# Large Deviations for Epidemiological Models

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## Introduction

We consider the vector of proportions in our model as

$$Z^N(t) = z_0 + \frac{1}{N}\sum_{j=1}^{k} h_j P_j\left(\int_0^t N\beta_j(Z^N(s))ds\right). \qquad (1)$$

Again, the $P_j$'s are mutually independent standard Poisson processes. The process $Z^N(t)$ lives in the set

$$A = \{z \in \mathbb{R}_+^d;\ \sum_{i=1}^d z_i \le 1\}.$$

We shall denote by $D_{T,A}$ the set of functions defined on $[0,T]$ with values in $A$ which are right continuous with left limits at every $t$, and $\mathcal{AC}_{T,A}$ will denote the subset of absolutely continuous functions. For $\phi, \psi \in D_{T,A}$, we define $\|\phi - \psi\|_T = \sup_{0 \le t \le T} |\phi_t - \psi_t|$. Let $\mathbb{P}^N$ denote the law of $Z^N$, i.e.

$$\mathbb{P}^N(B) = \mathbb{P}(Z^N \in B),\ \forall B \in \mathcal{B},$$

where $\mathcal{B}$ denote the Borel $\sigma$–field of $D_{T,A}$.

We want to show that the collection of probability measures $\{\mathbb{P}^N,\ N \ge 1\}$ satisfies a Large Deviations Principle, in the sense that there exists a rate function $I_T$ (to be defined below) such that

$$-\inf_{\phi \in G} I_T(\phi) \le \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}(Z^N \in G),\ \text{if } G \subset D_{T,A} \text{ is open},$$
$$-\inf_{\phi \in F} I_T(\phi) \ge \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}(Z^N \in F),\ \text{if } F \subset D_{T,A} \text{ is closed}.$$

The main difficulty in proving such a large deviations principle comes from the fact that some of the rates $\beta_j$ may vanish at the boundary of the set $A$. To each $a > 0$ (small enough) we associate the sets

$$B^a = \{z \in A,\ z^i \ge a, 1 \le i \le d,\ \text{and } 1 - \sum_{i=1}^d z^i \ge a\},$$
$$R^a = \{\phi \in \mathcal{AC}_{T,A},\ \phi_t \in B^a,\ 0 \le t \le T\}.$$

We suppose that there exists a collection of mappings $\Phi_a : A \to A$, defined for each $a > 0$, which are such that $z^a = \Phi_a(z)$ satisfies for each $a > 0$

$$|z - z^a| \le \lambda_3 a$$
$$d(z^a, \partial A) \ge \rho a := a',$$

for some $0 < \rho < \lambda_3$. Hence $\Phi_a$ maps $A$ into $B^{a'}$.

**Remark 1.** *Since $A$ is convex, we define $\Phi_a(z) = z + a(z_0 - z)$, for some fixed $z_0 \in \mathring{A}$. The same definition is even possible for many non necessarily convex sets, provided $A$ is compact, and there is a point $z_0$ in its interior which is such that each segment joining $z_0$ and any point $z \in \partial A$ does not touch any other point of the boundary $\partial A$.*

We shall assume everywhere below that $\Phi_a(z) = z + a(z_0 - z)$ and

$$\nabla \Phi_a = (1 - a)I.$$

Let for any $a > 0$

$$C_a = \inf_{1 \le j \le k} \inf_{z \in B^a} \beta_j(z).$$

It is plain that $C_a > 0$ for $a > 0$, and $C_a \to 0$, as $a \to 0$. We shall assume

**Assumptions A**

A1 The rate functions $\beta_j$ are Lipschitz continuous with the Lipschitz constant equal to $C$, and bounded by a constant $\theta$.

A2 There exist two constants $\lambda_1, \lambda_2 > 0$ such that whenever $z \in A$ is such that $\beta_j(z) < \lambda_1$, $\beta_j(z^a) > \beta_j(z)$ for all $0 < a < \lambda_2$.

A3 There exists $\nu \in (0, 1/2)$ such that $\lim_{a \to 0} a^\nu \log C_a = 0$.

# 1 Law of Large Numbers and Girsanov theorem

We reformulate the Law of Large Numbers in the above notations

Theorem **2.** *Let $Z^N$ be given the solution of* (1)*. If the assumption 1 is satisfied, then for all $T > 0$,*

$$\|Z^N - Y\|_T \to 0 \text{ a.s. as } N \to \infty,$$

*where $Y_t$ is the unique solution of the ODE*

$$Y(t) = z_0 + \int_0^t b(Y(s))ds, \tag{1}$$

*with $b(z) = \sum_{j=1}^k \beta_j(z) h_j$.*

We shall need the following Girsanov theorem. Let $Q$ denote the number of jumps of the of $Z^N$ in the interval $[0,T]$, $\tau_p$ be the time of the $p$–th jump, and define

$$\delta_p(j) = \begin{cases} 1 & \text{, if the } p\text{–th jump is in the direction } h_j, \\ 0 & \text{, otherwise.} \end{cases}$$

We shall denote $\mathcal{F}_t^N = \sigma\{Z_s^N,\ 0 \le s \le t\}$. Consider another set of rates $\widetilde{\beta}_j(z)$, $1 \le j \le k$.

Theorem **3.** *Assume that* $\{x,\ \widetilde{\beta}\}_j(x) = 0\} \subset \{s,\ \beta_j(x) = 0\}$. *Let* $\widetilde{\mathbb{P}}^N$ *denote the law of* $Z^N$ *when the rates are* $\widetilde{\beta}_j$. *Then on the* $\sigma$ *algebra* $\mathcal{F}_t^N$, $\mathbb{P}^N|_{\mathcal{F}_T^N} << \widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}$, *and*

$$\begin{aligned} \Delta_T^N &= \frac{\mathbb{P}^N|_{\mathcal{F}_T^N}}{\widetilde{\mathbb{P}}^N|_{\mathcal{F}_T^N}} \\ &= \left( \prod_{p=1}^{Q} \prod_{j=1}^{k} \left[ \frac{\beta_j(Z^N(\tau_p^-))}{\widetilde{\beta}_j(Z^N(\tau_p^-))} \right]^{\delta_p(j)} \right) \exp\left( N \sum_{j=1}^{k} \int_0^T [\tilde{\beta}_j(Z^N(t)) - \beta_j(Z^N(t))] dt \right). \end{aligned}$$

## 2 The rate function

For any $\phi \in \mathcal{AC}_{T,A}$, let $\mathcal{A}_d(\phi)$ the set of vector valued Borel measurable functions $\mu$ such that for all $1 \le j \le k$, $\mu_t^j \ge 0$ and

$$\frac{d\phi_t}{dt} = \sum_{j=1}^{k} \mu_j^j h_j,\ t \text{ a.e.}$$

We define the rate function

$$I_T(\phi) = \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} I_T(\phi|\mu), & \text{if } \phi \in \mathcal{AC}_{T,A}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where

$$I_T(\phi|\mu) = \int_0^T \sum_{j=1}^{k} f(\mu_t^j, \beta_j(\phi_t)) dt,$$

with $f(\nu, \omega) = \nu \log(\nu/\omega) - \nu + \omega$, where we use the convention $\log(\nu/0) = +\infty$ for $\nu > 0$, while $0 \log(0/0) = 0 \log(0) = 0$.

Another possible definition leads to

$$\widetilde{I}_T(\phi) = \begin{cases} \inf_{\mu \in \mathcal{A}_d(\phi)} \int_0^T L(\phi_t, \phi'_t) dt, & \text{if } \phi \in \mathcal{AC}_{T,A}, \\ +\infty, & \text{otherwise,} \end{cases}$$

where for all $z \in A$, $y \in \mathbb{R}^d$,

$$L(x, y) = \sup_{\theta \in \mathbb{R}^d} \ell(z, y, \theta)$$

with

$$\ell(z, y, \theta) = \langle \theta, y \rangle - \sum_{j=1}^{k} \beta_j(z) \left( e^{\langle \theta, h_j \rangle} - 1 \right).$$

Recall the definition

**Definition 4.** *A rate function $I$ is a semi–continuous mapping $I : D_{T,A} \to [0, \infty]$ (i.e. its level sets $\Psi_I(\alpha) = \{\phi,\ I_T(\phi) \leq \alpha\}$ are closed subsets of $D_{T,A}$). A good rate function is a rate function whose level sets are compact.*

We have (see Kratz, Pardoux [2], Pardoux, Samegni [4])

**Proposition 5.** *$I_T = \widetilde{I}_T$ is a good rate function.*

# 3 Preliminary Lemmas

**Lemma 6.** *Suppose that the $\beta_j$, $j = 1, ..., k$ are bounded by $\theta$. If $I_T(\phi|\mu) \leq s$ then for all $0 \leq t_1, t_2 \leq T$ such that $t_2 - t_1 \leq 1/\theta$,*

$$\int_{t_1}^{t_2} \mu_t^j dt \leq \frac{s+1}{-\log(\theta(t_2 - t_1))} \quad \forall j = 1, ..., k.$$

Proof We have

$$\int_0^T f(\mu_t^j, \beta_j(\phi_t)) dt \leq I_T(\phi|\mu) \leq s.$$

moreover, the function $h(x) = x\log(x/\theta) - x$ is convex in $x$ so that for all $0 \le t_1, t_2 \le T$

$$
\begin{aligned}
& h\Big(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \mu_t^j dt\Big) \le \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} h(\mu_t^j) dt \\
& \le \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \Big(\mu_t^j \log\frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t)\Big) dt \\
& \le \frac{s}{t_2 - t_1}.
\end{aligned}
$$

It is easy to show that for all $\alpha > 0$, $h(x) \ge \alpha x - \theta \exp\{\alpha\}$ and then for all $\alpha > 0$

$$\int_{t_1}^{t_2} \mu_t^j dt \le \frac{1}{\alpha}(s + (t_2 - t_1)\theta \exp\{\alpha\}).$$

Therefore If $t_2 - t_1 < 1/\theta$ taking $\alpha = -\log(\theta(t_2 - t_1))$, the result follows. $\square$

For $\phi \in D_{T,A}$ let $\phi^a$ be defined by $\phi_t^a = \Phi_a(\phi_t)$. Clearly $\phi^a \in R^{a'}$.

**Lemma 7.** *Let $\phi$ be such that $I_T(\phi) < \infty$. We have $\limsup_{a\to 0} I_T(\phi^a) \le I_T(\phi)$.*

Proof Since $I_T(\phi) < \infty$, $\forall \eta > 0$ there exists $\mu$ such that $I_T(\phi|\mu) \le I_T(\phi) + \eta$. Let $\mu^a = (1-a)\mu$ so that $\mu^a$ is an allowed choice for $\phi^a$. We will show that

$$I_T(\phi^a|\mu^a) \to I_T(\phi|\mu) \quad \text{as} \quad a \to 0, \tag{1}$$

which clearly implies the result since

$$
\begin{aligned}
\limsup_{a\to 0} I_T(\phi^a) &\le \limsup_{a\to 0} I_T(\phi^a|\mu^a) \\
&= I_T(\phi|\mu) \le I_T(\phi) + \eta.
\end{aligned}
$$

By the convexity of $f(\nu, \omega)$ in $\nu$ and because $0 \le \mu_t^{j,a} \le \mu_t^j$, we have

$$
\begin{aligned}
& 0 \le f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \le f(0, \beta_j(\phi_t^a)) + f(\mu_t^j, \beta_j(\phi_t^a)) \\
& \le \theta + f(\mu_t^j, \beta_j(\phi_t^a)).
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
f(\mu_t^j, \beta_j(\phi_t^a)) &= \mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t^a)} - \mu_t^j + \beta_j(\phi_t^a) \\
&= \mu_t^j \log \frac{\mu_t^j}{\beta_j(\phi_t)} - \mu_t^j + \beta_j(\phi_t) + \mu_t^j \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} + \beta_j(\phi_t^a) - \beta_j(\phi_t) \\
&\leq f(\mu_t^j, \beta_j(\phi_t)) + 2\theta + \mu_t^j \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)}.
\end{aligned}
$$

If $\beta_j(\phi_t) < \lambda_1$ then $\beta_j(\phi_t) \leq \beta_j(\phi_t^a)$ and $\log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} < 0$.
If $\beta_j(\phi_t) \geq \lambda_1$ then using the Lipschitz continuity of the rates $\beta_j$ we have

$$
\begin{aligned}
\log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t^a)} &\leq \log \frac{\beta_j(\phi_t)}{\beta_j(\phi_t) - Ca} \leq \log \frac{\lambda_1}{\lambda_1 - Ca} \\
&\leq \log \frac{1}{1 - Ca/\lambda_1} < \frac{2Ca}{\lambda_1} < \frac{2C\lambda_2}{\lambda_1}.
\end{aligned}
$$

Since $\log(1/(1-x)) < 2x$ for $0 < x < 1/2$; here, we take $a$ small enough to ensure $Ca < \lambda_1/2$. Finally for all $a < (\lambda_1/2C) \wedge \lambda_2$

$$
0 \leq f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \leq f(\mu_t^j, \beta_j(\phi_t)) + 3\theta + \frac{2C\lambda_2}{\lambda_1} \mu_t^j.
$$

