# Approximation of epidemic models by diffusion processes and their statistical inferencedes

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A primer on statistical inference for diffusion processes

This is mainly based on lectures on Statistics of diffusion processes of V. Genon-Catalot (MAP5, Université Paris-Descartes).

# Continuous observation on a finite time interval [0,T]

On the probability space :$(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$

$$d\xi_t = b(\theta_0; t, \xi_t)dt + \sigma(t, \xi_t)dW_t, \xi_0 = \eta$$

$\sigma(t, \xi_t)$ identified from this observation $\Rightarrow$
Assumption: $\sigma(t, \xi_t)$ known.
$(B_t)$ : $p$- dimensional Brownian motion ,
$\eta$ $\mathcal{F}_0$-measurable ;
$\theta_0 \in \Theta$ compact subset of $\mathbb{R}^k$.

Aim: study of estimators of $\theta_0$ depending on the observation $(\xi_t, t \in [0, T])$.

# Probability distribution of a continuous time process

- $C_T = \{x = (x(t)) : [0, T] \to \mathbb{R}^p$ continuous$\}$,
- $\mathcal{C}_T$: Borelian filtration associated with the uniform topology
- Coordinate function: $X_t : C_T \to \mathbb{R}^p$, $X_t(x) = x(t)$.
- $(X_t)$: canonical process $\Rightarrow$ anonical filtration: $\mathcal{C}_t = \sigma(X_s, s \leq t)$.

Diffusion process $(\xi_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, $d\xi_t = b(t, \xi_t)dt + \sigma(t, \xi_t)dW_t, \xi_0 = \eta$.
$\Rightarrow \forall \omega, t \to \xi_t(\omega)$ is continuous $[0, T] \Rightarrow \xi^T := (\xi_t(\omega), t \in [0, T]) \in C_T$.

## Distribution of $(\xi_t, t \in [0, T])$ on $(C_T, \mathcal{C}_T)$

- $P^T_{b,\sigma}$ = probability distribution image of $\mathbb{P}$ by the r.v. $\xi^T$.
- $A_i$ borelian sets in $\mathbb{R}^p$, $A = \{x \in C_T,\ x(t_1) \in A_1, \ldots, x(t_k) \in A_k)$,
- $\mathbb{P}(\xi^T \in A) = P^T_b(X_{t_1} \in A_1, \ldots, X_{t_k} \in A_k\}$.

Wiener measure $W^T$: distribution of $(B_t, t \in [0, T]$ on $(C_T, \mathcal{C}_T)$.

# Likelihood for continuously observed diffusions on [0,T]

Consider the parametric model associated to the diffusion on $\mathbb{R}$

$\star$ $d\xi_t = b(\theta_0; \xi_t)dt + \sigma(\xi_t)dB_t, \xi_0 = x_0$.

$\star$ $\sigma(x), b(\theta, x)$ known; $x_0$ known; $\theta$ unknown $\Rightarrow \theta \in \Theta$.

$\star$ $P_\theta^T$ : distribution on $(C_T, \mathcal{C}_T)$ of $(\xi_t)$.

$\star$ $P_0^T$ distribution of $\xi_t = x_0 + \int_0^t \sigma(\xi_s)dB_s$

Assumptions ensuring existence, uniqueness of solutions of the SDE+..

$\star$ Additional asssumptions

### Theorem

For all $\theta$, the distributions $P_\theta^T$ and $P_0^T$ are equivalent and

$$\frac{dP_\theta^T}{dP_0^T}(X) = \exp[\int_0^T \frac{b(\theta, X_t)}{\sigma^2(X_t)}dX_t - \frac{1}{2}\int_0^T \frac{b^2(\theta, X_t)}{\sigma^2(X_t)}dt].$$

Above formula: stochastic integral w.r.t. the canonical process $(X_t)$

Under $P_0^T$, $\int_0^t \frac{dX_s}{\sigma^2(X_s)}ds$ is a standard Brownian motion,

Under $P_\theta^T$, $\int_0^t \frac{dX_s - b(\theta, X_s)ds}{\sigma^2(X_s)}$ is a Brownian motion.

# Comments and extensions

• Diffusions having distinct diffusion coefficients $\sigma(x)$, $\sigma'(x) \Rightarrow P_\sigma$ and $P_{\sigma'}$ are singular distributions on $(C_T, \mathcal{C}_T)$.
• Diffusion having distinct starting point $x_0, x_0'$ have singular distributions.

