Martingales

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Martingales in discrete time

- We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an increasing sequence $\{\mathcal{F}_n, \; n \geq 0\}$ of sub–σ–algebras of $\mathcal{F}$. We have

Definition

A sequence $\{X_n, \; n \geq 0\}$ of r.v.'s is a called a martingale if

1. For all $n \geq 0$, X_n is $\mathcal{F}_n$–measurable and integrable,
2. For all $n \geq 0$, $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a. s.

A sub–martingale is a sequence which satisfies the first condition and $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$. A super–martingale is a sequence which satisfies the first condition and $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$.

- We deduce from Jensen's inequality for conditional expectation

Proposition

If $\{X_n, \; n \geq 0\}$ is a martingale, $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function such that $\varphi(X_n)$ is integrable for all $n \geq 0$, then $\{\varphi(X_n), \; n \geq 0\}$ is a sub–martingale.

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- We need the notion

Definition

A stopping time τ is an $\mathbb{N} \cup \{+\infty\}$–valued r.v. which satisfies $\{\tau = n\} \in \mathcal{F}_n$, for all $n \geq 0$. Moreover

$$\mathcal{F}_T = \{B \in \mathcal{F},\ B \cap \{T = n\} \in \mathcal{F}_n,\ \forall n\}.$$

- We have Doob's optional sampling theorem :

Theorem

If $\{X_n,\ n \geq 0\}$ is a martingale (resp. a sub–martingale), and τ_1, τ_2 two stopping times s.t. $\tau_1 \leq \tau_2 \leq N$ a. s., then X_{τ_i} is $\mathcal{F}_{\tau_i}$ measurable and integrable, $i = 1, 2$ and moreover

$$\mathbb{E}(X_{\tau_2}|\mathcal{F}_{\tau_1}) = X_{\tau_1}$$
$$(resp.\ \ \mathbb{E}(X_{\tau_2}|\mathcal{F}_{\tau_1}) \geq X_{\tau_1}).$$

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- Let $A \in \mathcal{F}_{\tau_1}$.

$$A \cap \{\tau_1 < k \le \tau_2\} = A \cap \{\tau_1 \le k-1\} \cap \{\tau_2 \le k-1\}^c \in \mathcal{F}_{k-1}.$$

- Let $\Delta_k = X_k - X_{k-1}$. We have

$$\begin{aligned}\int_A (X_{\tau_2} - X_{\tau_1}) d\mathbb{P} &= \int_A \sum_{k=1}^n \mathbf{1}_{\{\tau_1 < k \le \tau_2\}} \Delta_k d\mathbb{P} \\ &= \sum_{k=1}^n \int_{A \cap \{\tau_1 < k \le \tau_2\}} \Delta_k d\mathbb{P} \\ &= 0\end{aligned}$$

or ≥ 0 in case $\{X_n,\ n \ge 0\}$ is a sub–martingale.

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If $X_1, \ldots, X_n$ is a sub–martingale, then for all $\alpha > 0$,

$$\mathbb{P}(\max_{1 \le i \le n} X_i \ge \alpha) \le \frac{1}{\alpha}\mathbb{E}(X_n^+).$$

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- Let $\tau = \inf\{0 \le k \le n,\ X_k \ge \alpha\}$, $M_k = \max_{1\le i\le k} X_i$.

$$\{M_n \ge \alpha\} \cap \{\tau \le k\} = \{M_k \ge \alpha\} \in \mathcal{F}_k.$$

- Hence $\{M_n \ge \alpha\} \in \mathcal{F}_\tau$, and from Doob's optional sampling theorem,

$$\begin{aligned}\alpha\mathbb{P}(M_n \ge \alpha) &\le \int_{\{M_n\ge\alpha\}} X_\tau d\mathbb{P}\\ &\le \int_{\{M_n\ge\alpha\}} X_n d\mathbb{P}\\ &\le \int_{\{M_n\ge\alpha\}} X_n^+ d\mathbb{P}\\ &\le \mathbb{E}(X_n^+).\end{aligned}$$

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Proposition

If $M_1, \ldots, M_n$ is a martingale, then

$$\mathbb{E}\left[\sup_{0\le k\le n} |M_k|^2\right] \le 4\mathbb{E}\left[|M_n|^2\right].$$

- Let $X_k = |M_k|$. $X_1, \ldots, X_n$ is a sub–martingale. It follows from the proof of the above inequality that, with the notation $X_k^* = \sup_{0\le k\le n} X_k$,

$$\mathbb{P}(X_n^* > \lambda) \le \frac{1}{\lambda}\mathbb{E}\left(X_n \mathbf{1}_{X_n^* > \lambda}\right).$$

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Consequently

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\begin{aligned}
\int_0^\infty \lambda \mathbb{P}(X_n^* > \lambda) d\lambda &\le \int_0^\infty \mathbb{E}\left(X_n \mathbf{1}_{X_n^* > \lambda}\right) d\lambda \\
\mathbb{E}\left(\int_0^{X_n^*} \lambda d\lambda\right) &\le \mathbb{E}\left(X_n \int_0^{X_n^*} d\lambda\right) \\
\frac{1}{2}\mathbb{E}\left[|X_n^*|^2\right] &\le \mathbb{E}(X_n X_n^*) \\
&\le \sqrt{E(|X_n|^2)}\sqrt{E(|X_n^*|^2)},
\end{aligned}
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Continuous time martingales

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- Suppose $\{M_t,\ t \geq 0\}$ is a right–continuous martingale. For any $n \geq 1$, $0 = t_0 < t_1 < \cdots < t_n$, $(M_{t_0}, M_{t_1}, \ldots, M_{t_n})$ is a discrete time martingale.

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