By Lemma 6 $\mu_t^j$ is integrable, we have bounded $f(\mu_t^{j,a}, \beta_j(\phi_t^a))$ for $0 < a < (\lambda_1/2C) \wedge \lambda_2$ by an integrable function. Since $f(\mu_t^{j,a}, \beta_j(\phi_t^a)) \to f(\mu_t^j, \beta_j(\phi_t))$ the dominated convergence theorem implies that

$$
\int_0^T f(\mu_t^{j,a}, \beta_j(\phi_t^a)) dt \to \int_0^T f(\mu_t^j, \beta_j(\phi_t)) dt \quad \text{as} \quad a \to 0,
$$

from which (1) follows, hence the result. □

**Lemma 8.** *Let $a > 0$ and $\phi \in R^a$ such that $I_T(\phi) < \infty$. For all $\eta > 0$ there exists $L > 0$ and $\phi^L \in R^{a/2}$ such that $\|\phi - \phi^L\|_T < a/2$ and $I_T(\phi^L|\mu^L) \leq I_T(\phi) + \eta$ where $\mu^L \in \mathcal{A}_d(\phi^L)$ such that $\mu_t^{L,j} < L$, $j = 1, ..., k$.*

Proof Let $\eta > 0$ and $\mu \in \mathcal{A}_d(\phi)$ such that $I_T(\phi|\mu) < I_T(\phi) + \eta/2$. For $L > 0$ let $\mu_t^{L,j} = \mu_t^j \wedge L$ and let $\phi^L$ be the solution of the ODE

$$
\frac{d\phi_t^L}{dt} = \sum_{j=1}^k \mu_t^{L,j} h_j, \quad \phi_0^L = \phi_0.
$$

It is plain that for $L$ sufficiently large $\phi^L$ is close to $\phi$ in supnorm. hence there exiosts $L_a > 0$ such that for all $L > L_a$ $\|\phi^L - \phi\| < \frac{a}{2}$. Since $\phi \in R^a$ the above also ensures that $\phi^L \in R^{a/2}$. To show the convergence of $I_T(\phi^L|\mu^L)$ to $I_T(\phi|\mu)$ we need to remark first using the convexity of $f(\nu, \omega)$ in $\nu$ that we have

$$f(\mu_t^{L,j}, \beta_j(\phi_t^L)) \le f(0, \beta_j(\phi_t^L)) + f(\mu_t^j, \beta_j(\phi_t^L)).$$

Since $\phi \in R^a$, $C_a \le \beta_j(\phi_t) \le \theta$ and $C_{a/2} \le \beta_j(\phi_t^L) \le \theta$ for all $L > L_a$, notice that

$$\frac{\partial f(\nu, \omega)}{\partial \omega} = -\frac{\nu}{\omega} + 1$$

and therefore on the interval $[K_a, \theta]$ where $K_a = C_a \wedge C_{a/2}$

$$|f(\mu_t^j, \beta_j(\phi_t^L)) - f(\mu_t^j, \beta_j(\phi_t))| < \bar{C}(\mu_t^j + 1)$$

for some constant $\bar{C} > 0$. Since $\mu_t^j$ and $f(\mu_t^j, \beta_j(\phi_t))$ are integrable the dominated convergence theorem implies that

$$\int_0^T f(\mu_t^{L,j}, \beta_j(\phi_t^L))dt \to \int_0^T f(\mu_t^j, \beta_j(\phi_t))dt \quad \text{as} \quad L \to \infty.$$

□

Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbf{N}$ and let the $\phi^\epsilon$ be the polygonal approximation of $\phi$ defined for $t \in [\ell\epsilon, (\ell+1)\epsilon)$ by

$$\phi_t^\epsilon = \phi_{\ell\epsilon} \frac{(\ell+1)\epsilon - t}{\epsilon} + \phi_{(\ell+1)\epsilon} \frac{t - \ell\epsilon}{\epsilon}. \tag{2}$$

**Lemma 9.** *Fix $\eta > 0$. Let $a \in (0,1)$ and $\phi \in R^a$ such that $I_T(\phi) < \infty$. Suppose that $\mu \in \mathcal{A}_d(\phi)$ such that $\mu_t^j < L$, $j = 1, ..., k$ for some $L > 0$ and $I_T(\phi|\mu) < \infty$ then there exists $a_\eta$ such that for all $a < a_\eta$ there exists an $\epsilon_a > 0$ such that for all $\epsilon < \epsilon_a$, $\phi^\epsilon \in R^a$ and $\|\phi - \phi^\epsilon\|_T < a/2$. Moreover, there exists $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ such that $\mu_t^{\epsilon,j} < L$, $j = 1, ..., k$ and $I_T(\phi^\epsilon|\mu^\epsilon) \le I_T(\phi|\mu) + \eta$.*

Proof Since $\phi$ is uniformly continuous on $[0, T]$ there exists an $\epsilon_a$ such that $\forall \epsilon < \epsilon_a$

$$\sup_{|t-t'|<2\epsilon} |\phi_t - \phi_{t'}| < \frac{ae^{-a^{-\nu}}}{2}$$

and then there exists $\bar{a}_\eta$ be such that for all $a < \bar{a}_\eta$, $e^{-a^{-\nu}} < 1$. We have then for all $a < \bar{a}_\eta$, $\|\phi - \phi^\epsilon\|_T < a/2$ and $\phi^\epsilon \in R^a$. For $t \in ]\ell\epsilon, (\ell+1)\epsilon[$

$$\frac{d\phi_t^\epsilon}{dt} = \frac{\phi_{(\ell+1)\epsilon} - \phi_{\ell\epsilon}}{\epsilon} = \frac{1}{\epsilon}\sum_{j=1}^{k} h_j \int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_t^j dt$$

therefore for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$, $\mu_t^\epsilon$ defined by

$$\mu_t^{\epsilon,j} = \frac{1}{\epsilon}\int_{\ell\epsilon}^{(\ell+1)\epsilon} \mu_t^j dt, j = 1, ..., k$$

is such that $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ and is constant over $[\ell\epsilon, (\ell+1)\epsilon[$. We also note that $\mu_t^{\epsilon,j} \leq L$ for all $j = 1, ..., k$. Moreover if $0 < \nu \leq L$ and $\omega \geq C_a$ then

$$\Big|\frac{\partial f(\nu,\omega)}{\partial\omega}\Big| = | - \frac{\nu}{\omega} + 1| \leq \frac{L}{C_a} + 1.$$

By the assumption A3, there exists $\tilde{a}_\eta > 0$ such that for all $a < \tilde{a}_\eta$

$$\frac{L}{C_a} + 1 \leq \frac{L}{e^{-a^{-\nu}}} + 1$$

Then for $t \in [\ell\epsilon, (\ell+1)\epsilon[$ and $a < \bar{a}_\eta$, $\tilde{a}_\eta$

$$|f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon)) - f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon}))| \leq \frac{1}{2}C(L+1)a = Va$$
$$|f(\mu_t^j, \beta_j(\phi_t)) - f(\mu_t^j, \beta_j(\phi_{\ell\epsilon}))| \leq \frac{1}{2}C(L+1)a = Va.$$

The above imply that

$$\begin{aligned}\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_t^\epsilon))dt &\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon}))dt + \epsilon Va \\ &= \epsilon f(\mu_{\ell\epsilon}^{\epsilon,j}, \beta_j(\phi_{\ell\epsilon})) + \epsilon Va \\ &\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_{\ell\epsilon}))dt + \epsilon Va \\ &\leq \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\phi_t))dt + 2Va\epsilon\end{aligned}$$

where the second inequality come from Jensen's inequality. Therefore

$$I_T(\phi^\epsilon|\mu^\epsilon) \leq I_T(\phi|\mu) + 2VTa$$

We can now choose $a < \min\{\bar{a}_\eta, \tilde{a}_\eta, \eta/2VT\}$ to have our result. □

The next lemma exploits a large deviation estimate for Poisson r.v.'s.

**Lemma 10.** *Let $Y_1$,$Y_2$,...be independent Poisson random variables with mean $\theta\epsilon$. For all $N \in \mathbb{N}$, let*

$$\bar{Y}^N = \frac{1}{N}\sum_{n=0}^{N} Y_n.$$

*For any $s > 0$ there exist $K, \epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that taking $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$ we have*

$$\mathbb{P}^N(\bar{Y}^N > g(\epsilon)) < \exp\{-sN\}$$

*for all $\epsilon < \epsilon_0$ and $N > N_0$.*

Proof We apply the Gramer's theorem that we can find in [1] (chapter 2) to have that there exist $N_0 \in \mathbb{N}$ such that

$$\limsup_{N\to\infty} \frac{1}{N}\log(\mathbb{P}^N(\bar{Y}^N > g(\epsilon))) \leq -\inf_{x\geq g(\epsilon)} \Lambda^*_\epsilon(x)$$

where $\Lambda^*_\epsilon(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda_\epsilon(\lambda)\}$ with

$$\Lambda_\epsilon(\lambda) = \log(\mathbb{E}(e^{\lambda Y_1}) = \theta\epsilon(e^\lambda - 1).$$

We deduce that

$$\Lambda^*_\epsilon(x) = x\log\frac{x}{\theta\epsilon} - x + \theta\epsilon.$$

This last function is convex then it reaches his infimuim in $x = \theta\epsilon$ and as $\lim_{\epsilon\to 0}\frac{g(\epsilon)}{\theta\epsilon} = +\infty$ there exists $\epsilon_1 > 0$ such that $g(\epsilon) > \theta\epsilon$ for all $\epsilon < \epsilon_1$ and then

$$\begin{aligned}\inf_{x\geq g(\epsilon)} \Lambda^*_\epsilon(x) &= g(\epsilon)\log\frac{g(\epsilon)}{\theta\epsilon} - g(\epsilon) + \theta\epsilon \\ &= g(\epsilon)\log(g(\epsilon)) - g(\epsilon)\log(\theta\epsilon) - g(\epsilon) + \theta\epsilon \\ &\approx K\sqrt{\log(1/\epsilon)} \to \infty \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

Then there exists $\epsilon_2 > 0$ such that $\inf_{x\geq g(\epsilon)}\Lambda^*_\epsilon(x) > s$ for all $\epsilon < \epsilon_2$.

Taking $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, we have the lemma. □

# 4 The Lower Bound

For a path $\phi$ let $F_\delta(\phi) = \{\psi : \|\psi - \phi\|_T < \delta\}$. We first prove that for all fixed path $\phi$ and any $\eta > 0$, $\delta > 0$ there exists $N_{\eta,\delta}$, such that for all $N > N_{\eta,\delta}$

$$(1) \qquad \mathbb{P}^N(F_\delta(\phi)) = \tilde{\mathbb{E}}\Big(\Delta_T^N \mathbf{1}_{\{Z^N \in F_\delta(\phi)\}}\Big) \geq \exp\{-N(I_{T,x}(\phi) + \eta)\}.$$

To this end, it is enough to prove (1) considering $\phi \in \mathcal{AC}_{T,A}$ because the inequality is true when $I_{T,x}(\phi) = \infty$. We apply some lemmas of the preceding section to show that it is enough to consider some suitable paths $\phi$ with the $\mu \in \mathcal{A}_d(\phi)$.

We have the

**Lemma 11.** *For any $a > 0$, $\epsilon > 0$ let $\phi \in R^a$ for $a > 0$. For $\epsilon > 0$ let $\phi^\epsilon$ be its polygonal approximation defined by* (2)*. Suppose that for all $\eta > 0$,$\delta > 0$ there exists $N_{\eta,\delta}$ such that for all $N \geq N_{\eta,\delta}$*

$$(2) \qquad \mathbb{P}(\|Z^N - \phi^\epsilon\|_T < \delta) \geq \exp\{-N(I_T(\phi^\epsilon|\mu^\epsilon) + \eta)\}$$

*where $\mu^\epsilon \in \mathcal{A}_d(\phi^\epsilon)$ such that $\mu_t^{\epsilon,j} \leq L$ for all $j = 1, ..., k$ for some $L > 0$. Then for all fixed $\phi \in \mathcal{AC}_{T,A}$, and any $\eta > 0$, $\delta > 0$ there exists $N_{\eta,\delta}$ such that for all $N > N_{\eta,\delta}$,*

$$\mathbb{P}^N(F_\delta(\phi)) = \mathbb{P}(\|Z^N - \phi\|_T < \delta) \geq \exp\{-N(I_T(\phi) + \eta)\}.$$

Proof For $\delta, \eta > 0$ let $\phi \in \mathcal{AC}_{T,A}$ such that $I_T(\phi) < \infty$ then using Lemma 7 we have that there exists $a_\eta > 0$ such that for all $a < a_\eta$ there exists $\phi^a \in R^a$ such that $\|\phi - \phi^a\|_T < a$ and $I_T(\phi^a) \leq I_T(\phi) + \eta/4$. As $I_T(\phi^a) < \infty$, we deduce from Lemma 8 that there exists $L > 0$ and $\phi^{a,L} \in R^{a/2}$ such that $\|\phi^a - \phi^{a,L}\|_T < a/2$ and $I_T(\phi^{a,L}|\mu^{a,L}) \leq I_T(\phi^a) + \eta/4$ where $\mu^{a,L} \in \mathcal{A}_d(\phi^{a,L})$ such that $\mu_t^{a,L,j} < L$, $j = 1, ..., k$. Now we can deduce from Lemma 9 that for all $\epsilon > 0$ the polygonal approximation $\phi^{a,L,\epsilon}$ of $\phi^{a,L}$ satisfies $\|\phi^{a,L} - \phi^{a,L,\epsilon}\|_T < a/4$ and $I_T(\phi^{a,L,\epsilon}|\mu^{a,L,\epsilon}) \leq I_T(\phi^{a,L}|\mu^{a,L}) + \eta/4$ where $\mu^{a,L,\epsilon} \in \mathcal{A}_d(\phi^{a,L,\epsilon})$ is such that $\mu_t^{a,L,\epsilon,j} < L$, $j = 1, ..., k$. Now we choose $a$ such that $2a < \delta/2$ and we have

$$
\begin{aligned}
\mathbb{P}\Big(\|Z^N-\phi\|_T<\delta\Big) &\geq \mathbb{P}\Big(\|Z^N-\phi\|_T<\frac{\delta}{2}+2a\Big)\\
&\geq \mathbb{P}\Big(\|Z^N-\phi^a\|_T<\frac{\delta}{2}+a\Big)\\
&\geq \mathbb{P}\Big(\|Z^N-\phi^{a,L}\|_T<\frac{\delta}{2}+\frac{a}{2}\Big)\\
&\geq \mathbb{P}\Big(\|Z^N-\phi^{a,L,\epsilon}\|_T<\frac{\delta}{2}\Big)\\
&\geq \exp\{-N(I_T(\phi^{a,L,\epsilon}|\mu^{a,L,\epsilon})+\eta/4)\}\\
&\geq \exp\{-N(I_T(\phi^{a,L}|\mu^{a,L})+\eta/2)\}\\
&\geq \exp\{-N(I_T(\phi^{a})+3\eta/4)\}\\
&\geq \exp\{-N(I_T(\phi)+\eta)\}
\end{aligned}
$$

where we have used (2) at the $5^{th}$ inequality. □

The goal of the next lemma is to show the inequality (2).