$(\xi_t)$: time-dependent multidimensional diffusion process
• $b(\theta, x) \to b(\theta, t, x)$; $\sigma^2(x) \to \Sigma(t, x) = \sigma(t, x)\,{}^t\sigma(t, x)$.
(Karatzas & Shreve for conditions ensuring existence and uniqueness of solutions.
• On $(C_T = C([0, T], \mathbb{R}^p), \mathcal{C}_T)$,

$$\begin{aligned} \frac{dP_\theta^T}{dP_0^T}(X) &= exp(\int_0^T \Sigma^{-1}(t, X_t) b(\theta; t, X_t) dX_t \\ &- \frac{1}{2}\int_0^T {}^t b(\theta; t, X_t)\Sigma^{-1}(t, X_t) b(\theta, t, X_t) dt). \end{aligned}$$

(Liptser & Shiryaev).

# Maximum Likelihood Estimator

• Canonical statistical model: $(C_T, \mathcal{C}_T), (P^T_{\theta,\sigma}, \theta \in \Theta)$ $\star$ The likelihood function associated to the observation $(\xi_t = \xi_t^{\theta_0})$:
$\star$ $\theta \to L_T(\theta)$, with
$\star$ $\ell_T(\theta) = \log L_T(\theta) = \int_0^T \frac{b(\theta,\xi_t)}{\sigma^2(\xi_t)} d\xi_t - \frac{1}{2} \int_0^T \frac{b^2(\theta,\xi_t)}{\sigma^2(\xi_t)} dt.$
$\star$ M.L.E. $\hat{\theta}_T$ s.t. $\ell_T(\hat{\theta}_T) = sup\{\ell_T(\theta), \theta \in \Theta\}$.

Properties of the MLE as $T \to \infty$: no general theory.

Example : $d\xi_t = \theta_0 f(t)dt + \sigma(t)dB_t;\ \xi_0 = 0,\ f, \sigma(.) > 0.$

• $\ell_T(\theta) = \theta \int_0^T \frac{f(s)}{\sigma^2(s)} d\xi_s - \frac{\theta^2}{2} \int_0^T \frac{f^2(s)}{\sigma^2(s)} ds \Rightarrow \hat{\theta}_T = \frac{\int_0^T \frac{f(s)}{\sigma^2(s)} d\xi_s}{\int_0^T \frac{f^2(s)}{\sigma^2(s)} ds}$. $\star$ Under $P_{\theta_0}$,

$\hat{\theta}_T = \theta_0 + \frac{\int_0^T \frac{f(s)}{\sigma(s)} dB_s}{\int_0^T \frac{f^2(s)}{\sigma^2(s)} ds} \Rightarrow \hat{\theta}_T \sim \mathcal{N}(\theta_0, I_T^{-1})$ with $I_T = \int_0^T \frac{f^2(s)}{\sigma^2(s)} ds.$

Asymptotic behaviour as $T \to \infty$ depends on $I_T$.
$\star$ $f(t) = 1, \sigma(t) = \sqrt{(1 + t^2)} \to I_T = Arctan T \to \pi/2$: MLE not consistent.

# Ergodic diffusion processes

Diffusion on $\mathbb{R}^p$:

$$d\xi_t = b(\theta, \xi_t)dt + \sigma(\xi_t)dB_t;$$

Assumptions:

- for $\theta \in \Theta$, $(\xi_t)$ positive recurrent diffusion process.
- Stationary distribution on $\mathbb{R}^p$: $\lambda(\theta; x)dx$.

Continuous observation on $[0, T]$ with $T \to \infty$

Assumptions ensuring that the statistical model is regular (Ibragimov Hasminskii)

MLE: Consistent estimator $\hat{\theta}_T$ of $\theta_0$,

$\sqrt{T}(\hat{\theta}_T - \theta_0) \to \mathcal{N}_k(0, I^{-1}(\theta_0))$

$I(\theta) = (I(\theta)_{i,j}, 1 \leq i, j \leq k)$ ,

$I(\theta)_{i,j} = \int_{\mathbb{R}^p} {}^t(\frac{\partial b(\theta,x)}{\partial \theta_i})\Sigma^{-1}(x)\frac{\partial b(\theta,x)}{\partial \theta_j}\lambda(\theta, x)dx$

see Kutoyants (2004)

# Discrete observations on a fixed time interval [0,T]

$$d\xi_t = b(t, \xi_t)dt + \sigma(\theta_0; t\xi_t)dB_t, \xi_0 = \eta$$

$b(t, x)$ known or unknown function, $\theta_0$ unknown parameter to estimate.

Observations at times $t_i^n = iT/n, i = 0, \dots n$ .
Asymptotics: $T > 0$ fixed and $n \rightarrow \infty$

(1) Only parameters in the diffusion coefficient can be estimated
(2) No consistent estimators for parameters in the drift.
(3) Estimation of $\theta_0 \Rightarrow$ Statistical model: Local Asymptotic Mixed Normal (Dohnal (JAP,1987),Genon-Catalot & Jacod (1993),Gobet (2001)
$\hat{\theta}_n$ converges of $\theta_0$ at rate $\sqrt{n}$ ; $\sqrt{n}(\hat{\theta}_n - \theta_0)$ : non Gaussian but Mixed variance Gaussian law.

