Continuity of the Feynman–Kac formula for a generalized parabolic equation

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Abstract

It is well–known since the work of Pardoux and Peng [12] that Backward Stochastic Differential Equations provide probabilistic formulae for the solution of (systems of) second order elliptic and parabolic equations, thus providing an extension of the Feynman–Kac formula to semilinear PDEs, see also Pardoux and Răşcanu [14]. This method was applied to the class of PDEs with a nonlinear Neumann boundary condition first by Pardoux and Zhang [15]. However, the proof of continuity of the extended Feynman–Kac formula with respect to x (resp. to (t, x)) is not correct in that paper.

Here we consider a more general situation, where both the equation and the boundary condition involve the (possibly multivalued) gradient of a convex function. We prove the required continuity. The result for the class of equations studied in [15] is a Corollary of our main results.

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1 Introduction

The 1998 paper of Pardoux and Zhang [15] has initiated the topics of the probabilistic study of semilinear parabolic and elliptic systems of second order partial differential equations with nonlinear Neumann boundary condition. The idea is to prove that an associated Backward Stochastic Differential Equation allows to define a certain function of (t, x) (or in the elliptic case of x alone), which is continuous, and is a viscosity solution of a certain system of parabolic or elliptic PDEs. Several papers, see [18, 19, 2, 16, 17, 1], have used the above results

However, the continuity is not really proved in [15]. It is claimed that it follows from several estimates given in earlier sections of the paper, but this is not really fair. In [10] Maticiuc and Rascanu give a proof of the continuity result under some additional assumption.

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In [6] the continuity is shown in the case where all coefficients are Lipschitz continuous. The difficulty is that not only the solution of forward SDE depends upon its starting point x (resp. (t,x)), but also its local time on the boundary, which regulates the reflection.

In this paper, we will give the proof of continuity for a class of problems which is more general than the one considered in [15], and deduce the continuity statements from that paper as a Corollary.

More precisely, the aim of this paper is to prove the continuity of the function $(t,x) \mapsto Y_t^{t,x} \overset{def}{=} u(t,x) = (u_1(t,x),\dots,u_m(t,x))^* : [0,T]\times\overline{D} \to \mathbb{R}^m$, candidate for being the viscosity solution of the following system of partial differential equations with a generalized nonlinear Robin boundary condition and involving multivalued subdifferential operators of some lower semicontinuous convex functions $\varphi,\psi : \mathbb{R}^m \to]-\infty,+\infty]$

$$\begin{cases} -\dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}_t u(t,x) + \partial\varphi\big(u(t,x)\big) \ni F\big(t,x,u(t,x),(\nabla u(t,x))^* g(t,x)\big), \\ \hfill t\in(0,T),\ x\in D, \\ \dfrac{\partial u(t,x)}{\partial n} + \partial\psi\big(u(t,x)\big) \ni G\big(t,x,u(t,x)\big), \\ \hfill t\in(0,T),\ x\in Bd\left(\overline{D}\right), \\ u(T,x)=\kappa(x),\quad x\in\overline{D}, \end{cases} \tag{1}$$

where $\mathcal{L}_t v$, with $v\in C^2\left(\mathbb{R}^d,\mathbb{R}^m\right)$, is a column vector with the coordinates $\left(\mathcal{L}_t v\right)_i$, $i\in\overline{1,m}$, given by

$$\begin{aligned} \left(\mathcal{L}_t v\right)_i(x) &= \frac{1}{2}\mathrm{Tr}\big[g(t,x)g^*(t,x)D^2 v_i(x)\big] + \langle f(t,x),\nabla v_i(x)\rangle \\ &= \frac{1}{2}\sum_{j,l=1}^{d}(gg^*)_{j,l}(t,x)\frac{\partial^2 v_i(x)}{\partial x_j\partial x_l} + \sum_{j=1}^{d} f_j(t,x)\frac{\partial v_i(x)}{\partial x_j} \end{aligned} \tag{2}$$

∇u is the matrix $d\times m$ with the columns $\nabla u_i = \left(\frac{\partial u_i}{\partial x_1},\dots,\frac{\partial u_i}{\partial x_d}\right)^*$, $i\in\overline{1,m}$, and D is an open connected bounded subset of $\mathbb{R}^d$ of the form

$$\begin{aligned} &(i) \qquad D=\left\{x\in\mathbb{R}^d : \phi(x)<0\right\},\ \text{where } \phi\in C_b^3\left(\mathbb{R}^d\right), \\ &(ii) \qquad Bd\left(\overline{D}\right)=\left\{x\in\mathbb{R}^d : \phi(x)=0\right\} \ \text{and} \\ &\qquad\qquad |\nabla\phi(x)| = 1\ \forall\, x\in Bd\left(\overline{D}\right). \end{aligned} \tag{3}$$

The outward normal derivative of $u(t,\cdot)$ at the point $x\in Bd\left(\overline{D}\right)$ is the column vector

$$\frac{\partial u(t,x)}{\partial n} = \left(\frac{\partial u_1(t,x)}{\partial n},\dots,\frac{\partial u_m(t,x)}{\partial n}\right)^*$$

given by

$$\frac{\partial u_i(t,x)}{\partial n} = \sum_{j=1}^{d}\frac{\partial\phi(x)}{\partial x_j}\frac{\partial u_i(t,x)}{\partial x_j} = (\nabla u_i(t,x))^*\ \nabla\phi(x),\ i\in\overline{1,m};$$

hence

$$\frac{\partial u(t,x)}{\partial n} = (\nabla u(t,x))^*\ \nabla\phi(x).$$

2 Assumptions and formulation of the problem

Consider the stochastic basis $\left(\Omega,\mathcal{F},\mathbb{P},\left(\mathcal{F}_s^t\right)_{s\geq 0}\right)$, where the filtration is generated by a $k-$dimensional Brownian motion $(B_r)_{r\geq 0}$ as follows: $\mathcal{F}_s^t=\mathcal{N}$ if $0\leq s\leq t$ and

$$\mathcal{F}_s^t=\sigma\left\{B_r-B_t : t\leq r\leq s\right\}\vee\mathcal{N},\quad \text{if } s>t,$$

where $\mathcal{N}$ is the family of $\mathbb{P}-$negligible subsets of Ω.

Denote $S_d^p\left[0,T\right],\ p\geq 0$, the space of (equivalence classes of) progressively measurable continuous stochastic processes $X:\Omega\times[0,T]\rightarrow\mathbb{R}^d$ such that :

$$\mathbb{E}\sup_{t\in[0,T]}|X_t|^p<+\infty,\ \text{if } p>0.$$

By $\Lambda_d^p\left(0,T\right),\ p\geq 0$, denote the space of (equivalent classes of) progressively measurable stochastic processes $X:\Omega\times\left]0,T\right[\rightarrow\mathbb{R}^d$ such that

$$\int_0^T\left|X_t\right|^2dt<+\infty,\quad \mathbb{P}-a.s.\ \omega\in\Omega,\quad \text{if } p=0,$$

and

$$\mathbb{E}\left(\int_0^T\left|X_t\right|^2dt\right)^{p/2}<+\infty,\quad \text{if } p>0.$$

Let $f\left(\cdot,\cdot\right):\mathbb{R}_+\times\mathbb{R}^d\rightarrow\mathbb{R}^d$ and $g\left(\cdot,\cdot\right):\mathbb{R}_+\times\mathbb{R}^d\rightarrow\mathbb{R}^{d\times k}$ are continuous functions and satisfy: there exist $\mu_f\in\mathbb{R}$ and $\ell_g>0$ such that for all $u,v\in\mathbb{R}^d$

$$\begin{aligned} &(i) \qquad \langle u-v,f(t,u)-f(t,v)\rangle\ \leq\mu_f|u-v|^2,\\ &(ii) \qquad |g(t,u)-g(t,v)|\leq\ell_g|u-v|. \end{aligned}\tag{4}$$

By Theorem 4.54 and Corollary 4.56 from Pardoux & Răşcanu [14] we infer that for any $(t,x)\in[0,T]\times\overline{D}$ fixed, there exists a unique pair $\left(X^{t,x},A^{t,x}\right):\Omega\times\left[0,\infty\right[\rightarrow\mathbb{R}^d\times\mathbb{R}$ of continuous progressively measurable stochastic processes such that, $\mathbb{P}-a.s.$:

$$\begin{cases} (j) & X_s^{t,x}\in\overline{D} \text{ and } X_{s\wedge t}^{t,x}=x \text{ for all } s\geq 0,\\ (jj) & 0=A_u^{t,x}\leq A_s^{t,x}\leq A_v^{t,x} \text{ for all } 0\leq u\leq t\leq s\leq v,\\ (jjj) & X_s^{t,x}+\displaystyle\int_t^s\nabla\phi\left(X_r^{t,x}\right)dA_r^{t,x}=x+\int_t^s f\left(r,X_r^{t,x}\right)dr\\ & \qquad\qquad +\displaystyle\int_t^s g\left(r,X_r^{t,x}\right)dB_r,\ \ \forall\ s\geq t,\\ (jv) & A_s^{t,x}=\displaystyle\int_t^s\mathbf{1}_{Bd(\overline{D})}\left(X_r^{t,x}\right)dA_r^{t,x}\ ,\ \ \forall\ s\geq t. \end{cases}\tag{5}$$

Moreover by (4.112) from [14]

$$\begin{aligned} A_s^{t,x}=\int_t^s\mathcal{L}_r\phi(X_r^{t,x})dr+\int_t^s\left\langle\nabla\phi(X_r^{t,x}),g(r,X_r^{t,x})dB_r\right\rangle\\ -\left[\phi(X_s^{t,x})-\phi\left(x\right)\right], \end{aligned}$$

with $\mathcal{L}_r$ defined by (2).

For every $p \geq 1$, by Proposition 4.55 and Corollary 4.56 from [14],

$$\begin{array}{ll} (j) & (t,x) \mapsto \left(X_{\cdot}^{t,x}, A_{\cdot}^{t,x}\right) : [0,T] \times \overline{D} \to S_d^p[0,T] \times S_1^p[0,T] \\ & \quad\quad\quad\quad \text{is a continuous mapping,} \\ (jj) & \sup\limits_{(t,x)\in[0,T]\times\overline{D}} \left(\sup\limits_{s\in[0,T]} \mathbb{E}e^{\lambda A_s^{t,x}}\right) \leq \exp\left(C + C\,\lambda^2\right), \end{array} \tag{6}$$

for some $C > 0$ and every $\lambda > 0$. Moreover for every pair of continuous functions $h_1, h_2 : [0,T] \times \overline{D} \to \mathbb{R}$ the mapping

$$(t,x) \mapsto \mathbb{E}\int_t^T h_1(s, X_s^{t,x})ds + \mathbb{E}\int_t^T h_2(s, X_s^{t,x})\, dA_s^{t,x} : [0,T] \times \overline{D} \to \mathbb{R}$$

is a.s. continuous.

