_r \\ \qquad - \displaystyle\int_s^{t_{m-1}} \sum_{l\in L} H_r^{t,\xi,i}(l)dM_r(l) \\ \mathbf{E}\displaystyle\int_t^{t_{m-1}} (|X_r^{t,\xi,i}|^2 + |Y_r^{t,\xi,i}|^2 + |H_r^{t,\xi,i}|^2 + |Z_r^{t,\xi,i}|^2)\,dr < +\infty \end{cases}$$

analogously, we prove

$$\begin{aligned} \mathbf{E}\sup_{t_{m-2}\leq s\leq t_{m-1}} |X_s^{t,\xi,i} - X_s^{t_{m-2},X_{t_{m-2}},i_2}|^2 &= \mathbf{E}\sup_{t_{m-2}\leq s\leq t_{m-1}} |Y_s - Y_s^{t_{m-2},X_{t_{m-2}},i_2}|^2 \\ &= \mathbf{E}\sum_{l\in L}\int_{t_{m-2}}^{t_{m-1}} (H_s(l) - H_s^{t_{m-2},X_{t_{m-2}},i_2}(l))^2\,ds \\ &= \mathbf{E}\int_{t_{m-2}}^{t_{m-1}} |Z_s - Z_s^{t_{m-2},X_{t_{m-2}},i_2}|^2\,ds \quad = 0 \end{aligned}$$

and in particular

$$Y_{t_{m-2}} = \theta_{i_2}(t_{m-2},X_{t_{m-2}}) = \theta_{N^{t,i_1}_{t_{m-2}}}(t_{m-2},X_{t_{m-2}}) \qquad \text{where} \qquad i_2 = N^{t,i_1}_{t_{m-1}}.$$

Applying an induction we get the desired result. ∎

Theorem 3.1 (Global solution). *Under the assumptions* **(H4)**, $\forall\, T>0$, $\forall\, i\in\mathcal{K}$, *for every* $\mathcal{G}_0$*-measurable random vector* ξ *with finite second moment the system* $(E^{0,\xi})$ *has a unique solution.*

Proof : Assume that **(H4)** holds and $T>0$. Let $i\in\mathcal{K}$ be fixed and ξ a $\mathcal{G}_0$-measurable random vector with finite second moment.
With the previous subdivision of $[0,T]$, we define jor all $j\in\{0,...,m-1\}$,

$$n_0 = i;\ \text{ and } \forall\, 1\leq j\leq m,\ \ n_j = N_{t_j}^{t_{j-1},n_{j-1}}$$

Thanks to proposition 3.2, the system

$$\begin{cases} \forall\, s\in[0,t_1], \\ X_s = \xi + \displaystyle\int_0^s b(r,\Theta_r^{(0)},N_r^{0,i})dr + \int_0^s \sigma(r,X_r,Y_r,N_r^{0,i})dB_r \\ Y_s = \theta(t_1,X_{t_1}^{(0)},N_{t_1}^{0,i}) + \displaystyle\int_s^{t_1} f(r,\Theta_r^{(0)},N_r^{0,i})dr - \int_s^{t_1} Z_r dB_r - \int_s^{t_1}\sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{0,\xi,i}) = (X_s^{(0)},Y_s^{(0)},H_s^{(0)},Z_s^{(0)})_{0\leq s\leq t_1}$ in $\mathcal{B}^2_{[0,t_1]}$ satisfying

$$\begin{aligned} Y_{t_1}^{(0)} &= \theta(t_1,X_{t_1}^{(0)},N_{t_1}^{0,i}) \\ \forall\, l\in L,\ \ H_{t_1}^{(0)}(l) &= \theta(t_1,X_{t_1}^{(0)},N_{t_1-}^{0,i}+l) \ - \ \theta(t_1,X_{t_1}^{(0)},N_{t_1-}^{0,i}) \end{aligned}$$