**Lemma 12.** *For $a>0$, $\epsilon>0$, let $\phi\in R^a$ be linear on each intervals $[\ell\epsilon,(\ell+1)\epsilon[$, $0\leq\ell\leq\frac{T}{\epsilon}$. Consider the $\mu\in\mathcal{A}_d(\phi)$ that is constant over these time intervals and such that all the components of $\mu$ are bounded above by some constant $L>0$. Then we have that for any $\eta>0$, and suitable small $\delta>0$(thus the inequality stay true for all delta $>0$) there exists $N_{\eta,\delta}\in\mathbb{N}$ such that for all $N>N_{\eta,\delta}$*

$$\mathbb{P}(\|Z^N-\phi\|_T<\delta)\geq \exp\{-N(I_T(\phi|\mu)+\eta)\}.$$

Proof Define the events $B_j$, $j=1,...,k$ for controlling the likelihood ratio. For $\xi>0$ let

$$B_j=\Big\{\Big|\sum_{p=1}^{Q}\delta_p(j)\log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^j_{\lfloor\tau_p/\epsilon\rfloor\epsilon}}\Big)-N\sum_{\ell=1}^{T/\epsilon}\mu^j_{\ell\epsilon}\log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big)\epsilon\Big|\leq N\xi\Big\}$$

We have on $\{Z^N \in F_\delta(\phi)\} \cap (\bigcap_{j=1}^k B_j) = \{Z^N \in F_\delta(\phi)\} \cap B$

$$\Delta_T^N = \exp\Big\{ \sum_{p=1}^{Q} \sum_{j=1}^{k} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^j_{\tau_p^-}}\Big) + N\int_0^T \sum_{j=1}^{k} (\mu_t^j - \beta_j(Z^N(t)))dt\Big\}$$
$$\geq \exp\Big\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu^j_{\ell\epsilon} \log\Big(\frac{\mu^j_{\ell\epsilon}}{\beta_j(\phi_{\ell\epsilon})}\Big)\epsilon + N\int_0^T \sum_{j=1}^{k} (\mu_t^j - \beta_j(Z^N(t)))dt - kN\xi\Big\}$$
$$\geq \exp\Big\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu^j_{\ell\epsilon} \log\Big(\frac{\mu^j_{\ell\epsilon}}{\beta_j(\phi_{\ell\epsilon})}\Big)\epsilon + N\int_0^T \sum_{j=1}^{k} (\mu_t^j - \beta_j(\phi_t))dt - N(kTC\delta + k\xi)\Big\}$$
$$\geq \exp\Big\{ -N \sum_{\ell=1}^{T/\epsilon} \sum_{j=1}^{k} \mu^j_{\ell\epsilon} \log\Big(\frac{\mu^j_{\ell\epsilon}}{\beta_j(\phi_{\ell\epsilon})}\Big)\epsilon + N\int_0^T \sum_{j=1}^{k} (\mu_t^j - \beta_j(\phi_t))dt - N.O(\delta + \xi)\Big\}$$

We note here that the first inequality is true because the $\mu_t^j$ is constant on the intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and the second one come from the Lipschitz continuity of the rates $\beta_j$. Since the integrand is continuous, we deduce from the convergence of the Riemann sums that when $\epsilon$ is small enough we have

$$\Delta_T^N \geq \exp\Big\{ -N\int_0^T \sum_{j=1}^{k} \Big[\mu_t^j \log\Big(\frac{\mu_t^j}{\beta_j(\phi_t)}\Big) - \mu_t^j + \beta_j(\phi_t)\Big]dt - NO(\delta+\xi)\Big\}$$
$$\geq \exp\{-N(I_T(\phi|\mu) + O(\delta+\xi))\} \quad \text{on the event} \quad \{Z^N \in F_\delta(\phi)\} \cap B.$$

Then for any $\eta > 0$, there exists $\delta > 0$ and $\xi > 0$ such that for $N$ large enough we have

$$\Delta_T^N \geq \exp\{-N(I_T(\phi|\mu) + \eta/2)\}$$

Moreover

$$\mathbb{P}^N(F_\delta(\phi)) = \widetilde{\mathbb{E}}\Big(\Delta_T^N.\mathbf{1}_{\{Z^N \in F_\delta(\phi)\}}\Big)$$
$$\geq \widetilde{\mathbb{E}}\Big(\Delta_T^N.\mathbf{1}_{\{\{Z^N \in F_\delta(\phi)\}\cap B\}}\Big)$$
$$\geq \exp\{-N(I_T(\phi|\mu) + \eta/2)\}\widetilde{\mathbb{P}}(\{Z^N \in F_\delta(\phi)\} \cap B)$$

To finish this proof it is enough to show the following lemma:

**Lemma 13.** *Let $\phi \in R^a$ linear over the intervals $[\ell\epsilon, (\ell+1)\epsilon[$,*

$$\lim_{N\to\infty} \widetilde{\mathbb{P}}(\{Z^N \in F_\delta(\phi)\} \cap B) = 1$$

Proof It suffices to prove that $\lim_{N\to\infty} \widetilde{\mathbb{P}}^N(F_\delta(\phi)) = 1$ and that for all $j = 1, ..., k$, $\lim_{N\to\infty} \widetilde{\mathbb{P}}(\{Z^N \in F_\delta(\phi)\} \cap B_j^c) = 0$. The first limit follows from Theorem 2 for processes under $\widetilde{\mathbb{P}}^N$.

We now show that $\widetilde{\mathbb{P}}^N(F_\delta(\phi) \cap B_j^c) \to 0$ as $N \to \infty$ for $1 \le j \le k$ . We have $\sup_p |Z^N(\tau_p) - \phi_{\tau_p}| < \delta$ on $\{Z^N \in F_\delta(\phi)\}$ and we can choose $\epsilon$ small enough such that $\sup_p |\phi_{\tau_p} - \phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon}| < \delta$ and thus $\sup_p |Z^N(\tau_p) - \phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon}| < 2\delta$.

Note that we have on $\{Z^N \in F_\delta(\phi)\}$

$$\begin{aligned}\Big|\sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big)\Big| &\le \Big|\sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\beta_j(\phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}\Big)\Big| \\ &\le \frac{2CQ\delta}{C_a},\end{aligned}$$

since $|\beta_j(Z^N(\tau_p^-)) - \beta_j(\phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon})| < 2C\delta$. Let $m_\ell$ be the number of jumps in the interval $[(\ell-1)\epsilon, \ell\epsilon[$. We have

$$\begin{aligned}&\Big|\sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - N\sum_{\ell=1}^{T/\epsilon} \mu^j_{\ell\epsilon} \log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big)\epsilon\Big| \\ &\le \Big|\sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - N\sum_{\ell=1}^{T/\epsilon} \mu^j_{\ell\epsilon} \log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big)\epsilon\Big| \\ &+ \Big|\sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(Z^N(\tau_p^-))}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big) - \sum_{p=1}^{Q} \delta_p(j) \log\Big(\frac{\beta_j(\phi_{\lfloor \tau_p/\epsilon \rfloor \epsilon})}{\mu^j_{\lfloor \tau_p/\epsilon \rfloor \epsilon}}\Big)\Big| \\ &\le \Big|\sum_{\ell=1}^{T/\epsilon} \log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu^j_{\ell\epsilon}\epsilon\Big)\Big| + \frac{2CQ\delta}{C_a}.\end{aligned}$$

As the rate of jumps are constant on the interval $[(\ell-1)\epsilon, \ell\epsilon[$ under $\widetilde{\mathbb{P}}^N$, $\sum_{p=1}^{m_\ell} \delta_p(j)$ is the number of jumps of a Poisson process $P_j$ on this interval. So it is a Poisson random variable with mean $N\mu^j_{\ell\epsilon}\epsilon$. We deduce of Chebyshev's inequality that

$$\widetilde{\mathbb{P}}\Big(\Big|\log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu^j_{\ell\epsilon}\epsilon\Big)\Big| > \frac{N\xi\epsilon}{2T}\Big) \le \frac{4T^2 \sup_{\ell \le T/\epsilon}\Big(\log^2\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu^j_{\ell\epsilon}}\Big) N\mu^j_{\ell\epsilon}\epsilon\Big)}{N^2\xi^2\epsilon^2}.$$

As $\beta_j(\phi_t) \geq C_a$ and $\mu_t^j \leq L$ we have $\sup_{\ell \leq T/\epsilon}\Big(\log^2\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu_{\ell\epsilon}^j}\Big)\mu_{\ell\epsilon}^j\Big) \leq C(L,a)$. Thus

$$\begin{aligned}\widetilde{\mathbb{P}}(\{Z^N \in F_\delta(\phi)\} \cap B_j^c) &\leq \widetilde{\mathbb{P}}\Big(\Big|\sum_{\ell=1}^{T/\epsilon} \log\Big(\frac{\beta_j(\phi_{\ell\epsilon})}{\mu_{\ell\epsilon}^j}\Big)\Big(\sum_{p=1}^{m_\ell} \delta_p(j) - N\mu_{\ell\epsilon}^j \epsilon\Big)\Big| + \frac{2CQ\delta}{C_a} > N\xi\Big) \\ &\leq \frac{4T^2 C(L,a)}{N\xi^2\epsilon} + \widetilde{\mathbb{P}}\Big(\frac{2CQ\delta}{C_a} \geq \frac{N\xi}{2}\Big).\end{aligned}$$

The number of jumps during the period time $T$ under $\widetilde{\mathbb{P}}^N$ is the sums of the Poisson random variables with mean $N\sum_{j=1}^k \mu_{\ell\epsilon}^j \epsilon$. we take $\xi = \frac{8C\delta}{C_a}\sum_{\ell=1}^{T/\epsilon}\sum_{j=1}^k \mu_{\ell\epsilon}^j \epsilon$ where $\delta$ is chosen such that $\delta/C_a$ is small. Therefore, as long as $\sum_{\ell=1}^{T/\epsilon}\sum_{j=1}^k \mu_{\ell\epsilon}^j > 0$, the law of large number for Poisson variables give us

$$\widetilde{\mathbb{P}}\Big(\frac{2CQ\delta}{C_a} \geq \frac{N\xi}{2}\Big) = \widetilde{\mathbb{P}}\Big(\frac{Q}{N} \geq 2\sum_{\ell=1}^{T/\epsilon}\sum_{j=1}^{k} \mu_{\ell\epsilon}^j \epsilon\Big) \to 0$$

as $N \to \infty$. □

We finish the proof of the lower bound by the following theorem

Theorem **14.** *For all open set $G \in D_{T,A}$,*

$$\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N(G) \geq -\inf_{\phi\in G} I_T(\phi).$$

Proof It is enough to assume that (1) is true and show (14). To this end let $I = \inf_{\phi\in G} I_T(\phi) < \infty$ then, for $\eta > 0$ there exists a $\phi^\eta \in G$ such that $I_T(\phi^\eta) \leq I + \eta$. Moreover we can choose $\delta = \delta(\phi^\eta)$ small enough such that $F_\delta(\phi^\eta) \subset G$. And then $\mathbb{P}^N(F_\delta(\phi^\eta)) \leq \mathbb{P}^N(G)$. This implies from the inequality (1) that for all $\eta > 0$,

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N(G) &\geq \liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N(F_\delta(\phi^\eta)) \\ &\geq -I_T(\phi^\eta) \\ &\geq -I - \eta\end{aligned}$$

and then

$$\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{P}^N(G) \geq -I.$$

□

Specifying the starting point, we can reformulate the above result as

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}(Z^{N,x} \in G) \geq - \inf_{\phi \in G, \phi_0 = x} I_T(\phi).$$

We need in fact the stronger statement

Theorem **15.** *For all open set $G \in D_{T,A}$ such that all trajectories in $G$ remain in a compact set which does not intersect the boundary $\partial A$, for any compact set $K \subset \mathring{A}$,*

$$\liminf_{N\to\infty} \frac{1}{N} \log \inf_{x\in K} \mathbb{P}(Z^{N,x} \in G) \geq - \sup_{x\in K} \inf_{\phi \in G, \phi_0 = x} I_T(\phi).$$

# 5 The Upper Bound

For all $\phi \in D_{T,A}$ and $H \subset D_{T,A}$ we define

$$\rho_T(\phi, H) = \inf_{\psi \in H} \|\phi - \psi\|_T \tag{1}$$

and for all $\delta, s > 0$ we define the set

$$H_\delta(s) = \{\phi \in D_{T,A} : \rho_T(\phi, \Phi(s)) \geq \delta\}$$

where $\Phi(s) = \{\phi \in D_{T,A} : I_T(\phi) \leq s\}$. We start by proving the following lemma which will be enough to conclude the upper bound.