By the Kolmogorov criterion (choosing a proper version)

$$\begin{array}{l} (t,x,s) \mapsto \left(X_s^{t,x}(\omega), A_s^{t,x}(\omega)\right) : [0,T] \times \overline{D} \times [0,T] \to \mathbb{R}^d \times \mathbb{R} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{is continuous, } \mathbb{P} - a.s.\ \omega \in \Omega \end{array} \tag{7}$$

and consequently if $(t_n, x_n) \to (t,x)$, then (based also on (5-j), the boundedness of $\overline{D}$ and (6-jj)) we infer that for all $q > 0$, as $n \to \infty$,

$$\left|X_{t_n}^{t_n,x_n} - X_t^{t_n,x_n}\right| + \left|A_{t_n}^{t_n,x_n} - A_t^{t_n,x_n}\right| \to 0, \quad \mathbb{P} - a.s. \text{ and in } L^q(\Omega, \mathcal{F}, \mathbb{P}). \tag{8}$$

Moreover for all $q > 0$:

$$\lim_{\delta \searrow 0} \mathbb{E}\left[\sup\left\{\left|X_r^{t,x} - X_s^{t,x}\right|^q + \left|A_r^{t,x} - A_s^{t,x}\right|^q : r,s \in [0,T],\ |r-s| \leq \delta\right\}\right] = 0$$

Let $T > 0$ be fixed. We now consider $\left(Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x}, V_r^{t,x}\right)_{r\in[t,T]}$ the $\mathbb{R}^m \times \mathbb{R}^{m\times k} \times \mathbb{R}^m \times \mathbb{R}^m$-valued stochastic process solution of the backward stochastic variational inequality (BSVI):

$$\begin{aligned} & -dY_s^{t,x} + \partial\varphi\left(Y_s^{t,x}\right)ds + \partial\psi\left(Y_s^{t,x}\right)dA_s^{t,x} \ni F\left(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}\right)ds \\ & \quad\quad\quad\quad + G\left(s, X_s^{t,x}, Y_s^{t,x}\right)dA_s^{t,x} - Z_s^{t,x}dB_s, \quad s \in [t,T),\ d\mathbb{P}\text{-a.s.,} \\ & Y_T^{t,x} = \kappa\left(X_T^{t,x}\right), \end{aligned}$$

that is

$$\begin{cases} Y_s^{t,x} + \displaystyle\int_s^T \left(U_r^{t,x} dr + V_r^{t,x} dA_r^{t,x}\right) = \kappa\left(X_T^{t,x}\right) + \int_s^T F\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr, \\ \quad + \displaystyle\int_s^T G\left(r, X_r^{t,x}, Y_r^{t,x}\right) dA_r^{t,x} - \int_s^T Z_r^{t,x} dB_r, \ \forall\, s \in [t,T]\,,\ d\mathbb{P}\text{-a.s.}, \\ \displaystyle\int_u^v \left\langle U_r^{t,x}, S_r - Y_r^{t,x} \right\rangle dr + \int_u^v \varphi\left(Y_r^{t,x}\right) dr \le \int_u^v \varphi\left(S_r\right) dr, \quad d\mathbb{P}\text{-a.s. on } \Omega, \\ \qquad\qquad \text{for all } u,v \in [t,T]\,,\ u \le v,\ \text{for all continuous stochastic process } S; \\ \displaystyle\int_u^v \left\langle V_r^{t,x}, S_r - Y_r^{t,x} \right\rangle dA_r^{t,x} + \int_u^v \psi\left(Y_r^{t,x}\right) dA_r^{t,x} \le \int_u^v \psi\left(S_r\right) dA_r^{t,x}, \quad d\mathbb{P}\text{-a.s. on } \Omega, \\ \qquad\qquad \text{for all } u,v \in [t,T]\,,\ u \le v,\ \text{for all continuous stochastic process } S. \end{cases} \tag{9}$$

where $F : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \times \mathbb{R}^{m\times k} \to \mathbb{R}^m$, $G : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \to \mathbb{R}^m$ and $\kappa : \overline{D} \to \mathbb{R}^m$ are continuous. Assume that there exist $b_F, b_G, \ell_F > 0$ and $\mu_F, \mu_G \in \mathbb{R}$ (which can depend on T) such that $\forall t \in [0,T]$, $\forall x \in \overline{D}$, $y, \tilde{y} \in \mathbb{R}^m$, $z, \tilde{z} \in \mathbb{R}^{m\times k}$:

$$\begin{aligned} &(i) && \langle y - \tilde{y}, F(t,x,y,z) - F(t,x,\tilde{y},z)\rangle \le \mu_F |y-\tilde{y}|^2, \\ &(ii) && |F(t,x,y,z) - F(t,x,y,\tilde{z})| \le \ell_F |z - \tilde{z}|, \\ &(iii) && |F(t,x,y,0)| \le b_F\left(1 + |y|\right), \\ &(iv) && \langle y - \tilde{y}, G(t,x,y) - G(t,x,\tilde{y})\rangle \le \mu_G |y-\tilde{y}|^2, \\ &(v) && |G(t,x,y)| \le b_G\left(1+|y|\right). \end{aligned} \tag{10}$$

We also assume that

$$\begin{aligned} &(i) && \varphi, \psi : \mathbb{R}^m \to (-\infty, +\infty] \ \text{ are proper convex l.s.c. functions} \\ &(ii) && \exists u_0 \in int\left(Dom\left(\varphi\right)\right) \cap int\left(Dom\left(\psi\right)\right) \ \text{such that} \\ &&& \qquad\qquad \varphi\left(y\right) \ge \varphi\left(u_0\right) \ \text{and}\ \psi\left(y\right) \ge \psi\left(u_0\right),\ \forall\, y \in \mathbb{R}. \end{aligned} \tag{11}$$

where $Dom\left(\varphi\right) = \{y \in \mathbb{R}^m : \varphi\left(y\right) < \infty\}$ and similarly for $Dom\left(\psi\right)$.

We also introduce some *compatibility conditions* :
there exists $M > 0$ such that

$$(a) \qquad \sup_{x\in\overline{D}} \left|\varphi\left(\kappa(x)\right)\right| + \sup_{x\in\overline{D}} \left|\psi\left(\kappa(x)\right)\right| = M < \infty \tag{12}$$

and there exists $c > 0$ such that for all $\varepsilon > 0$, $t \in [0,T]$, $x \in \overline{D}$, $y \in \mathbb{R}^m$, $z \in \mathbb{R}^{m\times k}$,

$$\begin{aligned} &(b) && \langle \nabla\varphi_\varepsilon\left(y\right), \nabla\psi_\varepsilon\left(y\right)\rangle \ge 0, \\ &(d) && \langle \nabla\varphi_\varepsilon\left(y\right), G\left(t,x,y\right)\rangle \le c\left|\nabla\psi_\varepsilon\left(y\right)\right|\left[1 + \left|G\left(t,x,y\right)\right|\right], \\ &(e) && \langle \nabla\psi_\varepsilon\left(y\right), F\left(t,x,y,z\right)\rangle \le c\left|\nabla\varphi_\varepsilon\left(y\right)\right|\left[1 + \left|F\left(t,x,y,z\right)\right|\right], \\ &(f) && -\langle \nabla\varphi_\varepsilon\left(y\right), G\left(t,x,u_0\right)\rangle \le c\left|\nabla\psi_\varepsilon\left(y\right)\right|\left[1 + \left|G\left(t,x,u_0\right)\right|\right], \\ &(g) && -\langle \nabla\psi_\varepsilon\left(y\right), F\left(t,x,u_0,0\right)\rangle \le c\left|\nabla\varphi_\varepsilon\left(y\right)\right|\left[1 + \left|F\left(t,x,u_0,0\right)\right|\right] \end{aligned} \tag{13}$$

where $\nabla\varphi_\varepsilon(y)$, $\nabla\psi_\varepsilon(y)$ are the unique solutions u and v, respectively, of equations

$$\partial\varphi(y-\varepsilon u)\ni u \quad \text{and} \quad \partial\psi(y-\varepsilon v)\ni v.$$

(the Moreau-Yosida approximations: see the Annex below).

We remark that the compatibility assumptions are satisfied if, for example,

(a) $\varphi=\psi$,
or in the one dimensional case (i.e. $m=1$)

(b) If $\varphi,\psi:\mathbb{R}\to(-\infty,+\infty]$ are the convex indicator functions

$$\varphi(y)=\begin{cases}0, & \text{if } y\in[a,\infty),\\ +\infty, & \text{if } y\notin[a,\infty),\end{cases} \quad \text{and } \psi(y)=\begin{cases}0, & \text{if } y\in(-\infty,b],\\ +\infty, & \text{if } y\notin(-\infty,b],\end{cases}$$

where $-\infty\leq a<b\leq+\infty$, then

$$\nabla\varphi_\varepsilon(y)=\frac{-(a-y)^+}{\varepsilon} \text{ and } \nabla\psi_\varepsilon(y)=\frac{(y-b)^+}{\varepsilon}.$$

In this case the compatibility assumptions (13) are satisfied in particular if there exists $u_0\in(a,b)$ such that for all $(t,x)\in[0,T]\times\overline{D}$ and for all $z\in\mathbb{R}^{1\times k}$:

$$\begin{aligned}&G(t,x,y)\geq 0,\ \text{for all } y<a,\\ &F(t,x,y,z)\leq 0,\ \text{for all } y>b,\\ &G(t,x,u_0)\leq 0 \quad \text{and} \quad F(t,x,u_0,0)\geq 0,\end{aligned}$$

Remark that the backward stochastic variational inequality (9) satisfies the assumptions of Theorem 5.69 from [14] Therefore (9) has a unique progressively measurable solution $\left(Y^{t,x},Z^{t,x},U^{t,x},V^{t,x}\right)$, with $Y^{t,x}$ having continuous trajectories, such that for all $\lambda\geq 0$, $(t,x)\in[0,T]\times\overline{D}$,

$$\mathbb{E}\sup_{r\in[t,T]}e^{2\lambda A_r^{t,x}}\left|Y_r^{t,x}\right|^2+\mathbb{E}\left(\int_t^T e^{2\lambda A_r^{t,x}}\left|Z_r^{t,x}\right|^2dr\right)<\infty.$$

We extend the stochastic processes from (9) on $[0,t]$ by the deterministic solution of the following backward "stochastic" variational inequality ($F=0$, $G=0$) (which again has a unique solution)

$$\begin{cases}A_s^{t,x}=0,\ Z_s^{t,x}=0,\ \forall\, s\in[0,t]\,,\\ Y_s^{t,x}+\displaystyle\int_s^t U_r^{t,x}dr+\int_s^t V_r^{t,x}dr=Y_t^{t,x},\ \forall\, s\in[0,t]\,,\\ U_r^{t,x}\in\partial\varphi\left(Y_r^{t,x}\right) \text{ and } V_r^{t,x}\in\partial\psi\left(Y_r^{t,x}\right) \quad \text{a.e. on } [0,t]\,.\end{cases} \tag{14}$$

Now we can write (9) as follows

$$\begin{cases} Y_s^{t,x} + \int_s^T \left(U_r^{t,x} dr + V_r^{t,x} dA_r^{t,x}\right) = \kappa\left(X_T^{t,x}\right) + \int_s^T \mathbf{1}_{[t,T]}(r)\ F\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr \\ \qquad\qquad + \int_s^T \mathbf{1}_{[t,T]}(r)\, G\left(r, X_r^{t,x}, Y_r^{t,x}\right) dA_r^{t,x} - \int_s^T Z_r^{t,x} dB_r,\ \forall\, s \in [0,T], \\ \int_u^v U_r^{t,x}\left(S_r - Y_r^{t,x}\right) dr + \int_u^v \varphi\left(Y_r^{t,x}\right) dr \le \int_u^v \varphi(S_r)\, dr, \quad d\mathbb{P}\text{-a.s on } \Omega, \\ \qquad \text{for all } u, v \in [0,T],\ u \le v,\ \text{for any } \mathbb{R}^m\text{-valued continuous stochastic process } S; \\ \int_u^v V_r^{t,x}\left(S_r - Y_r^{t,x}\right) dA_r^{t,x} + \int_u^v \psi\left(Y_r^{t,x}\right) dA_r^{t,x} \le \int_u^v \psi(S_r)\, dA_r^{t,x}, \quad d\mathbb{P}\text{-a.s on } \Omega, \\ \qquad \text{for any } u, v \in [0,T],\ u \le v,\ \text{for all } \mathbb{R}^m\text{-valued continuous stochastic process } S; \\ . \end{cases} \tag{15}$$

(since in particular it is plain that $A_s^{t,x} = 0$, $\forall\, s \in [0,t]$).

