Coming down from infinity for some population models

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19th june. CIRM, Luminy, Probability and biological evolution.

Some motivations in population dynamics or genetics

Coming down from infinity"=" regulation for large initial population.

- The effect of the competition arising in a large population [think of trees having a huge number of seeds].
  The short time behavior of genealogies in large population (such as Lambda coalescent, see Aldous, Schweinsberg, Berestycki, Berestycki, Limic, ...).

- Minimal conditions for persistence in a varying environment (WIP with Sylvie Méléard), scaling limits of individual based models.

- Geometric convergence to stationary distribution,
  Uniqueness of Quasi-Stationary Distribution (see [Van Dorn, Cattiaux & al] ...).
  Speed of convergence to the QSD (see [Champagnat, Villemonais, 15]).

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Outline

- Coming down from infinity for birth and death processes and competition for one specie [joint work with S. Méléard and M. Richard].
- Comparing a stochastic process to a dynamical system with non-expansive vector field and coming down from infinity.
- Some example in dimension 2 : (stochastic) Lotka Volterra competition model.

Model

Evolution of the population size $(X_t : t \geq 0)$ as a jump process :

$$k \to k+1 \qquad \textit{birth} \text{ at rate } \lambda_k$$

$$k \to k-1 \qquad \textit{death} \text{ at rate } \mu_k$$

We work under the extinction condition [Karlin McGregor 57]

$$\sum_{k \geq 1} \frac{1}{\lambda_k \pi_k} = \infty, \tag{1}$$

where

$$\pi_1 = \frac{1}{\mu_1} \quad \text{and for} \quad k \geq 2,\ \pi_k = \frac{\lambda_1 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}.$$

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Coming down from infinity

Let $T_n = \inf\{t \geq 0 : X_t = n\}$ and

$$S = \lim_{n\to\infty} \mathbb{E}_n(T_0) = \sum_{i\geq 1} \pi_i + \sum_{n\geq 1} \frac{1}{\lambda_n \pi_n} \sum_{i\geq n+1} \pi_i.$$

Proposition

The process comes down from infinity, in the sense that

$$\exists m, t > 0 : \inf_{k\in\mathbb{N}} \mathbb{P}_k(T_m < t) > 0$$

iff

$$S < \infty.$$

The weak limit of $\mathbb{P}_n$ in $\mathcal{P}(\mathbb{D}([0,\infty), \mathbb{N}\cup\{\infty\}))$ as $n \to \infty$ exists and is denoted by $\mathbb{P}_\infty$ and, as soon as the process comes down from infinity,

$$\forall t > 0 : X_t < \infty, \quad \text{while } X_0 = \infty \quad \mathbb{P}_\infty \text{ a.s.}$$

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How does X come down from infinity ?

We assume that $\quad \mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \overset{n\to\infty}{\longrightarrow} \alpha.$

Theorem

i) If $\alpha > 0$ and $\lambda_n/\mu_n \to \ell \in [0,1)$, then

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow[n\to+\infty]{(\mathrm{d})} \sum_{k\geq 0} \alpha\,(1-\alpha)^k\, Z_k,$$

where $(Z_k)_k$ i.i.d. r.v. whose Laplace transform $G_{\ell,\alpha}$ is characterized by

$$\forall a>0, \quad G_{\ell,\alpha}(a)\left[\ell(1-G_{\ell,\alpha}(a(1-\alpha))) + 1 + a(1-\ell(1-\alpha))\right] = 1.$$

ii) If $\alpha = 0$ (+L^2 assumption), then

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \overset{n\to\infty}{\longrightarrow} 1 \qquad \text{in } \mathbb{P}_\infty - \text{probability.}$$

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A.s. convergence and central limit theorem under additional assumptions for *ii*).

Proofs relies on the decomposition of T_n as the infinite sum of independent r.v.+
-Convergence of Laplace exponent as fixed point following proofs for continuous fractions for i).
-Klesov asymptotic results for sum of i.i.d. r.v. for ii).

Examples

- If $\mu_n = \exp(\beta n)$ and $\lambda_n/\mu_n \to \ell$, then $T_n/\mathbb{E}_\infty(T_n) \to Z_{\ell,1-\exp(-\beta)}$ in distribution.
- If $\mu_n = \exp(n/\log n)\log n$, then $T_n/\mathbb{E}_\infty(T_n) \to 1$ in $\mathbb{P}_\infty$ but not a.s.
- If $\mu_n = cn^\varrho$ ($\varrho > 1$) and $\lambda_n/\mu_n \to 0$, then the a.s. convergence and C.L.T. hold.