**Lemma 16.** *for any $\delta$, $\eta$, $s > 0$ there exists $N_0 \in \mathbf{N}$ such that*

$$\mathbb{P}^N(H_\delta(s)) \leq \exp\{-N(s-\eta)\} \tag{2}$$

*whenever $N \geq N_0$.*

Proof Let $Z^{N,a}(t) = \Phi_a(Z^N(t))$ then $\|Z^N - Z^{N,a}\| < a'$ and for all $a$ small enough,

$$\begin{aligned}\mathbb{P}^N(H_\delta(s)) &= \mathbb{P}(\rho_T\Big(Z^N, \Phi(s)) \geq \delta\Big) \\ &\leq \mathbb{P}\Big(\rho_T(Z^{N,a}, \Phi(s)) \geq \frac{\delta}{d}\Big).\end{aligned}$$

We now approximate the paths $Z^N$ by smoother paths. Let $\epsilon > 0$ be such that $T/\epsilon \in \mathbf{N}$. We construct a polygonal approximation to $Z^{N,a}$ defined for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$ by

$$\Upsilon_t = \Upsilon_t^{a,\epsilon} = Z^{N,a}(\ell\epsilon)\frac{(\ell+1)\epsilon - t}{\epsilon} + Z^{N,a}((\ell+1)\epsilon)\frac{t-\ell\epsilon}{\epsilon}.$$

The event $\{\|Z^{N,a} - \Upsilon\|_T < \frac{\delta}{2d}\} \cap \{\rho_T(Z^{N,a}, \Phi(s)) \geq \frac{\delta}{d}\}$ is contained in $\{\rho_T(\Upsilon, \Phi(s)) \geq \frac{\delta}{2d}\}$ and

$$\mathbb{P}\Big(\rho_T(Z^{N,a}, \Phi(s)) \geq \frac{\delta}{d}\Big) \leq \mathbb{P}\Big(\rho_T(\Upsilon, \Phi(s)) \geq \frac{\delta}{2d}\Big) + \mathbb{P}\Big(\{\|Z^{N,a} - \Upsilon\|_T \geq \frac{\delta}{2d}\}\Big) \tag{3}$$

$$\leq \mathbb{P}(I_T(\Upsilon) \geq s) + \mathbb{P}\Big(\|Z^{N,a} - \Upsilon\|_T \geq \frac{\delta}{2d}\Big)$$

We try now to bound $\mathbb{P}(I_T(\Upsilon) \geq s)$. For any choice $\mu \in \mathcal{A}_d(\Upsilon)$ we have $I_T(\Upsilon) \leq I_T(\Upsilon|\mu)$ and

$$\mathbb{P}(I_T(\Upsilon) \geq s) \leq \mathbb{P}(I_T(\Upsilon|\mu) \geq s).$$

Let $\mu_t^j$, $j = 1, ..., k$ be constant on the intervals $[\ell\epsilon, (\ell+1)\epsilon[$ and equal to

$$\mu_t^j = \frac{1-a/d^2}{N\epsilon}\Big[P_j\Big(N\int_0^{(\ell+1)\epsilon} \beta_j(Z^N(s)ds\Big) - P_j\Big(N\int_0^{\ell\epsilon} \beta_j(Z^N(s)ds\Big)\Big] \tag{4}$$

Since $\Upsilon$ is piecewise linear, for $t \in ]\ell\epsilon, (\ell+1)\epsilon[$

$$\frac{d\Upsilon_t^i}{dt} = \frac{(1-a/d^2)}{\epsilon}(Z_i^N((\ell+1)\epsilon) - Z_i^N(\ell\epsilon)) = \sum_{j=1}^k \mu_t^j h_j^i.$$

Then the $\mu_t^j$ given by (4) belong to $\mathcal{A}_d(\Upsilon)$.

To control the change in $\Upsilon$ over the intervals of length $\epsilon$ define $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$ where $K > 0$ is fixed, and define a collection of events $B = \{B_\epsilon\}_{\epsilon>0}$

$$B_\epsilon = \bigcap_{\ell=0}^{T/\epsilon - 1} B_\epsilon^\ell$$

where

$$B_\epsilon^\ell = \Big\{\sup_{\ell\epsilon \leq t_1, t_2 \leq (\ell+1)\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| \leq g(\epsilon) \quad \text{for} \quad i = 1, ..., d\Big\}.$$

We have

$$\mathbb{P}(I_T(\Upsilon|\mu) > s) \leq \mathbb{P}(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) + \mathbb{P}(B_\epsilon^c) \tag{5}$$

and using the Chebyshev inequality we have that for all $0 < \alpha < 1$

$$\mathbb{P}(\{I_T(\Upsilon|\mu) > s\} \cap B_\epsilon) \leq \frac{\mathbb{E}(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon})}{\exp\{\alpha N s\}}. \tag{6}$$

We need to show that the expectation above is appropriately small for $\alpha$ arbitrarily close to 1. For this we first prove the following lemma

**Lemma 17.** *For all $0 < \alpha < 1$, $j = 1, ..., k$ and $\ell = 0, ..., T/\epsilon - 1$, there exist $Z_j^-$ and $Z_j^+$ which conditionally upon $\mathcal{F}_\ell$ are Poisson random variables with mean $N\epsilon\beta_\ell^{j-} = N\epsilon(\beta_j(Z^N(\ell\epsilon)) - Cdg(\epsilon))_+$ and $N\epsilon\beta_\ell^{j+} = N\epsilon(\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon))$ respectively such that if*

$$\Theta_j^\ell = \exp\left\{\alpha N \int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\right\}\mathbf{1}_{B_\epsilon^\ell}$$

*and*

$$\Xi_j^\ell = \exp\{2\alpha N C d g(\epsilon)\epsilon\} \times \Big[\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a/d^2)Z_j^-}{\epsilon N}, \beta_\ell^{a,j-}\Big)\Big\} + \exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a/d^2)Z_j^+}{\epsilon N}, \beta_\ell^{a,j-}\Big)\Big\}\Big]$$

*with $\beta_\ell^{a,j-} = \beta_j(\Upsilon_{\ell\epsilon} - Cdg(\epsilon))$, then*

$$\Theta_j^\ell \leq \Xi_j^\ell \quad a.s \tag{7}$$

Proof On $B_\epsilon^\ell$, with $\epsilon$ such that $g(\epsilon) < 1$ and $t \in [\ell\epsilon, (\ell+1)\epsilon]$, using the Lipshitz continuity of the rates $\beta_j$ we have

$$|\beta_j(Z^N(t)) - \beta_j(Z^N(\ell\epsilon))| \leq C|Z^N(t) - Z^N(\ell\epsilon)| \leq Cdg(\epsilon), \quad j = 1, ..., k$$

Then we have

$$\Big|N\int_{p\epsilon}^{(\ell+1)\epsilon} \beta_j(Z^N(t))dt - N\epsilon\beta_j(Z^N(\ell\epsilon))\Big| \leq N\epsilon C d g(\epsilon), \quad j = 1, ..., k.$$

As $\mu_t^j$, $j = 1, ..., k$ satisfy (4), we can write

$$(8) \qquad \frac{(1-a/d^2)Z_j^-}{\epsilon N} \le \mu_{\ell\epsilon}^j \le \frac{(1-a/d^2)Z_j^+}{\epsilon N} \quad \text{a.s.}$$

where for example

$$Z_j^- = P_j\Big(N\int_0^{\ell\epsilon}\beta_j(Z^N(s))ds + \epsilon N(\beta_j(Z^N(\ell\epsilon)) - Cdg(\epsilon))_+\Big) - P_j\Big(N\int_0^{\ell\epsilon}\beta_j(Z^N(s))ds\Big)$$

$$Z_j^+ = P_j\Big(N\int_0^{\ell\epsilon}\beta_j(Z^N(s))ds + \epsilon N(\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon))\Big) - P_j\Big(N\int_0^{\ell\epsilon}\beta_j(Z^N(s))ds\Big).$$

Moreover it is easy to see that on $B_\epsilon^\ell$ we have

$$\max_{1\le i\le d}|\Upsilon_t^i - \Upsilon_{\ell\epsilon}^i| < (1-a/d^2)g(\epsilon) < g(\epsilon) \quad \text{for} \quad t\in[\ell\epsilon,(\ell+1)\epsilon].$$

And then

$$|\beta_j(\Upsilon_t) - \beta_j(\Upsilon_{\ell\epsilon})| \le C|\Upsilon_t - \Upsilon_{\ell\epsilon}| \le Cdg(\epsilon)$$

we deduce that

$$\beta_j(\Upsilon_t) \ge \beta_j(\Upsilon_{\ell\epsilon}) - Cdg(\epsilon) = \beta_\ell^{a,j-}$$

and

$$\beta_j(\Upsilon_t) \le \beta_j(\Upsilon_{\ell\epsilon}) + Cdg(\epsilon) = \beta_\ell^{a,j-} + 2Cdg(\epsilon).$$

Thus

$$\begin{aligned}
f(\mu_t^j, \beta_j(\Upsilon_t)) &= \mu_t^j \log\frac{\mu_t^j}{\beta_j(\Upsilon_t)} - \mu_t^j + \beta_j(\Upsilon_t)\\
&\le \mu_t^j\log\frac{\mu_t^j}{\beta_\ell^{a,j-}} - \mu_t^j + \beta_\ell^{a,j-} + 2Cdg(\epsilon) + \mu_t^j\log\frac{\beta_\ell^{a,j-}}{\beta_j(\Upsilon_t)}\\
&\le f(\mu_t^j, \beta_\ell^{a,j-}) + 2Cdg(\epsilon) \quad \text{since} \quad \log\frac{\beta_\ell^{a,j-}}{\beta_j(\Upsilon_t)} < 0.
\end{aligned}$$

As $\mu_t^j = \mu_{\ell\epsilon}^j$ is constant over the interval $[\ell\epsilon, (\ell+1)\epsilon[$, we deduce that on $B_\epsilon^\ell$
(9)

$$\exp\Big\{\alpha N\int_{\ell\epsilon}^{(\ell+1)\epsilon} f(\mu_t^j, \beta_j(\Upsilon_t))dt\Big\} \le \exp\{\alpha N\epsilon f(\mu_{\ell\epsilon}^j, \beta_\ell^{a,j-}) + 2\alpha NCd\epsilon g(\epsilon)\},$$

From (8), (9) and the convexity of $f(\nu,\omega)$ in $\nu$ we deduce the inequality of lemma. $\square$

The next proposition gives us a bound for the conditionnal expectation the right hand side of the inequality (7).

**Proposition 18.** *Let* $a = h(\epsilon) = \Big[ -\log g^{1/2}(\epsilon) \Big]^{-\frac{1}{\nu}}$. *For all* $0 < \alpha < 1$ *there exist* $\epsilon_\alpha$, $K_\alpha$ *and* $\tilde{K}$ *such that for all* $\epsilon \leq \epsilon_\alpha$ *we have*

$$\max_{q=-,+} \Big\{ \mathbb{E}\Big( \exp\Big\{ \alpha N \epsilon f\Big( \frac{(1-a/d^2)Z_j^q}{\epsilon N}, \beta_\ell^{a,j-} \Big) \Big\} \Big| \mathcal{F}_\ell \Big) \Big\}$$
$$\leq K_\alpha \exp\{ N\epsilon \tilde{K}(1 - \alpha + 2h(\epsilon) + 2dg(\epsilon)) \},$$

*where* $\mathcal{F}_\ell$ *is the* $\sigma-$*algebra generated by the process* $Z^N(.)$ *up until time* $\ell\epsilon$.