If we denote

$$K_s^{t,x} = \int_0^s \left(U_r^{t,x} dr + V_r^{t,x} dA_r^{t,x}\right), \quad \forall\, s \in [0,T],$$

then as measures on $[0,T]$ we have

$$dK_r^{t,x} = U_r^{t,x} dr + V_r^{t,x} dA_r^{t,x} \in \partial\varphi\left(Y_r^{t,x}\right) dr + \partial\psi\left(Y_r^{t,x}\right) dA_r^{t,x}$$

and from the monotonicity of the subdifferential operators we have for all $(t,x), (\tau,y) \in [0,T] \times \overline{D}$,

$$\left\langle Y_r^{t,x} - Y_r^{\tau,y}, dK_r^{t,x} - dK_r^{\tau,y}\right\rangle \ge 0,\ \text{as measure on } [0,T]. \tag{16}$$

We highlight (see [11], or [14] Proposition 5.46) that for every $p \ge 2$ there exists a positive constant $\hat{C}_p$ *depending only upon* p such that for all $t \in [0,T]$, $x \in \overline{D}$, $s \in [t,T]$ and $\lambda \ge \max\left\{\left(\mu_F + \ell_F^2\right), \mu_G\right\}$

$$\begin{aligned} &\mathbb{E} \sup_{r\in[0,T]} e^{p\lambda(r + A_r^{t,x})} \left|Y_r^{t,x} - u_0\right|^p + \mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})} \left|Z_r^{t,x}\right|^2 dr\right)^{p/2} \\ &\quad + \mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})} \left[\varphi\left(Y_r^{t,x}\right) - \varphi(u_0)\right] dr\right)^{p/2} \\ &\quad + \mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})} \left[\psi\left(Y_r^{t,x}\right) - \psi(u_0)\right] dA_r^{t,x}\right)^{p/2} \\ &\le \hat{C}_p\, \mathbb{E}\bigg[e^{p\lambda(T+A_T^{t,x})} \left|\kappa\left(X_T^{t,x}\right) - u_0\right|^p \\ &\quad + \left(\int_0^T e^{\lambda(r+A_r^{t,x})} \left|F\left(r, X_r^{t,x}, u_0, 0\right)\right| dr\right)^p \\ &\quad + \left(\int_0^T e^{\lambda(r+A_r^{t,x})} \left|G\left(r, X_r^{t,x}, u_0\right)\right| dA_r^{t,x}\right)^p\bigg]. \end{aligned} \tag{17}$$

Since $[0,T]\times\overline{D}$ is bounded, $X_r^{t,x}\in\overline{D}$ for all $r\in[0,T]$ and the functions κ, F and G are continuous, there exists a constant C_1 *independent of* (t,x) such that for all $r\in[0,T]$

$$\left|\kappa\left(X_T^{t,x}\right)\right|+\left|F\left(r,X_r^{t,x},u_0,0\right)\right|+\left|G\left(r,X_r^{t,x},u_0\right)\right|\leq C_1,\quad \mathbb{P}-a.s. \tag{18}$$

Taking in account the estimate (6-jj) we have that for every$\lambda\geq\left(\mu_F+\ell_F^2\right)\vee\mu_G$ and $p>0$ there exists a constant C_2 *independent of* (t,x) such that

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,T]}e^{p\lambda(r+A_r^{t,x})}\left|Y_r^{t,x}\right|^p+\mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|Z_r^{t,x}\right|^2dr\right)^{p/2}\\&+\mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})}\varphi\left(Y_r^{t,x}\right)dr\right)^{p/2}\\&+\mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|\psi\left(Y_r^{t,x}\right)\right|dA_r^{t,x}\right)^{p/2}\\&\leq C_2\end{aligned} \tag{19}$$

Moreover for another constant C_3 *independent of* (t,x) we have

$$\mathbb{E}\Big(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|U_r^{t,x}\right|^2dr\Big)+\mathbb{E}\Big(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|V_r^{t,x}\right|^2dA_r^{t,x}\Big)\leq C_3 \tag{20}$$

Since $|G(t,x,y)|\leq b_G\left(1+|y|\right)$ and $|F(t,x,y,z)|\leq\ell_F|z|+b_F\left(1+|y|\right)$, then every $p>0$ there exists a positive constant C_4 *independent of* $r,s,t,\tau,\theta\in[0,T]$ *and* $x,y,z\in\overline{D}$ such that

$$\begin{aligned}&\mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|F\left(r,X_r^{t,x},Y_r^{\tau,y},Z_r^{\tau,y}\right)\right|^2dr\right)^p\\&\quad+\mathbb{E}\left(\int_0^T e^{2\lambda(r+A_r^{t,x})}\left|G\left(r,X_r^{t,x},Y_r^{\tau,y}\right)\right|^2dA_r^{t,x}\right)^p\leq C_4\end{aligned} \tag{21}$$

It is clear that the inequalities (19), (20) and (21) are satisfied for all $\lambda\geq0$.

We define

$$u(t,x)=Y_t^{t,x},\quad (t,x)\in[0,T]\times\overline{D}, \tag{22}$$

which is a deterministic quantity since $Y_t^{t,x}$ is $\mathcal{F}_t^t\equiv\mathcal{N}$–measurable. In the next section we shall prove that $(t,x)\mapsto u(t,x):[0,T]\times\overline{D}\to\mathbb{R}^m$ is a continuous function

We remark that from the Markov property, we have

$$u(s,X_s^{t,x})=Y_s^{t,x}.$$

Remark 1 *We note that in the particular case where $\varphi=\psi\equiv0$, we are in the situation which was studied in [15].*

3 Continuity

We present here the main result of this paper. The proof will rely upon several Lemmas which will be proved later in this section.

Theorem 2 *Under the above assumptions, the mapping* $(t,x) \mapsto u(t,x) = Y_t^{t,x} : [0,T] \times \overline{\mathcal{D}} \to \mathbb{R}^m$ *is continuous.*

Proof. Let $(t_n, x_n)_{n\geq 1}, (t,x) \in [0,T] \times \overline{D}$ be such that $(t_n, x_n) \to (t,x)$, as $n \to \infty$.

Denote $\Theta_s^n = \Theta_s^{t_n,x_n}$ and $\Theta_s = \Theta_s^0 = \Theta_s^{t,x}$ for $\Theta = X, A, Y, Z, U, V, K$. From (19) and the continuity of the trajectories of Y^n, for all $q > 0$, $n \geq 0$,

$$\lim_{\delta \searrow 0} \mathbb{E}\left[\sup\left\{|Y_r^n - Y_s^n|^q : r, s \in [0,T], \ |r-s| \leq \delta\right\}\right] = 0.$$

We have

$$Y_s^n - Y_s = \kappa(X_T^n) - \kappa(X_T) + \int_s^T d\mathcal{K}_r^n - \int_s^T (Z_r^n - Z_r)\, dB_r$$

where

$$\begin{aligned} d\mathcal{K}_r^n &= d(K_r - K_r^n) \\ &+ \Big[\mathbf{1}_{[t_n,T]}(r)\ F(r, X_r^n, Y_r^n, Z_r^n) - \mathbf{1}_{[t,T]}(r)\ F(r, X_r, Y_r, Z_r)\Big] dr \\ &+ \Big[\mathbf{1}_{[t_n,T]}(r)\ G(r, X_r^n, Y_r^n)\, dA_r^n - \mathbf{1}_{[t,T]}(r)\ G(r, X_r, Y_r)\, dA_r\Big]. \end{aligned}$$

with $dK_r^n = U_r^n dr + V_r^n dA_r^n \in \partial\varphi(Y_r^n)\, dr + \partial\psi(Y_r^n)\, dA_r^n$ and $dK_r = U_r dr + V_r dA_r \in \partial\varphi(Y_r)\, dr + \partial\psi(Y_r)\, dA_r$. Remark that by (16) it holds

$$\langle Y_r^n - Y_r, dK_r - dK_r^n \rangle \leq 0, \quad \text{as a signed measure on } [0,T].$$

It is easy to verify that:

$$\begin{aligned} &\left\langle Y_r^n - Y_r, \mathbf{1}_{[t_n,T]}(r)\ F(r, X_r^n, Y_r^n, Z_r^n) - \mathbf{1}_{[t,T]}(r)\ F(r, X_r, Y_r, Z_r) \right\rangle dr \\ &\leq \left\langle Y_r^n - Y_r, \mathbf{1}_{[t_n,T]}(r) \left[F(r, X_r^n, Y_r^n, Z_r^n) - F(r, X_r^n, Y_r^n, Z_r)\right] \right\rangle dr \\ &+ \left\langle Y_r^n - Y_r, \mathbf{1}_{[t_n,T]}(r) \left[F(r, X_r^n, Y_r^n, Z_r) - F(r, X_r^n, Y_r, Z_r)\right] \right\rangle dr \\ &+ \left\langle Y_r^n - Y_r, \mathbf{1}_{[t_n,T]}(r)\, F(r, X_r^n, Y_r, Z_r) - \mathbf{1}_{[t,T]}(r)\, F(r, X_r, Y_r, Z_r) \right\rangle dr \\ &\leq \ell_F |Y_r^n - Y_r|\, |Z_r^n - Z_r|\, dr + \mu_F |Y_r^n - Y_r|^2 dr \\ &+ |Y_r^n - Y_r| \left|\mathbf{1}_{[t_n,T]}(r)\, F(r, X_r^n, Y_r, Z_r) - \mathbf{1}_{[t,T]}(r)\, F(r, X_r, Y_r, Z_r)\right| dr \\ &\leq \left(\mu_F + \ell_F^2\right) |Y_r^n - Y_r|^2 dr + \frac{1}{4} |Z_r^n - Z_r|^2 dr \\ &+ |Y_r^n - Y_r| \left|\mathbf{1}_{[t_n,T]}(r)\, F(r, X_r^n, Y_r, Z_r) - \mathbf{1}_{[t,T]}(r)\, F(r, X_r, Y_r, Z_r)\right| dr \end{aligned}$$

and

$$\begin{aligned} &\left\langle Y_r^n - Y_r, \mathbf{1}_{[t_n,T]}(r)\ G(r, X_r^n, Y_r^n)\, dA_r^n - \mathbf{1}_{[t,T]}(r)\ G(r, X_r, Y_r)\, dA_r \right\rangle \\ &\leq \left\langle Y_r^n - Y_r\ , \mathbf{1}_{[t_n,T]}(r)\ \ \left[G(r, X_r^n, Y_r^n) - G(r, X_r^n, Y_r)\right] dA_r^n \right\rangle \\ &+ \left\langle Y_r^n - Y_r\ , \left[\mathbf{1}_{[t_n,T]}(r)\, G(r, X_r^n, Y_r) - \mathbf{1}_{[t,T]}(r)\, G(r, X_r, Y_r)\right] dA_r^n \right\rangle \\ &+ \left\langle Y_r^n - Y_r\ , \mathbf{1}_{[t,T]}(r)\, G(r, X_r, Y_r)\, (dA_r^n - dA_r) \right\rangle \\ &\leq \mu_G |Y_r^n - Y_r|^2 dA_r^n \\ &+ |Y_r^n - Y_r| \left|\mathbf{1}_{[t_n,T]}(r)\, G(r, X_r^n, Y_r) - \mathbf{1}_{[t,T]}(r)\, G(r, X_r, Y_r)\right| dA_r^n \\ &+ \left\langle Y_r^n - Y_r\ , \mathbf{1}_{[t,T]}(r)\, G(r, X_r, Y_r)\, (dA_r^n - dA_r) \right\rangle \end{aligned}$$