The speed of coming down from infinity

Define the speed

$$v_t := \inf\{n \geq 0;\ \mathbb{E}_\infty(T_n) \leq t\}$$

Corollary

Assuming also that $\limsup_{n\to\infty} \lambda_n/\mu_n < 1$, *then*

$$\frac{X_t}{v_t} \xrightarrow{t\downarrow 0} 1 \qquad \textit{in} \quad \mathbb{P}_\infty - probability.$$

Proof using the maximal height of the excursions of X during $[T_{n+1}, T_n)$ + inversion technic.

Example : $\mu_n \sim cn^\varrho$, then a.s. convergence and C.L.T. for

$$t^{1/(\varrho-1)} X_t \qquad \text{as } t \downarrow 0.$$

$\varrho = 2$, $\lambda_k = 0$ yields Aldous speed of coming down from infinity for Kingman Coalescent (or logistic pure death process).

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Random Pertubation of a dynamical system

Let X be a càdlàg process on $E \subset \mathbb{R}^d$ such that

$$X_t = x_0 + \int_0^t \psi(X_s)ds + R_t,$$

where ψ satisfies for each $x, y \in D \subset \mathbb{R}^d$ ($E \subset D$ and D open),

$$(\psi(x) - \psi(y)).(x - y) \leq L \parallel x - y \parallel_2^2 \qquad [L \;\; non - expansivity]$$

and

$$R_t = A_t + M_t^c + M_t^d \qquad (R_0 = 0)$$

where A_t is càdlàg adapted with finite variations, M_t^c is a continuous local martingale and M_t^d is a totally discontinuous local martingale.

$$X_t = x_0 + \int_0^t \psi(X_s)ds + R_t$$

and x the dynamical system associated with ψ

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Proposition

As long as the dynamical system x_t is in D (i.e. for $T \leq T_D(x_0)$),

$$\left\{\sup_{t\leq T} \parallel X_t - x_t \parallel_2 \geq \epsilon\right\} \subset \left\{T_L^R(\epsilon) \leq T\right\}$$

where $$T_L^R(\epsilon) := \inf\left\{t \geq 0 : \mathbb{1}_{\sup_{s\leq t-} \|X_s - x_s\|_2 \leq \epsilon} \widetilde{R}_t \geq (\epsilon \exp(-2LT))^2 \right\}$$

and $$\widetilde{R}_t = 2\int_0^t (X_{s-} - x_s).dR_s + \|< M_t^c >\|_1 + \sum_{s\leq t} \parallel \Delta R_s \parallel_2^2 .$$

Sketch of proof

Taking the L^1 norm of the quadratic variation of $X - x$ (or using Itô's formula),

$$\| X_t - x_t \|_2^2 = 2\int_0^t (X_s - x_s).(\psi(X_s) - \psi(x_s))ds + 2\int_0^t (X_{s-} - x_s).dR_s + \|< M_t^c >\|_1 + \sum_{s\leq t} \| X_s - X_{s-} \|_2^2 .$$

As ψ is L non-expansive on D, for each $t \leq T_D(x_0)$, noting $S_t = \sup_{s\leq t} \| X_s - x_s \|_2$,

$$\mathbb{1}_{S_{t-}\leq\epsilon} S_t^2 \leq \mathbb{1}_{S_{t-}\leq\epsilon}\Big[2L\int_0^t \| X_s - x_s \|_2^2\, ds + 2\int_0^t (X_{s-} - x_s).dR_s + \|< M_t^c >\|_1 + \sum_{s\leq t} \| X_s - X_{s-} \|_2^2\Big].$$

+Gronwall lemma to get $\mathbb{1}_{S_{t-}\leq\epsilon} S_t^2 < \epsilon$ for $t < T_L^R(\eta)$.

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Inequality for martingale

Writing

$$M_t = M_t^c + M_t^d$$

the local martingale and

$$S_t = \sup_{s\leq t} \| X_s - x_s \|_2,$$

we obtain by Markov inequality and (Doob) maximal inequality for martingales

$$\mathbb{P}_{x_0}(S_T > \epsilon)$$
$$\leq \frac{Ce^{4LT}}{\epsilon^2}\Big[\mathbb{E}_{x_0}\left(\| \int_0^T 1\!\!1_{S_{s-}\leq\epsilon} |dA|_s \|_1^2\right)$$
$$+\mathbb{E}_{x_0}\left(\sup_{s\leq T} 1\!\!1_{S_{s-}\leq\epsilon} \| < M_s^c > \|_1\right) + \mathbb{E}_{x_0}\left(\sum_{s\leq T} 1\!\!1_{S_{s-}\leq\epsilon} \| \Delta X_s \|_2^2\right)$$

