Branching processes

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Bienaymé–Galton–Watson processes

- Consider an ancestor (at generation 0) who has X_0 children, such that

$$\mathbb{P}(X_0 = k) = q_k, \ k \geq 0 \quad \text{et} \sum_{k \geq 0} q_k = 1.$$

- Each child of the ancestor belongs to generation 1. The i–th of those children has himself $X_{1,i}$ children, where the r.v.'s $\{X_{k,i}, \ k \geq 0, i \geq 1\}$ are i.i.d., all having the same law as X_0.
- If we define Z_n as the number of individuals in generation n, we have

$$Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}.$$

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- Let us compute the generating function of Z_n : $g_n(s) = \mathbb{E}[s^{Z_n}]$.
- We obtain
$$g_n(s) = g \circ \cdots \circ g(s).$$
- Now
$$\begin{aligned}\mathbb{P}(Z_n = 0) &= g^{\circ n}(0)\\ &= g\left[g^{\circ(n-1)}(0)\right].\end{aligned}$$
- Hence if $z_n = \mathbb{P}(Z_n = 0)$, $z_n = g(z_{n-1})$, and $z_1 = q_0$. We have $z_n \uparrow z_\infty$, where $z_\infty = \mathbb{P}(Z_n = 0 \text{ from some } n \text{ on})$.

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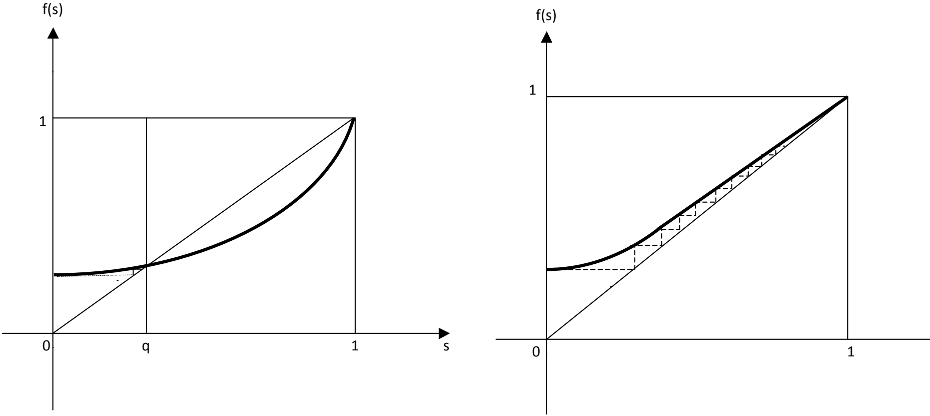

Figure: Graphs of g in case $m > 1$ (left) and in case $m \leq 1$ (right).

Extinction and non–extinction

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Proposition

*If $m \leq 1$, then $\mathbb{P}(Z_n = 0) \to 1$ as $n \to \infty$, and $z_\infty = 1$.
If $m > 1$, $\mathbb{P}(Z_n = 0) \to z_\infty$ as $n \to \infty$, where z_∞ is the smallest solution of the equation $z = g(z)$.*

- $W_n = m^{-n} \, Z_n$ is a martingale.

$$\mathbb{E}(W_{n+1}|Z_n) = m^{-n}\mathbb{E}\left(m^{-1}\sum_1^{Z_n} X_{n,i}|Z_n\right)$$
$$= m^{-n}Z_n$$
$$= W_n.$$

- One can show that $W_n \to W$ a.s. as $n \to \infty$, and moreover

$$\{W > 0\} = \{\text{the branching process does not go extinct}\}.$$

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Extinction and non–extinction, contin'd

If $E = \cup_{n\geq 1}\{Z_n = 0\}$, we have just shown that on E^c, $Z_n \to +\infty$, in fact at exponential speed.