Using the same result on $[t_1,t_2]$, we know that the system

$$\begin{cases} \forall\, s\in[t_1,t_2], \\ X_s = X^{(0)}_{t_1} + \displaystyle\int_{t_1}^{s} b(r,\Theta^{(1)}_r, N^{t_1,n_1}_r)dr + \int_{t_1}^{s} \sigma(r,X_r,Y_r,N^{t_1,n_1}_r)dB_r \\ Y_s = \theta(t_2, X^{(1)}_{t_2}, N^{t_1,n_1}_{t_2}) + \displaystyle\int_s^{t_2} f(r,\Theta^{(1)}_r, N^{t_1,n_1}_r)dr - \int_s^{t_2} Z_r dB_r - \int_s^{t_2}\sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{t_1,X^{(0)}_{t_1},n_1}) \triangleq (X^{(1)}_s, Y^{(1)}_s, H^{(1)}_s, Z^{(1)}_s)_{t_1\le s\le t_2}$ in $\mathcal{B}^2_{[t_1,t_2]}$ and applying the previous result, we have $\forall\, s\in[t_1,t_2]$,

$$\begin{aligned} Y^{(1)}_s &= \theta(s, X^{(1)}_s, N^{t_1,n_1}_s). \\ \forall\, l\in L,\ \ H^{(1)}_s(l) &= \theta(s, X^{(1)}_s, N^{t_1,n_1}_{s^-}+l) \ - \ \theta(s, X^{(1)}_s, N^{t_1,n_1}_{s^-}). \end{aligned}$$

Hence we can deduce

$$\begin{aligned} X^{(1)}_{t_1} &= X^{(0)}_{t_1} \\ Y^{(1)}_{t_1} &= \theta(t_1, X^{(1)}_{t_1}, n_1) = \theta(t_1, X^{(0)}_{t_1}, n_1) = Y^{(0)}_{t_1}. \\ \forall\, l\in L,\ \ H^{(1)}_{t_1}(l) &= \theta(t_1, X^{(1)}_{t_1}, n_1+l) \ - \ \theta(t_1, X^{(1)}_{t_1}, n_1) = H^{(0)}_{t_1}(l) \end{aligned}$$

So, applying again this argument on $[t_2,t_3]$, the system

$$\begin{cases} \forall\, s\in[t_2,t_3], \\ X_s = X^{(1)}_{t_2} + \displaystyle\int_{t_2}^{s} b(r,\Theta^{(2)}_r, N^{t_2,n_2}_r)dr + \int_{t_1}^{s} \sigma(r,X_r,Y_r,N^{t_2,n_2}_r)dB_r \\ Y_s = \theta(t_3, X^{(2)}_{t_3}, N^{t_2,n_2}_{t_3}) + \displaystyle\int_s^{t_3} f(r,\Theta^{(2)}_r, N^{t_2,n_2}_r)dr - \int_s^{t_3} Z_r dB_r - \int_s^{t_3}\sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

admits a unique solution $(\Theta_s^{t_2,X^{(1)}_{t_2},n_2}) = (X^{(2)}_s, Y^{(2)}_s, H^{(2)}_s, Z^{(2)}_s)_{t_2\le s\le t_3}$ dans $\mathcal{B}^2_{[t_2,t_3]}$ and applying again the previous result we get $\forall\, s\in[t_2,t_3],\ \ \theta(s, X^{(2)}_s, N^{t_2,n_2}_s) = Y^{(2)}_s$ and we deduce

$$\begin{aligned} X^{(2)}_{t_2} &= X^{(1)}_{t_2} \\ Y^{(2)}_{t_2} &= \theta(t_2, X^{(2)}_{t_2}, n_2) = \theta(t_2, X^{(1)}_{t_2}, n_2) = Y^{(1)}_{t_2}. \\ \forall\, l\in L,\ \ H^{(2)}_{t_2}(l) &= \theta(t_2, X^{(2)}_{t_2}, n_2+l) \ - \ \theta(t_2, X^{(1)}_{t_2}, n_2) = H^{(1)}_{t_2}(l) \end{aligned}$$