Stochastic differential equations

$X = (X^i : i = 1 \dots d) \in \mathbb{D}([0,\infty), E)$ satisfies

$$X_t^i = x_0 + \int_0^t b^i(X_s)ds + \int_0^t \sigma^i(X_s)dB_s^i + \int_0^t \int_{\mathcal{X}} H^i(X_{s-}, z)N(ds, dz)$$

where

- B is a d dimensional Brownian motion ;
- N is a punctual Poisson measure independent of B, with intensity $dsq(dz)$ and $\widetilde{N}$ its compensated measure

$$X_t = x_0 + \int_0^t \psi(X_s)ds + M_t$$

where M is a local martingale given by

$$M_t = \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z)\widetilde{N}(ds, dz)$$

and $< M_t >= \int_0^t \sigma(X_s)^2 ds + \int_0^t \int_{\mathcal{X}} H(X_{s-}, z)^2 dsq(dz)$

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Let F be a $\mathcal{C}^2$ function such that its Jacobian matrix J_F is invertible on D. We set

$$b_F(x) = b(x) + J_F(x)^{-1}\left(\int_{\mathcal{X}}[F(x+H(x,z)) - F(x)]q(dz)\right)$$

and the associated flow ϕ_F par

$$\phi_F(x_0,0) = x_0, \qquad \frac{\partial}{\partial t}\phi_F(x_0,t) = b_F(\phi_F(x_0,t)).$$

and

$$V_F(x) = (J_F(x)\sigma(x))^2 + \int_{\mathcal{X}}[F(x+K(x,z)) - F(x)]^2 q(dz).$$

is giving the bracket of the martingale part.

$$\phi_F(x_0,0)=x_0, \qquad \frac{\partial}{\partial t}\phi_F(x_0,t)=b_F(\phi_F(x_0,t))$$

$$\psi_F=(J_F b_F)\circ F^{-1}=\left(J_F b(.)+\int_{\mathcal{X}}[F(.+H(x,z))-F(.)]q(dz)\right)\circ F^{-1}$$

Theorem

We assume that ψ_F is L non-expansive on $F(D)$ (+some technical assumption). Then, for all $x_0\in D$

$$\mathbb{P}_{x_0}\left(\sup_{t\leq T\wedge T_D(x_0)}\parallel F(X_t)-F(\phi_F(x_0,t))\parallel_2>\epsilon\right)$$
$$\leq\frac{Ce^{4LT}}{\epsilon^2}\int_0^T\left[1+\bar{V}_{F,\epsilon}(x_0,s)\right]ds,$$

where $$\bar{V}_{F,\epsilon}(x_0,s)=\sup_{\substack{x\in E\\ \|F(x)-F(\phi_F(x_0,s))\|_2\leq\epsilon}}\parallel V_F(x)\parallel_1 .$$

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where $$\bar{V}_{F,\epsilon}(x_0,s)=\sup_{\substack{x\in E\\ \|F(x)-F(\phi_F(x_0,s))\|_2\leq\epsilon}} \| V_F(x) \|_1 .$$

Stochastically monotone model

Definition

For all $x_0 \leq x_1$*,* $t \geq 0$*,* $a \in \mathbb{R}$*,*

$$\mathbb{P}_{x_0}(X_t \geq a) \leq \mathbb{P}_{x_1}(X_t \geq a)$$

Examples : birth and death process, Λ coalescent ;
random catastrophes IF the rate of catastrophe does not depend (or decreases)on the size of the population, diffusions ...

We also assume that F goes to ∞ and $b_F(x)$ is negative for x large enough.

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Criteria for instantaneous coming down from infinity

Proposition (In progress.)

Assume that X is stochastically monotone, ψ_F is L non expansive and

$$\int_0^{.} \sup_{x_0 \in E} \bar{V}_F(x_0, s) < \infty.$$

The sequence $\mathbb{P}_x$ converges weakly in $\mathcal{P}(\mathbb{D}_{E\cup\{\infty\}}([0,T]))$ as $x \to \infty$ ($x \in E$) to $\mathbb{P}_\infty$.