Hence for $\lambda \geq \left(\mu_F + \ell_F^2\right) \vee \mu_G$

$$\begin{aligned}\langle Y_r^n - Y_r \, , d\mathcal{K}_r^n \rangle &\leq \frac{1}{4} \left| Z_r^n - Z_r \right|^2 dr + \left| Y_r^n - Y_r \right|^2 \lambda \left(dr + dA_r^n \right) \\ &\quad + \left| Y_r^n - Y_r \right| dL_r^{(n)} + dR_r^{(n)},\end{aligned}$$

with

$$\begin{aligned}dL_r^{(n)} &= \left| \mathbf{1}_{[t_n,T]}(r) \, F\left(r, X_r^n, Y_r, Z_r\right) - \mathbf{1}_{[t,T]}(r) \, F\left(r, X_r, Y_r, Z_r\right) \right| dr \\ &\quad + \left| \mathbf{1}_{[t_n,T]}(r) \, G\left(r, X_r^n, Y_r\right) - \mathbf{1}_{[t,T]}(r) \, G\left(r, X_r, Y_r\right) \right| dA_r^n\end{aligned} \tag{23}$$

and

$$dR_r^{(n)} = \left\langle Y_r^n - Y_r \, , \mathbf{1}_{[t,T]}(r) \, G\left(r, X_r, Y_r\right) \left(dA_r^n - dA_r \right) \right\rangle \tag{24}$$

Then by Lemma 15 below with $a = 1/2$, we have

$$\begin{aligned}&\mathbb{E} \sup_{r \in [0,T]} e^{2\lambda(r + A_r^n)} \left| Y_r^n - Y_r \right|^2 + \mathbb{E} \left(\int_0^T e^{2\lambda(r + A_r^n)} \left| Z_r^n - Z_r \right|^2 dr \right) \\ &\quad \leq C_a \, \mathbb{E} \Big[e^{2\lambda\left(T + A_T^n\right)} \left| \kappa\left(X_T^n\right) - \kappa\left(X_T\right) \right|^2 + \left(\int_0^T e^{\lambda(r + A_r^n)} dL_r^{(n)} \right)^2 \\ &\qquad + \int_0^T e^{2\lambda(r + A_r^n)} dR_r^{(n)} \Big].\end{aligned}$$

and consequently by Lemma 3, Lemma 4 and Lemma 6 below, we have

$$\limsup_{n\to\infty} \mathbb{E} \sup_{r \in [0,T]} \left| Y_r^n - Y_r \right|^2 \leq \limsup_{n\to\infty} \mathbb{E} \sup_{r \in [0,T]} e^{2\lambda(r + A_r^n)} \left| Y_r^n - Y_r \right|^2 = 0.$$

We now deduce

$$\begin{aligned}\left| Y_{t_n}^{t_n,x_n} - Y_t^{t,x} \right|^2 &\leq 2\mathbb{E} \left| Y_{t_n}^{t_n,x_n} - Y_{t_n}^{t,x} \right|^2 + 2\mathbb{E} \left| Y_{t_n}^{t,x} - Y_t^{t,x} \right|^2 \\ &\leq 2\mathbb{E} \sup_{r \in [0,T]} \left| Y_r^n - Y_r \right|^2 + 2\mathbb{E} \left| Y_{t_n}^{t,x} - Y_t^{t,x} \right|^2 \\ &\to 0, \quad \text{as } n \to \infty;\end{aligned}$$

hence the result. ■

Recall that the constants C_1, C_2, C_3 and C_4 appearing in (18), (19), (20) and (21) are uniform w.r.t. (t,x). Consequently those estimates are valid for $(X^n, A^n, Y^n, Z^n, U^n, V^n)$ for all $n \geq 0$, with the same constants, which are independent of n. This fact will be used repeatedly in the proofs below.

Lemma 3 *We have*

$$\lim_{n\to\infty} \mathbb{E} \left(e^{2\lambda\left(T + A_T^n\right)} \left| \kappa\left(X_T^n\right) - \kappa\left(X_T\right) \right|^2 \right) = 0$$

Proof. By Lebesgue's dominated convergence theorem and (7) (also taking in account the boundedness (6-jj) and (18)), we have

$$\begin{aligned}
&\mathbb{E}\left(e^{2\lambda\left(T+A_T^n\right)}\left|\kappa\left(X_T^n\right)-\kappa\left(X_T\right)\right|^2\right)\\
&\leq\left(\mathbb{E}e^{4\lambda\left(T+A_T^n\right)}\right)^{1/2}\left(\mathbb{E}\left|\kappa\left(X_T^n\right)-\kappa\left(X_T\right)\right|^4\right)^{1/2}\\
&\leq C_\lambda\left(\mathbb{E}\left|\kappa\left(X_T^n\right)-\kappa\left(X_T\right)\right|^4\right)^{1/2}\\
&\to 0,\quad \text{as } n\to\infty.
\end{aligned}$$

■

Lemma 4 *Let $L^{(n)}$ defined by (23). Then*

$$\int_0^T e^{\lambda(r+A_r^n)}dL_r^{(n)}\to 0$$

in mean square, as $n\to\infty$.

Proof. By (6-jj) we get

$$\mathbb{E}\left(\int_0^T e^{\lambda(r+A_r^n)}dL_r^{(n)}\right)^2\leq 3\ \left[\mathbb{E}\left(\Lambda_n\right)+\mathbb{E}\left(\Gamma_n\right)+\mathbb{E}\left(\Delta_n\right)\right],$$

where

$$\begin{aligned}
\Lambda_n&=\left(\int_0^T\left|\mathbf{1}_{[t_n,T]}(r)\,F\left(r,X_r^n,Y_r,Z_r\right)-\mathbf{1}_{[t,T]}(r)\,F\left(r,X_r,Y_r,Z_r\right)\right|^2dr\right)^2,\\
\Gamma_n&=\left(\int_0^T\left|G\left(r,X_r^n,Y_r\right)-G\left(r,X_r,Y_r\right)\right|^2dA_r^n\right)^2,\\
\Delta_n&=\left(\int_0^T\left|G\left(r,X_r,Y_r\right)\right|^2\left|\mathbf{1}_{[t_n,T]}(r)-\mathbf{1}_{[t,T]}(r)\right|^2dA_r^n\right)^2.
\end{aligned}\tag{25}$$

Step 1. $\mathbb{E}\left(\Lambda_n\right)\to 0:$
Since

$$\mathbf{1}_{[t_n,T]}(r)\,F\left(r,X_r^n,Y_r,Z_r\right)-\mathbf{1}_{[t,T]}(r)\,F\left(r,X_r,Y_r,Z_r\right)\to 0\quad \text{a.e. } r\in[0,T],$$

and

$$\begin{aligned}
&\left|\mathbf{1}_{[t_n,T]}(r)\,F\left(r,X_r^n,Y_r,Z_r\right)-\mathbf{1}_{[t,T]}(r)\,F\left(r,X_r,Y_r,Z_r\right)\right|^2\\
&\qquad\leq C\left(1+\left|Y_r\right|^2+\left|Z_r\right|^2\right),
\end{aligned}$$

then by Lebesgue's dominated convergence theorem $\mathbb{E}\Lambda_n\to 0$.

Step 2. $\mathbb{E}\left(\Gamma_n\right)\to 0:$

We have $\Gamma_n \to 0, \quad \mathbb{P} - a.s.$, because

$$\begin{aligned}\Gamma_n &= \left(\int_0^T \left| G\left(r, X_r^n, Y_r\right) - G\left(r, X_r, Y_r\right) \right|^2 dA_r^n \right)^2 \\ &\leq \left(A_T^n\right)^2 \sup_{r \in [0,T]} \left| G\left(r, X_r^n, Y_r\right) - G\left(r, X_r, Y_r\right) \right|^4 .\end{aligned}$$

Since for all $q > 1$

$$\begin{aligned}\mathbb{E}\, \Gamma_n^q &\leq C\, \mathbb{E} \left[\left(1 + \|Y\|_T^{4q} \right) |A_T^n|^{2q} \right] \\ &\leq C_1 \left(1 + \mathbb{E}\, \|Y\|_T^{8q} + \mathbb{E}\, |A_T^n|^{4q} \right) \\ &\leq C_2,\end{aligned}$$

then the sequence of random variables Γ_n is uniformly integrable and therefore $\mathbb{E}\left(\Gamma_n\right) \to 0.$

Step 3. $\mathbb{E}\left(\Delta_n\right) \to 0:$

We have

$$\begin{aligned}\Delta_n &= \left(\int_0^T \left| G\left(r, X_r, Y_r\right) \right|^2 \left| \mathbf{1}_{[t_n,T]}(r) - \mathbf{1}_{[t,T]}(r) \right|^2 dA_r^n \right)^2 \\ &\leq \left(\sup_{r \in [0,T]} \left| G\left(r, X_r, Y_r\right) \right|^4 \right) \left(\int_0^T \left| \mathbf{1}_{[t_n,T]}(r) - \mathbf{1}_{[t,T]}(r) \right|^2 dA_r^n \right)^2 \\ &= \left(\sup_{r \in [0,T]} \left| G\left(r, X_r, Y_r\right) \right|^4 \right) \left| A_{t_n}^n - A_t^n \right|^2 \\ &\to 0, \quad \mathbb{P} - a.s.,\end{aligned}$$

where we have used (8) on the last line. Moreover for $q > 1$,

$$\begin{aligned}\mathbb{E}\, \Delta_n^q &\leq \mathbb{E} \left[\sup_{r \in [0,T]} \left| G\left(r, X_r, Y_r\right) \right|^{4q} \ \left| A_{t_n}^n - A_t^n \right|^{2q} \right] \\ &\leq C \left(\mathbb{E} \sup_{r \in [0,T]} \left| G\left(r, X_r, Y_r\right) \right|^{8q} + \mathbb{E} \sup_{r \in [0,T]} |A_r^n|^{4q} \right) \\ &\leq C_1\end{aligned}$$

Consequently, by uniformly integrability, we conclude that $\mathbb{E}\left(\Delta_n\right) \to 0.$

■

Consider $N \in \mathbb{N}$, $N > T$ and the partition $\pi_N : 0 = r_0 < r_1 < \ldots < r_i < \ldots < r_N = T$ with $r_i = \frac{iT}{N}$. We denote $\lfloor r|N \rfloor = \max\left\{r_i : r_i \leq r\right\} = \left[\frac{rN}{T}\right] \frac{T}{N}$, where $[x]$ is the integer part of x. Given a continuous stochastic process $(H_t)_{t \in [0,T]}$, we define

$$H_r^N = \sum_{i=0}^{N-1} H_{r_i} \mathbf{1}_{[r_i, r_{i+1})}(r) + H_T\, \mathbf{1}_{\{T\}}(r) = H_{\lfloor r|N \rfloor}\, .$$

Lemma 5 *Let $1 < q < 2$. There exists a positive constant C independent of $(t,x),(t_n,x_n) \in [0,T]\times\overline{D}$ and $N \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \mathbb{E}\left(\int_0^T \left|Y_r^n - Y_r^{n,N}\right|^q (dA_r^n + dA_r)\right) \leq \frac{C}{N^{q/2}} + C\left[\mathbb{E}\max_{i=\overline{1,N}} \left(A_{r_i} - A_{r_{i-1}}\right)^{2q/(2-q)}\right]^{(2-q)/4}.$$