Using an induction we are able to build a quartet $(\Theta^{(j)}_s) = (X^{(j)}_s, Y^{(j)}_s, H^{(j)}_s, Z^{(j)}_s)_{j=0,\dots,m-1,\ s\in[t_j,t_{j+1}]}$ solution in $\mathcal{B}^2_{[t_j,t_{j+1}]}$ of the system

$$(\tilde{E})\begin{cases} \forall\, s\in[t_j,t_{j+1}], \\ X_s = X^{(j-1)}_{t_j} + \displaystyle\int_{t_j}^{s} b(r,\Theta^{(j)}_r, N^{t_j,n_j}_r)dr + \int_{t_j}^{s} \sigma(r,X_r,Y_r,N^{t_j,n_j}_r)dB_r \\ Y_s = \theta(t_{j+1}, X^{(j)}_{t_{j+1}}, N^{t_j,n_j}_{t_{j+1}}) + \displaystyle\int_s^{t_{j+1}} f(r,\Theta^{(j)}_r, N^{t_j,n_j}_r)dr - \int_s^{t_{j+1}} Z_r dB_r - \int_s^{t_{j+1}}\sum_{l\in L} H_r(l)dM_r(l) \end{cases}$$

which satisfies

$$X^{(j)}_{t_j} = X^{(j-1)}_{t_j}\ ; \qquad Y^{(j)}_{t_j} = \theta(t_j, X^{(j-1)}_{t_j}, n_j) = Y^{(j-1)}_{t_j}\ ; \qquad H^{(j)}_{t_j} = H^{(j-1)}_{t_j}.$$

Therefore the quartet of $\mathcal{G}_s$-measurable process $(\Theta_s)_{0\le s\le T} \triangleq (X_s, Y_s, H_s, Z_s)_{0\le s\le T}$ defined by putting

$$\forall\, j\in\{0,...,m-1\},\ \ \forall\, s\in[t_j,t_{j+1}],\ \ X_s = X^{(j)}_s,\ \ Y_s = Y^{(j)}_s,\ \ H_s = H^{(j)}_s,\ \ Z_s = Z^{(j)}_s$$

is solution of $(E^{0,\xi})$.
For the proof of uniqueness, let us consider the process $(\tilde{\Theta}_s)_{0\leq s\leq T}$ as another solution of $(E^{0,\xi})$. Using corollary 3.1, we have

$$\mathbf{E}\sup_{0\leq s\leq t_1}|\tilde{X}_s - X_s|^2 = \mathbf{E}\sup_{0\leq s\leq t_1}|\tilde{Y}_s - Y_s|^2 = \int_0^{t_1}\sum_{l\in L}(\tilde{H}_s(l) - H_s(l))^2 ds = \int_0^{t_1}|\tilde{Z}_s - Z_s|^2\, ds = 0$$

Hence we deduce

$$\tilde{X}_{t_1} = X_{t_1}.$$

Using a backward induction similarly to the construction of the solution $(\Theta_s)_{0\leq s\leq T}$, we prove that

$\forall\, t\in[0,T],\quad \tilde{X}_t = X_t\ ;\quad \tilde{Y}_t = Y_t\ ;\quad \tilde{H}_t = H_t\ ;\quad \tilde{Z}_t = Z_t$ ■

Corollary 3.2. *Assume that* **(H3)** *holds and for $1\leq i\leq k$, let $(Y_s^{t,x,i}, t\leq s\leq T)$ be the unique solution of $(E^{t,x})$ given by Theorem 3.1. Then the map θ defined by*

$$\theta(t,x) = (Y_t^{t,x,1}, Y_t^{t,x,2}, ..., Y_t^{t,x,k}),\quad (t,x)\in[0,T]\times\mathbf{R}^d \tag{3.9}$$

coincides with the unique solution of the system of PDEs $(\mathcal{E})$ given by Corollary 2.8.

Proof: Let us consider the previous subdivision of $[0,T]$ and θ defined by (3.9). Let n be the unique integer such that $t\in[t_n,t_{n+1}]$. Thanks to the assumptions **(H3)** and Proposition 2.2, the function θ is a solution of $(\mathcal{E})$ on $[t_n,t_{n+1}]$. Then using Corollary 2.8, we deduce $\forall\, t\in[t_n,t_{n+1}],\quad \theta(t,x) = \tilde{u}(t,x)$.
Using an induction, we complete the proof. ■

Moreover we have the following result whose proof is an adaptation of the ones given in [10] and [9]. However, for the notion of viscosity solution to make sense we need to assume that b does not depend on z.

Corollary 3.3. *Under the assumptions* **(H4)** *if moreover b does not depend on z, the map θ defined in the previous corollary provides a viscosity solution of the system of quasilinear PDEs $(\mathcal{E})$.*

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