Poisson process

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The standard Poisson process

- Let $\lambda > 0$ be given. A rate λ Poisson (counting) process is defined as

$$P_t = \sup\{k \geq 1,\ T_k \leq t\},$$

where $0 = T_0 < T_1 < T_2 < \cdots < T_k < \cdots < \infty$, the r.v.'s $\{T_k - T_{k-1},\ k \geq 1\}$ being independent and identically distributed, each following the law $\text{Exp}(\lambda)$.

- We have

Proposition

For all $n \geq 1$, $0 < t_1 < t_2 < \cdots < t_n$, the r.v.'s $P_{t_1}, P_{t_2} - P_{t_1}, \ldots, P_{t_n} - P_{t_{n-1}}$ are independent, and for all $1 \leq k \leq n$, $P_{t_k} - P_{t_{k-1}} \sim Poi[\lambda(t_k - t_{k-1})]$.

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Elementary convergence in law

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Lemma

For all $n \geq 1$, let U_n be a $B(n, p_n)$ random variable. If $np_n \to \lambda$ as $n \to \infty$, with $\lambda > 0$, then U_n converges in law towards $Poi(\lambda)$.

- To do :

Exercise

Let $\{P_t,\ t \geq 0\}$ be a rate λ Poisson process, and $\{T_k,\ k \geq 1\}$ the random points of this Poisson process, such that for all $t > 0$, $P_t = \sup\{k \geq 1,\ T_k \leq t\}$. Let $0 < p < 1$. Suppose that each T_k is selected with probability p, not selected with probability $1 - p$, independently of the others. Let P'_t denote the number of selected points on the interval $[0, t]$. Then $\{P'_t,\ t \geq 0\}$ is a rate λp Poisson process.

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Rate λ and rate 1 Poisson processes

- A Poisson process will be called standard if its rate is 1. If P is a standard Poisson process, then $\{P(\lambda t),\ t \geq 0\}$ is a rate λ Poisson process.
- A rate λ Poisson process ($\lambda > 0$) is a counting process $\{Q_t,\ t \geq 0\}$ such that $Q_t - \lambda t$ is a martingale.
- Let $\{P(t),\ t \geq 0\}$ be a standard Poisson process (i.e. with rate 1). Then $P(\lambda t) - \lambda t$ is martingale, and it is not hard to show that $\{P(\lambda t),\ t \geq 0\}$ is a rate λ Poisson process.

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Varying rate

- Let now $\{\lambda(t),\ t \geq 0\}$ be a measurable and locally integrable $\mathbb{R}_+$–valued function. Then the process $\{Q_t := P\left(\int_0^t \lambda(s)ds\right),\ t \geq 0\}$ is called a rate $\lambda(t)$ Poisson process. Clearly $M_t = Q_t - \int_0^t \lambda(s)ds$ is a martingale, i.e. M_t is $\mathcal{F}_t = \sigma\{Q_s,\ 0 \leq s \leq t\}$–measurable, and for $0 \leq s < t$, $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$.
- We now want to consider the case where λ is random. For that purpose, it is convenient to give an alternative definition of the above process Q_t.

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Poisson Random Measures

- Consider a standard Poisson random measure N on $\mathbb{R}_2^+$, which is defined as follows. N is the counting process associated to a random cloud of points in $\mathbb{R}_+^2$. One way to construct that cloud of points is as follows. We can consider $\mathbb{R}_+^2 = \cup_{i=1}^{\infty} A_i$, where the A_i's are disjoints squares with Lebesgue measure 1.
- Let K_i, $i \geq 1$ be i.i.d. Poisson r.v.'s with mean one. Let $\{X_j^i,\ j \geq 1, i \geq 1\}$ be independent random points of $\mathbb{R}_+^2$, which are such that for any $i \geq 1$, the X_j^i's are uniformly distributed in A_i.
- Then

$$N(dx) = \sum_{i=1}^{\infty} \sum_{j=1}^{K_i} \delta_{X_j^i}(dx).$$

$\lambda(t)$ denoting a positive valued measurable function, the above $\{Q_t,\ t \geq 0\}$ has the same law as

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- Let now $\{\lambda(t),\ t \geq 0\}$ be an $\mathbb{R}_+$–valued stochastic process, which is assumed to be predictable, in the following sense.
- Let for $t \geq 0$ $\mathcal{F}_t = \sigma\{N(A),\ A$ Borel subset of $[0,t] \times \mathbb{R}_+\}$, and consider the σ–algebra of subset of $[0,\infty) \times \Omega$ generated by the subsets of the form $\mathbf{1}_{(s,t]}\mathbf{1}_F$, where $0 \leq s < t$ and $F \in \mathcal{F}_s$, which is called the predictable σ–algebra.
- We assume moreover that $\mathbb{E}\int_0^t \lambda(s)ds < \infty$ for all $t > 0$. We now define as above

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Lemma

$Q_t - \int_0^t \lambda(s)ds$ is a martingale.

- Indication of proof :
  For any $\delta > 0$, let

$$Q_t^\delta = \int_0^t \int_0^{\lambda(s-\delta)} N(ds, du),$$

  where $\lambda(s) = 0$ for $s < 0$. It is not hard to show that $Q_t^\delta - \int_0^t \lambda(s-\delta)ds$ is a martingale which converges in $L^1(\Omega)$ to $Q_t - \int_0^t \lambda(s)ds$. The result follows.
- If we let $\sigma(t) = \inf\{r > 0,\ \int_0^r \lambda(s)ds > t\}$, we have that $P(t) := Q_{\sigma(t)}$ is a standard Poisson process, and it is plain that $Q_t = P\left(\int_0^t \lambda(s)ds\right)$.

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