- *(i) If $\int_{.}^{\infty} \frac{1}{-b_F(x)} < +\infty$, then*

$$\forall t > 0 : X_t < \infty \quad \text{and} \quad \lim_{t \downarrow 0+} F(X_t) - F(x_t) = 0 \quad \mathbb{P}_\infty \text{ a.s.}$$

- *(ii) Otherwise $\mathbb{P}_\infty(\forall t > 0 : X_t = +\infty) = 1.$*

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Two examples

- Λ coalescent. X=number of blocks.

$$F(x) = \log(x), \qquad \psi_F(x) \downarrow \qquad V_F(x) \text{ bounded}$$

and we recover [Berestycki, Berestycki, Limic 10]

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- Birth and death processes. $\mu_k = ck^\varrho$ ($\varrho > 1$), $\lambda_k - \lambda_n \leq C(k-n)$

$$F(x) = x^{1/2-\epsilon}, \quad \psi_F(x) \downarrow, \quad V_F(x) \sim x^{\varrho - 2\epsilon}, \quad \phi(x_0, t) \leq c.t^{1/(1-\varrho)}$$

and setting $\phi(\infty, t) = [c(\varrho - 1)t]^{1/(1-\varrho)}$

$$\lim_{t \downarrow 0} X_t^{1/2-\epsilon} - \phi(\infty, t)^{1/2-\epsilon} = 0 \quad \mathbb{P}_\infty \; a.s.$$

Possible extension to multiple births, random catastrophes, ... and (logistic) Feller diffusion...

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and setting $\phi(\infty,t)=[c(\varrho-1)t]^{1/(1-\varrho)}$

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Possible extension to multiple births, random catastrophes, ... and (logistic) Feller diffusion...

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\begin{aligned}
dX_t^1 &= X_t^1(\tau_1 - aX_t^1 - cX_t^2)dt + \sigma_1\sqrt{X_t^1}dB_t^1 \\
dX_t^2 &= X_t^2(\tau_2 - bX_t^2 - dX_t^1)dt + \sigma_2\sqrt{X_t^2}dB_t^2
\end{aligned}
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with intraspecific competition $a, b > 0$ and interspecific competition $c, d \geq 0$.

We compare this process to dynamical system whose flow $\phi_F = \phi$ given by

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Approximation by the flow coming down from infinity

Note that each component of X comes back to infinity and set

$$D_\epsilon = \{x \in (0,\infty)^2 : x_1 \geq 2\epsilon,\ x_2 \geq 2\epsilon\}$$

and

$$d_\beta(x,y) = |x_1^\beta - y_1^\beta| + |x_2^\beta - y_2^\beta|.$$

Proposition

For any $\beta \in [0,1)$ *and* $\epsilon > 0$*,*

$$\lim_{T\to 0} \sup_{x_0 \in D_\epsilon} \mathbb{P}_{x_0}\left(\sup_{t \leq T \wedge T_{D_\epsilon}(x_0)} d_\beta(X_t, x_t) \geq \epsilon \right) = 0$$

The proof consists in gluing a collections of domains (cones) where

$$F_{\beta,\gamma}(x) = (x_1^\beta, \gamma x_2^\beta)$$

is non-expansive and apply the previous result.

Approximation by the flow coming down from infinity

Note that each component of X comes back to infinity and set

$$D_\epsilon = \{x \in (0,\infty)^2 : x_1 \geq 2\epsilon,\ x_2 \geq 2\epsilon\}$$

and $$d_\beta(x,y) = |x_1^\beta - y_1^\beta| + |x_2^\beta - y_2^\beta|.$$

Proposition

For any $\beta \in [0,1)$ and $\epsilon > 0$,

$$\lim_{T\to 0} \sup_{x_0 \in D_\epsilon} \mathbb{P}_{x_0}\left(\sup_{t \leq T \wedge T_{D_\epsilon}(x_0)} d_\beta(X_t, x_t) \geq \epsilon \right) = 0$$

The proof consists in gluing a collections of domains (cones) where

$$F_{\beta,\gamma}(x) = (x_1^\beta, \gamma x_2^\beta)$$

is non-expansive and apply the previous result.

Classification using Poincaré compactification for flows

(i-intraspecifc) If $b > c$ and $a > d$, then there exists $x_\infty \in (0,\infty)^2$ such that for any $x_0 \in (0,\infty)^2$and $\eta > 0$,

$$\lim_{T\to 0}\lim_{r\to\infty}\mathbb{P}_{rx_0}\left(\sup_{\eta T\leq t\leq T} \| \; tX_t - x_\infty \; \|_2 \geq \epsilon\right) = 0.$$

(ii-interspecific) If $c > b$ and $d > a$, then for any $\epsilon > 0$ and $\beta \in (0, 1)$,