Proof. Since

$$Y_s^{n,N} + \int_{\lfloor s|N\rfloor}^s (U_r^n dr + V_r^n dA_r^n) = Y_s^n + \int_{\lfloor s|N\rfloor}^s \mathbf{1}_{[t_n,T]}(r)\ F(r, X_r^n, Y_r^n, Z_r^n)\, dr,$$
$$+\int_{\lfloor s|N\rfloor}^s \mathbf{1}_{[t_n,T]}(r)\, G(r, X_r^n, Y_r^n)\, dA_r^n - \int_{\lfloor s|N\rfloor}^s \langle Z_r^n, dB_r\rangle,\ \forall\, s \in [0,T],$$

then

$$\left|Y_s^{n,N} - Y_s^n\right|^q \leq \frac{C}{N^{q/2}}\left[\int_{\lfloor s|N\rfloor}^s \left(|U_r^n|^2 + |F(r, X_r^n, Y_r^n, Z_r^n)|^2\right) dr\right]^{q/2}$$
$$+ C\left(A_s^n - A_{\lfloor s|N\rfloor}^n\right)^{q/2}\left[\int_{\lfloor s|N\rfloor}^s \left(|V_r^n|^2 + |G(r, X_r^n, Y_r^n)|^2\right) dA_r^n\right]^{q/2}$$
$$+ C\left|\int_{\lfloor s|N\rfloor}^s \langle Z_r^n, dB_r\rangle\right|^q.$$

Hence

$$\mathbb{E}\left(\int_0^T \left|Y_r^n - Y_r^{n,N}\right|^q (dA_r^n + dA_r)\right) \leq \alpha_{n,N} + \beta_{n,N} + \gamma_{n,N}.$$

We have first

$$\alpha_{n,N} = \frac{C}{N^{q/2}}\mathbb{E}\Big[\int_0^T \big(\int_{\lfloor s|N\rfloor}^s (|U_r^n|^2 + |F(r, X_r^n, Y_r^n, Z_r^n)|^2)dr\big)^{q/2} (dA_s^n + dA_s)\Big]$$
$$\leq \frac{C}{N^{q/2}}\mathbb{E}\Big[(A_T^n + A_T)\big(\int_0^T (|U_r^n|^2 + |F(r, X_r^n, Y_r^n, Z_r^n)|^2)dr\big)^{q/2}\Big]$$
$$\leq \frac{C}{N^{q/2}}\left[\mathbb{E}\ (A_T^n + A_T)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\left(\mathbb{E}\textstyle\int_0^T |U_r^n|^2 dr + \mathbb{E}\int_0^T |F(r, X_r^n, Y_r^n, Z_r^n)|^2 dr\right)^{\frac{q}{2}}$$
$$\leq \frac{C}{N^{q/2}}.$$

Since $(A_s^n)_{s\geq 0}$ and $(A_s)_{s\geq 0}$ are increasing stochastic processes,

$$\beta_{n,N} = C\,\mathbb{E}\int_0^T \Big[\left(A_s^n - A_{\lfloor s|N\rfloor}^n\right)^{\frac{q}{2}}\big(\int_{\lfloor s|N\rfloor}^s \left(|V_r^n|^2 + |G(r, X_r^n, Y_r^n)|^2\right) dA_r^n\big)^{\frac{q}{2}}\Big] (dA_s^n + dA_s)$$

$$\leq C\,\mathbb{E}\,\Big[\Big(\int_0^T (|V_r^n|^2 + |G(r,X_r^n,Y_r^n)|^2)dA_r^n\Big)^{\frac{q}{2}}\sum_{i=1}^N\int_{r_{i-1}}^{r_i}(A_s^n - A_{\lfloor s|N\rfloor}^n)^{\frac{q}{2}}\,(dA_s^n + dA_s)\,\Big]$$

$$\leq C\Big[\mathbb{E}\,\Big(\sum_{i=1}^N (A_{r_i}^n - A_{r_{i-1}}^n)^{q/2}(A_{r_i}^n + A_{r_i} - A_{r_{i-1}}^n - A_{r_{i-1}})\Big)^{2/(2-q)}\Big]^{(2-q)/2}$$

Since by (6-j)

$$\lim_{n\to\infty}\mathbb{E}\sup_{r\in[0,T]}|A_r^n - A_r|^p = 0, \quad \text{for all } p>0,$$

and

$$\mathbb{E}\sup_{r\in[0,T]}|A_r|^p + \sup_{n\in\mathbb{N}}\left(\mathbb{E}\sup_{r\in[0,T]}|A_r^n|^p\right) < \infty, \quad \text{for all } p>0,$$

we infer that for all $N\in\mathbb{N}$

$$\begin{aligned}\limsup_{n\to\infty}\beta_{n,N} &\leq C\left[\mathbb{E}\,\left(\sum_{i=1}^N \left(A_{r_i} - A_{r_{i-1}}\right)^{q/2}\left(A_{r_i} - A_{r_{i-1}}\right)\right)^{2/(2-q)}\right]^{(2-q)/2} \\ &\leq C\,\left[\mathbb{E}\,\left(\max_{i=\overline{1,N}}\left(A_{r_i} - A_{r_{i-1}}\right)^{q/2} A_T\right)^{2/(2-q)}\right]^{(2-q)/2} \\ &\leq C_1\left[\mathbb{E}\,\max_{i=\overline{1,N}}\left(A_{r_i} - A_{r_{i-1}}\right)^{2q/(2-q)}\right]^{(2-q)/4}.\end{aligned}$$

We finally consider

$$\gamma_{n,N} = C\mathbb{E}\int_0^T\left|\int_{\lfloor s|N\rfloor}^s\langle Z_r^n, dB_r\rangle\right|^q (dA_s^n + dA_s)$$

$$= C\,\mathbb{E}\sum_{i=1}^N\int_{r_{i-1}}^{r_i}\left|\int_{\lfloor s|N\rfloor}^s\langle Z_r^n, dB_r\rangle\right|^q (dA_s^n + dA_s)$$

$$\leq C\sum_{i=1}^N\mathbb{E}\Big[\sup_{s\in[r_{i-1},r_i]}\Big|\int_{r_{i-1}}^s\langle Z_r^n, dB_r\rangle\Big|^q\left(A_{r_i}^n - A_{r_{i-1}}^n + A_{r_i} - A_{r_{i-1}}\right)\Big]$$

$$\leq C\sum_{i=1}^N\Big[\mathbb{E}\sup_{s\in[r_{i-1},r_i]}\Big|\int_{r_{i-1}}^s\langle Z_r^n, dB_r\rangle\Big|^2\Big]^{q/2}\left[\mathbb{E}\,\left(A_{r_i}^n - A_{r_{i-1}}^n + A_{r_i} - A_{r_{i-1}}\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}$$

$$\leq C_1\sum_{i=1}^N\left(\mathbb{E}\int_{r_{i-1}}^{r_i}|Z_r^n|^2\,dr\right)^{q/2}\left[\mathbb{E}\,\left(A_{r_i}^n - A_{r_{i-1}}^n + A_{r_i} - A_{r_{i-1}}\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}.$$

From the above and the following Hölder's inequality, for $1<q<2$,

$$\sum_{i=1}^N a_i^{q/2} b_i^{(2-q)/2} \leq \left(\sum_{i=1}^N a_i\right)^{q/2}\left(\sum_{i=1}^N b_i\right)^{(2-q)/2},$$

we deduce that

$$\gamma_{n,N} \le C_2 \left[\sum_{i=1}^{N} \mathbb{E} \left(A^n_{r_i} - A^n_{r_{i-1}} + A_{r_i} - A_{r_{i-1}} \right)^{2/(2-q)} \right]^{(2-q)/2}.$$

Hence for all $N \in \mathbb{N}$

$$\begin{aligned}
\limsup_{n\to\infty} \gamma_{n,N} &\le C \left[\sum_{i=1}^{N} \mathbb{E} \left(A_{r_i} - A_{r_{i-1}} \right)^{2/(2-q)} \right]^{(2-q)/2} \\
&\le C \left[\mathbb{E} \left(\max_{i=\overline{1,N}} \left(A_{r_i} - A_{r_{i-1}} \right)^{q/(2-q)} \sum_{i=1}^{N} \left(A_{r_i} - A_{r_{i-1}} \right) \right) \right]^{(2-q)/2} \\
&\le C_1 \left[\mathbb{E} \max_{i=\overline{1,N}} \left(A_{r_i} - A_{r_{i-1}} \right)^{2q/(2-q)} \right]^{(2-q)/4}.
\end{aligned}$$

The result follows. ■

Lemma 6 *Let $R^{(n)}$ defined by (24). Then*

$$\limsup_{n\to\infty} \mathbb{E} \int_0^T e^{2\lambda(r+A^n_r)} dR^{(n)}_r = 0.$$

Proof. Denote $G_r = G(r, X_r, Y_r)$ and $\|G\|_T = \sup_{r\in[0,T]} |G_r|$. Then

$$\begin{aligned}
(Y^n_r - Y_r) G(r, X_r, Y_r) = &\left(Y^{n,N}_r - Y^N_r \right) \left(G_r - G^N_r \right) + \left(Y^N_r - Y_r \right) G_r \\
&+ \left(Y^{n,N}_r - Y^N_r \right) G^N_r + \left(Y^n_r - Y^{n,N}_r \right) G_r
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathbb{E} \left(\int_0^T e^{2\lambda(r+A^n_r)} dR^{(n)}_r \right) \\
&= \mathbb{E} \int_0^T e^{2\lambda(r+A^n_r)} (Y^n_r - Y_r) \mathbf{1}_{[t,T]}(r) G(r, X_r, Y_r) (dA^n_r - dA_r) \\
&\le (2\lambda)^{-1} \mathbb{E} \left[\left((\|Y^n\|_T + \|Y\|_T) \left\|G - G^N\right\|_T + \left\|Y^N - Y\right\|_T \|G\|_T \right) e^{2\lambda\left(T + A^n_T + A_T\right)} \right] \\
&+ \mathbb{E} \left(e^{2\lambda\left(T+A^n_T\right)} \sum_{i=1}^{N} \left(Y^n_{r_{i-1}} - Y_{r_{i-1}} \right) G_{r_{i-1}} \left[\left(A^n_{r_i} - A_{r_i} \right) - \left(A^n_{r_{i-1}} - A_{r_{i-1}} \right) \right] \right) \\
&+ \mathbb{E} \left(e^{2\lambda\left(T+A^n_T\right)} \|G\|_T \int_0^T \left| Y^n_r - Y^{n,N}_r \right| (dA^n_r + dA_r) \right)
\end{aligned}$$

Let $1 < q < 2$. Using Hölder's inequality and the estimates (19) and (21), we obtain

$$\begin{aligned}
\mathbb{E} \left(\int_0^T e^{2\lambda(r+A^n_r)} dR^{(n)}_r \right) \le\ & C \sqrt{\mathbb{E} \|G - G^N\|^2_T} + \sqrt{\mathbb{E} \|Y^N - Y\|^2_T} \\
&+ C \sum_{i=1}^{N} \left[\mathbb{E} \left| \left(A^n_{r_i} - A_{r_i} \right) - \left(A^n_{r_{i-1}} - A_{r_{i-1}} \right) \right|^2 \right]^{1/2} \\
&+ C \left(\mathbb{E} \int_0^T \left| Y^n_r - Y^{n,N}_r \right|^q (dA^n_r + dA_r) \right)^{1/q}
\end{aligned}$$

By Lemma 5 we deduce that for all $N\in\mathbb{N}$

$$\limsup_{n\to\infty}\mathbb{E}\int_0^T e^{2\lambda(r+A_r^n)}dR_r^{(n)}\leq C\,\sqrt{\mathbb{E}\,\|G-G^N\|_T^2}+\sqrt{\mathbb{E}\,\|Y^N-Y\|_T^2}$$
$$+C\left[\frac{1}{N^{q/2}}+\left[\mathbb{E}\max_{i=\overline{1,N}}\left(A_{r_i}-A_{r_{i-1}}\right)^{2q/(2-q)}\right]^{(2-q)/4}\right]^{1/q}$$

and the result follows passing to limit as $N\to\infty$ in the last inequality.

