LLN and CLT for the final size of the epidemic

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Final size 1

- Define, for $0 \leq w \leq N+1$,

$$\mathcal{J}(w) = \frac{cp}{N} \sum_{i=0}^{[w]-1} \Delta T_{(i)}.$$

Recall that $i = 0$ is the index of the initially infected individual, $\Delta T_{(i)}$ is the infectious period of individual whose resistance level is $Q_{(i)}$.

- $\mathcal{J}(w)$ is the infection pressure produced by the first $[w]$ infected individuals (including number 0). For any integer k, $\mathcal{J}$ is of course constant on the interval $[k, k+1)$.
- Define for $v > 0$,

$$\mathcal{Q}(v) = \sum_{i=1}^{N} \mathbf{1}_{\{Q_i \leq v\}}.$$

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Final size 2

- The total number of infected individuals in the epidemic is

$$\begin{aligned} Y &= \min\left\{k \ge 0;\ Q_{(k+1)} > \frac{pc}{N}\sum_{i=0}^{k}\Delta T_i\right\} \\ &= \min\left\{k \ge 0;\ Q_{(k+1)} > \mathcal{J}(k+1)\right\} \\ &= \min\left\{w \ge 0;\ \mathcal{Q}(\mathcal{J}(w+1)) = w\right\}. \end{aligned}$$

- Suppose indeed that $Y = i$. Then

$$\begin{aligned} \mathcal{J}(j) > Q_{(j)}, &\quad \text{hence } \mathcal{Q}(\mathcal{J}(j)) \ge j, \quad \forall j \le i, \\ \text{and } \mathcal{J}(i+1) < Q_{(i+1)} &\quad \text{hence } \mathcal{Q}(\mathcal{J}(i+1)) < i+1. \end{aligned}$$

- In other words $Y = i$ iff i is the smallest integer such that

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- We now index $\mathcal{J}$ and $\mathcal{Q}$ by N, so that they become $\mathcal{J}_N$ and $\mathcal{Q}_N$. We now define

$$\overline{\mathcal{J}}_N(w) = \mathcal{J}_N(Nw)$$
$$\overline{\mathcal{Q}}_N(v) = \frac{\mathcal{Q}_N(v)}{N}.$$

- As $N \to \infty$,

$$\overline{\mathcal{J}}_N(w) \to cp\mathbb{E}(\Delta T)w = R_0 w, \quad \text{and}$$
$$\overline{\mathcal{Q}}_N(v) \to 1 - e^{-v} \quad \text{a.s.}$$

- Hence

$$\overline{\mathcal{Q}}_N \circ \overline{J}_N(w) \to 1 - e^{-R_0 w}.$$

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- We have

$$\frac{Y_N}{N} = \min\left\{\frac{w}{N} \ge 0;\ \mathcal{Q}_N(\mathcal{J}_N(w+1)) = w\right\}$$
$$= \min\left\{s \ge 0;\ \frac{1}{N}\mathcal{Q}_N\left(\mathcal{J}_N\left(N\left(s+\frac{1}{N}\right)\right)\right) = s\right\}$$
$$= \min\left\{s \ge 0;\ \overline{\mathcal{Q}}_N\left(\overline{\mathcal{J}}_N\left(s+\frac{1}{N}\right)\right) = s\right\}.$$

- Then Y_N/N converges a.s. towards the smallest positive solution of equation

$$1 - e^{-R_0 x} = x.$$

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  - If $R_0 \le 1$, the unique solution of this equation is $x = 0$.
  - If $R_0 > 1$, there is another solution $x > 0$.

  This solution $0 < x < 1$ is the size (measured as the proportion of the total population) of a “significant ” epidemic, if it goes off.

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CLT 1

- From the classical CLT, as $N \to \infty$,

$$\sqrt{N}(\overline{\mathcal{J}}_N(w) - R_0 w) = \frac{pc\sqrt{w}}{\sqrt{Nw}} \sum_{i=1}^{[Nw]} [\Delta T_i - \mathbb{E}(\Delta T_i)]$$
$$\Rightarrow A(w),$$

where $A(w) \sim \mathcal{N}(0, p^2 c^2 \text{Var}(\Delta T) w)$.

- One can in fact show that, as processes

$$\{\sqrt{N}(\overline{\mathcal{J}}_N(w) - R_0 w),\ 0 \le w \le 1\} \Rightarrow \{A(w),\ 0 \le w \le 1\},$$

where $\{A(w),\ 0 \le w \le 1\}$ is a Brownian motion, with $\text{Var}(A(w)) = r^2 R_0^2 w$, where $r^2 = (\mathbb{E}\Delta T)^{-2} \text{Var}(\Delta T)$.

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CLT 2

- Consider now $\overline{\mathcal{Q}}_N$.

$$\begin{aligned} B_N(v) &= \sqrt{N}(\overline{\mathcal{Q}}_N(v) - [1 - e^{-v}]) \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[\mathbf{1}_{\{Q_i \le v\}} - (1 - e^{-v}) \right] \\ &\Rightarrow B(v), \end{aligned}$$

where $B(v) \sim \mathcal{N}(0, e^{-v}(1 - e^{-v}))$.

- according to the Kolmogorov–Smirnov theorem, towards a time changed Brownian bridge. In simpler words, $\{B(v),\ v \ge 0\}$ is a centered Gaussian process with continuous trajectories whose covariance is specified by the identity $\mathbb{E}[B(u)B(v)] = e^{-u \vee v} - e^{-(u+v)}$.

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CLT 3

- Making use of the two above CLTs, we get

$$\begin{aligned} s &= 1 - e^{-\overline{\mathcal{J}}_N(s+N^{-1})} + N^{-1/2} B_N(\overline{\mathcal{J}}_N(s+N^{-1})) \\ &= 1 - \exp\left(-R_0(s+N^{-1}) + N^{-1/2} A_N(s+N^{-1})\right) \\ &\quad + N^{-1/2} B_N\left(R_0(s+N^{-1}) + N^{-1/2} A_N(s+N^{-1})\right). \end{aligned}$$

- Let $s = s^* + s_N N^{-1/2} + \circ(N^{-1/2})$, where s^* satisfies $e^{-R_0 s^*} = 1 - s^*$.
- We obtain

$$\begin{aligned} & s^* + s_N N^{-1/2} + \circ(N^{-1/2}) \\ &= 1 - \exp\left(-R_0 s^* - R_0 s_N N^{-1/2} - A_N(s^*) N^{-1/2} + \circ(N^{-1/2}\right) \\ &\quad + N^{-1/2} B_N(R_0 s^*) + \circ(N^{-1/2}) \\ &= 1 - e^{-R_0 s^*} + N^{-1/2} e^{-R_0 s^*}\left(R_0 s_N + A_N(s^*)\right) \\ &\quad + N^{-1/2} B_N(R_0 s^*) + \circ(N^{-1/2}). \end{aligned}$$

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CLT 4

- We simplfy the relation making use of $e^{-R_0 s^*} = 1 - s^*$, and then multiply by $N^{1/2}$; We get
- $$[1-(1-s^*)R_0]s_N = B_N(R_0 s^*) + (1-s^*)A_N(s^*).$$
- Hence $s_N \Rightarrow \Xi$, where
$$\Xi \sim \mathcal{N}\left(0, \frac{s^*(1-s^*)}{(1-(1-s^*)R_0)^2}\left(1+r^2(1-s^*)R_0^2\right)\right).$$
- Finally Y_N follows asymptotically the distribution
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