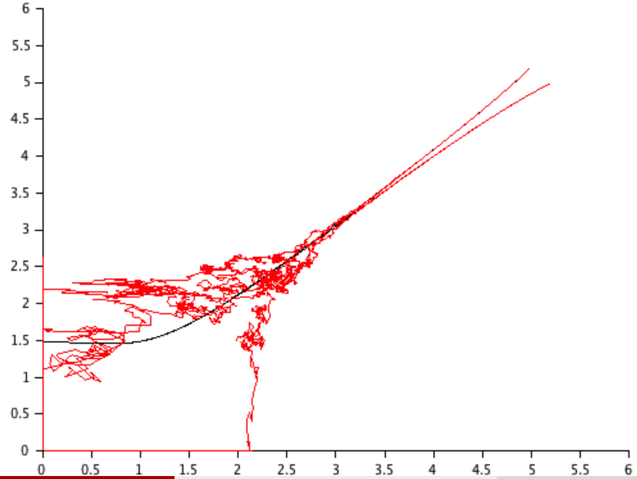
$$\lim_{T\to 0}\lim_{r\to\infty}\mathbb{P}_{rx_0}\left(\sup_{t\leq T} d_\beta(X_t, \phi(rx_0, t)) \geq \epsilon\right) = 0.$$

(iii-unbalanced) If $b > c$ and $d > a$, then for any $T > 0$,

$$\lim_{r\to\infty}\mathbb{P}_{rx_0}\left(\inf\{t \geq 0 : X_t^2 = 0\} \leq T\right) = 1$$

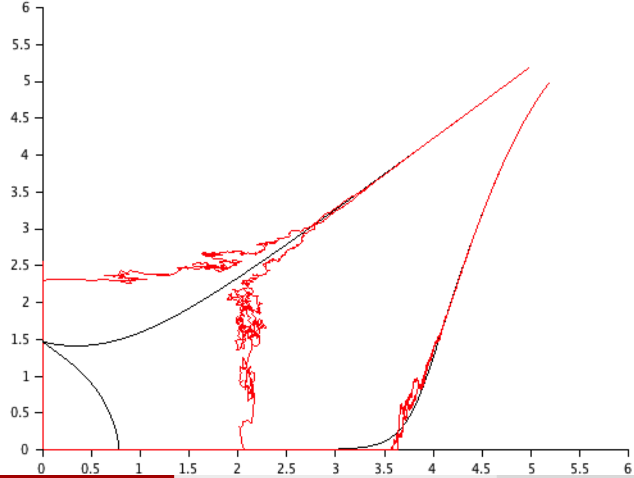
Simulations : $a = b = 1$, $c = 0.3$, $d = 0.5$ (i-intra)

2 simulations and the dynamic system for 2 initial large values (10^5).



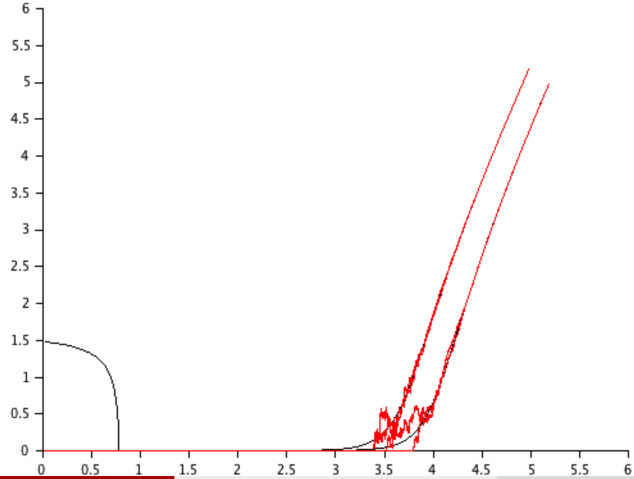
Simulations : $a = b = 1$, $c = 1.3$, $d = 1.4$ (ii-inter)

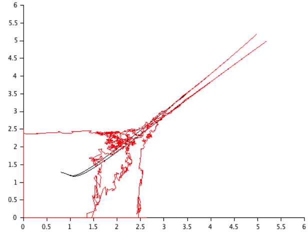
2 simulations and the dynamic system for 2 initial large values (10^5).

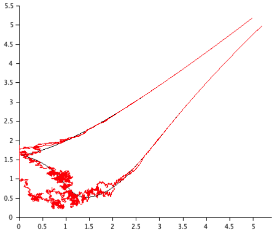


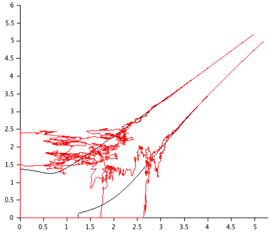
Simulations : $a = b = 1$, $c = 1/3$, $d = 3$

2 simulations and the dynamic system for 2 initial large values (10^5).









Thanks for your attention !