■

Theorem 2 in the particular case $\varphi=\psi\equiv 0$ yields the following

Corollary 7 *Proposition 4.1 from [15] and Corollary 14 from [9] hold true.*

4 Infinite horizon BSDEs: continuity

Let us consider the forward-backward problem (5) & (9) on the interval $[0,\infty)$ with f,g,F and G independent of time argument, $\kappa=0$ and $\varphi=\psi\equiv 0$, $u_0=0$, that is:

the forward reflected SDE starting from x at $t=0$:

$$\begin{aligned}
&(j)\quad X_s^x\in\overline{D}\ \text{ for all } s\geq 0,\\
&(jj)\quad 0=A_0^x\leq A_s^x\leq A_u^x \text{ for all } 0\leq s\leq u,\\
&(jjj)\quad X_s^x+\int_0^s\nabla\phi\left(X_r^x\right)dA_r^x=x+\int_0^s f\left(X_r^x\right)dr\\
&\qquad\qquad +\int_0^s g\left(X_r^x\right)dB_r,\quad\forall\, s\geq 0,\\
&(jv)\quad A_s^x=\int_0^s \mathbf{1}_{Bd(\overline{D})}\left(X_r^x\right)dA_r^x\ ,\quad\forall\, s\geq 0.
\end{aligned}$$

and the BSDE on $[0,\infty)$ with the final data 0 :

$$Y_s^x=\int_s^\infty F\left(X_r^x,Y_r^x,Z_r^x\right)dr+\int_s^\infty G\left(X_r^x,Y_r^x\right)dA_r^x-\int_s^\infty Z_r^x dB_r,\ \ s\geq 0,\tag{26}$$

Denote $(X_s^x,A_s^x,Y_s^{x;n},Z_s^{x;n})=\left(X_s^{0,x},A_s^{0,x},Y_s^{0,x},Z_s^{0,x}\right)$, $n\in\mathbb{N}$, the solution of the forward-backward problem (5)&(9) on the time interval $[0,n]$ with $(Y_s^{x;n},Z_s^{x;n})=0$, for $s>n$; hence

$$Y_s^{x;n}=\int_s^n F\left(X_r^x,Y_r^{x;n},Z_r^{x;n}\right)dr+\int_s^n G\left(X_r^x,Y_r^{x;n}\right)dA_r^x-\int_s^n Z_r^{x;n}dB_r,\ \ s\in[0,n]\,,\tag{27}$$

By Theorem 2 the mapping

$$x\longmapsto Y_0^{x;n}:\overline{D}\to\mathbb{R}^m \text{ is continuous.}\tag{28}$$

Estimates on the approximating equation (27) and the continuity result (28) yield:

Proposition 8 *Under the assumptions (10) and* $\max\left\{\left(\mu_F+\ell_F^2\right),\mu_G\right\}\leq\lambda<0$ *there exists a unique pair* $(Y^x,Z^x)\in S_m^0\left[0,T\right]\times\Lambda_{m\times k}^0\left(0,T\right)$ *solution of the BSDE (26) in the following sense:*

$$\begin{cases} (j) & Y_s^x=Y_T^x+\displaystyle\int_s^T F\left(X_r^x,Y_r^x,Z_r^x\right)dr+\int_s^T G\left(X_r^x,Y_r^x\right)dA_r^x-\int_s^T Z_r^x dB_r, \\ & \hfill \textit{for all } 0\leq s\leq T, \\ (jj) & \mathbb{E}\sup\limits_{r\geq 0}e^{2\lambda(r+A_r^x)}\left|Y_r^x\right|^2+\mathbb{E}\displaystyle\int_0^\infty e^{2\lambda(r+A_r^x)}\left|Z_r^x\right|^2dr<\infty, \\ (jjj) & \lim\limits_{N\to\infty}\mathbb{E}\sup\limits_{r\geq N}e^{2\lambda(r+A_r^x)}\left|Y_r^x\right|^2=0. \end{cases} \tag{29}$$

Moreover the mapping

$$x\longmapsto u\left(x\right)=Y_0^x:\overline{D}\to\mathbb{R}^m \textit{ is continuous.} \tag{30}$$

Proof. The existence and uniqueness result for the solution of (29) was proved by Pardoux and Zhang in [15], Theorem 2.1 (the result is also given in [14], Section 5.6.1). Proving here the continuity property (30) we obtain, once again, the existence of the solution; the uniqueness is a easy consequence of Lemma 15 via the assumptions (10) on F and G.

Using (10) we also deduce by Lemma 15 with $a=1/2$ (or directly from (17)) that for $0\leq s\leq n:$

$$\begin{aligned} &\mathbb{E}\sup_{r\in[s,n]}e^{2\lambda(r+A_r^x)}\left|Y_r^{x;n}\right|^2+\mathbb{E}\int_s^n e^{2\lambda(r+A_r^x)}\left|Z_r^{x;n}\right|^2dr \\ &\quad\leq C\,\mathbb{E}\Big[e^{2\lambda(n+A_n^x)}\left|Y_n^{x;n}\right|^2+\Big(\int_s^n e^{\lambda(r+A_r^x)}\left|F\left(X_r^x,0,0\right)\right|dr\Big)^2 \\ &\quad+\Big(\int_s^n e^{\lambda(r+A_r^x)}\left|G\left(X_r^x,0\right)\right|dA_r^x\Big)^2\Big] \\ &\quad\leq C'\,\mathbb{E}\,\left(\int_s^n e^{\lambda(r+A_r^x)}(dr+dA_r^x)\right)^2 \\ &\quad\leq\frac{C'}{|\lambda|}\,\mathbb{E}\,e^{2\lambda(s+A_s^x)} \\ &\quad\leq\frac{C'}{|\lambda|}\,e^{2\lambda s}, \end{aligned}$$

(we also used that $F(X_r^x,0,0)$ and $G(X_r^x,0)$ are uniformly bounded on the bounded domain $\overline{D}$).

Since $(Y_s^{x;n},Z_s^{x;n})=0$, for $s>n$ we infer that for all $s\geq 0$ and $n\in\mathbb{N}$,

$$\mathbb{E}\sup_{r\geq s}e^{2\lambda(r+A_r^x)}\left|Y_r^{x;n}\right|^2+\mathbb{E}\int_s^\infty e^{2\lambda(r+A_r^x)}\left|Z_r^{x;n}\right|^2dr\leq\frac{C}{|\lambda|}\,e^{2\lambda s}. \tag{31}$$

If $n,l\in\mathbb{N}$ and $s\in\left[0,n\right]$, then

$$Y_s^{x;n+l}-Y_s^{x;n}=Y_n^{x;n+l}+\int_s^n d\mathcal{K}_r-\int_s^n\left(Z_r^{x;n+l}-Z_r^{x;n}\right)dB_r,$$

where

$$d\mathcal{K}_r = \Big[F(X_r^x, Y_r^{x;n+l}, Z_r^{x;n+l}) - F\left(X_r^x, Y_r^{x;n}, Z_r^{x;n}\right)\Big]\, dr$$
$$- \Big[G(X_r^x, Y_r^{x;n+l}) - G\left(X_r^x, Y_r^{x;n}\right)\Big]\, dA_r^x.$$

By the assumptions (10) we have

$$\Big\langle Y_r^{x;n+l} - Y_r^{x;n}\,, d\mathcal{K}_r\Big\rangle$$
$$\leq \mu_F \left|Y_r^{x;n+l} - Y_r^{x;n}\right|^2 dr + \ell_F \left|Y_r^{x;n+l} - Y_r^{x;n}\right| \left|Z_r^{x;n+l} - Z_r^{x;n}\right| dr$$
$$+\mu_G \left|Y_r^{x;n+l} - Y_r^{x;n}\right|^2 dA_r^x$$
$$\leq \frac{1}{4}\left|Z_r^{x;n+l} - Z_r^{x;n}\right|^2 dr + \left|Y_r^{x;n+l} - Y_r^{x;n}\right|^2 \lambda\left(dr + dA_r^n\right).$$

Therefore by Lemma 15 (with $a = 1/2$) and (31) we get

$$\mathbb{E} \sup_{r\in[0,n]} e^{2\lambda(r+A_r^x)} \left|Y_r^{x;n+l} - Y_r^{x;n}\right|^2 + \mathbb{E}\int_0^n e^{2\lambda(r+A_r^x)} \left|Z_r^{x;n+l} - Z_r^{x;n}\right|^2 dr$$
$$\leq C\, \mathbb{E}\, e^{2\lambda(n+A_n^x)} \left|Y_n^{x;n+l}\right|^2$$
$$\leq \frac{C}{|\lambda|}\, e^{2\lambda n}.$$

Hence

$$\mathbb{E} \sup_{r\geq 0} e^{2\lambda(r+A_r^x)} \left|Y_r^{x;n+l} - Y_r^{x;n}\right|^2 + \mathbb{E}\int_0^\infty e^{2\lambda(r+A_r^x)} \left|Z_r^{x;n+l} - Z_r^{x;n}\right|^2 dr \leq \frac{C}{|\lambda|}\, e^{2\lambda n}$$

and consequently there exists $(Y_s^x, Z_s^x)_{s\geq 0}$ a pair of progressively measurable stochastic process, $(Y_s^x)_{s\geq 0}$ having continuous trajectories, such that for all $s \geq 0$

$$\mathbb{E} \sup_{r\geq s} e^{2\lambda(r+A_r^x)} \left|Y_r^x\right|^2 + \mathbb{E}\int_s^\infty e^{2\lambda(r+A_r^x)} \left|Z_r^x\right|^2 dr < \frac{C}{|\lambda|}\, e^{2\lambda s}$$

and

$$\mathbb{E} \sup_{r\geq 0} e^{2\lambda(r+A_r^x)} \left|Y_r^x - Y_r^{x;n}\right|^2 + \mathbb{E}\int_0^\infty e^{2\lambda(r+A_r^x)} \left|Z_r^x - Z_r^{x;n}\right|^2 dr$$
$$\leq \frac{C}{|\lambda|}\, e^{2\lambda n} \to 0, \quad \text{as } n\to\infty.$$

Since for all $0 \leq T \leq n$:

$$Y_s^{x;n} = Y_T^{x;n} + \int_s^T F\left(X_r^x, Y_r^{x;n}, Z_r^{x;n}\right) dr + \int_s^T G\left(X_r^x, Y_r^{x;n}\right) dA_r^x - \int_s^T Z_r^{x;n} dB_r,\ s\in[0,n],$$

then passing to limit as $n\to\infty$ (possibly along a subsequence) we obtain that $(Y_s^x, Z_s^x)_{s\geq 0}$ is a solution of (29).

Let $y,x\in\overline{D}$. Since

$$\begin{aligned}|Y_0^y-Y_0^x| &\leq |Y_0^y-Y_0^{y;n}|+|Y_0^{y;n}-Y_0^{x;n}|+|Y_0^{x;n}-Y_0^x|\\ &\leq \tfrac{2\sqrt{C}}{\sqrt{|\lambda|}}\,e^{\lambda n}+|Y_0^{y;n}-Y_0^{x;n}|\,,\quad \text{for all } n\in\mathbb{N}.\end{aligned}$$

and $\lambda<0$, the continuity property (30) follows from (28).

■

We finally deduce that

Remark 9 *Theorem 5.1 from [15] holds true.*

5 Viscosity solutions

5.1 Parabolic PDEs

We recall some results on the viscosity solutions of the PVI (1) from [13], [8], [9], [14]. At the same time, we formulate the definition of the notion of viscosity solution of our system of equations.