Proof Conditionally on $\mathcal{F}_\ell$, $Z_j^q$ is a Poisson variable with mean $N\epsilon\beta_\ell^{j,q}$. Moreover we have by the definition

$$\max\{ |\beta_\ell^{a,j-} - \beta_\ell^{j-}|, |\beta_\ell^{a,j-} - \beta_\ell^{j+}| \} \leq \tilde{C}(a + 2dg(\epsilon))$$

let $\tilde{\epsilon} = \epsilon/(1 - a/d^2)$ and $\tilde{\alpha} = (1 - a/d^2)\alpha$ then we have

$$\mathbb{E}\Big( \exp\Big\{ \alpha N\epsilon f\Big( \frac{(1-a/d^2)Z_j^q}{\epsilon N}, \beta_\ell^{a,j-} \Big) \Big\} | \mathcal{F}_\ell \Big) = \mathbb{E}\Big( \exp\Big\{ \alpha N\epsilon f\Big( \frac{Z_j^q}{\tilde{\epsilon} N}, \beta_\ell^{a,j-} \Big) \Big\} | \mathcal{F}_\ell \Big)$$
$$= \sum_{m \geq 0} \exp\Big\{ \alpha N\epsilon f\Big( \frac{m}{\tilde{\epsilon}N}, \beta_\ell^{a,j-} \Big) \Big\} \frac{(N\epsilon\beta_\ell^{j,q})^m \exp\{-N\epsilon\beta_\ell^{j,q}\}}{m!}$$
$$= \sum_{m \geq 0} \exp\Big\{ \alpha N\epsilon\Big( \frac{m}{\tilde{\epsilon}N} \log\Big( \frac{m}{\tilde{\epsilon}N\beta_\ell^{a,j-}} \Big) - \frac{m}{\tilde{\epsilon}N} + \beta_\ell^{a,j-} \Big) \Big\} \frac{(N\epsilon\beta_\ell^{j,q})^m \exp\{-N\epsilon\beta_\ell^{j,q}\}}{m!}$$
$$\leq \exp\{ N\epsilon\tilde{C}(a + 2dg(\epsilon)) \} \sum_{m \geq 0} \frac{m^{\tilde{\alpha}m} \exp\{-\tilde{\alpha}m\}}{m!} (N\epsilon\beta_\ell^{a,j-})^{m(1-\tilde{\alpha})} \Big( \frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j-}} \Big)^m \exp\{-N\epsilon\beta_\ell^{a,j-}(1-\alpha)\}$$

(10)

$$\leq \exp\{ N\epsilon C_1(a + 2dg(\epsilon)) \} \sum_{m \geq 0} \frac{m^{\tilde{\alpha}m} \exp\{-\tilde{\alpha}m\}}{m!} (N\epsilon\beta_\ell^{a,j-})^{m(1-\tilde{\alpha})} \Big( \frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j-}} \Big)^m \exp\{-N\epsilon\beta_\ell^{a,j-}(1-\tilde{\alpha})\}.$$

Moreover the function $v(x) = x^{m(1-\tilde{\alpha})} \exp\{-2x(1-\tilde{\alpha})\}$ reaches its maximum at $x = m/2$ thus we have

$$x^{m(1-\tilde{\alpha})} \exp\{-2x(1-\tilde{\alpha})\} \leq \Big( \frac{m}{2} \Big)^{m(1-\tilde{\alpha})} \exp\{-m(1-\tilde{\alpha})\} \quad \forall x$$

In particular

$$(N\epsilon\beta_\ell^{a,j-})^{m(1-\tilde{\alpha})} \exp\{-2N\epsilon\beta_\ell^{a,j-}(1-\tilde{\alpha})\} \leq \Big( \frac{m}{2} \Big)^{m(1-\tilde{\alpha})} \exp\{-m(1-\tilde{\alpha})\}.$$

Thus

$$\sum_{m\geq 0} \frac{m^{\tilde{\alpha} m}\exp\{-\tilde{\alpha} m\}}{m!}(N\epsilon\beta_\ell^{a,j-})^{m(1-\tilde{\alpha})}\Big(\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j-}}\Big)^m \exp\{-N\epsilon\beta_\ell^{a,j-}(1-\tilde{\alpha})\}$$

(11)

$$\leq \exp\{N\epsilon\beta_\ell^{a,j-}(1-\tilde{\alpha})\}\sum_{m\geq 0}\frac{m^m\exp\{-m\}}{m!}\Big(\frac{\beta_\ell^{j,q}/\beta_\ell^{a,j-}}{2^{(1-\tilde{\alpha})}}\Big)^m$$

Moreover for $q = -$ we have

$$\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j-}} \leq \frac{\beta_j(Z^N(\ell\epsilon))}{\beta_j(Z^{N,a}(\ell\epsilon)) - Cdg(\epsilon)}$$

If $\beta_j(Z^N(\ell\epsilon)) < \lambda_1$ we have using the Assumption A2 and A3

$$\begin{aligned}\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j-}} &\leq \frac{\beta_j(Z^{N,a}(\ell\epsilon))}{\beta_j(Z^{N,a}(\ell\epsilon)) - Cdg(\epsilon)} \leq \frac{C_a}{C_a - Cdg(\epsilon)}\\ &\leq \frac{1}{1-\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}} \to 1 \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

If $\beta_j(Z^N(\ell\epsilon)) \geq \lambda_1$, we have

$$\begin{aligned}\frac{\beta_\ell^{j,-}}{\beta_\ell^{a,j-}} &\leq \frac{\beta_j(Z^N(\ell\epsilon))}{\beta_j(Z^N(\ell\epsilon)) - C\bar{C}a - Cdg(\epsilon)} \leq \frac{\lambda_1}{\lambda_1 - C\bar{C}h(\epsilon) - Cdg(\epsilon)}\\ &\to 1 \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

And for $q = +$ We have

$$\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j-}} \leq \frac{\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon)}{\beta_j(Z^{N,a}(\ell\epsilon)) - Cdg(\epsilon)}$$

If $\beta_j(Z^N(p\epsilon)) < \lambda_1$ we have using the Assumptions A2 and A3

$$\begin{aligned}\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j-}} &\leq \frac{\beta_j(Z^{N,a}(\ell\epsilon)) + Cdg(\epsilon)}{\beta_j(Z^{N,a}(\ell\epsilon)) - Cdg(\epsilon)}\\ &\leq \frac{C_a + Cdg(\epsilon)}{C_a - Cdg(\epsilon)} \leq \frac{1+\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}}{1-\frac{Cdg(\epsilon)}{g^{1/2}(\epsilon)}} \to 1 \quad as \quad \epsilon \to 0.\end{aligned}$$

If $\beta_j(Z^N(\ell\epsilon)) \geq \lambda_1$, we have

$$\begin{aligned}\frac{\beta_\ell^{j,+}}{\beta_\ell^{a,j-}} &\leq \frac{\beta_j(Z^N(\ell\epsilon)) + Cdg(\epsilon)}{\beta_j(Z^N(\ell\epsilon)) - C\bar{C}h(\epsilon) - Cdg(\epsilon)} \\ &\leq \frac{\lambda_1 + Cdg(\epsilon)}{\lambda_1 - C\bar{C}h(\epsilon) - Cdg(\epsilon)} \to 1 \quad \text{as} \quad \epsilon \to 0.\end{aligned}$$

Then there exists $\epsilon_\alpha$ such that $\frac{\beta_\ell^{j,q}}{\beta_\ell^{a,j,q}} < 2^{(1-\alpha)/2} < 2^{(1-\tilde{\alpha})/2}$ for all $\epsilon < \epsilon_\alpha$.
Thus for $\epsilon$ small enough we have

$$\begin{aligned}&\exp\{N\epsilon\beta_\ell^{a,j-}(1-\tilde{\alpha})\}\sum_{m\geq 0}\frac{m^m e^{-m}}{m!}\Big(\frac{\beta_\ell^{j,q}/\beta_\ell^{a,j-}}{2^{(1-\tilde{\alpha})}}\Big)^m \\ &\leq e^{N\epsilon\theta(1-\tilde{\alpha})}\sum_{m\geq 0}\frac{m^m e^{-m}}{m!}\Big(\frac{1}{2^{(1-\alpha)/2}}\Big)^m \\ &= e^{N\epsilon\theta(1-\tilde{\alpha})}K_\alpha.\end{aligned} \tag{12}$$

Since the series above converges. We deduce from (10), (11) and (12) that

$$\begin{aligned}\mathbb{E}^N\Big(\exp\Big\{\alpha N\epsilon f\Big(\frac{(1-a/d^2)Z_j^q}{\epsilon N}, \beta_\ell^{a,j-}\Big)\Big\}|\mathcal{F}_\ell\Big) &\leq K_\alpha\exp\{N\epsilon C_2(1-\alpha+a)\}\exp\{N\epsilon\tilde{C}(a+cdg(\epsilon))\} \\ &\leq K_\alpha\exp\{N\epsilon\tilde{K}(1-\alpha+2h(\epsilon)+2dg(\epsilon))\}.\end{aligned}$$

□

Thus, we have

$$\mathbb{E}^N(\Theta_j^\ell|\mathcal{F}_\ell) \leq \mathbb{E}^N(\Xi_j^\ell|\mathcal{F}_\ell) \leq 2K_\alpha\exp\{N\epsilon\tilde{K}_1(1-\alpha+2h(\epsilon)+4dg(\epsilon))\}.$$

The next lemma gives us a upper bound for the quantity $\mathbb{E}^N\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big)$.

**Lemma 19.** *We have the following inequality*

$$\mathbb{E}^N\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) \leq (2K_\alpha)^{\frac{kT}{\epsilon}}\exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\} \tag{13}$$

Proof We know that $\Xi_j^\ell$, $j = 1, ..., k$ are independent given $\mathcal{F}_\ell$. Taking iterative conditional expectations with respect to $\mathcal{F}_{T/\epsilon-1}$,$\mathcal{F}_{T/\epsilon-2}$,...,$\mathcal{F}_1$, we get that for all $0 < \alpha < 1$ and $\epsilon < \epsilon_\alpha$

$$\begin{aligned}
\mathbb{E}^N\Big(\exp\{\alpha N I_T(\Upsilon|\mu)\}\mathbf{1}_{B_\epsilon}\Big) &= \mathbb{E}^N\Big(\prod_{\ell=0}^{T/\epsilon-1}\exp\Big\{\alpha N\int_{\ell\epsilon}^{(\ell+1)\epsilon}\sum_j f(\mu_t^j,\beta_j(\Upsilon_t))dt\Big\}\mathbf{1}_{B_\epsilon^\ell}\Big) \\
&= \mathbb{E}^N\Big(\mathbb{E}^N\Big(\prod_{\ell=0}^{T/\epsilon-1}\prod_{j=1}^{k}\Theta_j^\ell|\mathcal{F}_{T/\epsilon-1}\Big)\Big) \le \mathbb{E}^N\Big(\mathbb{E}^N\Big(\prod_{\ell=0}^{T/\epsilon-1}\prod_{j=1}^{k}\Xi_j^\ell|\mathcal{F}_{T/\epsilon-1}\Big)\Big) \\
&\le \mathbb{E}^N\Big(\prod_{\ell=0}^{T/\epsilon-2}\prod_{j=1}^{k}\Xi_j^\ell\mathbb{E}^N\Big(\prod_{j=1}^{k}\Xi_j^{T/\epsilon-1}|\mathcal{F}_{T/\epsilon-1}\Big)\Big) \\
&\le \prod_{p=0}^{T/\epsilon-1}(2K_\alpha)^k\exp\{kN\epsilon\tilde{\tilde{C}}(1-\alpha+h(\epsilon)+4dg(\epsilon))\} \\
&= (2K_\alpha)^{\frac{kT}{\epsilon}}\exp\{kNT\tilde{K}_1(1-\alpha+h(\epsilon)+4dg(\epsilon))\}
\end{aligned}$$

□

In the next Lemma, we give an upper bound of $\mathbb{P}(B_\epsilon^c)$.

**Lemma 20.** *There exists $\epsilon_0 > 0$, $N_0 \in \mathbb{N}$ and $K > 0$ such that*

$$\mathbb{P}(B_\epsilon^c) < \frac{dkT}{\epsilon}\exp\{-sN\} \tag{14}$$

*for all $\epsilon < \epsilon_0$ and $N > N_0$ where $g(\epsilon) = K\sqrt{\log^{-1}(\epsilon^{-1})}$.*

Proof For all $j = 1, ..., k$ and $\ell = 1, ..., T/\epsilon$ we can write

$$\int_0^{(\ell+1)\epsilon}\beta_j(Z_s^N)ds < \int_0^{\ell\epsilon}\beta_j(Z_s^N)ds + \theta\epsilon.$$

Moreover, we have

$$B_\epsilon^c = \bigcup_{i=1,\dots,d}\bigcup_{\ell=1,\dots,T/\epsilon}\Big\{\sup_{(\ell-1)\epsilon\le t_1,t_2\le\ell\epsilon}|Z_i^N(t_1)-Z_i^N(t_2)| > g(\epsilon)\Big\}.$$

Thus

$$\mathbb{P}(B_\epsilon^c) \le \sum_{i=1}^{d}\sum_{\ell=1}^{T/\epsilon}\mathbb{P}\Big\{\sup_{(\ell-1)\epsilon\le t_1,t_2\le\ell\epsilon}|Z_i^N(t_1)-Z_i^N(t_2)| > g(\epsilon)\Big\}.$$

Using (1) and noting $Z_i^N(.)$ the $i^{th}$ coordinate of $Z^N(.)$ we have

$$\sup_{(\ell-1)\epsilon\le t_1,t_2\le \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)|$$
$$= \sup_{(\ell-1)\epsilon\le t_1,t_2\le \ell\epsilon} \Big| \sum_j \frac{h_j^i}{N}\Big[P_j\Big(N\int_0^{t_1} \beta_j(Z^N(s))ds\Big) - P_j\Big(N\int_0^{t_2}\beta_j(Z^N(s))ds\Big)\Big]\Big|$$
$$\le \frac{1}{N}\sum_j \Big|P_j\Big(N\int_0^{\ell\epsilon}\beta_j(Z^N(s))ds\Big) - P_j\Big(N\int_0^{(\ell-1)\epsilon}\beta_j(Z^N(s))ds\Big)\Big|$$
$$\le \frac{1}{N}\sum_j \Big|P_j\Big(N\int_0^{(\ell-1)\epsilon}\beta_j(Z^N(s))ds + N\theta\epsilon\Big) - P_j\Big(N\int_0^{(\ell-1)\epsilon}\beta_j(Z^N(s))ds\Big)\Big|$$
$$\le \frac{1}{N}\sum_j Z_j.$$

Where $Z_j$ $j = 1, ..., k$ be independent Poisson random variables with means $N\theta\epsilon$. Then

$$\mathbb{P}\Big\{\sup_{(\ell-1)\epsilon\le t_1,t_2\le \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon)\Big\} \le k\mathbb{P}^N(N^{-1}Z_1 > g(\epsilon)/k)$$

And it follows from lemma 10 that there exist a constants $K > 0$, $\epsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that

$$\mathbb{P}\Big\{\sup_{(\ell-1)\epsilon\le t_1,t_2\le \ell\epsilon} |Z_i^N(t_1) - Z_i^N(t_2)| > g(\epsilon)\Big\} \le k\exp\{-sN\}$$

For all $\epsilon < \epsilon_0$ and $N > N_0$. And then

$$\mathbb{P}(B_\epsilon^c) < \frac{dkT}{\epsilon}\exp\{-sN\}.$$

□

Now, we find a bound for $\mathbb{P}(\|Z^{N,a} - \ell\|_T \ge \delta/d)$ in (3).