Remark No explicit likelihood (unknown transition densities of $\xi_t$; $\Rightarrow$ No attempt to complete the sample path but use of contrast functions.

# Discrete observations on $[0, T]$ with $T \to \infty$

(2) Discrete observations on $[0, T]$ with sampling interval $\Delta_n$
$d\xi_t = b(\alpha, t, \xi_t)dt + \sigma(\beta, t, \xi_t)dW_t, \xi_0 = \eta$.
Observations: $(\xi_{t_i}, i = 1, \dots n)$ with $t_i = i\Delta_n, T = n\Delta_n$.
Double asymptotics indexed as $n$ (nb of observations) $\to \infty$
$\Delta_n \to 0$ and $T = n\Delta_n \to \infty$.

(3) Statistical model. Observations space: $((\mathbb{R}^p)^n, \mathcal{B}(\mathbb{R}^p)^n))$.
$P^n_{(\alpha,\beta)}$ distribution of the $n$-uple $\Rightarrow P^n_{(\alpha,\beta)}$ and $P^n_{(\alpha',\beta')}$ equivalent.
Likelihood: depends on the transitions of the Markov chain: untractable
Other approaches: Estimating functions, contrast functions...

- Parameters in the drift coefficient $\alpha$ estimated at rate $\sqrt{n\Delta_n}$.
- Parameters in the diffusion coefficient estimated at rate $\sqrt{n}$.

# Diffusion processes with small diffusion coefficient

Model: Multidimensional diffusion process on $\mathbb{R}^p$
$d\xi_t = b(\alpha, \xi_t)dt + \epsilon\sigma(\beta, \xi_t)dB_t, \xi_0 = x_0.$
$P^{\epsilon,T}_{\alpha,\beta}$: distribution of $(\xi_t, 0 \leq t \leq T)$ on $(C_T, \mathcal{C}_T)$.

Continuous observation on $[0, T]$

- $\beta \neq \beta' \Rightarrow P^{\epsilon,T}_{\alpha,\beta}$ and $P^{\epsilon,T}_{\alpha,\beta'}$ are singular
- $\Rightarrow \beta$ identified from the continuous observation $(\xi_t, 0 \leq t \leq T)$.
- $\beta = \beta_0$ or fixed $\sigma(\beta_0, x) = \sigma(x)$

Asymptotic framework: $T$ fixed and $\epsilon \to 0$.(Kutoyants,1980)
$\star\ \ell_\epsilon(\alpha) = \frac{1}{\epsilon^2}\int_0^T \frac{b(\alpha,\xi_s)}{\sigma^2(\xi_s)}d\xi_s - \frac{1}{2\epsilon^2}\int_0^T \frac{b^2(\alpha,\xi_s)}{\sigma^2(\xi_s)}ds \Rightarrow$ MLE $\hat{\alpha}_\epsilon$
$\star\ \epsilon^{-1}\left(\hat{\alpha}_\epsilon - \alpha_0\right) \to \mathcal{N}\left(0, I(\alpha_0)^{-1}\right)$
$I(\alpha) = (I_{i,j}(\alpha))_{1\leq i,j\leq k} = \int_0^T {}^t\frac{\partial b}{\partial \alpha_i}(\alpha, x(\alpha, s))\Sigma^{-1}(x(\alpha, s))\frac{\partial b}{\partial \alpha_j}(\alpha, x(\alpha, s))$
$\star\ x(\alpha, t) = x_0 + \int_0^t b(\alpha, x(\alpha, s))ds$ and $\Sigma(x) = \sigma(x)\,{}^t\sigma(x)$.

# Discrete observations on a fixed interval $[0, T]$

Diffusion process on $\mathbb{R}^p$
$dX(t) = b(\alpha, X(t))dt + \epsilon\sigma(\beta, X(t))dB(t), X(0) = x_0.$
**Observations:** $\{X(t_k), k = 0, \dots, n\}$ with $t_k = k\Delta$; $T = n\Delta$.

**Two possible asymptotic frameworks**

1. $\epsilon \to 0$ and $\Delta$ fixed with $T = n\Delta \Rightarrow$ Fixed nb of observations $n$.
2. $\epsilon \to 0$ and $\Delta = \Delta_n \to 0$ with $n\Delta_n = T$ simultaneously. $\Rightarrow n \to \infty$.

Results in framework (2)

- Different rates of convergence for parameters in the drift and in the diffusion coefficient (Gloter & Sorensen, 2009).
- Estimation of $\alpha$ at rate $\epsilon^{-1}$, $\beta$ at rate $\sqrt{n} = \Delta_n^{-1/2}$.

In practice difficult to assess which framework is more appropriate $\Rightarrow$ Distinction between parameters in the drift term $\alpha$ and in the diffusion term $\beta$ necessary.