We assume that the assumptions from Section 1 and Section 2 are satisfied and we let the dimension of the Brownian motion be $k=d$.

Denote $\mathbb{S}^d$ the set of symmetric matrices from $\mathbb{R}^{d\times d}$.

Let $h:[0,T]\times\overline{D}\to\mathbb{R}$ be a continuous function.

A triple $(p,q,X)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ is a parabolic super-jet to h, at $(t,x)\in[0,T]\times\overline{D}$, if for all $(s,x')\in[0,T]\times\overline{D}$,

$$\begin{aligned}h(s,x')\leq h(t,x)+p(s-t)+\langle q,x'-x\rangle+\frac{1}{2}\langle X(x'-x),x'-x\rangle\\ +o(|s-t|+|x'-x|^2).\end{aligned}\tag{32}$$

The set of parabolic super-jets at (t,x) is denoted by $\mathcal{P}^{2,+}h(t,x)$; the set of parabolic sub-jets is defined by $\mathcal{P}_{\mathcal{O}}^{2,-}h=-\mathcal{P}_{\mathcal{O}}^{2,+}(-h)$.

First we consider the system (1) with the functions $\varphi,\psi:\mathbb{R}^m\to]-\infty,+\infty]$ decoupled on coordinates as follows $\varphi(u_1,\ldots,u_m)=\varphi_1(u_1)+\cdots+\varphi_m(u_m)$ and $\psi(u_1,\ldots,u_m)=\psi_1(u_1)+\cdots+\psi_m(u_m)$, where $\varphi_i,\psi_i:\mathbb{R}\to]-\infty,+\infty]$ are l.s.c. convex functions; hence $\partial\varphi(u_1,\ldots,u_m)=\partial\varphi_1(u_1)\times\cdots\times\partial\varphi_m(u_m)$ and similar for $\partial\psi$.

We also assume that F_i , the $i-$th coordinate of F, depends only on the $i-$th row of the matrix Z.

Consider the system

$$\left\{\begin{array}{ll}(a) & -\dfrac{\partial u_i(t,x)}{\partial t}-\mathcal{L}_t u_i(t,x)+\partial\varphi_i\big(u_i(t,x)\big)\ni F_i\big(t,x,u(t,x),(\nabla u_i(t,x))^*g(t,x)\big),\\ & \qquad\qquad\qquad\qquad t\in(0,T)\,,\ x\in D,\quad i\in\overline{1,m},\\ (b) & \dfrac{\partial u_i(t,x)}{\partial n}+\partial\psi_i\big(u_i(t,x)\big)\ni G_i\big(t,x,u(t,x)\big),\\ & \qquad\qquad\qquad\qquad t\in(0,T)\,,\ x\in Bd\left(\overline{D}\right),\quad i\in\overline{1,m},\\ (c) & u(T,x)=\kappa(x),\ \ x\in\overline{D},\end{array}\right.\tag{33}$$

where

$$\mathcal{L}_t u_i(t,x) = \frac{1}{2}\sum_{j,l=1}^{d} (gg^*)_{j,l}(t,x)\frac{\partial^2 u_i(t,x)}{\partial x_j \partial x_l} + \sum_{j=1}^{d} f_j(t,x)\frac{\partial u_i(t,x)}{\partial x_j}$$

Define $\Phi_i, \Gamma_i : [0,T]\times\overline{D}\times\mathbb{R}^m\times\mathbb{R}^d\times\mathbb{S}^d \to \mathbb{R}$, $i\in\overline{1,m}$, to be the functions:

$$\begin{aligned}\Phi_i(t,x,y,q,X) = &\ \frac{1}{2}\mathrm{Tr}\big((gg^*)(t,x)X\big) + \langle q, f(t,x)\rangle + F_i\big(t,x,y,q^*g(t,x)\big)\\ \Gamma_i(t,x,y,q) = &\ -\langle\nabla\phi(x),q\rangle + G_i(t,x,y).\end{aligned} \tag{34}$$

If $u=(u_1,\ldots,u_m)^* : [0,T]\times\overline{D}\to\mathbb{R}^m$, then for each $i\in\overline{1,m}$ we have

$$\Phi_i\left(t,x,u(t,x),\nabla u_i(t,x), D^2u_i(t,x)\right) = \mathcal{L}_t u_i(t,x) + F_i\big(t,x,u(t,x),(\nabla u_i(t,x))^* g(t,x)\big), \text{ and}$$

$$\Gamma_i(t,x,u(t,x),\nabla u_i(t,x)) = -\frac{\partial u_i(t,x)}{\partial n} + G_i(t,x,u(t,x)).$$

We put the notations $a\wedge b \overset{def}{=} \min\{a,b\}$ and $a\vee b \overset{def}{=} \max\{a,b\}$.
The following results hold.

Theorem 10 *(Pardoux, Zhang [15]: Theorem 4.3; Pardoux, Răşcanu [14] : Theorem 5.43) Consider the parabolic system (33) with $\varphi=\psi=0$. Then the continuous function $u:[0,T]\times\overline{D}\to\mathbb{R}^m$ defined by (22) is a viscosity solution of the parabolic partial differential system (33) i.e.*

$$u(T,x) = \kappa(x),\ \forall\, x\in\overline{D},$$

and u is a viscosity sub-solution that is, for any $i\in\overline{1,m}$:

$$\begin{aligned}&(a)\quad \text{for any } (t,x)\in(0,T)\times\overline{D}, \text{ any } (p,q,X)\in\mathcal{P}^{2,+}u_i(t,x):\\ &\qquad\qquad p+\Phi_i(t,x,u(t,x),q,X)\geq 0,\\ &(b)\quad \text{for any } (t,x)\in(0,T)\times Bd\left(\overline{D}\right), \text{ any } (p,q,X)\in\mathcal{P}^{2,+}u_i(t,x):\\ &\qquad\qquad [p+\Phi_i(t,x,u(t,x),q,X)]\vee\Gamma_i(t,x,u(t,x),q)\geq 0,\end{aligned}$$

together with u is a viscosity super-solution that is, for any $i\in\overline{1,m}$:

$$\begin{aligned}&(c)\quad \text{for any } (t,x)\in(0,T)\times\overline{D}, \text{ any } (p,q,X)\in\mathcal{P}^{2,-}u_i(t,x):\\ &\qquad\qquad p+\Phi_i(t,x,u(t,x),q,X)\leq 0,\\ &(d)\quad \text{for any } (t,x)\in(0,T)\times Bd\left(\overline{D}\right), \text{ any } (p,q,X)\in\mathcal{P}^{2,-}u_i(t,x):\\ &\qquad\qquad [p+\Phi_i(t,x,u(t,x),q,X)]\wedge\Gamma_i(t,x,u(t,x),q)\leq 0.\end{aligned}$$

Theorem 11 *(Maticiuc, Răşcanu [9]: Theorem 5; Pardoux, Răşcanu [14] : Theorem 5.81) The continuous function $u:[0,T]\times\overline{D}\to\mathbb{R}^m$ defined by (22) is a viscosity solution of the parabolic differential system (33) on $\overline{D}$ i.e.*

$$\left|\begin{aligned}&u(T,x)=\kappa(x),\ \forall\, x\in\overline{D},\\ &u(t,x)\in Dom(\varphi),\quad \forall (t,x)\in(0,T)\times\overline{D},\\ &u(t,x)\in Dom(\psi),\quad \forall (t,x)\in(0,T)\times Bd\left(\overline{D}\right),\end{aligned}\right.$$

and u is a viscosity sub-solution that is, for any $i \in \overline{1,m}$:

$$\begin{array}{ll}(a) & \textit{for any } (t,x) \in (0,T) \times \overline{D}, \textit{ any } (p,q,X) \in \mathcal{P}^{2,+}u_i(t,x) : \\ & \qquad p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \geq \left(\varphi_i\right)'_-\left(u_i\left(t,x\right)\right), \\ (b) & \textit{for any } (t,x) \in (0,T) \times Bd\left(\overline{D}\right), \textit{ any } (p,q,X) \in \mathcal{P}^{2,+}u_i(t,x) : \\ & \qquad p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \geq \left(\varphi_i\right)'_-\left(u_i\left(t,x\right)\right), \ \ \textit{or} \\ & \qquad \Gamma_i\left(t,x,u\left(t,x\right),q\right) \geq \left(\psi_i\right)'_-\left(u_i\left(t,x\right)\right)\end{array}$$

together with u is a viscosity super-solution that is, for any $i \in \overline{1,m}$:

$$\begin{array}{ll}(c) & \textit{for any } (t,x) \in (0,T) \times \overline{D}, \textit{ any } (p,q,X) \in \mathcal{P}^{2,-}u_i(t,x) : \\ & \qquad p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \leq \left(\varphi_i\right)'_+\left(u_i\left(t,x\right)\right), \\ (d) & \textit{for any } (t,x) \in (0,T) \times Bd\left(\overline{D}\right), \textit{ any } (p,q,X) \in \mathcal{P}^{2,-}u_i(t,x) : \\ & \qquad p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \leq \left(\varphi_i\right)'_+\left(u_i\left(t,x\right)\right), \ \ \textit{or} \\ & \qquad \Gamma_i\left(t,x,u\left(t,x\right),q\right) \leq \left(\psi_i\right)'_+\left(u_i\left(t,x\right)\right)\end{array}$$

Theorem 12 *(Pardoux, Răşcanu [13] : Theorem 4.1) Assume that $D = \mathbb{R}^d$ (the system (33) is on $\mathbb{R}^d$ without boundary condition and in (5) and (9) $A^{t,x} = 0$, $G = 0$, $\psi = 0$). Then the continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^m$ defined by (22) is a viscosity solution of the parabolic differential system (33-(a) & (c)) on $\mathbb{R}^d$ i.e.*

$$\left| \begin{array}{l} u(T,x) = \kappa\left(x\right), \ \forall\, x \in \mathbb{R}^d, \\ u(t,x) \in Dom\left(\varphi\right), \ \ \forall (t,x) \in (0,T) \times \mathbb{R}^d, \end{array} \right.$$

and for any $i \in \overline{1,m}$, any $(t,x) \in (0,T) \times \mathbb{R}^d$:

$$\begin{array}{ll} p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \geq \left(\varphi_i\right)'_-\left(u_i\left(t,x\right)\right), & \textit{for all } (p,q,X) \in \mathcal{P}^{2,+}u_i(t,x), \textit{ and} \\ p + \Phi_i\left(t,x,u\left(t,x\right),q,X\right) \leq \left(\varphi_i\right)'_+\left(u_i\left(t,x\right)\right), & \textit{for all } (p,q,X) \in \mathcal{P}^{2,-}u_i(t,x). \end{array}$$

We highlight that in [13] and [9] the results are given for $m = 1$, but with the same proof the results hold too for the quasi-decoupled system (33).