**Lemma 21.** *For all $\delta > 0$ there exist $\epsilon_\alpha > 0$, $N_0 \in \mathbb{N}$ such that*

$$\mathbb{P}(\|Z^{N,a} - \Upsilon\|_T > \delta) < \frac{dkT}{\epsilon}\exp\{-sN\}, \tag{15}$$

*for all $\epsilon < \epsilon_\alpha$ and $N > N_0$.*

PROOF Using (1) we write for all $t \in [\ell\epsilon, (\ell+1)\epsilon[$

$$
\begin{aligned}
|Z_i^{N,a}(t) - \Upsilon_t^i| &\leq \sum_j \frac{1}{N} \Big| P_j\Big(N \int_0^{(\ell+1)\epsilon} \beta_j(Z^N(s))ds\Big) - P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds\Big) \Big| \\
&\leq \frac{1}{N} \sum_j \Big| P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds + N\theta\epsilon\Big) - P_j\Big(N \int_0^{\ell\epsilon} \beta_j(Z^N(s))ds\Big) \Big| \\
&\leq \frac{1}{N} \sum_j Z_j
\end{aligned}
$$

where the $Z_j$ are as in the proof of the last lemma. Let $\epsilon_1$ be the maximal $\epsilon$ such that $\delta/kd > g(\epsilon)$. Then we have from lemma 10 that for all $\epsilon < \epsilon_\alpha = \min\{\epsilon_0, \epsilon_1\}$ and $N > N_0$

$$
\begin{aligned}
\mathbb{P}(\|Z^{N,a} - \Upsilon\|_T > \delta) &\leq \mathbb{P}^N\Big( \bigcup_{i=1}^d \{|Z_i^{N,a}(t) - \Upsilon_t^i| > \frac{\delta}{d}\} \quad \text{for some} \quad t \in [0,T] \Big) \\
&\leq \frac{T}{\epsilon} \max_{0 \leq \ell \leq T/\epsilon - 1} \mathbb{P}\Big( \bigcup_{i=1}^d \{|Z_i^{N,a}(t) - \Upsilon_t^i| > \frac{\delta}{d}\} \quad \text{for some} \quad t \in [\ell\epsilon, (\ell+1)\epsilon[ \Big) \\
&\leq \frac{dkT}{\epsilon} \mathbb{P}(Z_1/N > \delta/kd) \leq \frac{dkT}{\epsilon} \exp\{-sN\}.
\end{aligned}
$$

□

The end of the proof of the lemma 16 can be done by using (13), (14), (15). We have thus for all $\delta > 0$, $0 < \alpha < 1$, $\epsilon < \min\{\epsilon_0, \epsilon_{\frac{\delta}{2d}}, \epsilon_1\}$ and $a = h(\epsilon) = \Big[ -\log g^{1/2}(\epsilon) \Big]^{-\frac{1}{\nu}}$,

$$
\begin{aligned}
\mathbb{P}(\rho_T(Z^N, \Phi(s)) \geq \delta) &\leq \mathbb{P}(I_T(\Upsilon|\mu) \geq s) + \mathbb{P}(\|Z^{N,a} - \Upsilon\|_T \geq \delta/d) \\
&\leq \frac{\mathbb{E}(\exp\{\alpha N I_T(\Upsilon|\mu)\} \mathbf{1}_{B_\epsilon})}{\exp\{\alpha N s\}} + \mathbb{P}(B_\epsilon^c) + \mathbb{P}(\|Z^{N,a} - \Upsilon\|_T \geq \delta/2d) \\
&\leq (2K_\alpha)^{\frac{kT}{\epsilon}} \exp\{kNT\tilde{K}_1(1 - \alpha + h(\epsilon) + 4dg(\epsilon))\} \\
&\times \exp\{-\alpha N s\} + \frac{2dTk}{\epsilon} \exp\{-sN\}.
\end{aligned}
$$

Here, we take $1-\alpha$ and $\epsilon$ small enough to ensure that $kT\tilde{K}_1(1 - \alpha + h(\epsilon) + 4dg(\epsilon)) < \eta/4$ and $(1-\alpha)s < \eta/4$.We also take $N$ large enogh so that

$kT\log(2K_\alpha)/N\epsilon < \eta/4$ and $\log(2dkT/\epsilon)/N < \eta/4$ and we have

$$\begin{aligned}\mathbb{P}(\rho_T(Z^N,\Phi(s)) \geq \delta) &\leq \exp\{-N(s-3\eta/4)\} + 2dT/\epsilon.\exp\{-sN\} \\ &\leq dkT/\epsilon.\exp\{-N(s-3\eta/4)\} \leq \exp\{-N(s-\eta)\}.\end{aligned}$$

Thus

$$\mathbb{P}^N(H_\delta(s)) \leq \exp\{-N(s-\eta)\}.$$

□

We finally have

Theorem **22.** *For all closed set $F \in D_{T,A}$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}^N(F) \leq -\inf_{\phi\in F} I_T(\phi).$$

PROOF All we have to show is that (2) implies the Theorem. To this end let $F \in D_{T,A}$ a closed set, choose $\eta > 0$ and put $s = \inf\{I_T(\phi) : \phi \in F\} - \eta/2$. The closed set $F$ does not intersect the compact set $\Phi(s)$. Therefore $\delta = \inf_{\phi\in F}\inf_{\psi\in\Phi(s)} \|\phi - \psi\|_T > 0$. We use the inequality (2) to have for any $\delta, \eta, s > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\begin{aligned}\mathbb{P}^N(F) &\leq \mathbb{P}^N(H_\delta(s)) \\ &\leq \exp\{-N(s-\eta/2)\} \\ &\leq \exp\{-N(\inf_{\phi\in F} I_T(\phi) - \eta)\}\end{aligned}$$

then

$$\limsup_{N\to\infty} \frac{1}{N}\mathbb{P}^N(F) \leq \inf_{\phi\in F} I_T(\phi).$$

□

We need a slightly stronger version

Theorem **23.** *For all closed set $F \in D_{T,A}$ such that all trajectories in $F$ remain in a compact set which does not intersect the boundary $\partial A$, for any compact set $K \subset \mathring{A}$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{x\in K} \mathbb{P}(Z^{N,x} \in F) \leq -\inf_{x\in K} \inf_{\phi\in F, \phi_0 = x} I_T(\phi).$$

# 6 Time of exit from a domain

We let $O \subsetneq A$ be relatively open in $A$ (with $O = \tilde{O} \cap A$ for $\tilde{O} \subset \mathbf{R}^d$ open) and $x^* \in O$ be a stable equilibrium of (1). By a slight abuse of notation, we say that

$$\widetilde{\partial O} := \partial \tilde{O} \cap A$$

is the *boundary* of $O$. For $y, z \in A$, we define the following functionals.

$$\begin{aligned} V(x,z,T) &:= \inf_{\phi \in D([0,T];A), \phi(0)=x, \phi(T)=z} I_{T,x}(\phi) \\ V(x,z) &:= \inf_{T>0} V(x,z) \\ \bar{V} &:= \inf_{z \in \widetilde{\partial O}} V(x^*, z). \end{aligned}$$

In other words, $\bar{V}$ is the minimal energy required to leave the domain $O$ when starting from $x^*$. We urge the reader to consider the two examples in section 6.4.

**Assumptions B**

B1 $x^*$ is the only stable equilibrium point of (1) in $O$ and the solution $Y^x$ of (1) with $x = Y^x(0) \in O$ satisfies

$$Y^x(t) \in O \text{ for all } t > 0 \text{ and } \lim_{t \to \infty} Y^x(t) = x^*.$$

B2 For a solution $Y^x$ of (1) with $x = Y^x(0) \in \widetilde{\partial O}$, we have

$$\lim_{t \to \infty} Y^x(t) = x^*.$$

B3 $\bar{V} < \infty$.

B4 For all $\rho > 0$ there exist constants $T(\rho)$, $\epsilon(\rho) > 0$ with $T(\rho), \epsilon(\rho) \downarrow 0$ as $\rho \downarrow 0$ such that for all $z \in \widetilde{\partial O} \cup \{x^*\}$ and all $x, y \in \overline{B(z,\rho)} \cap A$ there exists an

$$\phi = \phi(\rho, x, y) : [0, T(\rho)] \to A \quad \text{ with } \phi(0) = x, \phi(T(\rho)) = y \text{ and } I_{T(\rho)}(\phi) < \epsilon(\rho).$$

B5 For all $z \in \widetilde{\partial O}$ there exists an $\eta_0 > 0$ such that for all $\eta < \eta_0$ there exists a $\tilde{z} = \tilde{z}(\eta) \in A \setminus \bar{O}$ with $|z - \tilde{z}| > \eta$.

Let us shortly comment on Assumption B By B1, $O$ is a subset of the domain of attraction of $x^*$. B2 is violated by the applications we have in mind: we are interested in situations where $\widetilde{\partial O}$ is the *characteristic boundary* of $O$, i.e., the boundary separating two regions of attraction of equilibria of (1). In order to relax this assumption, we require an approximation argument later. By B3, it is possible to reach the boundary with finite energy. This assumption is always satisfied for the epidemiological models we consider. For $z = x^*$, B4 is also always satisfied in our models as the rates $\beta_j$ are bounded from above and away from zero in small neighborhoods of $x^*$; hence, the function $\phi(x, y, \rho)$ can, e.g., be chosen to be linear with speed one.

We are interested in the following quantity:

$$\tau^{N,x} := \tau^N := \inf\{t > 0 | Z^{N,x}(t) \notin O\},$$

i.e., the first time that $Z^{N,x}$ exits $O$.

## 6.1 Auxiliary results

Assumptions A4 + B4 yield.

**Lemma 24.** *Assume that Assumptions A and B hold. Then for any $\delta > 0$, there exists an $\rho_0 > 0$ such that for all $\rho < \rho_0$,*

$$\sup_{z \in \widetilde{\partial O} \cup x^*, x, y \in \overline{B(z,\rho)}} \inf_{T \in [0,1]} V(x, y, T) < \delta.$$

We have moreover.

**Lemma 25.** *Assume that Assumptions A and B hold. Then, for any $\eta > 0$ there exists a $\rho_0$ such that for all $\rho < \rho_0$ there exists a $T_0 < \infty$ such that*

$$\liminf_{N \to \infty} \frac{1}{N} \log \inf_{x \in \overline{B(x^*,\rho)}} \mathbb{P}[\tau^{N,x} \leq T_0] > -(\bar{V} + \eta).$$

Proof Let $x \in \overline{B(x^*, \rho)}$. We use Lemma 24 with $\delta = \eta/4$ (and we let $\rho$ be small enough for Lemma 24 to hold). We construct a continuous path $\psi^x$ with $\psi^x(0) = x$, $\psi^x(t_x) = x^*$ ($t_x \leq 1$) and $I_{t_x,x}(\psi^x) \leq \eta/4$. We then use Assumption B3. For $T_1 < \infty$, we can construct a path $\phi \in C[0, T_1]$ such that $\phi(0) = x^*$, $\phi(T_1) = z \in \widetilde{\partial O}$ and $I_{T_1,0}(\phi) \leq \bar{V} + \eta/4$. Subsequently, we use Lemma 24 and obtain a path $\tilde{\psi}$ with $\tilde{\psi}(0) = z$, $\tilde{\psi}(s_x) \notin O$ ($s \leq 1$),

$I_{s,z}(\tilde{\psi}) \le \eta/4$ and $d(\bar{z}, O) =: \Delta > 0$.[1] We finally let $\theta^x$ be the solution of the ODE (1) with $\theta^x(0) = \bar{z}$ on $[0, 2 - t_x - s]$, consequently $I_{2-t_x-s,\bar{z}}(\theta^x) = 0$.

We concatenate the paths $\psi^x$, $\phi$, $\tilde{\psi}$ and $\theta^x$ and obtain the path $\phi^x \in C[0, T_0]$ ($T_0 = T_1 + 2$ independent of $x$) with $I_{T_0,x}(\phi^x) \le \bar{V} + \eta/2$.

Finally, we define

$$\Psi := \bigcup_{x \in \overline{B(x^*, \rho)}} \big\{\psi \in D([0, T_0]; A) | \|\psi - \phi^x\| < \Delta/2\big\};$$

hence $\Psi \subset D([0, T_0]; A)$ is open, $(\phi^x)_{x \in \overline{B(x^*,\rho)}} \subset \Psi$ and $\{Z^{N,x} \in \Psi\} \subset \{\tau^{N,x} \le T_0\}$. We now use Theorem 15.