Consider now the parabolic multivalued system (1) with $D = \mathbb{R}^d$ and F independent of the last argument w that is $F\left(t,x,y,w\right) \equiv F\left(t,x,y\right) \in \mathbb{R}^m$ for all $(t,x,y,w) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m\times m}$:

$$\begin{cases} -\dfrac{\partial u(t,x)}{\partial t} - \mathcal{L}_t u\left(t,x\right) + \partial\varphi\big(u(t,x)\big) \ni F\big(t,x,u(t,x)\big), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in (0,T)\,, \ x \in \mathbb{R}^d, \\ u(T,x) = \kappa(x), \ \ x \in \mathbb{R}^d, \end{cases} \tag{35}$$

Let $z \in \mathbb{R}^m$ and $\Phi_z : [0,T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$

$$\Phi_z\left(t,x,y,q,X\right) = \frac{1}{2}\mathrm{Tr}\big((gg^*)(t,x)X\big) + \langle q, f(t,x)\rangle + \langle F\left(t,x,y\right), z\rangle$$

Theorem 13 *(Maticiuc, Pardoux, Rǎşcanu, Zalinescu [8]: Theorem 6, Theorem 14) The continuous function $u : [0,T] \times \overline{D} \to \mathbb{R}^m$ defined by (22) is a viscosity solution of the parabolic differential system (35) i.e.*

$$\left| \begin{array}{l} u(T,x) = \kappa(x), \ \forall\, x \in \mathbb{R}^d, \\ u(t,x) \in Dom(\varphi), \ \ \forall (t,x) \in (0,T) \times \mathbb{R}^d, \end{array} \right.$$

and

$$\begin{array}{c} \text{for any } (t,x) \in (0,T) \times \mathbb{R}^d, \text{ any } z \in \mathbb{R}^m, \text{ any } (p,q,X) \in \mathcal{P}^{2,+} \langle u(t,x), z \rangle : \\ p + \Phi_z(t,x,u(t,x),q,X) \geq \varphi'_-\big(u(t,x), z\big). \end{array} \tag{36}$$

We remark that

(r_1) the condition (36) is equivalent to:

$$\begin{array}{c} \text{for any } (t,x) \in (0,T) \times \mathbb{R}^d, \text{ any } z \in \mathbb{R}^m, \text{ any } (p,q,X) \in \mathcal{P}^{2,-} \langle u(t,x), z \rangle : \\ p + \Phi_z(t,x,u(t,x),q,X) \leq \varphi'_+\big(u(t,x), z\big). \end{array}$$

(r_2) in one dimensional case $(m = 1)$ condition (36) means the sub-solution for $z > 0$ and a super-solution for $z < 0$.

We highlight that in supplementary assumptions the uniqueness of the viscosity solutions holds too in each case presented here above in this subsection. Moreover the uniqueness of the viscosity solution of the parabolic variational inequality (35) holds in a larger class of functions u (a weaker inequality (36)).

5.2 Elliptic PDEs

Assume the hypotheses from Sections 1 and 2 are satisfied and moreover f, g, F and G are independent of time argument, $\kappa = 0$, $\varphi = \psi \equiv 0$, $u_0 = 0$ and F_i the $i-$th coordinate of F, depends only on the $i-$th row of the matrix Z.

If $h : \overline{D} \to \mathbb{R}$ is a continuous function, then a pair $(q,X) \in \mathbb{R}^d \times \mathbb{S}^d$ is a elliptic super-jet to h, at $x \in \overline{D}$, if for all $x' \in \overline{D}$,

$$h(x') \leq h(x) + \langle q, x' - x \rangle + \frac{1}{2} \langle X(x'-x), x'-x \rangle + o(|x'-x|^2);$$

The set of elliptic super-jets at x is denoted by $\mathcal{P}^{2,+}h(x)$; the set of elliptic sub-jets is defined by $\mathcal{P}^{2,-}_{\mathcal{O}} h = -\mathcal{P}^{2,+}_{\mathcal{O}}(-h)$.

Consider the semi-linear elliptic partial differential system with nonlinear Robin boundary condition:

$$\left\{ \begin{array}{l} -\mathcal{L}u_i(x) = F_i(x, u(x), (\nabla u_i(x))^* g(x)), \quad x \in D, \quad i \in \overline{1,m}, \\ \dfrac{\partial u_i}{\partial n}(x) = G_i(x, u(x)), \ \ x \in Bd\left(\overline{D}\right), \quad i \in \overline{1,m}. \end{array} \right. \tag{37}$$

where

$$\mathcal{L}u_i(x) = \frac{1}{2}\sum_{j,l=1}^{d}(gg^*)_{j,l}(t,x)\frac{\partial^2 u_i(x)}{\partial x_j \partial x_l} + \sum_{j=1}^{d} f_j(t,x)\frac{\partial u_i(x)}{\partial x_j}.$$

Define Φ_i and Γ_i as in (34).

Proposition 14 *(E. Pardoux, S. Zhang [15]: Theorem 5.3) The continuous function $x \longmapsto u(x) : \overline{D} \to \mathbb{R}^m$ given by (30) is a viscosity solution of the elliptic partial differential system (37) i.e.: and u is a viscosity sub-solution that is, for any $i \in \overline{1,m}$:*

$$\begin{aligned}
&(a) \qquad \Phi_i(x,u(x),q,X) \geq 0, \ \ \textit{for any } x \in \overline{D}, \textit{any } (q,X) \in \mathcal{P}^{2,+}u_i(x),\\
&(b) \qquad \Phi_i(x,u(x),q,X) \vee \Gamma_i(x,u(x),q) \geq 0\\
&\qquad\qquad\qquad\qquad \textit{for any } x \in Bd\left(\overline{D}\right), \textit{any } (q,X) \in \mathcal{P}^{2,+}u_i(x),
\end{aligned}$$

together with u is a viscosity super-solution that is, for any $i \in \overline{1,m}$:

$$\begin{aligned}
&(c) \qquad \Phi_i(x,u(x),q,X) \leq 0, \ \ \textit{for any } x \in \overline{D}, \textit{any } (q,X) \in \mathcal{P}^{2,-}u_i(x),\\
&(d) \qquad \Phi_i(x,u(x),q,X) \wedge \Gamma_i(x,u(x),q) \leq 0\\
&\qquad\qquad\qquad\qquad \textit{for any } x \in Bd\left(\overline{D}\right), \textit{any } (q,X) \in \mathcal{P}^{2,-}u_i(x),
\end{aligned}$$

6 Annex

6.1 Convex functions

Let $\varphi : \mathbb{R}^m \to]-\infty,+\infty]$ be a proper convex lower semicontinuous function. We denote $\mathrm{Dom}(\varphi) = \{y \in \mathbb{R}^m : \varphi(y) < \infty\}$; φ is a proper function if $\mathrm{Dom}(\varphi) \neq \emptyset$.

The subdifferential (multivalued) operator $\partial\varphi$ is defined by

$$\partial\varphi(y) := \{\hat{y} \in \mathbb{R}^m : \langle \hat{y}, v-y\rangle + \varphi(y) \leq \varphi(v), \ \forall\, v \in \mathbb{R}^m\};$$

$\partial\varphi : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a maximal monotone operator. We have

$$\mathrm{Dom}(\partial\varphi) \overset{def}{=} \{y \in \mathbb{R}^m : \partial\varphi(y) \neq \emptyset\} \subset \mathrm{Dom}(\varphi).$$

Recall that $\overline{\mathrm{Dom}(\partial\varphi)} = \overline{\mathrm{Dom}(\varphi)}$ and $int(\mathrm{Dom}(\partial\varphi)) = int(\mathrm{Dom}(\varphi))$.

For all $y \in \mathrm{Dom}(\varphi)$ and $z \in \mathbb{R}^m$ we have

$$\varphi'_-(y,z) \overset{def}{=} \lim_{t \nearrow 0} \uparrow \frac{\varphi(y+tz)-\varphi(y)}{t} \leq \lim_{t \searrow 0} \downarrow \frac{\varphi(y+tz)-\varphi(y)}{t} \overset{def}{=} \varphi'_+(y,z).$$

$\varphi'_-(y,z) = -\varphi'_+(y,-z)$. Moreover

$$\begin{aligned}
\hat{y} \in \partial\varphi(y) \quad &\Longleftrightarrow \quad \langle \hat{y}, z\rangle \geq \varphi'_-(y,z), \ \forall\, z \in \mathbb{R}^m,\\
&\Longleftrightarrow \quad \langle \hat{y}, z\rangle \leq \varphi'_+(y,z), \ \forall\, z \in \mathbb{R}^m.
\end{aligned}$$

If $m = 1$ we write $\varphi'_-(y) = \varphi'_-(y,1)$, $\varphi'_+(y) = \varphi'_+(y,1)$ and we have

$$\partial\varphi(y) = \left[\varphi'_-(y), \varphi'_+(y)\right] \cap \mathbb{R}.$$

Let $\varepsilon > 0$. The Moreau–Yosida regularization of φ is the function $\varphi_\varepsilon : \mathbb{R}^m \to \mathbb{R}$

$$\varphi_\varepsilon(y) \stackrel{def}{=} \inf\left\{\frac{1}{2\varepsilon}|y-z|^2 + \varphi(z) : z \in \mathbb{R}^m\right\}.$$

We mention that φ_ε is a C^1 convex function and (see e.g. Pardoux & Răşcanu [14], Annex B) for all $x, y \in \mathbb{R}^m$

$$\begin{array}{ll} (a) & \varphi_\varepsilon(x) = \dfrac{\varepsilon}{2}|\nabla\varphi_\varepsilon(x)|^2 + \varphi(x - \varepsilon\nabla\varphi_\varepsilon(x)), \\ (b) & \nabla\varphi_\varepsilon(x) = \partial\varphi_\varepsilon(x) \in \partial\varphi(x - \varepsilon\nabla\varphi_\varepsilon(x)), \\ (c) & |\nabla\varphi_\varepsilon(x) - \nabla\varphi_\varepsilon(y)| \le \dfrac{1}{\varepsilon}|x-y|. \end{array} \tag{38}$$

6.2 A backward stochastic inequality

From Proposition 6.80 (Annex C) in Pardoux & Răşcanu [14] we have

Lemma 15 *Let $(Y, Z) \in S_m^0 \times \Lambda_{m\times k}^0$ satisfying*

$$Y_t = Y_T + \int_t^T d\mathcal{K}_r - \int_t^T Z_r dB_r,\ 0 \le t \le T, \quad \mathbb{P}-a.s.,$$

where $\mathcal{K} \in S_m^0$ and $\mathcal{K}_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$, $\mathbb{P}-a.s.\ \omega \in \Omega$.
Assume be given

- ▲ *L is a non-decreasing stochastic process, $L_0 = 0$,*
- ▲ *R is a stochastic process, $R_0 = 0$ and $R_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$, $\mathbb{P}-a.s.\ \omega \in \Omega$,*
- ▲ *V a continuous stochastic process, $V_0 = 0$, $V_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$, $\mathbb{P}-a.s.\ \omega \in \Omega$, and*

$$\mathbb{E}\left(\int_0^T e^{2V_r} dR_r\right)^- < \infty$$

If $a < 1$ and

$$\begin{array}{ll} (i) & \langle Y_r, d\mathcal{K}_r\rangle \le \dfrac{a}{2}|Z_r|^2 dr + \left(|Y_r|^2 dV_r + |Y_r| dL_r + dR_r\right) \\ & \qquad\qquad\qquad\qquad \text{as measures on } [0,T], \\ (ii) & \mathbb{E}\sup\limits_{r\in[\tau,\sigma]} e^{2V_r}|Y_r|^2 < \infty, \end{array} \tag{39}$$

then there exists a positive constant C_a, depending only a, such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{r\in[0,T]}\left|e^{V_r}Y_r\right|^2\right) + \mathbb{E}\left(\int_0^T e^{2V_r}|Z_r|^2 dr\right) \\ &\le C_a\,\mathbb{E}\left[\left|e^{V_T}Y_T\right|^2 + \left(\int_0^T e^{V_r} dL_s\right)^2 + \int_0^T e^{2V_r} dR_r\right]. \end{aligned} \tag{40}$$

We remark that the proof of Lemma 15 follows the proof of Proposition 6.80 [14], with a single small change : in the definition of the localization stopping time, we delete the term containing R, and therefore we do not need to restrict us to the case where R is non-decreasing.

7 Erratum

In this paper we have corrected the proofs of continuity of the function $(t,x) \mapsto u(t,x) = Y_t^{t,x}$ from the papers [9] (Proposition 13 and Corollary 14) and [15] (Proposition 4.1 and Theorem 5.1).

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