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N} \log \inf_{x \in \overline{B(x^*,\rho)}} \mathbb{P}[Z^{N,x} \in \Psi] &\ge - \sup_{x \in \overline{B(x^*,\rho)}} \inf_{\phi \in \Psi} I_{T_0,x}(\phi) \\
&\ge - \sup_{x \in \overline{B(x^*,\rho)}} I_{T_0,x}(\phi^x) \\
&> -(\bar{V} + \eta).
\end{aligned}$$

□

We also require the following result

**Lemma 26.** *Assume that Assumptions A and B hold. Let $\rho > 0$ such that $\overline{B(x^*, \rho)} \subset O$ and*

$$\sigma_\rho^{N,x} := \inf\{t > 0 | Z_t^{N,x} \in \overline{B(x^*, \rho)} \text{ or } Z_t^{N,x} \notin O\}.$$

*Then*

$$\lim_{t\to\infty} \limsup_{N\to\infty} \frac{1}{N} \log \sup_{x \in O} \mathbb{P}[\sigma_\rho^{N,x} > t] = -\infty.$$

Proof Note first that for $x \in \overline{B(x^*, \rho)}$, $\sigma_\rho^{N,x} = 0$; we hence assume from now on that $x \notin \overline{B(x^*, \rho)}$. For $t > 0$, we define the closed set $\Psi_t \subset D([0, t]; A)$,

$$\Psi_t := \{\phi \in D([0, t]; A) | \phi(s) \in \overline{O \setminus B(x^*, \rho)} \text{ for all } s \in [0, t]\};$$

hence for all $x, N$,

$$\{\sigma_\rho^{N,x} > t\} \subset \{Z^{N,x} \in \Psi_t\}.$$

[1] The additional assumption B5 is required here.

By Theorem 23, this implies for all $t > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{x\in\overline{O\setminus B(x^*,\rho)}} \mathbb{P}[\sigma_\rho^{N,x} > t] \leq \limsup_{N\to\infty} \frac{1}{N} \log \sup_{x\in\overline{O\setminus B(x^*,\rho)}} \mathbb{P}[Z^{\epsilon,x} \in \Psi_t]$$
$$\leq - \inf_{\phi\in\Psi_t} I_{t,\phi(0)}(\phi).$$

It hence suffices to show that

$$\lim_{t\to\infty} \inf_{\phi\in\Psi_t} I_{t,\phi(0)}(\phi) = \infty. \tag{1}$$

To this end, consider $x \in \overline{O \setminus B(x^*,\rho)}$ and recall that $Y^x$ is the solution of (1) (on $[0,t]$ for all $t > 0$). By Assumption B2, there exists a $T_x < \infty$ such that $Y^x(T_x) \in \overline{B(x^*, 3\rho)}$. We have (here $B$ denotes the Lipschitz constant of $b$),

$$|\phi^x(t)-\phi^y(t)| \leq |x-y|+\int_0^t |b(\phi^x(s))-b(\phi^y(s))|ds \leq +|x-y|+\int_0^t B|\phi^x(s)-\phi^y(s)|ds$$

and therefore by Gronwall's inequality $|Y^x(T_x) - Y^y(T_x)| \leq |x-y|e^{T_xB}$; consequently, there exists a neighborhood $W_x$ of $x$ such that for all $y \in W_x$, $Y^y(T_x) \in \overline{B(x^*, 3\rho)}$. By the compactness of $\overline{O \setminus B(x^*,\rho)}$, there exists a finite open subcover $\cup_{i=1}^k W_{x_i} \supset \overline{O \setminus B(x^*,\rho)}$; for $T := \max_{i=1,\dots,k} T_{x_i}$ and $y \in \overline{O \setminus B(x^*,\rho)}$ this implies that $Y^y(s) \in \overline{B(x^*, 2/3\rho)}$ for some $s \leq T$.

Assume now that (1) is false. Then there exits an $M < \infty$ such that for all $n \in \mathbf{N}$ there exists an $\phi_n \in \Psi_{nT}$ with $I_{nT}(\phi_n) \leq M$. The function $\phi_n$ is concatenated by functions $\phi_{n,k} \in \Psi_T$ and we obtain

$$M \geq I_{nT}(\phi_n) = \sum_{k=1}^n I_T(\phi_{n,k}) \geq n \min_{k=1,\dots,n} I_T(\phi_{n,k}).$$

Hence there exists a sequence $(\psi_k)_k \subset \Psi_T$ with $\lim_{k\to\infty} I_T(\psi_k) = 0$. Note now that the set

$$\phi(t) := \{\phi \in C[0,T] | I_{T,\phi(0)}(\phi) \leq 1, \phi(s) \in \overline{O \setminus B(x^*,\rho)} \text{ for all } s \in [0,T]\} \subset \Psi_T$$

is compact (as a subset of $(C[0,T], \|\cdot\|_\infty)$); hence there exists a subsequence $(\psi_{k_l})_l$ of $(\psi_k)_k$ such that $\lim_{l\to\infty} \psi_{k_l} =: \psi^* \in \phi(t)$ in $(C[0,T], \|\cdot\|_\infty)$. By the lower semi-continuity of $I_T$, this implies

$$0 = \liminf_{l\to\infty} I_T(\psi_{n_l}) \geq I_T(\psi^*),$$

which in turn implies that $\psi^*$ solves (1) for $x = \psi^*(0)$. But then, $\psi^*(s) \in \overline{B(x^*, 2/3\rho)}$ for some $s \le T$, a contradiction to $\psi^* \in \Psi_T$. $\square$

**Lemma 27.** *Assume that Assumptions A and B hold. Let $C \subset A \setminus O$ be closed. Then,*

$$\lim_{\rho\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \sup_{x\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,x}_{\sigma_\rho} \in C] \le -\inf_{z\in C} V(x^*, z).$$

Proof We can assume without loss of generality that $\inf_{z\in C} V(x^*, z) > 0$ (else the assertion is trivial). For $\inf_{z\in C} V(x^*, z) > \delta > 0$, we define

$$V^\delta_C := (\inf_{z\in C} V(x^*, z) - \delta) \wedge 1/\delta > 0.$$

By Lemma 24, there exists a $\rho_0 = \rho_0(\delta) > 0$ such that for all $0 < \rho < \rho_0$,

$$\sup_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} V(x^*, y) < \delta;$$

hence

(2)
$$\inf_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)},\ z\in C} V(y, z) \ge \inf_{z\in C} V(x^*, z) - \sup_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} V(x^*, y) > V^\delta_C.$$

For $T > 0$, we define the closed set $\Phi^T \subset D([0, T]; A)$ by

$$\Phi^T := \Phi := \{\phi \in D([0, T]; A) | \phi(t) \in C \text{ for some } t \in [0, T]\}.$$

We then have for $y \in \overline{B(x^*, 3\rho) \setminus B(x^*, 2\rho)}$,

$$\mathbb{P}[Z^{N,y}_{\sigma_\rho} \in C] \le \mathbb{P}[\sigma^{N,y}_\rho > T] + \mathbb{P}[Z^{N,y} \in \Phi^T]. \tag{3}$$

In the following, we bound the two parts in Inequality (3) from above.

For the second part, we note first that (cf. Inequality (2))

$$\inf_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)},\ \phi\in\Phi^T} I_{T,y}(\phi) \ge \inf_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)},\ z\in C} V(y, z) > V^\delta_C;$$

hence, we obtain by Theorem 23

(4)
$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y} \in \Phi^T] \le - \inf_{y\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)},\ \phi\in\Phi^T} I_{T,y}(\phi) < -V^\delta_C.$$

For the first part in Inequality (3), we use Lemma 26: There exists a $0 < T_0 < \infty$ such that for all $T \geq T_0$

$$(5) \qquad \limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in \overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[\sigma^{N,y} > T] < -V_C^\delta.$$

We let $T \geq T_0$ and $\rho < \rho_0$ and combine Inequalities (3), (4) and (5). Hence there exists an $N_0 > 0$ such that for all $N > N_0$,

$$\begin{aligned}
&\frac{1}{N} \log \sup_{y\in \overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y}_{\sigma_\rho} \in C] \\
&\quad \leq \frac{1}{N} \log \Big( \sup_{y\in \overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[\sigma^{N,y}_\rho > T] + \sup_{y\in \overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y} \in \Phi^T] \Big) \\
&\quad < \frac{1}{N} \log \left(2e^{-NV_C^\delta}\right) = \frac{1}{N}\log 2 - V_C^\delta;
\end{aligned}$$

and

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in \overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,x}_{\sigma_\rho} \in C] \leq -V_C^\delta.$$

Taking the limit $\delta \to 0$ finishes the proof. □

We have moreover

**Lemma 28.** *Assume that Assumptions A and B hold. Then, for all $\rho > 0$ such that $\overline{B(x^*,\rho)} \subset O$ and for all $x \in O$,*

$$\lim_{N\to\infty} \mathbb{P}[Z^{N,x}_{\sigma_\rho} \in \overline{B(x^*,\rho)}] = 1.$$

Proof Let $x \in O \setminus \overline{B(x^*,\rho)}$ (the case $x \in \overline{B(x^*,\rho)}$ is clear). Let furthermore $T := \inf\{t \geq 0 | \phi(t) \in B(x^*,\rho/2)\}$. Since $Y^x$ is continuous and never reaches $\widetilde{\partial O}$ (Assumption B1), we have $\inf_{t\geq 0} d(Y^x(t), \widetilde{\partial O}) =: \Delta > 0$. Hence we have the following implication:

$$\sup_{t\in[0,T]} |Z^{N,x}_t - Y^x(t)| \leq \frac{\Delta}{2} \Rightarrow Z^{N,x}_{\sigma_\rho} \in \overline{B(x^*,\rho)}.$$

In other words,

$$(6) \qquad \mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin \overline{B(x^*,\rho)}] \leq \mathbb{P}\Big[\sup_{t\in[0,T]} |Z^{N,x}_t - Y^x(t)| > \frac{\Delta}{2}\Big].$$

The right hand side of Inequality (6) converges to zero as $N \to \infty$ by Theorem 2. □

**Lemma 29.** *Assume that Assumptions A and B hold. Then, for all $\rho, c > 0$, there exists a constant $T = T(c,\rho) < \infty$ such that*

$$\limsup_{N\to\infty} \frac{1}{N} \log \sup_{x\in O} \mathbb{P}[\sup_{t\in[0,T]} |Z_t^{N,x} - x| \geq \rho] < -c.$$

Proof Let $\rho, c > 0$ be fixed. For $T, N > 0$ and $x \in O$ we have

$$\begin{aligned}
\mathbb{P}[\sup_{t\in[0,T]} |Z_t^{N,x} - x| \geq \rho] &= \mathbb{P}\Big[\sup_{t\in[0,T]} \frac{1}{N} \Big|\sum_j h_j P_j\Big(N \int_0^t \beta_j(Z_s^{N,x}) ds\Big)\Big| \geq \rho\Big] \\
&\leq \mathbb{P}\Big[\sum_j P_j(N\bar\beta T) \geq N\rho\bar h^{-1}\Big] \\
&\leq k\mathbb{P}\Big[P(N \underbrace{\bar\beta T}_{=:c_1(T)}) \geq N \underbrace{\rho\bar h^{-1}k^{-1}}_{=:c_2}\Big] \qquad (7)
\end{aligned}$$

for a standard Poisson process $P$. We now let

$$T < T_0 := \frac{e^{-1}c_2}{2\bar\beta} \wedge \frac{e^{-c/c_2-1}c_2}{\bar\beta} \quad \text{and} \quad N > N_0 := 1/c_2 \wedge \frac{\log 2k}{c_1(T)}. \qquad (8)$$

We then obtain (note that $Nc_2 > 1$ and $\frac{e}{c_2}c_1(T) < 1/2$ by (8))

$$\begin{aligned}
k\mathbb{P}\Big[P(Nc_1(T)) \geq Nc_2\Big] &= ke^{-Nc_1(T)} \sum_{m\geq Nc_2} \frac{N^m c_1(T)^m}{m!} \\
&< ke^{-Nc_1(T)} \sum_{m\geq Nc_2} \frac{\big(eN\big)^m c_1(T)^m}{m^m\sqrt{2\pi m}} \qquad (9) \\
&\leq \frac{1}{2} \sum_{m\geq Nc_2} \frac{\big(eN\big)^m c_1(T)^m}{\big(Nc_2\big)^m} \\
&\leq \frac{1}{2} \frac{\big(\frac{e}{c_2}c_1(T)\big)^{Nc_2}}{1 - \frac{e}{c_2}c_1(T)} \\
&\leq \big(\frac{e}{c_2}c_1(T)\big)^{Nc_2}; \qquad (10)
\end{aligned}$$

here we applied Stirling's formula, $m! > \sqrt{2\pi m}(m/e)^m$, in Inequality (9). Finally, we have

$$\big(\frac{e}{c_2}c_1(T)\big)^{Nc_2} = \Big(\big(\frac{e}{c_2}c_1(T)\big)^{-c_2}\Big)^{-N} < (e^c)^{-N} = e^{-Nc} \qquad (11)$$

by (8). The assertion now follows by combining the Inequalities (7), (10) and (11). □

## 6.2 Main results

We can now establish

Theorem **30.** *Assume that Assumptions A and B hold. Then, for all* $x \in O$ *and* $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\big[e^{(\bar{V}-\delta)N} < \tau^{N,x} < e^{(\bar{V}+\delta)N}\big] = 1.$$

Proof Upper bound of exit time:

We fix $\delta > 0$ and apply Lemma 25 to $\eta := \delta/4$. Hence, for $\rho < \rho_0$ there exists a $T_0 < \infty$ and an $N_0 > 0$ such that for $N > N_0$,

$$\underline{\inf}_{x\in B(x^*,\rho)} \mathbb{P}[\tau^{N,x} \le T_0] > e^{-N(\bar{V}+\eta)}.$$

Furthermore, by Lemma 26 there exists a $T_1 < \infty$ and $N_1 > 0$ such that for all $N > N_1$,

$$\inf_{x\in O} \mathbb{P}[\sigma_\rho^{N,x} \le T_1] > 1 - e^{-2N\eta}.$$

For $T := T_0 + T_1$ and $N > N_0 \vee N_1 \vee 1/\eta$, we hence obtain

$$\begin{aligned} q^N := q &:= \inf_{x\in O} \mathbb{P}[\tau^{N,x} \le T] \\ &\ge \inf_{x\in O} \mathbb{P}[\sigma_\rho^{N,x} \le T_1] \underline{\inf}_{y\in B(x^*,\rho)} \mathbb{P}[\tau^{N,y} \le T_0] \\ &> (1 - e^{-2N\eta})e^{-N(\bar{V}+\eta)} \\ &\ge e^{-N(\bar{V}+2\eta)}. \end{aligned} \tag{12}$$

This yields for $k \in \mathbf{N}$

$$\begin{aligned} \mathbb{P}[\tau^{N,x} > (k+1)T] &= \big(1 - \mathbb{P}[\tau^{N,x} \le (k+1)T|\tau^{N,x} > kT]\big)\mathbb{P}[\tau^{N,x} > kT] \\ &\le (1-q)\mathbb{P}[\tau^{N,x} > kT] \end{aligned}$$

and hence inductively

$$\sup_{x\in O} \mathbb{P}[\tau^{N,x} > kT] \le (1-q)^k.$$

This implies

(13)

$$\sup_{x\in O}\mathbb{E}[\tau^{N,x}] \le T\big(1+\sum_{k=1}^{\infty}\sup_{x\in O}\mathbb{P}[\tau^{N,x}>kT]\big) \le T\sum_{k=0}^{\infty}(1-q)^k = \frac{T}{q} \overset{(12)}{\le} Te^{N(\bar{V}+2\eta)};$$

by Chebychev's Inequality we obtain

$$\mathbb{P}[\tau^{N,x} \ge e^{N(\bar{V}+\delta)}] \le e^{-N(\bar{V}+\delta)}\mathbb{E}[\tau^{N,x}] \le Te^{-\delta N/2}$$

which approaches zero as $N\to\infty$ as required.

Lower bound of exit time:

For $\rho>0$ such that $\overline{B(x^*,3\rho)}\subset O$, we define recursively $\theta_0 := 0$ and for $m\in\mathbf{N}_0$,

$$\tau_m^x := \tau_m := \inf\{t\ge\theta_m^x | Z_t^{N,x}\in\overline{B(x^*,\rho)} \text{ or } Z_t^{N,x}\notin O\},$$
$$\theta_{m+1}^x := \theta_{m+1} := \inf\{t\ge\tau_m^x | Z_t^{N,x}\in\overline{B(x^*,3\rho)\setminus B(x^*,2\rho)}\},$$

with the convention $\theta_{m+1} := \infty$ if $Z_{\tau_m}^N \notin O$. Note that we have $\tau^{N,x} = \tau_m^x$ for some $m\in\mathbf{N}_0$.

For fixed $T_0>0$ and $k\in\mathbf{N}$ we have the following implication: If for all $m=0,\dots,k$, $\tau_m\ne\tau^N$ and for all $m=1\dots,k$, $\tau_m-\tau_{m-1}>T_0$, then

$$\tau^N > \tau_k = \sum_{m=1}^{k}(\tau_m-\tau_{m-1}) > kT_0.$$

In particular, we have for $k := \lfloor T_0^{-1}e^{N(\bar{V}-\delta)}\rfloor + 1$ (note that $\theta_m-\tau_{m-1}\le\tau_m-\tau_{m-1}$),

$$\begin{aligned}\mathbb{P}[\tau^{N,x}\le e^{N(\bar{V}-\delta)}] &\le \mathbb{P}[\tau^{N,x}\le kT_0]\\ &\le \sum_{m=0}^{k}\mathbb{P}[\tau^{N,x}=\tau_m^x] + \sum_{m=1}^{k}\mathbb{P}[\theta_m^x-\tau_{m-1}^x\le T_0]\\ &= \mathbb{P}[\tau^{N,x}=\tau_0^x] + \sum_{m=1}^{k}\mathbb{P}[\tau^{N,x}=\tau_m^x] + \sum_{m=1}^{k}\mathbb{P}[\theta_m^x-\tau_{m-1}^x\le T_0]. \qquad (14)\end{aligned}$$

In the following, we bound the three parts in (14) from above. To this end, we assume $\bar{V}>0$ for now. The simpler case $\bar{V}=0$ is treated below.

For the first part, we have

$$\mathbb{P}[\tau^{N,x} = \tau_0^x] = \mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin O]. \tag{15}$$

For the second part, we use the fact that $Z^{N,x}$ is a strong Markov process and that the $\tau_m$'s are stopping times. We obtain for $m \geq 1$ and $x \in O$,

$$\mathbb{P}[\tau^{N,x} = \tau_m^x] \leq \sup_{y \in \overline{B(x^*,3\rho) \setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y}_{\sigma_\rho} \notin O]. \tag{16}$$

Similarly, we obtain for the third part for $m \geq 1$ and $x \in O$,

$$\mathbb{P}[\theta_m^x - \tau_{m-1}^x \leq T_0] \leq \sup_{y \in O} \mathbb{P}[\sup_{t \in [0,T_0]} |Z_t^{N,y} - y| \geq \rho]. \tag{17}$$

The upper bounds in (16) and (17) can now be bounded by using the Lemma 27 and 29, respectively. We fix $\delta > 0$. By Lemma 27 (for $C = A \setminus O$), there exists a $\rho = \rho(\delta) > 0$ and an $N_1 = N_1(\rho, \delta) > 0$ such that for all $N > N_1$,

$$\sup_{y \in \overline{B(x^*,3\rho) \setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y}_{\sigma_\rho} \notin O] \leq \exp\big(-N(\bar{V} - \delta/2)\big). \tag{18}$$

By Lemma 29 (for $\rho = \rho(\delta)$ as above and $c = \bar{V}$), there exists a constant $T_0 = T(\rho, \bar{V}) < \infty$ and an $N_2 = N_2(\rho, \delta) > 0$ such that for all $N > N_2$,

$$\sup_{y \in O} \mathbb{P}[\sup_{t \in [0,T_0]} |Z_t^{N,y} - y| \geq \rho] \leq \exp\big(-N(\bar{V} - \delta/2)\big). \tag{19}$$

We now let $N > N_1 \vee N_2$ (and large enough for $T_0^{-1} \exp\big(N(\bar{V} - \delta)\big) > 1$ for the specific $T_0$ above). Then by Inequality (14),

$$\begin{aligned}
\mathbb{P}[\tau^{N,x} \leq e^{N(\bar{V}-\delta)}] &\overset{(15),(16),(17)}{\leq} \mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin O] + k \sup_{y \in \overline{B(x^*,3\rho) \setminus B(x^*,2\rho)}} \mathbb{P}[Z^{N,y}_{\sigma_\rho} \notin O] \\
&\qquad + k \sup_{y \in O} \mathbb{P}[\sup_{t \in [0,T_0]} |Z_t^{N,y} - y| \geq \rho] \\
&\overset{(18),(19)}{\leq} \mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin O] + 4T_0^{-1} \exp\big(-N\delta/2\big). \qquad (20)
\end{aligned}$$

The right-hand side of Inequality (20) tends to zero as $\epsilon \to 0$ by Lemma 28, finishing the proof for $\bar{V} > 0$.

Finally, let us assume that $\bar{V} = 0$ and that the assertion is false for a given $x \in O$. Then there exists a $\mu_0 \in (0, 1/2)$ and a $\delta_0 > 0$ such that for all $\bar{N} > 0$ there exists an $N > \bar{N}$ with

$$\mu_0 \leq \mathbb{P}[\tau^{N,x} \leq e^{-N\delta_0}].$$

We fix $\rho > 0$ such that $\overline{B(x^*, 2\rho)} \subset O$. Using the strong Markov property of $Z$ and the fact that $\sigma_\rho$ is a stopping time again, we have that for all $\bar{N} > 0$ there exists an $N > \bar{N}$ with

$$\begin{aligned} \mu_0 &\leq \mathbb{P}[\tau^{N,x} \leq e^{-N\delta_0}] \\ &\leq \mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin \overline{B(x^*, \rho)}] + \sup_{y \in O} \mathbb{P}[\sup_{t \in [0, e^{-N\delta_0}]} |Z^{N,y}_t - y| \geq \rho]. \end{aligned} \tag{21}$$

By Lemma 28, there exists an $N_0$ such that for all $N > N_0$,

$$\mathbb{P}[Z^{N,x}_{\sigma_\rho} \notin \overline{B(x^*, \rho)}] \leq \frac{\mu_0}{2}. \tag{22}$$

We now set $c := -2\epsilon_0 \log \frac{\mu_0}{2}$. Then by Lemma 29, there exists a $T = T(c, \rho) > 0$ and an $N_1 > N_0$ such that for all $N > N_1$,

$$e^{-N\delta_0} < T \tag{23}$$

and

$$\sup_{y \in O} \mathbb{P}[\sup_{t \in [0,T]} |Z^{N,y}_t - y| \geq \rho] \leq e^{-Nc/2} < \frac{\mu_0}{2}. \tag{24}$$

Combining Inequalities (22), (23) and (24) yields a contradiction to Inequality (21), finishing the proof. □

### 6.3 The case of a characteristic boundary

Since we are mainly interested in studying the time of exit form the basin of attraction of one local equilibrium to that of another, we need to consider situations which do not satisfy the above assumptions. More precisely, we want to suppress the assumptions B3 and B5, and keep assumptions B1, B2 and B4. In the examples which we have in mind, there exists a collection of open sets $\{O_\rho,\ \rho > 0\}$ which is such that

- $\overline{O}_\rho \subset O$ for any $\rho > 0$.

- $d(O_\rho, \tilde{\partial O}) \to 0$, as $\rho \to 0$.
- $O_\rho$ satisfies assumptions B1,..B5 for any $\rho > 0$.

Let then $O$ be a domain satisfying assumptions (B1, B2 and B4, and we assume that there exist a sequence $\{O_\rho,\ \rho > 0\}$ satisfying the three above conditions.

If we define $\bar{V}_\rho$ as $\bar{V}$, but with $O$ replaced by $O_\rho$, it follows from Lemma 24 that $\bar{V}_\rho \to \bar{V}$ as $\rho \to 0$. By an obvious monotonicity property, the lower bound

$$\lim_{\epsilon \to 0} \mathbb{P}\big[\tau^{N,x} > e^{(\bar{V}-\delta)N}\big] = 1$$

follows immediately from Theorem 30.

## 6.4 Applications

Consider the following two epidemiological models with several equilibria, both

1. the $SIV$ model studied by Kribs–Zaleta and Velasco–Hernández :

$$\begin{aligned}
\frac{ds}{dt}(t) &= \mu(1 - s(t)) + \alpha i(t) - \beta s(t) i(t) - \eta s(t) + \theta v(t),\ t > 0,\\
\frac{di}{dt}(t) &= -\mu i(t) + \beta s(t) i(t) - \alpha i(t) + r\beta v(t) i(t),\ t > 0,\\
\frac{dv}{dt}(t) &= -\mu v(t) + \eta s(t) - \theta v(t) - r\beta v(t) i(t),\ t > 0;
\end{aligned}$$

2. and the $S_0 I S_1$ model of Safan, Heesterbeek and Dietz

$$\begin{aligned}
\frac{ds_0}{dt}(t) &= \mu(1 - s_0(t)) - \beta s_0(t) i(t),\ t > 0,\\
\frac{di}{dt}(t) &= -\mu i(t) + \beta s_0(t) i(t) - \alpha i(t) + r\beta s_1(t) i(t),\ t > 0,\\
\frac{ds_1}{dt}(t) &= -\mu s_1(t) + \alpha i(t) - r\beta s_1(t) i(t),\ t > 0.
\end{aligned}$$

In those two above models, one can choose the parameters in such a way that both the DFE and one of the endemic equilibria are locally stable. Denote by $\mathcal{O}$ the basing of attraction of the endemic equilibrium. Let us denote by $\tau^{N,x}$ the time it takes for the stochastic system, starting from $x \in \mathcal{O}$, to exit $\mathcal{O}$ ($\simeq$ the time to reach the DFE). Theorem 30 extended to the case of a characteristic boundary implies that For any $x \in \mathcal{O}$, $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}(e^{(\overline{V}-\delta)N} < \tau^{N,x} < e^{(\overline{V}+\delta)N}) = 1.$$

**Numerical computation of $\overline{V}$**

1. In the SIV model with $\beta = 3.6$, $\alpha = 1$, $\theta = 0.02$, $\mu = 0.03$, $\eta = 0.3$ and $r = 0.1$, we get $\overline{V} = 0.39$.

   This gives rather astronomical values of $\tau^N$, even for $N = 100$ !

2. In the $S_0IS_1$ model with $\beta = 3$, $\alpha = 5$, $\mu = 0.015$ and $r = 2$, we get $\overline{V} = 0.0745$.

   This means that for $N = 100$, $\tau^N \simeq 1720$, and for larger $N$, the value of $\tau^N$ is huge !

3. We have not yet checked how $\overline{V}$ depends upon the parameters !

It would be interesting to understand how those results would be modified if we incorporate heterogeneity (nonhomogeneous mixing, spatial dispersion, ...).

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# References

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